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Categorical relationships of fuzzy topological systems with fuzzy topological spaces and underlying algebras

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Abstract. This paper deals with categorical relationships between fuzzy topological systems, fuzzy topological spaces and their underlying algebraic structures. Two kinds of fuzzy topological systems are considered. Adjointness of functors, equivalence and duality of categories are studied. Some of the results generalize some ideas and observations of S. Vickers made in the context of geometric logic. The sum and product of fuzzy topological systems of one kind are defined. Connections of this categorical study with future development of logics are indicated.

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1. Introduction

A topological system is a mathematical concept introduced in [33] in the year 1989. It is a triple $(X, \models, A)$, where X is a non-empty set, A is a frame and $\models$ is a binary relation from X to A. $x \models a$ where $x \in X$ and $a \in A$ is read as 'x satisfies a'. A frame is a lattice which admits arbitrary joins and finite meets. A typical example of a frame is a topology τ over a set X [20]. Though intersection of an arbitrary collection of sets exists in the original power set lattice $\wp(X)$ of X, the subset τ of $\wp(X)$ may not (in general) be closed with respect to arbitrary intersections. Vickers introduced the notion of topological system in the context of the so-called geometric logic, which was further studied in [32]. Topological systems and frames form the categories **TopSys** and **Frm** respectively. These are linked by two adjoint functors [33].

Now, the notion of satisfaction may be graded. From various standpoints it is reasonable to assume that in some situations x satisfies a to some extent or to a degree. In other words, the binary relation $\models$ may be a fuzzy relation [34]. The extent to which x satisfies a shall be denoted by $grade(x \models a)$ or simply $gr(x \models a)$, which is an element of some suitable value set. The value set in fuzzy literature has been generalized from the unit interval [0,1] in [34] to a lattice [10]. In the present work, the value set shall be chosen as either the unit interval [0,1] or the set $\bar{n} = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \frac{3}{n-1}, ..., \frac{n-2}{n-1}, 1\}$ (where n is a positive integer) in both cases with the natural ordering. Thus, we arrive at the notion of a *fuzzy topological system*. In fact, by a fuzzy topological system, we shall understand a triple $(X, \models, A)$, where X is a non-empty set, A is an algebra, which is at least a lattice with arbitrary join and finite meet, and $\models$ is a fuzzy relation from X to A, grades of relatedness being assigned from a suitable value set. In this paper, we take two such fuzzy topological systems. The first has a frame A and [0,1] as the value set. An attempt in this direction was taken in [31], but there are differences with our approach which will be shown later. The other fuzzy topological system has an $Ł_n^c$-algebra as A and $\bar{n}$ as the value set. An $Ł_n^c$-algebra is an MV_n-algebra enriched by constants [19]. In [19], a categorical duality has been established between $Ł_n^c$-algebras and $\bar{n}$-valued fuzzy topological spaces, showing thereby a duality between one kind of fuzzy topology and Łukasiewicz n-valued logic. The definition of both kinds of fuzzy topological systems that we shall present here was introduced in 2011 [15].

There exists a huge literature on fuzzy topological spaces, [4, 5, 8, 13, 17, 24, 25] to mention a few. This area of study appeared immediately after fuzzy set theory was introduced by Lotfi Zadeh in 1965. One is naturally inclined to study the relationships between fuzzy topological systems and existing fuzzy topological spaces, particularly with respect to the viewpoint of categorical duality and equivalence. As has been observed by Vickers [33], topological spaces make a special kind of topological system, and we shall see here that fuzzy topological spaces make a special kind of fuzzy topological system.

Generalization of topological systems to fuzzy topological systems may be important from another angle too. In first-order logic, (semantic) consequence relation is defined in terms of satisfaction. When the satisfaction relation is fuzzy, the corresponding consequence relation may be either crisp or fuzzy. In the first case, we get many valued logic [2, 23] and in the second case, logic of graded consequence [3, 6]. The logic of graded consequence falls within the broad category of fuzzy logic, but is marked by its distinction from the variety of fuzzy logics in [12, 21, 22] with respect to the nature of logical consequence [6]. Thus, fuzzy topological systems may be viewed as abstract generalizations of fuzzy logics with graded consequence. However, we shall not delve into this issue in this paper – this is included as one of our future goals. Our objective here is rather narrow. We take two value sets as indicated before with standard algebraic structures imposed on them. Then, based on these value sets, fuzzy topological systems are defined. There already exist fuzzy topological spaces with respect to these value sets. We shall study some categorical relationships between the categories of fuzzy topological systems and of fuzzy topological spaces. Duality between the category of one kind of fuzzy topological systems and $Ł_n^c$-algebras shall be established. This category will be shown to be equivalent to the category of $\bar{n}$-fuzzy topological spaces, which are Kolmogorov, compact and zero-dimensional. As a consequence, duality between the category of these topological spaces and the category of Ł_n^c-algebras will be established. This result constitutes another proof of the above mentioned duality proved in [19]. It may be recalled that the celebrated Stone duality theorem [16] of algebraic logic states that there exists a categorical duality between Boolean algebras and zero dimensional compact Hausdorff spaces [16, 20].

We have also defined sum and product of fuzzy topological systems of the first kind. These definitions are different from those given in [31], and we consider them to be more appropriate in the fuzzy context where the satisfaction relation is graded.

The paper is organized as follows: in Section 2, we present the preliminary notions required for the purpose of this work. Section 3 contains the relationships among the categories of fuzzy topological systems, fuzzy topological spaces and frames, where fuzzy topology is defined in the sense of Chang [8]. Section 4 contains sums and products of fuzzy topological systems of the above kind. In Section 5, dualities among the categories of $\bar{n}$-fuzzy Boolean systems (i.e., $\bar{n}$-valued fuzzy topological systems), Ł_n^c-algebras and $\bar{n}$-fuzzy Boolean Space (i.e., $\bar{n}$-valued fuzzy topological spaces of Löwen's kind with some additional properties) shall be established. Section 6 contains some concluding remarks.

2. Preliminaries

For definitions of a category, dual or opposite category, functors and natural transformations we refer to [1].

Let $G : \mathbb{A} \longrightarrow \mathbb{B}$ be a functor, and let B be a $\mathbb{B}$-object.

Definition 2.1. [G-structured arrow and G-costructured arrow]

(1) A G-structured arrow with domain B is a pair (f, A) consisting of an $\mathbb{A}$-object A and a $\mathbb{B}$-morphism $f : B \longrightarrow GA$.

(2) A G-costructured arrow with codomain B is a pair (A, f) consisting of an $\mathbb{A}$-object A and a $\mathbb{B}$-morphism $f : GA \longrightarrow B$.

Definition 2.2. [G-universal arrow and G-couniversal arrow]

(1) A G-structured arrow (g, A) with domain B is called G-universal for B provided that for each G-structured arrow (g', A') with domain B, there exists a unique $\mathbb{A}$-morphism $\hat{f} : A \longrightarrow A'$ with $g' = G(\hat{f}) \circ g$.

