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FS-closure operators and FS-interior operators

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Abstract. Notions of FS-closure operators, FS-interior operators and their components are introduced. Various properties of FS-closure systems and FS-interior systems are studied and established a relation between them. A set of necessary and sufficient conditions under which an FS-closure operator and an FS-interior operator induce same fuzzy sequential topology on the underlying set have been obtained.

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1. Introduction

After the introduction of fuzzy sets by L. A. Zadeh in 1965 ([17]), C. L. Chang introduced the concept of fuzzy topology on a non empty set in 1968 ([6]). The concept of fuzzy sequential topological spaces (FSTS) were introduced in ([13]). In fuzzy set theory, fuzzy closure operators and fuzzy closure systems have been studied by Mashour and Ghanim ([10]), G. Gerla ([8]), Bandler and Kohout ([1]), R. Belohlavek ([2]), whereas fuzzy interior operators and fuzzy interior systems have appeared in the studies of R. Belohlavek and T. Funiokova ([3]), Bandler and Kohout ([1]).

Closure and interior operators on an ordinary set belong to the very fundamental mathematical structures with direct applications on the many fields like topology, logic etc. Being motivated by the importance of closure and interior operators, we introduce the concept of FS-closure and FS-interior operators on a set. Books ([5], [7] [9], [11]) and the articles ([4], [12], [14], [15], [16]) may provide a suitable background for the present work as some basic ideas have been derived from these sources. We begin with some basic definitions and results of ([13]) and ([16]). Let X be a non empty set and $I = [0, 1]$ be the closed unit interval in the set of real

numbers. Let $A_f(s) = \{A_f^n\}_n$ and $B_f(s) = \{B_f^n\}_n$ be sequences of fuzzy sets in X called fuzzy sequential sets in X and we define

(i) $A_f(s) \vee B_f(s) = \{A_f^n \vee B_f^n\}_n$ (union),

(ii) $A_f(s) \wedge B_f(s) = \{A_f^n \wedge B_f^n\}_n$ (intersection),

(iii) $A_f(s) \leq B_f(s)$ if and only if $A_f^n \leq B_f^n$ for all $n \in \mathbb{N}$,

(iv) $A_f(s) \leq_w B_f(s)$ if and only if there exists $n \in \mathbb{N}$ such that $A_f^n \leq B_f^n$,

(v) $A_f(s) = B_f(s)$ if and only if $A_f^n = B_f^n$ for all $n \in \mathbb{N}$,

(vi) $A_f(s)(x) = \{A_f^n(x)\}_n$, $x \in X$,

(vii) $A_f(s)(x) \geq_M r$ if and only if $A_f^n(x) \geq r_n$ for all $n \in M$, where $r = \{r_n\}_n$ is a sequence in I. In particular if $M = \mathbb{N}$, we write $A_f(s)(x) \geq r$,

(viii) $X_f^l(s) = \{X_f^n\}_n$ where $l \in I$ and $X_f^n(x) = l$, for all $x \in X$, $n \in \mathbb{N}$,

(ix) $(A_f(s))^c = \{1 - A_f^n\}_n = \{(A_f^n)^c\}_n$, called complement of $A_f(s)$,

(x) A fuzzy sequential set $P_f(s) = \{p_f^n\}_n$ is called a fuzzy sequential point if there exists $x \in X$ and a non zero sequence $r = \{r_n\}_n$ in I such that

$$\begin{aligned} p_f^n(t) &= r_n, \text{ if } t = x, \\ &= 0, \text{ if } t \in X - \{x\}, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

If M be the collection of all $n \in \mathbb{N}$ such that $r_n \neq 0$, then we can write the above expression as

$$\begin{aligned} p_f^n(x) &= r_n, \text{ whenever } n \in M, \\ &= 0, \text{ whenever } n \in \mathbb{N} - M. \end{aligned}$$

The point x is called the support, M is called the base and r is called the sequential grade of membership of x in the fuzzy sequential point $P_f(s)$ and we write $P_f(s) = (p_{fx}^M, r)$. If further $M = \{n\}$, $n \in \mathbb{N}$, then the fuzzy sequential point is called a simple fuzzy sequential point and it is denoted by (p_{fx}^n, r_n). A fuzzy sequential point is called complete if its base is the set of natural numbers. A fuzzy sequential point $P_f(s) = (p_{fx}^M, r)$ is said to belong to $A_f(s)$ if and only if $P_f(s) \leq A_f(s)$ and we write $P_f(s) \in A_f(s)$. It is said to belong weakly to $A_f(s)$, symbolically $P_f(s) \in_w A_f(s)$, if and only if there exists $n \in M$ such that $p_f^n(x) \leq A_f^n(x)$.

Definition 1.1 ([13]). A family $\delta(s)$ of fuzzy sequential sets on a non empty set X satisfying the properties

(i) $X_f^r(s) \in \delta(s)$ for $r = 0$ and 1,

(ii) $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$ and

(iii) for any family $\{A_{fj}(s) \in \delta(s),\ j \in J\}$, $\bigvee_{j \in J} A_{fj}(s) \in \delta(s)$

is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets in X. Complement of an open fuzzy sequential set in X is called closed fuzzy sequential set in X.

Definition 1.2 ([13]). If $(X, \delta(s))$ is an FSTS, then (X, δ_n) is a fuzzy topological space (FTS), where $\delta_n = \{A_f^n;\ A_f^n(s) = \{A_f^n\}_n \in \delta(s)\}$, $n \in \mathbb{N}$. (X, δ_n), where $n \in \mathbb{N}$, is called the nth component FTS of the FSTS $(X, \delta(s))$.

Proposition 1.3 ([13]). *Let $A_f(s)=\{A_f^n\}_n$ be an open (closed) fuzzy sequential set in the FSTS $(X,\delta(s))$, then for each $n \in \mathbb{N}$, A_f^n is an open (closed) fuzzy set in (X,δ_n) but the converse is not necessarily true.*

Proposition 1.4 ([13]). *If δ be a fuzzy topology (FT) on a non empty set X, then $\delta^{\mathbb{N}}$ forms an FST on X.*

Definition 1.5 ([13]). Let $A_f(s)$ be any fuzzy sequential set in an FSTS $(X,\delta(s))$. The closure $\overline{A_f(s)}$ and interior $\overset{o}{A}_f(s)$ of $A_f(s)$ are defined as

$$\overline{A_f(s)} = \wedge\{C_f(s);\, A_f(s) \leq C_f(s),\, (C_f(s))^c \in \delta(s)\},$$

$$\overset{o}{A}_f(s) = \vee\{O_f(s);\, O_f(s) \leq A_f(s),\, O_f(s) \in \delta(s)\}.$$

Definition 1.6 ([13]). A fuzzy sequential set $A_f(s)$ in an FSTS $(X,\, \delta(s))$ is called a neighbourhood (in short nbd) of a fuzzy sequential point $P_f(s)$ if and only if there exists $B_f(s) \in \delta(s)$ such that $P_f(s) \in B_f(s) \leq A_f(s)$. A nbd $A_f(s)$ is called open if and only if $A_f(s) \in \delta(s)$.

2. Definition and results

Definition 2.1. Let X be a non empty set. An operator $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is said to be an FS-closure operator on X if it satisfies the following conditions:

(FSC1) $\mathbf{Cl}(X_f^0(s)) = X_f^0(s)$.

(FSC2) $A_f(s) \leq \mathbf{Cl}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.

(FSC3) $\mathbf{Cl}(\mathbf{Cl}(A_f(s))) = \mathbf{Cl}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.

