Fuzzy implicative filters of \(BE\)-algebras

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Received 16 April 2013; Accepted 9 May 2013

Abstract. The concept of fuzzy implicative filters is introduced in \(BE\)-algebras. A characterization of fuzzy implicative filters is derived in terms of fuzzy filters of a \(BE\)-algebra. Some properties of fuzzy relations and homomorphisms are studied with respect to a fuzzy implicative filter.

2010 AMS Classification: 06F35, 03G25, 08A30

Keywords: \(BE\)-algebra, Implicative filter, Fuzzy filter, Fuzzy implicative filter, Level subset, Fuzzy relation.

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1. Introduction


In this paper, the fuzzification of implicative filters of \(BE\)-algebras is considered and discussed the related properties. We discuss a characterization of fuzzy implicative filters of \(BE\)-algebras in terms of fuzzy level filters. It is also observed that every fuzzy implicative filter of a \(BE\)-algebra is a fuzzy filter but not the converse. Some equivalent conditions are derived for a fuzzy filter of a \(BE\)-algebra to become a fuzzy implicative filter. An extension property of fuzzy implicative filters is also
studied. The properties of homomorphic images of fuzzy implicative filters are studied. Finally, fuzzy relations including the cartesian products of fuzzy implicative filters are discussed in BE-algebras.

2. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [1], [7], [8], [9] and [10] for the ready reference of the reader.

Definition 2.1 ([7]). An algebra \((X, \ast, 1)\) of type \((2, 0)\) is called a BE-algebra if it satisfies the following properties:
\[
\begin{align*}
(1) & \quad x \ast x = 1 \\
(2) & \quad x \ast 1 = 1 \\
(3) & \quad 1 \ast x = x \\
(4) & \quad x \ast (y \ast z) = y \ast (x \ast z)
\end{align*}
\]
for all \(x, y, z \in X\).

Theorem 2.2 ([7]). Let \((X, \ast, 1)\) be a BE-algebra. Then we have the following:
\[
\begin{align*}
(1) & \quad x \ast (y \ast x) = 1 \\
(2) & \quad x \ast ((x \ast y) \ast y)) = 1
\end{align*}
\]

We introduce a relation \(\leq\) on a BE-algebra \(X\) by \(x \leq y\) implies \(x \ast y = 1\). A BE-algebra \(X\) is called self-distributive if \(x \ast (y \ast z) = (x \ast y) \ast (x \ast z)\) for all \(x, y, z \in X\).

Definition 2.3 ([8]). A BE-algebra \((X, \ast, 1)\) is said to be transitive if for all \(x, y, z \in X\), it satisfies \(y \ast z \leq (x \ast y) \ast (x \ast z)\).

Definition 2.4 ([1]). Let \((X, \ast, 1)\) be a BE-algebra. A non-empty subset \(F\) of \(X\) is called a filter of \(X\) if, for all \(x, y \in X\), it satisfies the following properties:
\[
\begin{align*}
(a) & \quad 1 \in F \\
(b) & \quad x \ast y \in F \text{ imply that } y \in F
\end{align*}
\]

Definition 2.5 ([10]). Let \((X_1, \ast, 1)\) and \((X_2, \circ, 1')\) be two BE-algebras. Then a mapping \(f : X_1 \longrightarrow X_2\) is called a homomorphism if \(f(x \ast y) = f(x) \circ f(y)\) for all \(x, y \in X_1\).

It it clear that if \(f : X_1 \longrightarrow X_2\) is a homomorphism, then \(f(1) = 1'\).

Definition 2.6 ([10]). Let \(X\) be a set. Then a fuzzy set in \(X\) is a function \(\mu : X \longrightarrow [0, 1]\).

Definition 2.7 ([9]). A fuzzy set \(\mu\) in \(X\) is called a fuzzy filter of \(X\) if it satisfies:
\[
\begin{align*}
(F_1) & \quad \mu(1) \geq \mu(x) \\
(F_2) & \quad \mu(y) \geq \min\{\mu(x), \mu(x \ast y)\} \text{ for all } x, y \in X
\end{align*}
\]

Lemma 2.8 ([9]). Let \(\mu\) be a fuzzy filter of a BE-algebra \(X\). Then the following conditions hold for all \(x, y \in X\):
\[
\begin{align*}
(1) & \quad \mu(x \ast y) = \mu(1) \text{ implies } \mu(x) \leq \mu(y) \\
(2) & \quad x \leq y \text{ implies } \mu(x) \leq \mu(y)
\end{align*}
\]
Definition 2.9 ([9]). Let \( \mu \) be a fuzzy set in a BE-algebra \( X \). For any \( \alpha \in [0,1] \), the set \( \mu_\alpha = \{ x \in X \mid \mu(x) \geq \alpha \} \) is called a level subset of \( \mu \).

Definition 2.10 ([9]). Let \( \mu \) be a fuzzy filter of a BE-algebra \( X \). Then the filters \( \mu_\alpha = \{ x \in X \mid \mu(x) \geq \alpha \} \), \( \alpha \in [0,1] \), are called level filters of \( X \).