(2) A G-costructured arrow (A, g) with codomain B is called G-couniversal for B provided that for each G-costructured arrow (A', g') with codomain B, there exists a unique $\mathbb{A}$-morphism $\hat{f} : A' \longrightarrow A$ with $g' = g \circ G(\hat{f})$.

Definition 2.3. [Left Adjoint and Right Adjoint]

(1) A functor $G : \mathbb{A} \longrightarrow \mathbb{B}$ is said to be left adjoint provided that for every $\mathbb{B}$-object B, there exists a G-couniversal arrow with codomain B.

As a consequence, there exists a natural transformation $\eta : id_A \longrightarrow FG$ (id_A is the identity morphism from A to A), where $F : \mathbb{B} \longrightarrow \mathbb{A}$ is a functor s.t. for given $f : A \longrightarrow FB$ there exists a unique $\mathbb{B}$-morphism $\hat{f} : GA \longrightarrow B$ s.t. the triangle

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & FGA \arrow[d, "F\hat{f}"] \\ & FB \end{tikzcd}$$

commutes.
This η is called the unit of the adjunction.
Hence, we have the diagram of unit as follows:

$$\begin{array}{l|r} \mathbb{A} & \mathbb{B} \\ \hline \begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "f"'] & FGA \arrow[d, "F\hat{f}"] \\ & FB \end{tikzcd} & \begin{tikzcd} GA \arrow[d, "\hat{f}"] \\ B \end{tikzcd} \end{array}$$

(2) A functor $G : \mathbb{A} \longrightarrow \mathbb{B}$ is said to be right adjoint provided that for every $\mathbb{B}$-object B, there exists a G-universal arrow with domain B.
From the definition above, it follows that there exists a natural transformation $\xi : FG \longrightarrow id_A$ (id_A is the identity morphism from A to A), where $F : \mathbb{B} \longrightarrow \mathbb{A}$ is a functor s.t. for given $f' : FB \longrightarrow A$, there exists a unique $\mathbb{B}$-morphism $\hat{f} : B \longrightarrow GA$ s.t the triangle

$$\begin{tikzcd} FGA \arrow[r, "\xi_A"] & A \\ FB \arrow[u, "F\hat{f}"] \arrow[ur, "f'"'] & \end{tikzcd}$$

commutes.
This ξ is called the co-unit of the adjunction.
Hence, we have the diagram of co-unit as follows:

$$\begin{array}{l|r} \mathbb{A} & \mathbb{B} \\ \hline \begin{tikzcd} FGA \arrow[r, "\xi"] & A \\ FB \arrow[u, "F\hat{f}"] \arrow[ur, "f'"'] & \end{tikzcd} & \begin{tikzcd} GA \\ B \arrow[u, "\hat{f}"] \end{tikzcd} \end{array}$$

Definition 2.4 ([33])**.** [Frame] A frame is a partially ordered set such that

(1) every subset has a join,
(2) every finite subset has a meet, and
(3) binary meets distribute over arbitrary joins:
$x \wedge \bigvee Y = \bigvee\{x \wedge y : y \in Y\}$

Definition 2.5. [Frame homomorphism] A function between two frames is a frame homomorphism if and only if it preserves arbitrary joins and finite meets.

Definition 2.6. Given a lattice L, an L-fuzzy subset $\tilde{A}$ of X is given by the membership function $\tilde{A} : X \longrightarrow L$.
For each $x \in X$, the value of $\tilde{A}(x)$ is called the grade of membership of x in the L-fuzzy subset $\tilde{A}$. It will also be denoted by $gr(x \in \tilde{A})$.

Definition 2.7. Let f be a mapping from X to Y, and let $\tilde{B}$ be a L-fuzzy subset of Y. Then $f^{-1}(\tilde{B})$ is a L-fuzzy subset of X given by $f^{-1}(\tilde{B})(x) = \tilde{B}(f(x))$.

Henceforth a $[0,1]$-fuzzy subset will be called simply a fuzzy subset.

Definition 2.8 ([8])**.** Let X be a set, and τ be a collection of fuzzy subsets of X s.t.

(1) $\tilde{\phi}$, $\tilde{X} \in \tau$, where $\tilde{\phi}(x) = 0$, for all $x \in X$ and $\tilde{X}(x) = 1$, for all $x \in X$;
(2) $\tilde{A}_i \in \tau$ for $i \in I \Rightarrow \bigcup_{i \in I} \tilde{A}_i \in \tau$, where $\bigcup_{i \in I} \tilde{A}_i(x) = sup_{i \in I}(\tilde{A}_i(x))$;
(3) $\tilde{A}_1$, $\tilde{A}_2 \in \tau \Rightarrow \tilde{A}_1 \cap \tilde{A}_2 \in \tau$, where $\tilde{A}_1 \cap \tilde{A}_2 = \tilde{A}_1(x) \wedge \tilde{A}_2(x)$.

Then (X, τ) is called a fuzzy topological space. τ is called a fuzzy topology over X.

Elements of τ are called fuzzy open sets of fuzzy topological space (X, τ).

Definition 2.9 ([17])**.** Let X be a set, and $\tau \subset [0,1]^X$ be a collection of fuzzy subsets of X s.t.

(1) For all $r \in [0,1]$, $\tilde{r} \in \tau$, where $\tilde{r}$ is the constant map with value r;
(2) $\tilde{A}_i \in \tau$ for $i \in I \Rightarrow \bigcup_{i \in I} \tilde{A}_i \in \tau$;
(3) $\tilde{A}_1$, $\tilde{A}_2 \in \tau \Rightarrow \tilde{A}_1 \cap \tilde{A}_2 \in \tau$.

This pair (X, τ) is called a fuzzy topological space in the sense of Löwen which is of late called "stratified".

Definition 2.10. Let (X, τ_1) and (Y, τ_2) be two fuzzy topological spaces. A function $f : X \longrightarrow Y$ is said to be fuzzy continuous if and only if for every fuzzy open set $\tilde{B}$ of Y, $f^{-1}(\tilde{B})$ is an open set of X.

3. Categories: Fuzzy Top System, Fuzzy Top, Frm and their interrelationships

We first define a fuzzy topological system with value set [0,1]

3.1. Categories.

[0, 1]-TopSys.

Definition 3.1. [Fuzzy topological systems] A fuzzy topological system is a triple $(X, \models, A)$, where X is a non-empty set, A is a frame and $\models$ is a fuzzy relation from X to A such that

(1) if S is a finite subset of A, then $gr(x \models \bigwedge S) = inf\{gr(x \models s) : s \in S\}$;
(2) if S is any subset of A, then $gr(x \models \bigvee S) = sup\{gr(x \models s) : s \in S\}$.

It is easy to deduce that $gr(x \models \top) = 1$ and $gr(x \models \bot) = 0$, for any $x \in X$, where $\top$ and $\bot$ denotes the top and the bottom elements of the frame A, respectively.
In [31] a fuzzy topological system is defined by the following coditions viz.