(FSC4) $\mathbf{Cl}(A_f(s) \vee B_f(s)) = \mathbf{Cl}(A_f(s)) \vee \mathbf{Cl}(B_f(s))$ for all $A_f(s),\, B_f(s) \in (I^X)^{\mathbb{N}}$.

Example 2.2. For any FSTS $(X,\delta(s))$, closure of an fs-set (fuzzy sequential set) is an FS-closure operator on X.

Example 2.3. Let X be a non empty set. The operator $\mathbf{C} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by $\mathbf{C}(A_f(s)) = A_f(s) \vee D_f(s)$ whenever $A_f(s) \neq X_f^0(s)$ and $\mathbf{C}(X_f^0(s)) = X_f^0(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X, is an FS-closure operator on X.

Theorem 2.4. *If $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X, then*

(i) *$\boldsymbol{Cl}$ is monotonic increasing, that is, $A_f(s) \leq B_f(s) \Rightarrow \boldsymbol{Cl}(A_f(s)) \leq \boldsymbol{Cl}(B_f(s))$ for all $A_f(s),\, B_f(s) \in (I^X)^{\mathbb{N}}$.*

(ii) *$A_f(s) \leq \boldsymbol{Cl}(B_f(s)) \Rightarrow \boldsymbol{Cl}(A_f(s)) \leq \boldsymbol{Cl}(B_f(s))$ for all $A_f(s),\, B_f(s) \in (I^X)^{\mathbb{N}}$.*

Proof. Proof is omitted. □

Theorem 2.5. *Let X be a non empty set and $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an operator on X satisfying $(FSC1)$, $(FSC2)$ and $(FSC4)$, then*

a) *The collection $\delta'(s) = \{(A_f(s))^c;\, A_f(s) \in (I^X)^{\mathbb{N}}$ and $\boldsymbol{Cl}(A_f(s)) = A_f(s)\}$ forms an FST on X.*

b) *If $\boldsymbol{Cl}$ also satisfies $(FSC3)$, then for all $A_f(s) \in (I^X)^{\mathbf{N}}$ we have $\overline{A_f(s)} = \boldsymbol{Cl}(A_f(s))$, where $\overline{A_f(s)}$ is the closure of $A_f(s)$ in $\delta'(s)$.*

Proof. Proof is omitted. □

Remark 2.6. From **Theorem** 2.5 it follows that if $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X then $\delta'(s) = \{(A_f(s))^c;\ A_f(s) \in (I^X)^{\mathbb{N}}$ and $\mathbf{Cl}(A_f(s)) = \overline{A_f(s)}\}$ forms an FST on X. Also $\overline{A_f(s)} = \mathbf{Cl}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbf{N}}$, where $\overline{A_f(s)}$ is the closure of $A_f(s)$ in $\delta'(s)$. This FST $\delta'(s)$ is called the fuzzy sequential topology induced by the FS-closure operator $\mathbf{Cl}$ and we denote it by $\delta_{\mathbf{Cl}}(s)$.

Remark 2.7. Example 2.8 shows that if an operator $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on a non empty set X, satisfies $(FSC1)$, $(FSC2)$ and $(FSC4)$ but does not satisfy $(FSC3)$, then $\delta_{\mathbf{Cl}}(s)$ forms an FST on X but $\overline{A_f(s)}$ may not be equal to $\mathbf{Cl}(A_f(s))$, $A_f(s) \in (I^X)^{\mathbb{N}}$.

Example 2.8. Let $X = \{a\}$. Let $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be defined by

$$\mathbf{Cl}(A_f(s)) = \{A_f^n \vee A_f^{n+1}\}_{n=1}^{\infty}\ \forall A_f(s) = \{A_f^n\}_{n=1}^{\infty} \in (I^X)^{\mathbb{N}}.$$

Then $\mathbf{Cl}$ is an operator on X satisfying $(FSC1)$, $(FSC2)$ and $(FSC4)$ and hence $(X, \delta_{\mathbf{Cl}}(s))$ forms an FSTS. Further $\mathbf{Cl}$ does not satisfy $(FSC3)$ and in $(X, \delta_{\mathbf{Cl}}(s))$, $\mathbf{Cl}(B_f(s)) \neq \overline{B_f(s)}$ if $B_f(s) = \{B_f^n\}_{n=1}^{\infty}$ where $B_f^1 = p_a^{0.2}$, $B_f^2 = p_a^{0.4}$, $B_f^3 = p_a^{0.5}$, $B_f^n = \overline{0}\ \forall\ n \neq 1, 2, 3$

Definition 2.9. Let X be a non empty set and $\mathbf{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X. A function $(\mathbf{Cl})_f^n$: $I^X \to I^X$ defined by $(\mathbf{Cl})_f^n(A) = n^{th}$ term of $\mathbf{Cl}({}_{nA}X_f^0(s))$, where ${}_{nA}X_f^0(s)$ denotes an fs-set whose n^{th} term is A and others are $\overline{0}$, is called the nth component of $\mathbf{Cl}$, $n \in \mathbb{N}$.

Theorem 2.10. *Let X be a non empty set. If $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X, then each component $(\boldsymbol{Cl})_f^n$: $I^X \to I^X$, $n \in \mathbb{N}$ is a fuzzy closure operator. Also $(\delta_{\boldsymbol{Cl}})_n = \delta_{(\boldsymbol{Cl})_f^n}$ where $(\delta_{\boldsymbol{Cl}})_n$ is the nth component fuzzy topology of FST $\delta_{\boldsymbol{Cl}}(s)$ and $\delta_{(\boldsymbol{Cl})_f^n}$ is the fuzzy topology induced by the component $(\boldsymbol{Cl})_f^n$ of $\boldsymbol{Cl}$.*

Proof. $(\mathbf{Cl})_f^n(\overline{0}) = \overline{0}$ by definition. Let $A \in I^X$, then ${}_{nA}X_f^0(s) \leq \mathbf{Cl}({}_{nA}X_f^0(s)) \Rightarrow A \leq (\mathbf{Cl})_f^n(A)$. Hence $(\mathbf{Cl})_f^n(A) \leq (\mathbf{Cl})_f^n((\mathbf{Cl})_f^n(A))$. Also

$$\begin{aligned} & \mathbf{Cl}(\mathbf{Cl}({}_{nA}X_f^0(s))) = \mathbf{Cl}({}_{nA}X_f^0(s)) \\ \Rightarrow\ & \mathbf{Cl}({}_{n(\mathbf{Cl})_f^n(A)}X_f^0(s)) \leq \mathbf{Cl}({}_{nA}X_f^0(s)) \\ \Rightarrow\ & (\mathbf{Cl})_f^n((\mathbf{Cl})_f^n(A)) \leq (\mathbf{Cl})_f^n(A) \end{aligned}$$

Hence $(\mathbf{Cl})_f^n((\mathbf{Cl})_f^n(A)) = (\mathbf{Cl})_f^n(A)$.
Again let A, B $\in I^X$, then

$$\begin{aligned} & \mathbf{Cl}({}_{nA}X_f^0(s) \vee_{nB} X_f^0(s)) = \mathbf{Cl}({}_{nA}X_f^0(s)) \vee \mathbf{Cl}({}_{nB}X_f^0(s)) \\ \Rightarrow\ & \mathbf{Cl}({}_{n(A\vee B)}X_f^0(s)) = \mathbf{Cl}({}_{nA}X_f^0(s)) \vee \mathbf{Cl}({}_{nB}X_f^0(s)) \\ \Rightarrow\ & (\mathbf{Cl})_f^n(A \vee B) = (\mathbf{Cl})_f^n(A) \vee (\mathbf{Cl})_f^n(B) \end{aligned}$$