Theorem 2.11 ([9]). A fuzzy set \( \mu \) of a BE-algebra \( X \) is a fuzzy filter in \( X \) if and only if it satisfies the following conditions:

1. \( \mu(1) \geq \mu(x) \) for all \( x \in X \)
2. \( \mu(x * z) \geq \min \{ \mu(x * (y * z)), \mu(y) \} \) for all \( x, y, z \in X \)

3. Fuzzy implicative filters of BE-algebras

In this section, we introduce the concept of implicative filters in BE-algebras and then we discuss the fuzzification of implicative filters in BE-algebras.

Definition 3.1. A non-empty subset \( F \) of a BE-algebra \( X \) is called an implicative filter if, for all \( x, y, z \in X \), it satisfies the following conditions.

1. \( 1 \in F \)
2. \( x * (y * z) \in F \) and \( x * y \in F \) imply that \( x * z \in F \)

Definition 3.2. A fuzzy set \( \mu \) of a BE-algebra \( X \) is called a fuzzy implicative filter if it satisfies, for all \( x, y, z \in X \), the following conditions.

1. \( \mu(1) \geq \mu(x) \)
2. \( \mu(x * z) \geq \min \{ \mu(x * (y * z)), \mu(x * y) \} \)

If we replace \( x \) of the above Definitions 3.1 and 3.2 by the element 1, then it can be easily observed that every implicative filter is a filter as well as every fuzzy implicative filter is a fuzzy filter. However, every fuzzy filter is not a fuzzy implicative filter as shown in the following example.

Example 3.3. Let \( X = \{1, a, b, c\} \) be a non-empty set. Define a binary operation \(*\) on \( X \) as follows:

\[
\begin{array}{cccc}
* & 1 & a & b & c & d \\
1 & 1 & a & b & c & d \\
a & 1 & 1 & b & c & b \\
b & 1 & a & 1 & b & a \\
c & 1 & a & 1 & a & a \\
d & 1 & 1 & 1 & b & 1 \\
\end{array}
\]

Then it can be easily verified that \((X, *, 1)\) is a BE-algebra. Define a fuzzy set \( \mu \) on \( X \) as follows:

\[
\mu(x) = \begin{cases} 
0.9 & \text{if } x = a, 1 \\
0.2 & \text{otherwise} 
\end{cases}
\]

for all \( x \in X \). Then clearly \( \mu \) is a fuzzy filter of \( X \), but \( \mu \) is not a fuzzy implicative filter of \( X \) since \( \mu(b * c) \not\geq \min \{ \mu(b * (d * c)), \mu(b * d) \} \).

Theorem 3.4. Let \( X \) be a distributive BE-algebra. Then every fuzzy filter is a fuzzy implicative filter.
Proof. Let $F$ be a fuzzy filter of a distributive $BE$-algebra $X$. Then clearly $\mu(1) \geq \mu(x)$ for all $x \in X$. Let $x, y, z \in X$. Since $F$ is a fuzzy filter, by Theorem 2.11, we get $\mu(x \ast z) \geq \min\{\mu((x \ast y) \ast (x \ast z)), \mu(x \ast y)\} = \min\{\mu(x \ast (y \ast z)), \mu(x \ast y)\}$, because of $X$ is distributive. Therefore $\mu$ is a fuzzy implicative filter in $X$. \qed

We now derive a sufficient condition for every fuzzy filter of a transitive $BE$-algebra to become a fuzzy implicative filter.

**Theorem 3.5.** Let $\mu$ be a fuzzy filter of a transitive $BE$-algebra $X$ such that

$$\mu(y \ast z) \geq \min\{\mu(x), \mu((x \ast (y \ast z)))\}$$

for all $x, y, z \in X$.

*Proof.* Assume that $\mu$ is a fuzzy filter of $X$ satisfying the given condition. Let $x, y, z \in X$. Since $X$ is a transitive $BE$-algebra, we get

$$\mu(x \ast (y \ast z)) = \mu((x \ast (y \ast z))) \leq \mu(x \ast y) \ast (x \ast z)$$

Hence by assumed condition, we get

$$\mu(x \ast z) \geq \min\{\mu(x \ast y), \mu((x \ast y) \ast (x \ast z))\} \geq \min\{\mu(x \ast y), \mu(x \ast (y \ast z))\}$$

Therefore $\mu$ is a fuzzy implicative filter in $X$. \qed

In the following, we derive a set of equivalent conditions for every fuzzy filter of a transitive $BE$-algebra to become a fuzzy implicative filter.

**Theorem 3.6.** Let $\mu$ be a fuzzy filter of a transitive $BE$-algebra $X$. Then the following conditions are equivalent.

1. $\mu$ is a fuzzy implicative filter in $X$
2. $\mu(x \ast y) = \mu((x \ast (y \ast z)))$ for all $x, y \in X$
3. $\mu((x \ast (y \ast z))) \geq \mu(x \ast (y \ast z))$ for all $x, y, z \in X$

*Proof.* (1) $\Rightarrow$ (2) : Assume that $\mu$ is a fuzzy implicative filter in $X$. Let $x, y \in X$. Since $x \ast y \leq x \ast (x \ast y)$, we get $\mu(x \ast y) \leq \mu(x \ast (x \ast y))$. Since $\mu$ is a fuzzy filter in $X$, we get that

$$\mu(x \ast y) \geq \min\{\mu(x \ast (x \ast y)), \mu(x \ast x)\} = \min\{\mu(x \ast (x \ast y)), \mu(1)\} = \mu(x \ast (x \ast y))$$

Hence $\mu(x \ast y) \geq \mu(x \ast (x \ast y))$. Therefore $\mu(x \ast y) = \mu(x \ast (x \ast y))$.