(1) if S is a finite subset of A, then $gr(x \models \bigwedge S) \leq gr(x \models s)$ for all $s \in S$;

(2) if S is any subset of A, then $gr(x \models \bigvee S) \leq gr(x \models s)$ for some $s \in S$;
(3) $gr(x \models \top) = 1$ and $gr(x \models \bot) = 0$ for all $x \in X$.

The above conditions allow one to get a fuzzy topological space from a fuzzy topological system, but not conversely. However, using Definition 3.1 of fuzzy topological system, we can produce a functor(Ext) from the category of fuzzy topological systems to the category of fuzzy topological spaces and also a functor(J) from the category of fuzzy topological spaces to fuzzy topological systems. Not only that, we can show that these two functors are adjoint. Definition 3.1 is a natural one and its advantages over the other definition would be clear in the sequel. In the next section, the value set in the above definition will be taken as $\bar{n} = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \frac{3}{n-1}, ..., \frac{n-2}{n-1}, 1\}$. The resulting systems will be called $\bar{n}$-fuzzy topological systems.

Definition 3.2. Let $D = (X, \models, A)$ and $E = (Y, \models', B)$ be fuzzy topological systems. A continuous map $f : D \longrightarrow E$ is a pair (f_1, f_2) where,

(1) $f_1 : X \longrightarrow Y$ is a function;
(2) $f_2 : B \longrightarrow A$ is a frame homomorphism;
(3) $gr(x \models f_2(b)) = gr(f_1(x) \models' b)$, for all $x \in X$ and $b \in B$.

Definition 3.3. Let $D = (X, \models, A)$ be a fuzzy topological system. The identity map $I_D : D \longrightarrow D$ is a pair (I_1, I_2) defined by

$$I_1 : X \longrightarrow X$$
$$I_2 : A \longrightarrow A$$

Let $D = (X, \models', A)$, $E = (Y, \models'', B)$, $F = (Z, \models''', C)$. Let $(f_1, f_2) : D \longrightarrow E$ and $(g_1, g_2) : E \longrightarrow F$ be continuous maps.
The composition $(g_1, g_2) \circ (f_1, f_2) : D \longrightarrow F$ is defined by

$$g_1 \circ f_1 : X \longrightarrow Z$$
$$f_2 \circ g_2 : C \longrightarrow A,$$

i.e., $(g_1, g_2) \circ (f_1, f_2) = (g_1 \circ f_1, f_2 \circ g_2)$.

Theorem 3.4. *Fuzzy topological systems together with continuous maps form the category* $[\mathbf{0}, \mathbf{1}]$-**TopSys**.

It is known that fuzzy topological spaces together with fuzzy continuous maps form a category $[\mathbf{0}, \mathbf{1}]$-**Top** [24], and frames together with frame homomorphisms form the category **Frm** [1]. We shall now investigate the interrelations among the categories $[\mathbf{0}, \mathbf{1}]$-**TopSys**, $[\mathbf{0}, \mathbf{1}]$-**Top** and **Frm**.

3.2. **Functors.** In this subsection, we define various functors required to prove our desired results.

Functor Ext.

Definition 3.5. Let $(X, \models, A)$ be a fuzzy topological system, and $a \in A$. For each a, its extent in $(X, \models, A)$ is a mapping $ext(a)$ from X to $[0, 1]$ given by $ext(a)(x) = gr(x \models a)$. Also $ext(A) = \{ext(a)\}_{a \in A}$.

Lemma 3.6. *$ext(A)$ forms a fuzzy topology [8] on X.*

It is easy to show that $\tilde{\phi}$, $\tilde{X} \in ext(A)$ as we have $gr(x \models \perp) = 0$ and $gr(x \models \top) = 1$, for all $x \in X$ respectively.

As a consequence $(X, ext(A))$ forms a fuzzy topological space.

Lemma 3.7. *If $(f_1, f_2) : (X, \models', A) \longrightarrow (Y, \models'', B)$ is continuous then $f_1 : (X, ext(A)) \longrightarrow (Y, ext(B))$ is fuzzy continuous.*

Definition 3.8. Ext is a functor from $[\mathbf{0},\mathbf{1}]$-**TopSys** to $[\mathbf{0},\mathbf{1}]$-**Top** defined as follows. Ext acts on an object $(X, \models', A)$ as $Ext(X, \models', A) = (X, ext(A))$ and on a morphism (f_1, f_2) as $Ext(f_1, f_2) = f_1$.

The above two Lemmas 3.6 and 3.7 show that Ext is a functor.

Functor J from $[\mathbf{0},\mathbf{1}]$-Top to $[\mathbf{0},\mathbf{1}]$-TopSys.

Definition 3.9. J is a functor from $[\mathbf{0},\mathbf{1}]$-**Top** to $[\mathbf{0},\mathbf{1}]$-**TopSys** defined as follows. J acts on an object (X, τ) as $J(X, \tau) = (X, \in, \tau)$ where $gr(x \in \tilde{T}) = \tilde{T}(x)$ for $\tilde{T}$ in τ and on a morphism f as $J(f) = (f, f^{-1})$.

Lemma 3.10. *$(X, \in, \tau)$ is a fuzzy topological system.*

Lemma 3.11. *$J(f) = (f, f^{-1})$ is continuous provided f is fuzzy continuous.*

So J is a functor from $[\mathbf{0},\mathbf{1}]$-**Top** to $[\mathbf{0},\mathbf{1}]$-**TopSys**.

Functor fm from $[\mathbf{0},\mathbf{1}]$-TopSys to Frm$^{\mathbf{op}}$.

Definition 3.12. fm is a functor from $[\mathbf{0},\mathbf{1}]$-**TopSys** to **Frm$^{\mathbf{op}}$** defined as follows. fm acts on an object $(X, \models, A)$ as $fm(X, \models, A) = A$ and on a morphism (f_1, f_2) as $fm(f_1, f_2) = f_2$.

It is easy to see that fm is a functor.

Functor S from Frm$^{\mathbf{op}}$ to $[\mathbf{0},\mathbf{1}]$-TopSys.

Definition 3.13. Let A be a frame, $Hom(A, [0,1]) = \{frame\ hom\ v : A \longrightarrow [0,1]\}$.

Lemma 3.14. *$(Hom(A, [0,1]), \models_*, A)$, where A is a frame and $gr(v \models_* a) = v(a)$, is a fuzzy topological system.*

Lemma 3.15. *If $f : B \longrightarrow A$ is a frame homomorphism then $(\_ \circ f, f) : (Hom(A, [0,1]), \models_*, A) \longrightarrow (Hom(B, [0,1]), \models_*, B)$ is continuous.*

Recall that morphisms in **Frm$^{\mathbf{op}}$** are morphisms of **Frm** but acting in opposite direction. **Frm$^{\mathbf{op}}$** is also known as **Loc** [16], i.e. the category of locale.