Thus $(\mathbf{Cl})_f^n$ is a fuzzy closure operator.
For the next part, Let A$\in (\delta_{\mathbf{Cl}})_n$, then $\overline{1}$-A is a closed fuzzy set in $(X, (\delta_{\mathbf{Cl}})_n)$. Let

$B_f(s) = \{B_f^n\}_{n=1}^{\infty}$ be a closed fs-set in $(X, \delta_{\mathbf{Cl}}(s))$ such that $B_f^n = \overline{1} - A$. Now,

$$\begin{aligned} & {}_{n(\overline{1}-A)}X_f^0(s) \leq B_f(s) \\ \Rightarrow\ & \mathbf{Cl}({}_{n(\overline{1}-A)}X_f^0(s)) \leq \mathbf{Cl}(B_f(s)) \\ \Rightarrow\ & (\mathbf{Cl})_f^n(\overline{1} - A) \leq B_f^n = \overline{1} - A \\ \Rightarrow\ & (\mathbf{Cl})_f^n(\overline{1} - A) = \overline{1} - A \\ \Rightarrow\ & A \in \delta_{(\mathbf{Cl})_f^n} \end{aligned}$$

Also A$\in \delta_{(\mathbf{Cl})_f^n}$ implies $(\mathbf{Cl})_f^n(\overline{1} - A) = \overline{1} - A$. Let $B_f(s) = \mathbf{Cl}({}_{n(\overline{1}-A)}X_f^o(s))$, then $B_f(s)$ is a closed fs-set in $(X, \delta_{\mathbf{Cl}}(s))$ and its nth component is $\overline{1} - A$. Therefore $A \in (\delta_{\mathbf{Cl}})_n$. Hence the theorem. □

Theorem 2.11. *Let $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on a non empty set X and $A \subset X$. If $Char(A)$ denote the characteristic function of A, Then $\boldsymbol{Cl}_A$: $(I^A)^{\mathbb{N}} \to (I^A)^{\mathbb{N}}$ defined by*

$$\boldsymbol{Cl}_A(B_f(s)) \quad = \quad \{Char(A)\}_{n=1}^{\infty} \wedge \boldsymbol{Cl}(B_f(s))\ \forall B_f(s) \in (I^A)^{\mathbb{N}}.$$

is an FS-closure operator on A and $(\boldsymbol{Cl}_A)_f^n(B) = Char(A) \wedge (\boldsymbol{Cl})_f^n(B)$ for all $B \in I^A$.

Proof. Let $B_f(s) \in (I^A)^{\mathbb{N}}$. Now

$$\begin{aligned} & \mathbf{Cl}_A(\mathbf{Cl}_A(B_f(s)) = \mathbf{Cl}_A(\{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}(B_f(s))) \\ = \ & \{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}(\{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}(B_f(s))) \\ \leq \ & \{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}(\{Char(A)\}_{n=1}^{\infty}) \wedge \mathbf{Cl}(\mathbf{Cl}(B_f(s))) \\ = \ & \{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}(B_f(s)) \\ = \ & \mathbf{Cl}_A(B_f(s)) \end{aligned}$$

All the other conditions being straightforward, we an conclude that $\mathbf{Cl}_A$ is an FS-closure operator on A. Also for $B \in I^A$, $(\mathbf{Cl}_A)_f^n(B) = n^{th}$ component of $\mathbf{Cl}_A({}_{nB}X_f^0(s))$= n^{th} component of $\{Char(A)\}_{n=1}^{\infty} \wedge \mathbf{Cl}({}_{nB}X_f^0(s)) = Char(A) \wedge$ n^{th} component of $\mathbf{Cl}({}_{nB}X_f^0(s)) = Char(A) \wedge (\mathbf{Cl})_f^n(B)$. □

Theorem 2.12. *Let $\{\boldsymbol{Cl}_\lambda : (I^{X_\lambda})^{\mathbb{N}} \to (I^{X_\lambda})^{\mathbb{N}}; \lambda \in \Lambda\}$ be a family of FS-closure operators, where $X_\lambda \wedge X_\mu = \phi$ for all $\lambda, \mu \in \Lambda$. If $X = \vee_{\lambda\in\Lambda} X_\lambda$ and $Char(X_\lambda)$ denote the characteristic function of X_λ, then $\boldsymbol{C} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by $\boldsymbol{C}(A_f(s)) = \vee_{\lambda\in\Lambda} \boldsymbol{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s))$ is an FS-closure operator on X.*

Proof. For $A_f(s) \in (I^X)^{\mathbb{N}}$,

$$\begin{aligned} & \mathbf{C}(\mathbf{C}(A_f(s))) = \mathbf{C}(\vee_{\lambda\in\Lambda}\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s))) \\ = \ & \vee_{\lambda\in\Lambda}\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge (\vee_{\lambda\in\Lambda}\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s)))) \\ = \ & \vee_{\lambda\in\Lambda}\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge (\vee_{\lambda\in\Lambda}(\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty}) \wedge \mathbf{Cl}_\lambda(A_f(s))))) \\ = \ & \vee_{\lambda\in\Lambda}\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge \mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s))) \\ = \ & \vee_{\lambda\in\Lambda}\mathbf{Cl}_\lambda(\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s))) \\ = \ & \vee_{\lambda\in\Lambda}\mathbf{Cl}_\lambda(\{Char(X_\lambda)\}_{n=1}^{\infty} \wedge A_f(s)) \\ = \ & \mathbf{C}(A_f(s)) \end{aligned}$$

Other conditions being straightforward, it follows that $\mathbf{C}$ is an FS-closure operator. □

Definition 2.13. A collection $\zeta(s) = \{A_{\lambda f}(s) \in (I^X)^{\mathbb{N}}; \lambda \in \Lambda\}$ is called an FS-closure system if for each $A_f(s) \in (I^X)^{\mathbb{N}}$, $\wedge_{\lambda\in\Lambda, A_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s) \in \zeta(s)$

Theorem 2.14. *$\zeta(s)$ is an FS-closure system iff $\zeta(s)$ is closed under arbitrary intersection.*

Proof. Suppose $\zeta(s)$ is closed under arbitrary intersection. Let $A_f(s) \in (I^X)^{\mathbb{N}}$ and let $\{A_{\lambda f}(s); \lambda \in \Lambda\} \in \zeta(s)$ such that $A_f(s) \leq A_{\lambda f}(s)\ \forall\lambda \in \Lambda$. Then

$$\wedge_{\lambda\in\Lambda, A_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s) \in \zeta(s)$$

Conversely, suppose $\zeta(s)$ is an FS-closure system. Let $\{A_{\lambda f}(s); \lambda \in \Lambda\} \in \zeta(s)$ and let $A_f(s) = \wedge_{\lambda\in\Lambda} A_{\lambda f}(s)$. Then

$$\begin{aligned} & A_f(s) \leq A_{\lambda f}(s)\ \forall\lambda \in \Lambda \\ \Rightarrow\ & \wedge_{\lambda\in\Lambda} A_{\lambda f}(s) = \wedge_{\lambda\in\Lambda, A_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s) \in \zeta(s) \end{aligned}$$

Hence $\zeta(s)$ is closed under arbitrary intersection. □

Lemma 2.15. *Let $\zeta(s) = \{A_{\lambda f}(s) \in (I^X)^{\mathbb{N}}; \lambda \in \Lambda\}$ be an FS-closure system containing $X_f^0(s)$. Then $\boldsymbol{Cl}_{\zeta(s)} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by*

$$\boldsymbol{Cl}_{\zeta(s)}(A_f(s)) = \wedge_{\lambda\in\Lambda, A_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s) \text{ and}$$

$$\boldsymbol{Cl}_{\zeta(s)}(A_f(s) \vee B_f(s)) = \boldsymbol{Cl}_{\zeta(s)}(A_f(s)) \vee \boldsymbol{Cl}_{\zeta(s)}(B_f(s))\ \forall A_f(s),\ B_f(s) \in (I^X)^{\mathbb{N}}$$

is an FS-closure operator. Moreover for all $A_f(s) \in (I^X)^{\mathbb{N}}$, $A_f(s) \in \zeta(s)$ iff $A_f(s) = \boldsymbol{Cl}_{\zeta(s)}(A_f(s))$.