(2) $\Rightarrow$ (3) : Assume the condition (2). Let $x, y, z \in X$. Since $X$ is transitive, we get $y \ast z \leq (x \ast y) \ast (x \ast z)$ and hence we get that $x \ast (y \ast z) \leq x \ast ((x \ast y) \ast (x \ast z))$. Hence $\mu(x \ast (y \ast z)) \leq \mu(x \ast ((x \ast y) \ast (x \ast z)))$. Now, we get

$$\mu((x \ast (y \ast z))) = \mu((x \ast (x \ast y) \ast z))$$

by (2)

$$\geq \mu((x \ast (x \ast y) \ast (x \ast z)))$$

for all $x, y, z \in X$. Since $X$ is transitive, we get $y \ast z \leq (x \ast y) \ast (x \ast z)$ and hence we get that $x \ast (y \ast z) \leq x \ast ((x \ast y) \ast (x \ast z))$. Hence $\mu(x \ast (y \ast z)) \leq \mu(x \ast ((x \ast y) \ast (x \ast z)))$. Now, we get

$$\mu((x \ast (y \ast z))) = \mu((x \ast (x \ast y) \ast z)) = \mu((x \ast (x \ast y) \ast (x \ast z)))$$

by (2)

$$\geq \mu((x \ast (x \ast y) \ast (x \ast z)))$$

for all $x, y, z \in X$. Since $X$ is transitive, we get $y \ast z \leq (x \ast y) \ast (x \ast z)$ and hence we get that $x \ast (y \ast z) \leq x \ast ((x \ast y) \ast (x \ast z))$. Hence $\mu(x \ast (y \ast z)) \leq \mu(x \ast ((x \ast y) \ast (x \ast z)))$. Now, we get

$$\mu((x \ast (y \ast z))) = \mu((x \ast (x \ast y) \ast z)) = \mu((x \ast (x \ast y) \ast (x \ast z)))$$

by (2)

$$\geq \mu((x \ast (x \ast y) \ast (x \ast z)))$$

for all $x, y, z \in X$. Since $X$ is transitive, we get $y \ast z \leq (x \ast y) \ast (x \ast z)$ and hence we get that $x \ast (y \ast z) \leq x \ast ((x \ast y) \ast (x \ast z))$. Hence $\mu(x \ast (y \ast z)) \leq \mu(x \ast ((x \ast y) \ast (x \ast z)))$. Now, we get

$$\mu((x \ast (y \ast z))) = \mu((x \ast (x \ast y) \ast z)) = \mu((x \ast (x \ast y) \ast (x \ast z)))$$

by (2)
(3) ⇒ (1): Assume that \( \mu \) satisfies the condition (2). Let \( x, y, z \in X \). Since \( \mu \) is a fuzzy filter in \( X \), we have the following
\[
\mu(x \ast z) \geq \min\{\mu((x \ast y) \ast (x \ast z)), \mu(x \ast y)\} \\
\geq \min\{\mu(x \ast (y \ast z)), \mu(x \ast y)\}
\]
Therefore \( \mu \) is a fuzzy implicative filter in \( X \).

**Proposition 3.7.** Let \( \mu \) be a fuzzy set in a BE-algebra \( X \). Then \( \mu \) is a fuzzy implicative filter in \( X \) if and only if for each \( \alpha \in [0, 1] \), the level set \( \mu_\alpha \) is an implicative filter in \( X \) when \( \mu_\alpha \neq \emptyset \).

**Proof.** Assume that \( \mu \) is a fuzzy implicative filter of \( X \). Then \( \mu(1) \geq \mu(x) \) for all \( x \in X \). In particular, \( \mu(1) \geq \mu(x) \geq \alpha \) for all \( x \in \mu_\alpha \). Hence \( 1 \in \mu_\alpha \). Let \( x \ast (y \ast z), x \ast y \in \mu_\alpha \). Then \( \mu(x \ast (y \ast z)) \geq \alpha \) and \( \mu(x \ast y) \geq \alpha \). Since \( \mu \) is a fuzzy implicative filter, we get \( \mu(x \ast z) \geq \min\{\mu(x \ast (y \ast z)), \mu(x \ast y)\} \geq \alpha \). Thus \( x \ast z \in \mu_\alpha \). Therefore \( \mu_\alpha \) is an implicative filter in \( X \).