Definition 3.16. S is a functor from **Frm$^{\mathbf{op}}$** to $[\mathbf{0},\mathbf{1}]$-**TopSys** defined as follows. S acts on an object A as $S(A) = (Hom(A, [0,1]), \models_*, A)$ and on a morphism f as $S(f) = (\_ \circ f, f)$.

Previous two Lemmas 3.14 and 3.15 show that S is indeed a functor.

Lemma 3.17. *Ext is the right adjoint to the functor J.*

Proof Sketch. It is possible to prove the theorem by presenting the co-unit of the adjunction.
Recall that $J(X,\tau) = (X, \in, \tau)$ and $Ext(X, \models, A) = (X, ext(A))$.
$So, J(Ext(X, \models, A)) = (X, \in, ext(A))$.
Let us draw the diagram of co-unit

[0, 1]-TopSys | **[0, 1]-Top**

$$\begin{tikzcd} J(Ext(X,\models,A)) \arrow[r, "\xi_X"] & (X,\models,A) \\ J(Y,\tau') \arrow[u, "J(f)(\equiv (f_1, f_1^{-1}))"] \arrow[ur, "\hat{f}(\equiv (f_1,f_2))"'] & \end{tikzcd} \qquad \begin{tikzcd} Ext(X,\models,A) \\ (Y,\tau') \arrow[u, "f(\equiv f_1)"'] \end{tikzcd}$$

Hence co-unit is defined by $\xi_X = (id_X, ext^*)$

$$i.e.\ (X, \in, ext(A)) \xrightarrow[(id_X,\, ext^*)]{\xi_X} (X, \models, A)$$

where ext^* is a mapping from A to $ext(A)$ such that $ext^*(a) = ext(a)$, for all $a \in A$.

Now by routine check one can conclude that Ext is the right adjoint to the functor J. □

Lemma 3.18. *fm is the left adjoint to the functor S.*

Proof Sketch. It is possible to prove the theorem by presenting the unit of the adjunction.
Recall that $S(B) = (Hom(B, [0,1]), \models_*, B)$ where $gr(v \models_* a) = v(a)$.
Hence $S(fm(X, \models, A)) = (Hom(A, [0,1]), \models_*, A)$.

[0, 1]-TopSys | **Frm$^{\mathbf{op}}$**

$$\begin{tikzcd} (X,\models,A) \arrow[r, "\eta_A"] \arrow[dr, "f(\equiv (f_1,f_2))"'] & S(fm(X,\models,A)) \arrow[d, "S\hat{f}"] \\ & S(B) \end{tikzcd} \qquad \begin{tikzcd} fm(X,\models,A) \arrow[d, "\hat{f}(\equiv f_2)"] \\ B \end{tikzcd}$$

Then unit is defined by $\eta_A = (p*, id_A)$.

$$i.e.\ (X, \models, A) \xrightarrow[(p^*,\, id_A)]{\eta_A} S(fm(X, \models, A))$$

where
$p^* : X \longrightarrow Hom(A, [0,1])$,
$x \longmapsto p_x : A \longrightarrow [0,1]$ such that $p_x(a) = gr(x \models a)$. Now by routine check one can conclude that fm is the left adjoint to the functor S.

□

Remark 3.19. Results in Lemmas 3.17, 3.18 occur in more generality in [26, 27, 30]. To maintain continuity and clarity we have presented them in order and drawn the figures.

Theorem 3.20. *$Ext \circ S$ is the right adjoint to the functor $fm \circ J$.*

Proof. Follows from the combination of the adjoint situations in Lemmas 3.17, 3.18. □

The obtained functorial relationships can be illustrated by the following diagram:

$$\begin{tikzcd}
& \mathbf{[0,1]\text{-}TopSys} \arrow[dl, bend left, "Ext"] \arrow[dr, bend left, "fm"] & \\
\mathbf{[0,1]\text{-}Top} \arrow[ur, bend left, "J"] \arrow[rr, bend left, "fm \circ J"] & & \mathbf{Frm^{op}} \arrow[ul, bend left, "S"] \arrow[ll, bend left, "Ext \circ S"]
\end{tikzcd}$$

The obtained adjunction between the categories of fuzzy topological spaces and locales is a fuzzification of the well-known **Top-Loc** adjunction [16].

Remark 3.21. The above diagram corroborates the conclusion arrived at in Section 6 of [27] viz.
spatialization + **localification** = **fixed-basis approach**.

4. Sums and products of fuzzy topological systems

Definition 4.1. Let $\{D_\lambda\}$ be a family of fuzzy topological systems, where $D_\lambda \equiv (X_\lambda, \models_\lambda, A_\lambda)$. The fuzzy topological sum $\sum D_\lambda = (X, \models^*, A)$ is defined by,
1. $X = \bigcup X_\lambda$, union of the sets
2. $A = \prod A_\lambda$, the Cartesian product of the frames
3. $gr(z \models^* < a_\lambda >) = \bigvee_\lambda(\tilde{X_\lambda}(z) \wedge gr(z \models_\lambda a_\lambda))$, where $\tilde{X_\lambda}$ is the membership function of the set X_λ(characteristic function).

It should be noted that the Cartesian product of frames is a frame.

Lemma 4.2. *A sum of fuzzy topological systems is a fuzzy topological system.*

Proof. $\prod A_\lambda$ is a frame. It will be enough to show that
1. $gr(z \models^* < a_\lambda > \wedge < b_\lambda >) = gr(z \models^* < a_\lambda >) \wedge gr(z \models^* < b_\lambda >)$
2. $gr(z \models^* \bigvee_i < a^i_\lambda >) = \bigvee_i gr(z \models^* < a^i_\lambda >)$
Proof of 1 is straight forward.
2. $gr(z \models^* \bigvee_i < a^i_\lambda >) = \bigvee_\lambda(\tilde{X_\lambda}(z) \wedge gr(z \models_\lambda \bigvee_i a^i_\lambda)) = \bigvee_\lambda(\tilde{X_\lambda}(z) \wedge (\bigvee_i gr(z \models_\lambda a^i_\lambda))) = \bigvee_\lambda(\bigvee_i(\tilde{X_\lambda}(z) \wedge gr(z \models_\lambda a^i_\lambda))) = \bigvee_i(\bigvee_\lambda(\tilde{X_\lambda}(z) \wedge gr(z \models_\lambda a^i_\lambda)))$
$= \bigvee_i gr(z \models^* < a^i_\lambda >)$. □

Definition 4.3. Let A and B be two frames. The tensor product (coproduct) $A \otimes B$ is the frame presented as follows:
$Fr(a \otimes b : a \in A, b \in B \mid \bigwedge_i(a_i \otimes b_i) = (\bigwedge_i a_i) \otimes (\bigwedge_i b_i), \bigvee_i(a_i \otimes b) = (\bigvee_i a_i) \otimes b, \bigvee_i(a \otimes b_i) = a \otimes (\bigvee_i b_i))$.
We also define two injections $i_A : A \longrightarrow A \otimes B$ and $i_B : B \longrightarrow A \otimes B$ by
$i_A(a) = a \otimes 1$ and $i_B(b) = 1 \otimes b$.