Proof. Since $\mathbf{Cl}_{\zeta(s)}(A_f(s)) \in \zeta(s)$ for $A_f(s) \in (I^X)^{\mathbb{N}}$, we have

$$\mathbf{Cl}_{\zeta(s)}(\mathbf{Cl}_{\zeta(s)}(A_f(s))) \quad = \quad \wedge_{\lambda\in\Lambda, \mathbf{Cl}_{\zeta(s)}(A_f(s))\leq A_{\lambda f}(s)} A_{\lambda f}(s) \leq \mathbf{Cl}_{\zeta(s)}(A_f(s))$$

Hence $\mathbf{Cl}_{\zeta(s)}$ is an FS-closure operator.
Now, if $A_f(s) \in \zeta(s)$, then $A_f(s) = A_{\lambda f}(s)$ for some $\lambda \in \Lambda$ and

$$\mathbf{Cl}_{\zeta(s)}(A_f(s)) \quad = \quad \wedge_{i\in\Lambda, A_f(s)\leq A_{if}(s)} A_{if}(s) \leq A_{\lambda f}(s) = A_f(s)$$

Also $A_f(s) \leq \mathbf{Cl}_{\zeta(s)}((A_f(s))$. Hence $\mathbf{Cl}_{\zeta(s)}(A_f(s)) = A_f(s)$. Converse part follows from the definition of $\mathbf{Cl}_{\zeta(s)}$. □

Lemma 2.16. *Let $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator. Then*

$$\zeta_{\boldsymbol{Cl}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}}; A_f(s) = \boldsymbol{Cl}(A_f(s))\}$$

is an FS-closure system.

Proof. Let $B_f(s) \in (I^X)^{\mathbb{N}}$ and let $\{A_{\lambda f}(s); \lambda \in \Lambda\} \in \zeta(s)$ such that $B_f(s) \leq A_{\lambda f}(s)$ $\forall\lambda \in \Lambda$. Let $D_f(s) = \wedge_{\lambda\in\Lambda, B_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s)$. We know, $D_f(s) \leq \mathbf{Cl}(D_f(s))$. Again

$$\begin{aligned} & D_f(s) \leq A_{\lambda f}(s)\ \forall\lambda \in \Lambda \\ \Rightarrow\ & \mathbf{Cl}(D_f(s)) \leq \mathbf{Cl}(A_{\lambda f}(s))\ \forall\lambda \in \Lambda \\ \Rightarrow\ & \mathbf{Cl}(D_f(s)) \leq \wedge_{\lambda\in\Lambda, B_f(s)\leq A_{\lambda f}(s)} \mathbf{Cl}(A_{\lambda f}(s)) = \wedge_{\lambda\in\Lambda, B_f(s)\leq A_{\lambda f}(s)} A_{\lambda f}(s) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = D_f(s) \end{aligned}$$

Thus $D_f(s) = \mathbf{Cl}(D_f(s))$ and so $D_f(s) \in \zeta_{\mathbf{Cl}}(s)$. Hence $\zeta_{\mathbf{Cl}}(s)$ is an FS-closure system. □

Note 2.17. In **Lemma** 2.16, the FS-closure system $\zeta_{\mathbf{Cl}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}}; A_f(s) = \mathbf{Cl}(A_f(s))\}$ is called an FS-closure system generated by the FS-closure operator **Cl**.

Theorem 2.18. *Let* $\boldsymbol{Cl}$ *be an FS-closure operator and* $\zeta(s)$ *be an FS-closure system on* X *containing* $X_f^0(s)$*, then* $\zeta_{\boldsymbol{Cl}}(s)$ *and* $\boldsymbol{Cl}_{\zeta(s)}$ *are respectively FS-closure system and FS-closure operator on* X*. Also* $\boldsymbol{Cl} = \boldsymbol{Cl}_{\zeta_{\boldsymbol{Cl}}(s)}$ *and* $\zeta(s) = \zeta_{\boldsymbol{Cl}_{\zeta(s)}}(s)$*, that is, the mappings* $\boldsymbol{Cl} \to \boldsymbol{Cl}_{\zeta_{\boldsymbol{Cl}}(s)}$ *and* $\zeta(s) \to \zeta_{\boldsymbol{Cl}_{\zeta(s)}}(s)$ *are mutually inverse.*

Proof. The first part follows from **Lemma** 2.15 and **Lemma** 2.16. Let $A_f(s) \in (I^X)^{\mathbb{N}}$ and let $\{A_{\lambda f}(s); \lambda \in \Lambda\} \in \zeta_{\mathbf{Cl}}(s)$ such that $A_f(s) \leq A_{\lambda f}(s)\ \forall \lambda \in \Lambda$. Then $\mathbf{Cl}_{\zeta_{\mathbf{Cl}}(s)}(A_f(s)) = \wedge_{\lambda \in \Lambda, A_f(s) \leq A_{\lambda f}(s)} A_{\lambda f}(s)$. Now,

$$\begin{aligned} & A_f(s) \leq A_{\lambda f}(s)\ \forall \lambda \in \Lambda \\ \Rightarrow\ & \mathbf{Cl}(A_f(s)) \leq \mathbf{Cl}(A_{\lambda f}(s))\ \forall \lambda \in \Lambda \\ \Rightarrow\ & \mathbf{Cl}(A_f(s)) \leq A_{\lambda f}(s)\ \forall \lambda \in \Lambda \\ \Rightarrow\ & \mathbf{Cl}(A_f(s)) \leq \wedge_{\lambda \in \Lambda, A_f(s) \leq A_{\lambda f}(s)} A_{\lambda f}(s) = \mathbf{Cl}_{\zeta_{\mathbf{Cl}}(s)}(A_f(s)) \end{aligned}$$

Again,

$$\begin{aligned} & A_f(s) \leq \mathbf{Cl}(A_f(s)) \in \zeta_{\mathbf{Cl}}(s) \\ \Rightarrow\ & \mathbf{Cl}_{\zeta_{\mathbf{Cl}}(s)}(A_f(s)) = \wedge_{\lambda \in \Lambda, A_f(s) \leq A_{\lambda f}(s)} A_{\lambda f}(s) \leq \mathbf{Cl}(A_f(s)) \end{aligned}$$

Hence $\mathbf{Cl} = \mathbf{Cl}_{\zeta_{\mathbf{Cl}}(s)}$.
Also,

$$\begin{aligned} & A_f(s) \in \zeta_{\mathbf{Cl}_{\zeta(s)}}(s) \\ \Leftrightarrow\ & A_f(s) = \mathbf{Cl}_{\zeta(s)}(A_f(s)) \\ \Leftrightarrow\ & A_f(s) \in \zeta(s) \end{aligned}$$

Thus $\zeta(s) = \zeta_{\mathbf{Cl}_{\zeta(s)}}(s)$. □

Definition 2.19. Let X be a non empty set. An operator $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is said to be an FS-interior operator if it satisfies the following conditions:
(FSI1) $\mathbf{I}(X_f^1(s)) = X_f^1(s)$.
(FSI2) $\mathbf{I}(A_f(s)) \leq A_f(s)$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.
(FSI3) $\mathbf{I}(\mathbf{I}(A_f(s))) = \mathbf{I}(A_f(s)))$ for all $A_f(s)) \in (I^X)^{\mathbb{N}}$.
(FSI4) $\mathbf{I}(A_f(s) \wedge B_f(s)) = \mathbf{I}(A_f(s)) \wedge \mathbf{I}(B_f(s))$ for all $A_f(s)$, $B_f(s) \in (I^X)^{\mathbb{N}}$.