Conversely, assume that \( \mu_\alpha \) is an implicative filter of \( X \) for each \( \alpha \in [0, 1] \) with \( \mu_\alpha \neq \emptyset \). Suppose there exists \( x_0 \in X \) such that \( \mu(1) < \mu(x_0) \). Again, let \( \alpha_0 = \frac{1}{2}(\mu(1) + \mu(x_0)) \). Then \( \mu(1) < \alpha_0 \) and \( 0 \leq \alpha_0 < \mu(x_0) \leq 1 \). Hence \( x_0 \in \mu_\alpha_1 \) and \( \mu_\alpha_1 \neq \emptyset \). Since \( \mu_\alpha_1 \) is an implicative filter in \( X \), we get \( 1 \in \mu_\alpha_1 \) and hence \( \mu(1) \geq \alpha_0 \), which is a contradiction. Therefore \( \mu(1) \geq \mu(x) \) for all \( x \in X \). Let \( x, y, z \in X \) be such that \( \mu(x \ast (y \ast z)) = \alpha_1 \) and \( \mu(x \ast y) \geq \alpha_2 \). Then \( x \ast (y \ast z) \in \mu_\alpha_1 \) and \( x \ast y \in \mu_\alpha_2 \).

Without loss of generality, assume that \( \alpha_1 \leq \alpha_2 \). Then clearly \( \mu_\alpha_2 \subseteq \mu_\alpha_1 \). Hence \( x \ast y \in \mu_\alpha_1 \). Since \( \mu_\alpha_1 \) is an implicative filter in \( X \), we get \( x \ast z \in \mu_\alpha_1 \). Thus \( \mu(x \ast z) \geq \min\{\alpha_1, \alpha_2\} = \min\{\mu(x \ast (y \ast z)), \mu(x \ast y)\} \). Therefore \( \mu \) is a fuzzy implicative filter of \( X \).

**Theorem 3.8.** Let \( F \) be an implicative filter of a BE-algebra. Then there exists a fuzzy implicative filter \( \mu \) of \( X \) such that \( \mu_\alpha = F \) for some \( \alpha \in (0, 1) \).

**Proof.** Let \( \mu \) be a fuzzy set in a BE-algebra \( X \) defined by
\[
\mu(x) = \begin{cases} 
\alpha & \text{if } x \in F \\
0 & \text{otherwise}
\end{cases}
\]
where \( \alpha \) is a fixed number \( (0 < \alpha < 1) \). Since \( 1 \in F \), we get \( \mu(1) = \alpha \geq \mu(x) \) for all \( x \in X \). Let \( x, y, z \in X \). Suppose \( x \ast (y \ast z), x \ast y \in F \). Since \( F \) is an implicative filter, we get \( x \ast z \in F \). Then \( \mu(x \ast y) = \mu(x \ast (y \ast z)) \geq \min\{\mu(x \ast (y \ast z)), \mu(x \ast y)\} \). Hence \( \mu(x \ast z) \geq \min\{\mu(x \ast (y \ast z)), \mu(x \ast y)\} \). Suppose \( x \ast (y \ast z) \notin F \) and \( x \ast y \notin F \). Then \( \mu(x \ast y) = \mu(x \ast (y \ast z)) = 0 \). Hence \( \mu(x \ast z) \geq \min\{\mu(x \ast (y \ast z)), \mu(x \ast y)\} \). If exactly one of \( x \ast (y \ast z) \) and \( x \ast y \) is in \( F \), then exactly one of \( \mu(x \ast (y \ast z)) \) and \( \mu(x \ast y) \) is equal to 0. Hence \( \mu(x \ast z) \geq \min\{\mu(x \ast (y \ast z)), \mu(x \ast y)\} \). By summarizing the above results, we get \( \mu(x \ast z) \geq \min\{\mu(x \ast (y \ast z)), \mu(x \ast y)\} \) for all \( x, y, z \in X \). Therefore \( \mu \) is a fuzzy implicative filter of \( X \). Clearly \( \mu_\alpha = F \).

**Theorem 3.9.** Let \( \mu \) be a fuzzy implicative filter of a BE-algebra \( X \). Then two level implicative filters \( \mu_\alpha_1 \) and \( \mu_\alpha_2 \) (with \( \alpha_1 < \alpha_2 \)) of \( \mu \) are equal if and only if there is no \( x \in X \) such that \( \alpha_1 \leq \mu(x) < \alpha_2 \).
Assume that \( \mu_{\alpha_1} = \mu_{\alpha_2} \) for \( \alpha_1 < \alpha_2 \). Suppose there exists some \( x \in X \) such that \( \alpha_1 \leq \mu(x) < \alpha_2 \). Then \( \mu_{\alpha_2} \) is a proper subset of \( \mu_{\alpha_1} \), which is impossible. Conversely, assume that there is no \( x \in X \) such that \( \alpha_1 \leq \mu(x) < \alpha_2 \). Since \( \alpha_1 < \alpha_2 \), we get that \( \mu_{\alpha_2} \subseteq \mu_{\alpha_1} \). If \( x \in \mu_{\alpha_1} \), then \( \mu(x) \geq \alpha_1 \). Hence by assumption, we get that \( \mu(x) \geq \alpha_2 \). Hence \( x \in \mu_{\alpha_2} \) and so \( \mu_{\alpha_1} \subseteq \mu_{\alpha_2} \). Therefore \( \mu_{\alpha_1} = \mu_{\alpha_2} \). \( \square \)

**Theorem 3.10.** Let \( \mu \) be a fuzzy implicative filter of \( X \) with \( \text{Im}(\mu) = \{ \alpha_i \mid i \in \Delta \} \) and \( F = \{ \mu_{\alpha_i} \mid i \in \Delta \} \) where \( \Delta \) is an arbitrary indexed set. If \( \mu \) attains its infimum on all implicative filters of \( X \), then \( F \) contains all level implicative filters of \( \mu \).