Lemma 4.4. *Every element of $A \otimes B$ can be written in the form $\bigvee_i (a_i \otimes b_i)$ for some $a_i \in A$ and $b_i \in B$.*

Definition 4.5. Let $D = (X, \models, A), E = (Y, \models, B)$ be fuzzy topological systems. The fuzzy topological product $D \times E = (Z, \models^*, C)$, where
(i) $Z = X \times Y$ is the Cartesian product;
(ii) $C = A \otimes B$ is the tensor product of frames;
(iii) $gr((x, y) \models^* \bigvee_i (a_i \otimes b_i)) = \bigvee_i (gr(x \models a_i) \wedge gr(y \models b_i))$.

Lemma 4.6. *Product of two fuzzy topological systems is a fuzzy topological system.*

Proof. It will be enough to show that
1. $gr((x, y) \models^* \bigvee_i (a_i \otimes b_i) \wedge \bigvee_j (c_j \otimes d_j)) = gr((x, y) \models^* \bigvee_i (a_i \otimes b_i)) \wedge gr((x, y) \models^* \bigvee_j (c_j \otimes d_j))$.
2. $gr((x, y) \models^* \bigvee_j (\bigvee_i (a_i^j \otimes b_i^j))) = \bigvee_j gr((x, y) \models^* \bigvee_i (a_i^j \otimes b_i^j))$.
1. $gr((x, y) \models^* \bigvee_i (a_i \otimes b_i) \wedge \bigvee_j (c_j \otimes d_j)) = gr((x, y) \models^* \bigvee_{i,j} (a_i \wedge c_j) \otimes (b_i \wedge d_j)) = \bigvee_{i,j} (gr(x \models a_i \wedge c_j) \wedge gr(y \models b_i \wedge d_j)) = gr((x, y) \models^* \bigvee_i (a_i \otimes b_i)) \wedge gr((x, y) \models^* \bigvee_j (c_j \otimes d_j))$.
Proof of 2 is routine. □

5. $\bar{n}$-Fuzzy Boolean system, $\mathbf{Ł_n^c}$-Alg, $\bar{n}$-Fuzzy Boolean Space and their interrelationships

We first give the definitions of the related notions, which deal with $Ł_n^c$-algebras, and are taken from [19].

5.1. $Ł_n^c$-algebras.

Definition 5.1 ([18])**.** An **MV**-algebra is an algebra $\mathcal{A} = (A, \oplus, *, ^\perp, 0, 1)$ of type $(2, 2, 1, 0, 0)$, where $(A, \oplus, 0)$ is a commutative monoid and for any $x, y, z \in A$ the following axioms are satisfied:

1. $x \oplus 1 = 1$, 2. $(x^\perp)^\perp = x$, 3. $0^\perp = 1$,
4. $(x^\perp \oplus y)^\perp \oplus y = (y^\perp \oplus x)^\perp \oplus x$, 5. $x * y = (x^\perp \oplus y^\perp)^\perp$.

The original definition of **MV**-algebra was given in [7]. The present definition is by Mangani [18] which we take from [14].

Definition 5.2. For any $m \in \mathbb{N}$, we define
(i) $0x = 0$ and $(m + 1)x = mx \oplus x$.
(ii) $x^0 = 1$ and $x^{m+1} = x^m * x$.

Definition 5.3 ([11])**.** An $\mathbf{MV}_n$-algebra $(n \geq 2)$ is an **MV**-algebra $\mathcal{A} = (A, \oplus, *, ^\perp, 0, 1)$ whose operations fulfill the additional axioms

1. $(n - 1)x \oplus x = (n - 1)x$, 1′. $x^{(n-1)} * x = x^{(n-1)}$,

and if $n \geq 4$ the axioms

2. $[(jx) * (x^\perp \oplus [(j - 1)x]^\perp)]^{n-1} = 0$, 2′. $(n - 1)[x^j \oplus (x^\perp * [x^{j-1}]^\perp)] = 1$,

where $1 < j < n - 1$ and j does not divide $n - 1$.

Definition 5.4 ([19]). $\bar{n}$ denotes the set $\{0, \frac{1}{n-1}, \frac{2}{n-1}, \frac{3}{n-1}, ..., \frac{n-2}{n-1}, 1\}$ equipped with all constants $r \in \bar{n}$ and the operations $(\wedge, \vee, *, \oplus, \rightarrow, ^{\perp})$ are defined as follows:
$x \wedge y = min(x, y), x \vee y = max(x, y), x * y = max(0, x + y - 1), x \oplus y = min(1, x + y), x \rightarrow y = min(1, 1 - (x - y)), x^{\perp} = 1 - x$ and 0-ary operations (i.e. constants) $r \in \bar{n}$.

Definition 5.5. An $Ł_n^c$-algebra is an MV_n algebra enriched by n constants [19]. That is, it is an MV_n algebra $\mathcal{A} = (A, \wedge, \vee, *, \oplus, \rightarrow, ^{\perp}, 0, 1)$ in which the algebra $\bar{n}$ is embedded.

We shall denote the counterparts of $\frac{1}{n-1}, \frac{2}{n-1}, \frac{3}{n-1}, ..., \frac{n-2}{n-1}$ in A by these tokens, and henceforth, a general $Ł_n^c$-algebra will be written as $(A, \wedge, \vee, *, \oplus, \rightarrow, ^{\perp}, 0, \frac{1}{n-1}, \frac{2}{n-1}, \frac{3}{n-1}, ..., \frac{n-2}{n-1}, 1)$.
We note that $\bar{n}$ is an $Ł_n^c$-algebra and also that every $Ł_n^c$-algebra is a frame.

Definition 5.6 ([19]). $Ł_n^c$-homomorphism is a function between two $Ł_n^c$-algebras which preserves the operations.

The following results w.r.t. $Ł_n^c$-algebras may be obtained [19].