Example 2.20. For any FSTS $(X, \delta(s))$, interior of an fs-set is an FS-interior operator on X.

Example 2.21. Let X be a non empty set. The operator $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by $\mathbf{I}(A_f(s)) = A_f(s) \wedge D_f(s)$ whenever $A_f(s) \neq X_f^1(s)$ and $\mathbf{I}(X_f^1(s)) = X_f^1(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X, is an FS-interior operator on X.

Theorem 2.22. *If $\boldsymbol{I}\colon (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X, then*

(i) *$\boldsymbol{I}$ is monotonic increasing, that is, $A_f(s) \leq B_f(s) \Rightarrow \boldsymbol{I}(A_f(s)) \leq \boldsymbol{I}(B_f(s))$ for all $A_f(s)$, $B_f(s) \in (I^X)^{\mathbb{N}}$.*

(ii) *$\boldsymbol{I}(A_f(s)) \leq B_f(s) \Rightarrow \boldsymbol{I}(A_f(s)) \leq \boldsymbol{I}(B_f(s))$ for all $A_f(s)$, $B_f(s) \in (I^X)^{\mathbb{N}}$.*

Proof. Proof is omitted. □

Theorem 2.23. *Let X be a non empty set and $\boldsymbol{I}\colon (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an operator satisfying $(FSI1)$, $(FSI2)$ and $(FSI4)$, then*

a) *the collection $\delta(s) = \{A_f(s) \in (I^X)^{\mathbb{N}}; \boldsymbol{I}(A_f(s)) = A_f(s)\}$ forms an FST on X.*

b) *if $\boldsymbol{I}$ also satisfies $(FSI3)$, then for all $A_f(s) \in (I^X)^{\mathbf{N}}$ we have $\overset{o}{A}_f(s) = \boldsymbol{I}(A_f(s))$, where $\overset{o}{A}_f(s)$ is the interior of $A_f(s)$ in $\delta(s)$.*

Proof. Proof is omitted. □

Remark 2.24. From **Theorem** 2.23 it follows that if $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X then $\delta(s) = \{A_f(s) \in (I^X)^{\mathbb{N}}\,; \mathbf{I}(A_f(s)) = A_f(s)\}$ forms an FST on X. Also $\overset{o}{A}_f(s) = \mathbf{I}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbf{N}}$, where $\overset{o}{A}_f(s)$ is the interior of $A_f(s)$ in $\delta(s)$. This FST $\delta(s)$ is called the fuzzy sequential topology induced by the FS-interior operator $\mathbf{I}$ and we denote it by $\delta_{\mathbf{I}}(s)$.

Remark 2.25. Example 2.26 shows that if an operator $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on a non empty set X, satisfies $(FSI1)$, $(FSI2)$ and $(FSI4)$ but does not satisfy $(FSI3)$, then $\delta_{\mathbf{I}}(s)$ forms an FST on X but $\overset{o}{A}_f(s)$ may not be equal to $\mathbf{I}(A_f(s))$, $A_f(s) \in (I^X)^{\mathbb{N}}$.

Example 2.26. Let $X = \{a\}$. Let $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be defined by $\mathbf{I}(A_f(s)) = \{A_f^n \wedge A_f^{n+1}\}_{n=1}^{\infty}$ $\forall A_f(s) = \{A_f^n\}_{n=1}^{\infty} \in (I^X)^{\mathbb{N}}$. Then $\mathbf{I}$ is an operator on X satisfying $(FSI1)$, $(FSI2)$ and $(FSI4)$ and hence $(X, \delta_{\mathbf{I}}(s))$ forms an FSTS. Further $\mathbf{I}$ does not satisfy $(FSI3)$ and in $(X, \delta_{\mathbf{I}}(s))$, $\mathbf{I}(B_f(s)) \neq \overset{0}{B}_f(s)$ if $B_f(s) = \{B_f^n\}_{n=1}^{\infty}$ where $B_f^1 = p_a^{0.2}$, $B_f^2 = p_a^{0.4}$, $B_f^3 = p_a^{0.5}$, $B_f^n = \overline{0}$ $\forall\ n \neq 1, 2, 3$.

Definition 2.27. Let X be a non empty set and $\mathbf{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X. A function $(\mathbf{I})_f^n : I^X \to I^X$ defined by $(\mathbf{I})_f^n(A) = n^{th}$ term of $\mathbf{I}({}_{nA}X_f^1(s))$, where ${}_{nA}X_f^1(s)$ denotes an fs-set whose n^{th} term is A and others are $\overline{1}$, is called the nth component of $\mathbf{I}$, $n \in \mathbb{N}$.

Theorem 2.28. *Let X be a non empty set. If $\boldsymbol{I}\colon (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X, then each component $(\boldsymbol{I})_f^n : I^X \to I^X$, $n \in \mathbb{N}$ is a fuzzy interior operator. Also $(\delta_{\boldsymbol{I}})_n = \delta_{(\boldsymbol{I})_f^n}$ where $(\delta_{\boldsymbol{I}})_n$ is the nth component fuzzy topology of FST $\delta_{\boldsymbol{I}}(s)$ and $\delta_{(\boldsymbol{I})_f^n}$ is the fuzzy topology induced by the component $(\boldsymbol{I})_f^n$ of $\boldsymbol{I}$.*

Proof. $(\mathbf{I})_f^n(\overline{1}) = \overline{1}$ by definition. Let $A \in I^X$, then $\mathbf{I}({}_{nA}X_f^1(s)) \leq {}_{nA}X_f^1(s) \Rightarrow (\mathbf{I})_f^n(A) \leq A$. Hence $(\mathbf{I})_f^n((\mathbf{I})_f^n(A)) \leq (\mathbf{I})_f^n(A)$. Also

$$\begin{aligned} & \mathbf{I}(\mathbf{I}({}_{nA}X_f^1(s))) = \mathbf{I}({}_{nA}X_f^1(s)) \\ \Rightarrow\ & \mathbf{I}({}_{nA}X_f^1(s)) \leq \mathbf{I}({}_{n(\mathbf{I})_f^n(A)}X_f^1(s)) \\ \Rightarrow\ & (\mathbf{I})_f^n(A) \leq (\mathbf{I})_f^n((\mathbf{I})_f^n(A)) \end{aligned}$$

Hence $(\mathbf{I})^n_f((\mathbf{I})^n_f(A)) = (\mathbf{I})^n_f(A)$.
Again let A, B $\in I^X$, then

$$\begin{aligned} & \mathbf{I}({}_{nA}X^1_f(s) \wedge {}_{nB}X^1_f(s)) = \mathbf{I}({}_{nA}X^1_f(s)) \wedge \mathbf{I}({}_{nA}X^1_f(s)) \\ \Rightarrow\ & \mathbf{I}({}_{n(A\wedge B)}X^1_f) = \mathbf{I}({}_{nA}X^1_f(s)) \wedge \mathbf{I}({}_{nB}X^1_f(s)) \\ \Rightarrow\ & (\mathbf{I})^n_f(A \wedge B) = (\mathbf{I})^n_f(A) \wedge (\mathbf{I})^n_f(B) \end{aligned}$$