**Proof.** Suppose \( \mu \) attains its infimum on all implicative filters of \( X \). Let \( \mu_{\alpha} \) be a level implicative filter of \( \mu \). If \( \alpha = \alpha_i \) for some \( i \in \Delta \), then clearly \( \mu_{\alpha} \in F \). Assume that \( \alpha \neq \alpha_i \) for all \( i \in \Delta \). Then there exists no \( x \in X \) such that \( \mu(x) = \alpha \). Let \( F = \{ x \in X \mid \mu(x) > \alpha \} \). Clearly \( 1 \in F \). Let \( x, y, z \in X \) be such that \( x * y \in F \) and \( x * (y * z) \in F \). Then \( \mu(x * y) > \alpha \) and \( \mu(x * (y * z)) \). Since \( \mu \) is a fuzzy implicative filter in \( X \), we get

\[
\mu(x * z) \geq \min\{\mu(x * (t * z)), \mu(x * y)\} > \alpha
\]

Hence \( \mu(x * z) > \alpha \), which implies that \( x * z \in F \). Therefore \( F \) is an implicative filter of \( X \). By the hypothesis, there exists \( y \in F \) such that

\[
\mu(y) = \inf\{\mu(x) \mid x \in X\}
\]

Hence \( \mu(y) \in \text{Im}(\mu) \), which yields that \( \mu(y) = \alpha_i \) for some \( i \in \Delta \). It is clear that \( \alpha_i \geq \alpha \). Hence, by assumption, we get \( \alpha_i > \alpha \). Thus there exists no \( x \in X \) such that \( \alpha \leq \mu(x) < \alpha_i \). Hence by above Theorem 3.9, we get \( \mu_{\alpha} = \mu_{\alpha_i} \). Therefore \( \mu_{\alpha} \in F \). This completes the proof. \( \square \)

We now present an extension property of fuzzy implicative filters

**Theorem 3.11.** Let \( \mu \) and \( \nu \) be two fuzzy filters of a transitive \( BE \)-algebra \( X \) with \( \mu \leq \nu \) and \( \mu(1) = \nu(1) \). If \( \mu \) is a fuzzy implicative filter of \( X \), then so is \( \nu \).

**Proof.** Assume that \( \mu \) and \( \nu \) are two fuzzy filters of a transitive \( BE \)-algebra \( X \) such that \( \mu \leq \nu \) and \( \mu(1) = \nu(1) \). Suppose \( \mu \) is a fuzzy implicative filter of \( X \). Let \( x, y, z \in X \). Since \( \mu \subseteq \nu \) and \( \mu \) is a fuzzy implicative filter, by Theorem 3.6(3), we can obtain the following consequence.

\[
\nu((x * y) * (x * ((y * z) * z))) \geq \mu((x * y) * ((x * (y * z)) * z)) \\
\geq \mu(x * ((y * z) * (y * z))) \\
= \mu(x * (y * z) * (y * z)) \\
= \mu((x * (y * z)) * (x * (y * z))) \\
= \mu(1) \\
= \nu(1)
\]

whence

\[
\nu((x * (y * z)) * ((x * y) * (x * z))) = \nu((x * y) * ((x * (y * z)) * (x * z))) \\
= \nu((x * y) * (x * ((y * z) * (y * z)))) \\
\geq \nu(1)
\]

\( \geq 760 \)}
Hence \( \nu((x \ast (y \ast z)) \ast ((x \ast y) \ast (x \ast z))) = \nu(1) \). Since \( \nu \) is a fuzzy filter, by Lemma 2.8, we get that

\[
\nu((x \ast y) \ast (x \ast z)) \geq \nu(x \ast (y \ast z))
\]

By Theorem 3.6(3), it yields that \( \nu \) is a fuzzy implicative filter in \( X \).

**Definition 4.12.** Let \( f : X \to Y \) be a homomorphism of BE-algebras and \( \mu \) is a fuzzy set in \( Y \). Then define a mapping \( \mu^f : X \to [0, 1] \) such that \( \mu^f(x) = \mu(f(x)) \) for all \( x \in X \).

Clearly the above mapping \( \mu^f \) is well-defined and a fuzzy set in \( X \).

**Theorem 3.13.** Let \( f : X \to Y \) be an onto homomorphism of BE-algebras and \( \mu \) is a fuzzy set in \( Y \). Then \( \mu \) is a fuzzy implicative filter in \( Y \) if and only if \( \mu^f \) is a fuzzy implicative filter in \( X \).