Proposition 5.7 ([19]). (1) *Let A be an $Ł_n^c$-algebra. If $a, b \in A$ are idempotent, i.e $a * a = a$, $b * b = b$, then $a * b = a \wedge b$ and $a \oplus b = a \vee b$.*
(2) *Let A be an $Ł_n^c$-algebra, and $r \in \bar{n}$. There is an idempotent term $T_r(x)$ with one variable x such that, for any homomorphism $v : A \longrightarrow \bar{n}$ and any $x \in A$, the following holds:*
(i) $v(T_r(x)) = 1$ iff $v(x) = r$;
(ii) $v(T_r(x)) = 0$ iff $v(x) \neq r$.
Any homomorphism from one $Ł_n^c$-algebra to another preserves the operation $T_r(-)$.
(3) *Let A be an $Ł_n^c$-algebra, and $a_i \in A$ for $i \in I$ and I is a finite set. Then, $T_1(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} T_1(a_i)$ and $T_1(\bigwedge_{i\in I} a_i) = \bigwedge_{i\in I} T_1(a_i)$.*
(4) *Let A be an $Ł_n^c$-algebra. For any $a, b \in A$, the following holds: $\bigwedge_{r\in\bar{n}}(T_r(a) \leftrightarrow T_r(b)) \leq a \leftrightarrow b$, where $a \leftrightarrow b \equiv (a \rightarrow b) \wedge (b \rightarrow a)$.*
(5) *Let A be an $Ł_n^c$-algebra, and $r \in \bar{n}$. There is a term $S_r(x)$ with one variable x such that for any homomorphism $v : A \longrightarrow \bar{n}$, the following two conditions hold:*
(i) $v(S_r(x)) = r$ iff $v(x) = 1$;
(ii) $v(S_r(x)) = 0$ iff $v(x) \neq 1$.
Any homomorphism preserves the operation $S_r(-)$.

Definition 5.8 ([19]). Let A be an $Ł_n^c$-algebra. A non-empty subset F of A is called an $\bar{n}$-filter of A iff F is an upper set and is closed under $*$. An $\bar{n}$-filter F of A is called proper iff $F \neq A$.
A proper $\bar{n}$-filter P of A is prime iff for any $a, b \in A$, $a \vee b \in P$ implies either $a \in P$ or $b \in P$.

Definition 5.9 ([19]). Let A be an $Ł^c_n$-algebra. A subset X of A has finite intersection property (f.i.p.) with respect to $*$ iff for any non-empty subset $\{a_1, a_2, ..., a_n\}$, $a_1 * ...... * a_n \neq 0$.

Proposition 5.10 ([19]). (1) *Let A be an L^c_n-algebra, and F an $\bar{n}$-filter of A. Let $b \in A$ be such that $b \notin F$. Then there is a prime $\bar{n}$-filter P of A such that $F \subset P$ and $b \notin P$.*

(2) *Let A be an L^c_n-algebra, and X a subset of A. If X has f.i.p. with respect to $*$, then there is a prime $\bar{n}$-filter P of A with $X \subset P$.*

(3) *Let A be an L^c_n-algebra. For a prime $\bar{n}$-filter P of A, define $v_P : A \longrightarrow \bar{n}$ by $v_P(a) = r \Leftrightarrow T_r(a) \in P$. Then, v_P is a bijection from the set of all prime $\bar{n}$-filters of A to the set of all homomorphisms from A to $\bar{n}$ with $v_P^{-1}(\{1\}) = P$.*

5.2. **$\bar{n}$-valued fuzzy topology.** An $\bar{n}$-fuzzy set on a set X is defined as a function from X to $\bar{n}$. Let μ, λ be two $\bar{n}$-fuzzy sets. Operations on $\bar{n}$-fuzzy sets are defined pointwise in the usual way. Let X, Y be sets, and f a function from X to Y. For an $\bar{n}$-fuzzy set μ on X, define the direct image $f(\mu) : Y \longrightarrow \bar{n}$ of μ under f by $f(\mu)(y) = \bigvee\{\mu(x) : x \in f^{-1}(\{y\})\}$ for $y \in Y$.
For $f : X \longrightarrow Y$ and an $\bar{n}$-fuzzy set λ on Y, define the inverse image $f^{-1}(\lambda) : X \longrightarrow \bar{n}$ of λ under f by $f^{-1}(\lambda) = \lambda \circ f$.
Note: f^{-1} commutes with $\bigvee$, i.e., $f^{-1}(\bigvee_{i\in I} \mu_i) = \bigvee_{i\in I} f^{-1}(\mu_i)$ for $\bar{n}$-fuzzy sets μ_i on Y.

For sets X and Y, Y^X denotes the set of all functions from X to Y. By r we shall also denote the constant function with value $r \in \bar{n}$. The following definitions are standard ones restated from [19].

Definition 5.11 ([19]). For a set X, $\bar{n}^X$ is called the discrete $\bar{n}$-fuzzy topology on X. $(X, \bar{n}^X)$ is called a discrete $\bar{n}$-fuzzy topological space.

Definition 5.12 ([19]). Let (X, τ) be an $\bar{n}$-fuzzy space. Then an open basis $\mathcal{B}$ of (X, τ) is a subset of τ such that the following holds:
(i) $\mathcal{B}$ is closed under finite meets;
(ii) for any $\mu \in \tau$, there are $\mu_i \in \mathcal{B}$ for $i \in I$ such that $\mu = \bigvee_{i\in I} \mu_i$.

Definition 5.13 ([19]). An $\bar{n}$-fuzzy space (X, τ) is Kolmogorov iff for any $x, y \in X$ with $x \neq y$, there is an open $\bar{n}$-fuzzy set μ on (X, τ) with $\mu(x) \neq \mu(y)$.

Definition 5.14 ([19]). An $\bar{n}$-fuzzy space (X, τ) is Hausdorff iff for any $x, y \in X$ with $x \neq y$, there are $r \in \bar{n}$ and open $\bar{n}$-fuzzy sets μ, λ on (X, τ) such that $\mu(x) \geq r$, $\lambda(y) \geq r$ and $\mu \wedge \lambda < r$.

Definition 5.15 ([19]). Let (X, τ) be an $\bar{n}$-fuzzy topological space. An $\bar{n}$-fuzzy set λ on (X, τ) is compact iff if $\lambda \leq \bigvee_{i\in I} \mu_i$ for open $\bar{n}$-fuzzy sets μ_i on X, then there is a finite subset J of I s.t. $\lambda \leq \bigvee_{i\in J} \mu_i$.

(X, τ) is compact iff **1** is compact, where **1** is the constant map taking every element to 1.

In [19], Maruyama established the duality between $Ł^c_n$-algebra and $\bar{n}$-fuzzy Boolean space, which is a zero dimensional, compact, Kolmogorov $\bar{n}$-fuzzy topological space, via suitable functors. The pictorial presentation of Maruyama's work is as follows:

$$\begin{tikzcd} \mathbf{FBS_n} \arrow[r, bend left, "Cont"] & (\mathbf{Ł_n^c}\text{-}\mathbf{Alg})^{op} \arrow[l, bend left, "Spec"] \end{tikzcd}$$

In our work we introduce the notion of $\bar{n}$-fuzzy Boolean system. Consequently duality between $\bar{n}$-fuzzy Boolean system and $Ł_n^c$-algebra, as well as equivalence of $\bar{n}$-fuzzy Boolean system and $\bar{n}$-fuzzy Boolean space are established. As a result duality between $\bar{n}$-fuzzy Boolean space and $Ł_n^c$-algebra is shown.

5.3. **Categories.**

$\bar{n}$-Fuzzy Boolean systems.