Thus $(\mathbf{I})^n_f$ is a fuzzy interior operator.
For the next part, Let A$\in (\delta_{\mathbf{I}})_n$. Let $B_f(s) = \{B^n_f\}^\infty_{n=1}$ be an open fs-set in $(X, \delta_{\mathbf{I}}(s))$ such that $B^n_f = A$. Now,

$$\begin{aligned} & B_f(s) \leq {}_{nA}X^1_f(s) \\ \Rightarrow\ & \mathbf{I}(B_f(s)) \leq \mathbf{I}({}_{nA}X^1_f(s)) \\ \Rightarrow\ & B_f(s) \leq \mathbf{I}({}_{nA}X^1_f(s)) \\ \Rightarrow\ & A \leq (\mathbf{I})^n_f(A) \\ \Rightarrow\ & (\mathbf{I})^n_f(A) = A \\ \Rightarrow\ & A \in \delta_{(\mathbf{I})^n_f} \end{aligned}$$

Also $A \in \delta_{(\mathbf{I})^n_f}$ implies $(\mathbf{I})^n_f(A) = A$. Let $B_f(s) = \mathbf{I}({}_{nA}X^1_f(s))$, then $B_f(s)$ is an open fs-set in $(X, (\delta_{\mathbf{I}}(s))$ and its nth component is A. Therefore $A \in (\delta_{\mathbf{I}})_n$. Hence the theorem. □

Theorem 2.29. *Let $\boldsymbol{I}\colon (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on a non empty set X and $A \subset X$. If $Char(A)$ denote the characteristic function of A, then $\boldsymbol{I}_A$: $(I^A)^{\mathbb{N}} \to (I^A)^{\mathbb{N}}$ defined by*

$$\boldsymbol{I}_A(B_f(s)) \quad = \quad \{Char(A)\}^\infty_{n=1} \vee \boldsymbol{I}(B_f(s))\ \forall B_f(s) \in (I^A)^{\mathbb{N}}.$$

is an FS-interior operator on A and $(\boldsymbol{I}_A)^n_f(B) = Char(A) \vee (\boldsymbol{I})^n_f(B)$ for all $B \in I^A$.

Proof. Let $B_f(s) \in (I^A)^{\mathbb{N}}$. Now

$$\begin{aligned} & \mathbf{I}_A(B_f(s))) = \{Char(A)\}^\infty_{n=1} \vee \mathbf{I}(B_f(s)) \\ = \ & \{Char(A)\}^\infty_{n=1} \vee \mathbf{I}(\{Char(A)\}^\infty_{n=1}) \vee \mathbf{I}(\mathbf{I}(B_f(s))) \\ \leq \ & \{Char(A)\}^\infty_{n=1} \vee \mathbf{I}(\{Char(A)\}^\infty_{n=1} \vee \mathbf{I}(B_f(s))) \\ = \ & \mathbf{I}_A(\{Char(A)\}^\infty_{n=1} \vee \mathbf{I}(B_f(s))) \\ = \ & \mathbf{I}_A(\mathbf{I}_A(B_f(s))) \end{aligned}$$

All the other conditions being straightforward, we an conclude that $\mathbf{I}_A$ is an FS-interior operator. Also $(\mathbf{I}_A)^n_f(B) = n^{th}$ component of $\mathbf{I}_A({}_{nB}X^1_f(s)) = n^{th}$ component of $\{Char(A)\}^{\mathbb{N}} \vee \mathbf{I}({}_{nB}X^1_f(s)) = Char(A) \vee n^{th}$ component of $\mathbf{I}({}_{nB}X^1_f(s)) = Char(A) \vee (\mathbf{I})^n_f(B)$. □

Definition 2.30. A collection $\eta(s) = \{A_{jf}(s) \in (I^X)^{\mathbb{N}}; j \in J\}$ is called an FS-interior system if for each $A_f(s) \in (I^X)^{\mathbb{N}}$, $\vee_{j\in J, A_{jf}(s)\leq A_f(s)} A_{jf}(s) \in \eta(s)$.

Theorem 2.31. *$\eta(s)$ is an FS-interior system iff $\eta(s)$ is closed under arbitrary union.*

Proof. Suppose $\eta(s)$ is closed under arbitrary union. Let $A_f(s) \in (I^X)^{\mathbb{N}}$. Let $A_{jf}(s) \leq A_f(s)\ \forall j \in J$ where $A_{jf}(s) \in \eta(s)\ \forall j \in J$. Then

$$\vee_{j\in J, A_{jf}(s)\leq A_f(s)} A_{jf}(s) \in \eta(s)$$

Conversely, suppose $\eta(s)$ is an FS-interior system. Let $\{A_{jf}(s); j \in J\} \in \eta(s)$ and let $A_f(s) = \vee_{j\in J} A_{jf}(s)$. Then

$$\begin{aligned} & A_{jf}(s) \leq A_f(s)\ \forall j \in J \\ \Rightarrow\ & \vee_{j\in J} A_{jf}(s) = \vee_{j\in J, A_{jf}(s)\leq A_f(s)} A_{jf}(s) \in \eta(s) \end{aligned}$$

Hence $\eta(s)$ is closed under arbitrary union. □

Lemma 2.32. *Let $\eta(s) = \{A_{jf}(s) \in (I^X)^{\mathbb{N}}; j \in J\}$ be an FS-interior system containing $X^1_f(s)$. Then $\boldsymbol{I}_{\eta(s)} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by*

$$\begin{aligned} & \boldsymbol{I}_{\eta(s)}(A_f(s)) = \vee_{j\in J, A_{jf}(s)\leq A_f(s)} A_{jf}(s) \text{ and} \\ & \boldsymbol{I}_{\eta(s)}(A_f(s) \wedge B_f(s)) = \boldsymbol{I}_{\eta(s)}(A_f(s)) \wedge \boldsymbol{I}_{\eta(s)}(B_f(s))\ \forall A_f(s),\ B_f(s) \in (I^X)^{\mathbb{N}} \end{aligned}$$

is an FS-interior operator. Moreover for all $A_f(s) \in (I^X)^{\mathbb{N}}$, $A_f(s) \in \eta(s)$ iff $A_f(s) = \boldsymbol{I}_{\eta(s)}(A_f(s))$.

Proof. Proof of the first part is straightforward.
Now, if $A_f(s) \in \eta(s)$, then $A_f(s) = A_{jf}(s)$ for some $j \in J$ and

$$\mathbf{I}_{\eta(s)}(A_f(s)) = \vee_{i\in J, A_{if}(s)\leq A_f(s)} A_{if}(s) = A_f(s)$$

Converse part follows from the definition of $\mathbf{I}_{\eta(s)}$. □

Lemma 2.33. *Let $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator. Then*

$$\eta_{\boldsymbol{I}}(s) \quad = \quad \{A_f(s) \in (I^X)^{\mathbb{N}}; A_f(s) = \boldsymbol{I}(A_f(s))\}$$

is an FS-interior system.