**Proof.** Assume that \( \mu \) is a fuzzy implicative filter of \( Y \). For any \( x \in X \), we have

\[
\mu^f(1) = \mu(f(1)) = \mu(Y) \geq \mu(f(x)) = \mu^f(x).
\]

Let \( x, y, z \in X \). Then

\[
\mu^f(x \ast z) = \mu(f(x \ast z)) = \mu(f(x) \ast f(z)) \geq \min\{\mu(f(x) \ast (f(y) \ast f(z))), \mu(f(x) \ast f(y))\} = \min\{\mu(f(x \ast (y \ast z))), \mu(f(x \ast y))\} = \min\{\mu^f(x \ast (y \ast z)), \mu^f(x \ast y)\}
\]

Hence \( \mu^f \) is a fuzzy implicative filter of \( X \). Conversely, assume that \( \mu^f \) is a fuzzy implicative filter of \( X \). Let \( x \in Y \). Since \( f \) is onto, there exists \( y \in X \) such that \( f(y) = x \). Then \( \mu^f(1) = \mu(f(1)) = \mu^f(1) \geq \mu^f(y) = \mu(f(y)) = \mu(x) \). Let \( x, y, z \in Y \). Then there exist \( a, b, c \in X \) such that \( f(a) = x, f(b) = y \) and \( f(c) = z \). Hence we get

\[
\mu(x \ast y) = \mu(f(a) \ast f(c)) = \mu(f(a \ast c)) = \mu^f(a \ast c) \geq \min\{\mu^f(a \ast (b \ast c)), \mu^f(a \ast b)\} = \min\{\mu(f(a) \ast (f(b) \ast f(c))), \mu(f(a) \ast f(b))\} = \min\{\mu(x \ast (y \ast z)), \mu(x \ast y)\}
\]

Therefore \( \mu \) is a fuzzy implicative filter in \( Y \).

4. Cartesian products of fuzzy implicative filters

In this section, we discuss some properties of the cartesian products of fuzzy implicative filters of BE-algebras. The notion of fuzzy relations [2] are extended to the case of fuzzy implicative filters of BE-algebras.

**Definition 4.1.** A fuzzy relation on a set \( S \) is a fuzzy set \( \mu : S \times S \to [0, 1] \).

**Definition 4.2.** Let \( \mu \) be a fuzzy relation on a set \( S \) and \( \nu \) a fuzzy set in \( S \). Then \( \mu \) is a fuzzy relation on \( \nu \) if for all \( x, y \in S \),

\[
\mu(x, y) \geq \nu(x) \ast \nu(y)
\]
\[ \mu(x, y) \leq \min\{\nu(x), \nu(y)\} \]

**Definition 4.3.** Let \( \mu \) and \( \nu \) be two fuzzy sets in a \( BE \)-algebra \( X \). The cartesian product of \( \mu \) and \( \nu \) is defined by

\[ (\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} \]

for all \( x, y \in X \).

The following Lemma is a direct consequence of the above definitions.

**Lemma 4.4.** Let \( \mu \) and \( \nu \) be two fuzzy sets in a \( BE \)-algebra \( X \). Then the following hold.

1. \( \mu \times \nu \) is a fuzzy relation on \( X \).
2. \( (\mu \times \nu)_\alpha = \mu_\alpha \times \nu_\alpha \) for all \( \alpha \in [0, 1] \).

For any two \( BE \)-algebras \( X \) and \( Y \), define an operation \( * \) on \( X \times Y \) as follows:

\[ (x, y) * (x', y') = (x * x', y * y') \]

for all \( x, x' \in X \) and \( y, y' \in Y \).

Then it can be easily observed that \((X \times Y, *, (1, 1))\) is a \( BE \)-algebra.

**Proposition 4.5.** Let \( F \) and \( G \) be two implicative filters of the \( BE \)-algebras \( X \) and \( Y \) respectively. Then \( F \times G \) is an implicative filter in \( X \times Y \).

**Proof.** It can be routinely verified. \( \square \)

**Theorem 4.6.** Let \( \mu \) and \( \nu \) be two fuzzy implicative filters of a \( BE \)-algebra \( X \). Then \( \mu \times \nu \) is a fuzzy implicative filter in \( X \times X \).

**Proof.** Let \( (x, y) \in X \times X \). Since \( \mu \) and \( \nu \) are fuzzy implicative filters in \( X \), we get

\[ (\mu \times \nu)(1, 1) = \min\{\mu(1), \nu(1)\} \]
\[ \geq \min\{\mu(x), \nu(y)\} \]
\[ = (\mu \times \nu)(x, y) \]

for all \( x, y \in X \).

Let \( (x, x'), (y, y'), (z, z') \in X \times X \). Put \( t = x * (y * z) \) and \( t' = x' * (y' * z') \). Clearly

\[ (t, t') = (x, x') * ((y, y') * (z, z')) \].