Definition 5.16. [$\bar{n}$-fuzzy Boolean system] An $\bar{n}$-fuzzy Boolean system is a triple $(X, \models, A)$ where X is a non empty set, A is an $Ł_n^c$-algebra and $\models$ is an $\bar{n}$ valued fuzzy relation from X to A such that

(1) $gr(x \models a * b) = max(0, gr(x \models a) + gr(x \models b) - 1)$;
(2) $gr(x \models a^{\perp}) = 1 - gr(x \models a)$;
(3) $gr(x \models r) = r$ for all $r \in \bar{n}$;
(4) $x_1 \neq x_2 \Rightarrow gr(x_1 \models a) \neq gr(x_2 \models a)$ for some $a \in A$.

It turns out that an $\bar{n}$-fuzzy Boolean system is an $\bar{n}$-fuzzy topological system with certain additional conditions.

Definition 5.17. Let $D = (X, \models, A)$ and $E = (Y, \models', B)$ be $\bar{n}$-fuzzy Boolean systems. A continuous map $f : D \longrightarrow E$ is a pair (f_1, f_2), where

(1) $f_1 : X \longrightarrow Y$ is a function;
(2) $f_2 : B \longrightarrow A$ is $Ł_n^c$-homomorphism;
(3) $gr(x \models f_2(b)) = gr(f_1(x) \models' b)$, for all $x \in X$, $b \in B$.

Definition 5.18. Let $D = (X, \models, A)$ be $\bar{n}$-fuzzy Boolean system. The identity map $I_D : D \longrightarrow D$ is the pair (I_1, I_2) of identity maps-

$$I_1 : X \longrightarrow X$$
$$I_2 : A \longrightarrow A$$

Let $D = (X, \models', A)$, $E = (Y, \models'', B)$, $F = (Z, \models''', C)$. Let $(f_1, f_2) : D \longrightarrow E$ and $(g_1, g_2) : E \longrightarrow F$ be continuous maps.
The composition $(g_1, g_2) \circ (f_1, f_2) : D \longrightarrow F$ is defined by

$$g_1 \circ f_1 : X \longrightarrow Z$$
$$f_2 \circ g_2 : C \longrightarrow A$$

i.e $(g_1, g_2) \circ (f_1, f_2) = (g_1 \circ f_1, f_2 \circ g_2)$.

$\bar{n}$-fuzzy Boolean Systems together with continuous functions forms a category denoted $\mathbf{FBSys_n}$

Definition 5.19. A continuous map $f : D(\equiv (X, \models, A)) \longrightarrow E(\equiv (Y, \models', B))$ is a homeomorphism if and only if there is a map $g : E \longrightarrow D$ such that $g \circ f = id_D$ and $f \circ g = id_E$.
When there is a homeomorphism from D to E, say that D and E are homeomorphic.

This means that homeomorphic systems are structurally equivalent, i.e., (assuming that $f = (f_1, f_2)$ and $g = (g_1, g_2)$)

- X and Y are in bijective correspondence i.e there exists a bijection between X and Y;
- A and B are isomorphic $Ł_n^c$-algebras;
- $gr(x \models f_2(b)) = gr(f_1(x) \models' b)$ and $gr(g_1(y) \models a) = gr(y \models' g_2(b))$.

Note that since X and Y are in bijective correspondence, the last condition reduces to $gr(x \models f_2(b)) = gr(f_1(x) \models' b)$.

The category $\mathbf{Ł_n^c}$-Alg. $Ł_n^c$-algebras together with $Ł_n^c$-homomorphisms form the category $\mathbf{Ł_n^c}$**-Alg**.

$\bar{n}$-fuzzy Boolean spaces. Let $\bar{n}$ be equipped with the discrete $\bar{n}$-fuzzy topology.

Definition 5.20. [19] Let (X, τ) be an $\bar{n}$-fuzzy topological space. Then, $Cont(X, \tau)$ is defined as the set of all continuous functions from X to $\bar{n}$. We endow $Cont(X, \tau)$ with the operations $(\wedge, \vee, *, \oplus, \rightarrow, ^{\perp}, 0, \frac{1}{n-1}, \frac{2}{n-1}, ..., \frac{n-2}{n-1}, 1)$ defined point wise.

Lemma 5.21 ([19])**.** *Let (X, τ) be an $\bar{n}$-fuzzy topological space. Then $Cont(X, \tau)$ is closed under the operations* $(\wedge, \vee, *, \oplus, \rightarrow, ()^{\perp}, 0, \frac{1}{n-1}, \frac{2}{n-1}, \frac{3}{n-1}, ..., 1)$.

Definition 5.22 ([19])**.** An $\bar{n}$-fuzzy topological space (X, τ) is called zero-dimensional if and only if $Cont(X, \tau)$ forms an open basis of (X, τ).

Definition 5.23 ([19])**.** An $\bar{n}$-fuzzy topological space (X, τ) is called an $\bar{n}$-fuzzy Boolean space iff (X, τ) is zero dimensional, compact and Kolmogorov.

In [19], the category of $\bar{n}$-fuzzy Boolean spaces and continuous functions has been defined and called $\mathbf{FBS_n}$.

We shall now consider interrelation among the categories $\mathbf{FBSys_n}$, $\mathbf{FBS_n}$ and $\mathbf{Ł_n^c}$**-Alg**.

5.4. **Functors.**

Functor Ext.

Definition 5.24. Let $(X, \models, A)$ be an $\bar{n}$-fuzzy Boolean system, and $a \in A$. For each a, its extent in $(X, \models, A)$ is a mapping $ext(a)$ from X to $\bar{n}$ given by $ext(a)(x) = gr(x \models a)$.
In the set $ext(A) = \{ext(a) : a \in A\}$, the operations $(\vee, \wedge, *, \oplus, \rightarrow, ^{\perp}, 0, \frac{1}{n-1}, \frac{2}{n-1}, ..., \frac{n-2}{n-1}, 1)$ are defined pointwise. Thus ext is a homomorphism from A to $\bar{n}^X$.