Proof. Let $B_f(s) \in (I^X)^{\mathbb{N}}$. Let $D_f(s) = \vee_{j\in J, A_{jf}(s)\leq B_f(s)} A_{jf}(s)$, where $A_{jf}(s) \in \eta_{\mathbf{I}}(s)\ \forall j \in J$. We know, $\mathbf{I}(D_f(s)) \leq D_f(s)$. Again,

$$\begin{aligned} & A_{jf}(s) \leq D_f(s)\ \forall j \in J \text{ and for all } A_{jf}(s) \leq B_f(s) \\ \Rightarrow\ & \mathbf{I}(A_{jf}(s)) \leq \mathbf{I}(D_f(s))\ \forall j \in J \text{ and for all } A_{jf}(s) \leq B_f(s) \\ \Rightarrow\ & \vee_{j\in J, A_{jf}(s)\leq B_f(s)} \mathbf{I}(A_{jf}(s)) = \vee_{j\in J, A_{jf}(s)\leq B_f(s)} A_{jf}(s) = D_f(s) \leq \mathbf{I}(D_f(s)) \end{aligned}$$

Thus $D_f(s) = \mathbf{I}(D_f(s))$ and so $D_f(s) \in \eta_{\mathbf{I}}(s)$. Hence $\eta_{\mathbf{I}}(s)$ is a FS-interior system. □

Note 2.34. In **Lemma** 2.33, the FS-interior system $\eta_{\mathbf{I}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}}; A_f(s) = \mathbf{I}(A_f(s))\}$ is called an FS-interior system generated by the FS-interior operator $\mathbf{I}$.

Theorem 2.35. *Let $\boldsymbol{I}$ be an FS-interior operator and $\eta(s)$ be an FS-interior system on X containing $X^1_f(s)$, then $\eta_{\boldsymbol{I}}(s)$ and $\boldsymbol{I}_{\eta(s)}$ are respectively FS-interior system and FS-interior operator on X. Also $\boldsymbol{I} = \boldsymbol{I}_{\eta_{\boldsymbol{I}}(s)}$ and $\eta(s) = \eta_{\boldsymbol{I}_{\eta(s)}}(s)$, that is, the mappings $\boldsymbol{I} \to \boldsymbol{I}_{\eta_{\boldsymbol{I}}(s)}$ and $\eta(s) \to \eta_{\boldsymbol{I}_{\eta(s)}}(s)$ are mutually inverse.*

Proof. The first part follows from **Lemma** 2.32 and **Lemma** 2.33. Let $A_f(s) \in (I^X)^{\mathbb{N}}$, and let $\{A_{jf}(s); j \in J\} \in \eta_{\mathbf{I}}(s)$ such that $A_{jf}(s) \leq A_f(s)\ \forall j \in J$. Then $\mathbf{I}_{\eta_{\mathbf{I}}(s)}(A_f(s)) = \vee_{j\in J, A_{jf}(s)\leq A_f(s)(\in\eta_{\mathbf{I}}(s))}(A_{jf}(s))$. Now,

$$\begin{aligned} & A_{jf}(s) \leq A_f(s)\ \forall j \in J \\ \Rightarrow\ & \mathbf{I}(A_{jf}(s)) \leq \mathbf{I}(A_f(s))\ \forall j \in J \\ \Rightarrow\ & \vee_{j\in J, A_{jf}(s)\leq A_f(s)}\mathbf{I}(A_{jf}(s)) = \vee_{j\in J, A_{jf}(s)\leq A_f(s)}A_{jf}(s) \\ & \qquad\qquad\qquad\qquad = \mathbf{I}_{\eta_{\mathbf{I}}(s)}(A_f(s)) \leq \mathbf{I}(A_f(s)) \end{aligned}$$

Again,

$$\begin{aligned} & \mathbf{I}(A_f(s))(\leq A_f(s)) \in \eta_{\mathbf{I}}(s) \\ \Rightarrow\ & \mathbf{I}(A_f(s)) \leq \vee_{j\in J, A_{jf}(s)\leq A_f(s)(\in\eta_{\mathbf{I}}(s))}A_{jf}(s) = \mathbf{I}_{\eta_{\mathbf{I}}(s)}(A_f(s)). \end{aligned}$$

Hence $\mathbf{I} = \mathbf{I}_{\eta_{\mathbf{I}}(s)}$.
Also,

$$\begin{aligned} & A_f(s) \in \eta_{\mathbf{I}_{\eta(s)}}(s) \\ \Leftrightarrow\ & A_f(s) = \mathbf{I}_{\eta(s)}(A_f(s)) \\ \Leftrightarrow\ & A_f(s) \in \eta(s). \end{aligned}$$

Thus $\eta(s) = \eta_{\mathbf{I}_{\eta(s)}}(s)$. □

Definition 2.36. If $\mathbf{I}$ be an FS-interior operator on a non empty set X, then the collection $\{(A_f(s))^c \in (I^X)^{\mathbf{N}}; \mathbf{I}(A_f(s)) = A_f(s)\}$ forms an FS-closure system on X and we call it to be an FS-closure system generated by the FS-interior operator $\mathbf{I}$.

Definition 2.37. If $\mathbf{Cl}$ be an FS-closure operator on a non empty set X, then the collection $\{(A_f(s))^c \in (I^X)^{\mathbf{N}}; \mathbf{Cl}(A_f(s)) = A_f(s)\}$ forms an FS-interior system on X and we call it to be an FS-interior system generated by the FS-closure operator $\mathbf{Cl}$.

Theorem 2.38. *Let $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X, then the following conditions are equivalent:*
(i) $\delta_{\boldsymbol{I}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}};\ \boldsymbol{I}(A_f(s)) = A_f(s)\}$ *forms an FST on X.*
(ii) $\delta_{\boldsymbol{I}}(s) = \{A_f(s) \in (I^X)^{\mathbb{N}};\ \boldsymbol{I}(A_f(s)) = A_f(s)\}$ *forms an FS-interior system on X.*
(iii) $\{A_f(s);\ (A_f(s))^c \in \delta_{\boldsymbol{I}}(s)\}$ *forms an FS-closure system on X.*

Proof. Proof is omitted. □

Theorem 2.39. *Let $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X, then the following conditions are equivalent:*
(i) $\delta_{\boldsymbol{Cl}}(s) = \{(A_f(s))^c \in (I^X)^{\mathbb{N}};\ \boldsymbol{Cl}(A_f(s)) = A_f(s)\}$ *forms an FST on X.*
(ii) $\delta_{\boldsymbol{Cl}}(s) = \{(A_f(s))^c \in (I^X)^{\mathbb{N}};\ \boldsymbol{I}(A_f(s)) = A_f(s)\}$ *forms an FS-interior system on X.*
(iii) $\{A_f(s);\ (A_f(s))^c \in \delta_{\boldsymbol{Cl}}(s)\}$ *forms an FS-closure system on X.*

Proof. Proof is omitted. □

Theorem 2.40. *Let X be a non empty set. If $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X, then the operator $\boldsymbol{I_{Cl}} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by*

$$\boldsymbol{I_{Cl}}(A_f(s)) \quad = \quad X_f^1(s) - \boldsymbol{Cl}((A_f(s))^c) \ \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is an FS-interior operator on X. Again, if $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X, then the operator $\boldsymbol{Cl_I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by

$$\boldsymbol{Cl_I}(A_f(s)) \quad = \quad X_f^1(s) - \boldsymbol{I}((A_f(s))^c) \ \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is an FS-closure operator on X.

Proof. Proof is omitted. □

Note 2.41. It follows from **Theorem** 2.40 that given an FS-closure operator we can define an FS-interior operator and given an FS-interior operator we can define an FS-closure operator. In fact, there is a one to one correspondence between the collections of all FS-closure and FS-interior operators on a set (**Theorem** 2.42). We denote the collection of all FS-closure operators and the collection of all FS-interior operators on X by $\mathcal{C}_X$ and $\mathcal{I}_X$ respectively.