Since \( \mu \) and \( \nu \) are fuzzy implicative filters in \( X \), we can obtain the following consequence.

\[ (\mu \times \nu)((x, x') * (z, z')) = (\mu \times \nu)(x * z, x' * z') \]
\[ = (\mu \times \nu)(x * z, x' * z') \]
\[ \geq \min\{\min\{\mu(x * y), \mu(t)\}, \min\{\nu(x' * y'), \nu(t')\}\} \]
\[ = \min\{\min\{\mu(x * y), \nu(x' * y')\}, \min\{\mu(t), \nu(t')\}\} \]
\[ = \min\{\mu(x * y), \nu(x' * y')\}, \min\{\mu(t), \nu(t')\}\} \]
\[ = \min\{\mu(x * y), \nu(x' * y')\}, (\mu \times \nu)(t, t')\} \]
\[ = \min\{\mu(x * y), \nu(x' * y')\}, (\mu \times \nu)(t, t')\} \]

Therefore \( \mu \times \nu \) is a fuzzy implicative filter in \( X \times X \). \( \square \)

**Theorem 4.7.** Let \( \mu \) and \( \nu \) be two fuzzy sets in a \( BE \)-algebra \( X \) such that \( \mu \times \nu \) is a fuzzy filter of \( X \). Then we have the following:

1. \( \mu(x) \leq \mu(1) \) or \( \nu(x) \leq \nu(1) \) for all \( x \in X \).
2. \( \mu(x) \leq \mu(1) \) for all \( x \in X \), then either \( \mu(x) \leq \mu(1) \) or \( \nu(x) \leq \nu(1) \).
3. \( \nu(x) \leq \nu(1) \) for all \( x \in X \), then either \( \mu(x) \leq \mu(1) \) or \( \mu(x) \leq \mu(1) \).
4. either \( \mu \) or \( \nu \) is a fuzzy implicative filter of \( X \).
Proof. (1). Suppose $\mu(x) > \mu(1)$ and $\nu(y) > \nu(1)$ for some $x, y \in X$. Then 
$$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} > \min\{\mu(1), \nu(1)\} = (\mu \times \nu)(1, 1),$$
which is a contradiction. Hence either $\mu(x) \leq \mu(1)$ or $\nu(x) \leq \nu(1)$ for all $x \in X$.

(2). Assume that $\mu(x) \leq \mu(1)$ for all $x \in X$. Suppose $\mu(x) > \mu(1)$ and $\nu(y) > \nu(1)$
for some $x, y \in X$. Then 
$$(\mu \times \nu)(1, 1) = \min\{\mu(1), \nu(1)\} = \nu(1).$$
Hence 
$$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} > \nu(1) = (\mu \times \nu)(1, 1)$$
which is a contradiction. Therefore (2) holds.

(3). It can be obtained in a similar fashion.

(4). Since, by (1), either $\mu(x) \leq \mu(1)$ or $\nu(x) \leq \nu(1)$ for all $x \in X$. Without loss
of generality, we may assume that $\mu(x) \leq \mu(1)$ for all $x \in X$. From (2), we can get
either $\mu(x) \leq \nu(1)$ or $\nu(x) \leq \nu(1)$ for all $x \in X$.

**Case I:** Suppose $\mu(x) \leq \nu(1)$ for all $x \in X$. Then
$$(\mu \times \nu)(x, 1) = \min\{\mu(x), \nu(1)\} = \mu(x)$$
for all $x \in X$.

Since $\mu \times \nu$ is a fuzzy implicative filter in $X \times X$, we get $\mu(x) = \min\{\mu(x), \nu(1)\} = 
(\mu \times \nu)(x, 1) \leq (\mu \times \nu)(1, 1) = \mu(1)$. Also
$$\mu(x \ast z) = \min\{\mu(x \ast z), \nu(1)\}$$
$$= (\mu \times \nu)(x \ast z, 1)$$
$$= (\mu \times \nu)(x \ast z, x \ast 1)$$
$$\geq \min\{(\mu \times \nu)((x, x) \ast (y, 1)), (\mu \times \nu)((x, x) \ast ((y, 1) \ast (z, 1)))\}$$
$$= \min\{(\mu \times \nu)(x \ast y \ast 1), (\mu \times \nu)(x \ast (y \ast z), x \ast (1 \ast 1))\}$$
$$= \min\{\min\{\mu(x \ast y), \nu(y \ast 1)\}, \min\{\mu(x \ast (y \ast z)), \nu(x \ast (1 \ast 1))\}\}$$
$$= \min\{\min\{\mu(x \ast y), \nu(1)\}, \min\{\mu(x \ast (y \ast z)), \nu(1)\}\}$$
$$= \min\{\mu(x \ast y), \mu(x \ast (y \ast z))\}$$
Therefore $\mu$ is a fuzzy implicative filter in $X$.