Theorem 2.42. *Let X be a non empty set, then there exists a one to one correspondence between $\mathcal{C}_X$ and $\mathcal{I}_X$.*

Proof. $t : \mathcal{C}_X \to \mathcal{I}_X$ by

$$t(\mathbf{Cl}) \quad = \quad \mathbf{I_{Cl}} \ \forall \mathbf{Cl} \in \mathcal{C}_X$$

Then t is a well defined map. Now, for $\mathbf{Cl}_1, \mathbf{Cl}_2 \in \mathcal{C}_X$ such that $t(\mathbf{Cl}_1) = t(\mathbf{Cl}_2)$, we have $\mathbf{I_{Cl_1}} = \mathbf{I_{Cl_2}}$. Hence $\forall A_f(s) \in (I^X)^{\mathbb{N}}$,

$$\begin{aligned} \mathbf{I_{Cl_1}}((A_f(s))^c) &= \mathbf{I_{Cl_2}}((A_f(s))^c) \\ X_f^1(s) - \mathbf{Cl}_1(A_f(s)) &= X_f^1(s) - \mathbf{Cl}_2(A_f(s)) \\ \mathbf{Cl}_1(A_f(s)) &= \mathbf{Cl}_2(A_f(s)) \end{aligned}$$

Thus t is injective. Again for $\mathbf{I} \in \mathcal{I}_X$, there is $\mathbf{Cl_I} \in \mathcal{C}_X$ such that $\forall A_f(s) \in (I^X)^{\mathbb{N}}$

$$\mathbf{Cl_I}((A_f(s))^c) \quad = \quad X_f^1(s) - \mathbf{I}(A_f(s))$$

Now, $\forall A_f(s) \in (I^X)^{\mathbb{N}}$

$$\begin{aligned} \mathbf{I_{Cl_I}}((A_f(s)) &= X_f^1(s) - \mathbf{Cl_I}((A_f(s))^c) \\ &= X_f^1(s) - (X_f^1(s) - \mathbf{I}(A_f(s))) \\ &= \mathbf{I}(A_f(s)) \end{aligned}$$

Therefore t is surjective and this completes the theorem. □

Note 2.43. If $\mathbf{I}$ is the t-image of $\mathbf{Cl}$ under the bijection t defined in **Theorem** 2.42, then $\mathbf{I}$ and $\mathbf{Cl}$ are called t-associated to each other.

Theorem 2.44. *The FST's induced by $\boldsymbol{Cl}$ and $\boldsymbol{I_{Cl}}$ are identical and the FST's induced by $\boldsymbol{I}$ and $\boldsymbol{Cl_I}$ are identical.*

Proof. Proof is omitted. □

Now, if we define an FS-interior and an FS-closure operator, separately, on a non empty set, they will induce two fuzzy sequential topologies which may not be identical in general. In view of **Theorem** 2.42 and **Theorem** 2.44, we give a necessary and sufficient condition that the two fuzzy sequential topologies induced by an FS-interior operator and an FS-closure operator are identical.

Theorem 2.45. *Let X be a non empty set. If $\boldsymbol{Cl} \in \mathcal{C}_X$ and $\boldsymbol{I} \in \mathcal{I}_X$, then $\delta_{\boldsymbol{Cl}}(s)$ and $\delta_{\boldsymbol{I}}(s)$ are identical iff $\boldsymbol{Cl}$ and $\boldsymbol{I}$ are t-associated to each other.*

Proof. Suppose $\mathbf{Cl}$ and $\mathbf{I}$ are t-associated to each other. Then $t(\mathbf{Cl}) = \mathbf{I}_{\mathbf{Cl}} = \mathbf{I}$. Now,

$$\begin{aligned} & A_f(s) \in \delta_{\mathbf{I}}(s) \\ \Leftrightarrow\ & \mathbf{I}(A_f(s)) = A_f(s) \\ \Leftrightarrow\ & \mathbf{I}_{\mathbf{Cl}}(A_f(s)) = A_f(s) \\ \Leftrightarrow\ & X_f^1(s) - \mathbf{Cl}((A_f(s))^c) = A_f(s) \\ \Leftrightarrow\ & \mathbf{Cl}((A_f(s))^c) = (A_f(s))^c \\ \Leftrightarrow\ & A_f(s) \in \delta_{\mathbf{Cl}}(s). \end{aligned}$$

Thus $\delta_{\mathbf{I}}(s)$ and $\delta_{\mathbf{Cl}}(s)$ are identical.
Conversely, suppose $\delta_{\mathbf{I}}(s)$ and $\delta_{\mathbf{Cl}}(s)$ are identical. Let $A_f(s) \in (I^X)^{\mathbb{N}}$. Then

$$\begin{aligned} & (\mathbf{Cl}((A_f(s))^c))^c \in \delta_{\mathbf{I}}(s) \\ \Rightarrow\ & \mathbf{I}((\mathbf{Cl}((A_f(s))^c))^c) = (\mathbf{Cl}((A_f(s))^c))^c = X_f^1(s) - \mathbf{Cl}((A_f(s))^c) \end{aligned}$$

Now,

$$\begin{aligned} & (A_f(s))^c \leq \mathbf{Cl}((A_f(s))^c) \\ \Rightarrow\ & (\mathbf{Cl}((A_f(s))^c))^c \leq A_f(s) \\ \Rightarrow\ & \mathbf{I}((\mathbf{Cl}((A_f(s))^c))^c) \leq \mathbf{I}(A_f(s)) \\ \Rightarrow\ & X_f^1(s) - \mathbf{Cl}((A_f(s))^c) \leq \mathbf{I}(A_f(s)). \end{aligned}$$

Again,

$$\begin{aligned} & \mathbf{I}(A_f(s)) \in \delta_{\mathbf{Cl}}(s) \\ \Rightarrow\ & \mathbf{Cl}((\mathbf{I}(A_f(s)))^c) = (\mathbf{I}(A_f(s)))^c = X_f^1(s) - \mathbf{I}(A_f(s)). \end{aligned}$$

Also,

$$\begin{aligned} & \mathbf{I}(A_f(s)) \leq A_f(s) \\ \Rightarrow\ & (A_f(s))^c \leq (\mathbf{I}(A_f(s)))^c \\ \Rightarrow\ & \mathbf{Cl}((A_f(s))^c) \leq \mathbf{Cl}((\mathbf{I}(A_f(s)))^c) = X_f^1(s) - \mathbf{I}(A_f(s)) \\ \Rightarrow\ & \mathbf{I}(A_f(s)) \leq X_f^1(s) - \mathbf{Cl}((A_f(s))^c). \end{aligned}$$

Thus $\mathbf{I}(A_f(s)) = X_f^1(s) - \mathbf{Cl}((A_f(s))^c) = \mathbf{I}_{\mathbf{Cl}}(A_f(s))\ \forall\ A_f(s) \in (I^X)^{\mathbb{N}}$. Hence $\mathbf{I} = \mathbf{I}_{\mathbf{Cl}} = t(\mathbf{Cl})$. □

Theorem 2.46. *Let X be a non empty set. If $\boldsymbol{Cl} \in \mathcal{C}_X$, $\boldsymbol{I} \in \mathcal{I}_X$, then the following conditions are equivalent:*

(i) *$\boldsymbol{I}$ and $\boldsymbol{Cl}$ are t-associated to each other.*

(ii) *The FST's* $\delta_{\boldsymbol{I}}(s)$ *and* $\delta_{\boldsymbol{Cl}}(s)$ *are identical.*
(iii) *FS-closure systems generated by* $\boldsymbol{Cl}$ *and* $\boldsymbol{I}$ *are identical.*
(iv) *FS-interior systems generated by* $\boldsymbol{Cl}$ *and* $\boldsymbol{I}$ *are identical.*

Proof. Proof is omitted. □

Note 2.47. Theorem 2.46 gives two more necessary and sufficient conditions ((iii) and (iv)), that the fuzzy sequential topologies induced by an FS-interior operator and an FS-closure operator are identical.

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