**Case II:** Suppose $\nu(x) \leq \nu(1)$ for all $x \in X$. Suppose $\mu(x) \leq \nu(1)$ for all $x \in X$.

Then it leads to case I, from which we can conclude that $\mu$ is a fuzzy implicative
filter. Suppose $\mu(t) > \nu(1)$ for some $t \in X$. Then $\mu(1) \geq \mu(t) > \nu(1)$. Since
$\nu(x) \leq \nu(1)$ for all $x \in X$, it yields that $\mu(1) > \nu(x)$ for all $x \in X$. Hence
$$\nu(x \ast z) = \min\{\mu(1), \nu(x \ast z)\}$$
$$= (\mu \times \nu)(1, x \ast z)$$
$$= (\mu \times \nu)(x \ast 1, x \ast z)$$
$$= (\mu \times \nu)((x, x) \ast (1, z))$$
$$\geq \min\{(\mu \times \nu)((x, x) \ast (1, y)), (\mu \times \nu)((x, x) \ast ((1, y) \ast (1, z)))\}$$
$$= \min\{(\mu \times \nu)(x \ast 1, x \ast y), (\mu \times \nu)(x \ast (1 \ast 1), x \ast (y \ast z))\}$$
$$= \min\{\min\{\mu(1), \nu(x \ast y)\}, \min\{\mu(1), \nu(x \ast (y \ast z))\}\}$$
$$= \min\{\nu(x \ast y), \nu(x \ast (y \ast z))\}$$
Therefore $\nu$ is a fuzzy implicative filter in $X$. \qed
In the following, we present an example to show that if $\mu \times \nu$ is a fuzzy implicative filter of $X \times X$, then $\mu$ and $\nu$ both need not be fuzzy implicative filters of $X$.

**Example 4.8.** Let $X$ be a $BE$-algebra with $|X| > 2$ and let $\alpha, \beta \in [0, 1]$ be such that $0 \leq \alpha \leq \beta < 1$. Define fuzzy sets $\mu$ and $\nu : X \to [0, 1]$ by $\mu(x) = \alpha$ and $\nu(x) = \beta$ if $x = 1$ and $\nu(x) = 1$ if $x \neq 1$ for all $x \in X$, respectively. Then we get $(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\} = \alpha$ for all $(x, y) \in X \times X$. Hence $\mu \times \nu : X \times X \to [0, 1]$ is a constant function. Thus $\mu \times \nu$ is a fuzzy implicative filter of $X \times X$. Now $\mu$ is a fuzzy implicative filter of $X$ but $\nu$ is not a fuzzy implicative filter of $X$ because $\nu$ does satisfy $F_1$.

**Definition 4.9.** Let $\nu$ be a fuzzy set in a $BE$-algebra $X$. Then the strongest fuzzy relation $\mu_\nu$ is a fuzzy relation $X$ defined by

$$\mu_\nu(x, y) = \min\{\nu(x), \nu(y)\}$$

for all $x, y \in X$.

**Theorem 4.10.** Let $\nu$ be a fuzzy set in a $BE$-algebra $X$ and $\mu_\nu$ be the strongest fuzzy relation on $X$. Then $\nu$ is a fuzzy implicative filter in $X$ if and only if $\mu_\nu$ is a fuzzy implicative filter of $X \times X$.

**Proof.** Assume that $\nu$ is a fuzzy implicative filter of $X$. Then for any $(x, y) \in X \times X$, we have

$$\mu_\nu(x, y) = \min\{\nu(x), \nu(y)\} \leq \min\{\nu(1), \nu(1)\} = \mu_\nu(1, 1)$$

Let $(x, x'), (y, y')$ and $(z, z') \in X \times X$. Then we have the following:

$$\mu_\nu((x, x') \ast (z, z')) = \mu_\nu(x \ast z, x' \ast z')$$

$$= \min\{\nu(x \ast z), \nu(x' \ast z')\}$$

$$\geq \min\{\min\{\nu(x \ast y), \nu(t)\}, \min\{\nu(x' \ast y'), \nu(t')\}\}$$

where $t = x \ast (y \ast z)$ and $t' = x' \ast (y' \ast z')$

$$= \min\{\nu(x \ast y), \nu(x' \ast y')\}, \min\{\nu(t), \nu(t')\}\}$$

$$= \min\{\mu_\nu(x \ast y, x' \ast y'), \mu_\nu(x \ast (y \ast z), x' \ast (y' \ast z'))\}$$

$$= \min\{\mu_\nu((x, x') \ast (y, y')), \mu_\nu((x, x') \ast ((y, y') \ast (z, z')))\}$$

Therefore $\mu_\nu$ is a fuzzy implicative filter in $X \times X$.

Conversely, assume that $\mu_\nu$ is a fuzzy implicative filter in $X \times X$. Then

$$\nu(1) = \min\{\nu(1), \nu(1)\} = \mu_\nu(1, 1) \geq \mu_\nu(x, y) = \min\{\nu(x), \nu(y)\}$$
for all $x, y \in X$. Hence it yields that $\nu(x) \leq \nu(1)$ for all $x \in X$. Let $(x, x'), (y, y')$ and $(z, z') \in X \times X$. Then we have the following consequence.

$$
\nu(x * z) = \min \{\nu(x * z), \nu(1)\}
$$

$$
= \mu_\nu(x * z, 1)
$$

$$
= \mu_\nu(x * z, x * 1)
$$

$$
= \mu_\nu((x, x) * (z, 1))
$$

$$
\geq \min \{\mu_\nu((x, z) * (y, 1)), \mu_\nu((x, z) * ((y, 1) * (z, 1)))\}
$$

$$
= \min \{\mu_\nu(x * y, z * 1), \mu_\nu(x * (y * z), z * (1 * 1))\}
$$

$$
= \min \{\min \{\nu(x * y), \nu(1)\}, \min \{\nu(x * (y * z)), \nu(1)\}\}
$$

Therefore $\nu$ is a fuzzy implicative filter in $X$. □

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