Lemma 5.25. *$(X, ext(A))$ is compact.*

Proof. Let us assume that $\mathbf{1} = \bigvee_{i \in I} ext(a_i)$ for some $a_i \in A$, where $\mathbf{1}$ is the constant function defined on X, whose value is always 1. Now, we have, $\mathbf{1} = T_1 \circ \mathbf{1} = T_1 \circ \bigvee_{i \in I} ext(a_i) = \bigvee_{i \in I} T_1 \circ ext(a_i) = \bigvee_{i \in I} ext(T_1(a_i))$. So, $\mathbf{0} = (\bigvee_{i \in I} ext(T_1(a_i)))^{\perp} = \bigwedge_{i \in I} ext((T_1(a_i))^{\perp})$. Therefore, $0 = \mathbf{0}(x) = (\bigwedge_{i \in I} ext((T_1(a_i))^{\perp}))(x)$ for all x. For a fixed x, $0 = (\bigwedge_{i\in I} ext((T_1(a_i))^{\perp}))(x) = \bigwedge_{i\in I} ext((T_1(a_i))^{\perp})(x) = \bigwedge_{i\in I} gr(x \models (T_1(a_i))^{\perp})$. Let there be a homomorphism $v : A \longrightarrow \bar{n}(\subseteq A)$ such that$v((T_1(a_i))^{\perp}) = 1$ for all $i \in I$. Then $gr(x \models v((T_1(a_i))^{\perp})) = v((T_1(a_i))^{\perp}) = 1$ as $v((T_1(a_i))^{\perp}) \in \bar{n}$ for all $i \in I$. Therefore, $\bigwedge_{i\in I} gr(x \models v((T_1(a_i))^{\perp})) = 1$, which is a contradiction. So, there is no homomorphism $v : A \longrightarrow \bar{n}$ such that $v((T_1(a_i))^{\perp}) = 1$ for all $i \in I$. By Proposition 5.10.3 there is no prime $\bar{n}$-filter of A which contains $\{(T_1(a_i))^{\perp} : i \in I\}$. So by Proposition 5.10.2 $(T_1(a_i))^{\perp} : i \in I\}$ does not have f.i.p. with respect to $*$ and so there is a finite subset $\{i_1, ..., i_m\}$ of I s.t. $(T_1(a_{i_1}))^{\perp} * ..... * (T_1(a_{i_m}))^{\perp} = 0$, which yields $T_1(a_{i_1}) \oplus ...... \oplus T_1(a_{i_m}) = 1$. Now, $T_1(a_{i_k})$ is idempotent, and therefore, for any $k \in \{1, ..., m\}$ we have $T_1(a_{i_1}) \vee ..... \vee T_1(a_{i_m}) = 1$, i.e., $T_1(a_{i_1} \vee .... \vee a_{i_m}) = 1$. Since, $T_1(x) \leq x$, so $a_{i_1} \vee ..... \vee a_{i_m} = 1$, i.e., therefore $ext(a_{i_1} \vee ..... \vee a_{i_m}) = 1$. □

Lemma 5.26. *$(X, ext(A))$ is Kolmogorov.*

Proof. Let $x_1 \neq x_2$. Then there is $a \in A$ s.t. $gr(x_1 \models a) \neq gr(x_2 \models a)$, i.e., $ext(a)(x_1) \neq ext(a)(x_2)$. □

Lemma 5.27. *$(X, ext(A))$ is zero-dimensional.*

Proof. Here we have to show that $Cont(X, ext(A))$ forms an open basis of $(X, ext(A))$. We already have that for $f, g \in Cont(X, ext(A))$, $(f \wedge g)(x) = f(x) \wedge g(x)$. Now, if we can show that $ext(a) \in Cont(X, ext(A))$ for $a \in A$, then we can say that $Cont(X, \tau)$ is zero-dimensional. Let μ be an $\bar{n}$-fuzzy set on $\bar{n}$. We have, $((ext(a))^{-1}(\mu))(x) = (\mu \circ ext(a))(x) = \mu(ext(a)(x)) = \bigvee_{r\in\bar{n}}(S_{\mu(r)} \circ T_r)(ext(a)(x)) = \bigvee_{r\in\bar{n}}(S_{\mu(r)} \circ T_r)(gr(x \models a)) = \bigvee_{r\in\bar{n}}(S_{\mu(r)}(T_r(gr(x \models a)))) = \bigvee_{r\in\bar{n}} S_{\mu(r)}(gr(x \models T_r(a))) = \bigvee_{r\in\bar{n}} gr(x \models S_{\mu(r)}(T_r(a))) = gr(x \models \bigvee_{r\in\bar{n}} S_{\mu(r)}(T_r(a)))$. Therefore, $ext(\bigvee_{r\in\bar{n}} S_{\mu(r)}(T_r(a))) \in ext(A)$. Hence, $ext(a) \in Cont(X, ext(A))$. □

From Lemmas 5.25, 5.26, 5.27 we get the following Theorem

Theorem 5.28. *Let $(X, \models, A)$ be an $\bar{n}$-fuzzy Boolean system. Then $(X, ext(A))$ is an $\bar{n}$-fuzzy Boolean space.*

Definition 5.29. *Ext* is a functor from $\mathbf{FBSys_n}$ to $\mathbf{FBS_n}$ defined as follows. *Ext* acts on an object $(X, \models', A)$ as $Ext(X, \models', A) = (X, ext(A))$ where $ext(A) = \{ext(a) : a \in A\}$ and on a morphism (f_1, f_2) as $Ext(f_1, f_2) = f_1$.

Theorem 5.28 and the fact that "if (f_1, f_2) is continuous then f_1 is $\bar{n}$-fuzzy continuous" shows that *Ext* is a functor.

Functor J.

Lemma 5.30. *Let (X, τ) be an $\bar{n}$-fuzzy Boolean space. Then $(X, \in, Cont(X, \tau))$ is an $\bar{n}$-fuzzy Boolean system.*

Proof. Here X is a set and $Cont(X, \tau)$ is an $Ł_n^c$-algebra, where the operations of $Cont(X, \tau)$ are defined point wise. Now, we have to show that
1. $gr(x \in t_1 * t_2) = max(0, gr(x \in t_1) + gr(x \in t_2) - 1)$
2. $gr(x \in t_1^{\perp}) = 1 - gr(x \in t_1)$
3. $gr(x \in r) = r$ for all $r \in \bar{n}$
4. $x_1 \neq x_2 \Rightarrow gr(x_1 \in t) \neq gr(x_2 \in t)$ for some $t \in Cont(X, \tau)$.

1. $gr(x \in t_1 * t_2) = t_1 * t_2(x) = t_1(x) * t_2(x) = max(0, gr(x \in t_1) + gr(x \in t_2) - 1)$.

2. $gr(x \in t_1^{\perp}) = t_1^{\perp}(x) = 1 - t_1(x) = 1 - gr(x \in t_1)$.
3. $gr(x \in r) = r(x) = r$ for all $r \in \bar{n}$.
4. As (X, τ) is Kolmogorov and zero-dimensional, for $x_1 \neq x_2$, there exists $t \in Cont(X, \tau)$ such that $t(x_1) \neq t(x_2)$.

Hence, we have,
$x_1 \neq x_2 \Rightarrow gr(x_1 \in t) \neq gr(x_2 \in t)$ for some $t \in Cont(X, \tau)$, which completes the proof. □

Lemma 5.31. *$(f, \_ \circ f)$ is continuous provided that f is $\bar{n}$ fuzzy continuous.*

Proof. Here $f : X \longrightarrow Y$ is a function and clearly $\_ \circ f : Cont(Y, \tau_2) \longrightarrow Cont(X, \tau_1)$ is an $Ł_n^c$-hom. It suffices to show that $gr(x \in t_2 \circ f) = gr(f(x) \in t_2)$. Now, $gr(x \in t_2 \circ f) = t_2 \circ f(x) = t_2(f(x)) = gr(f(x) \in t_2)$. Hence complete the proof. □

Definition 5.32. J is a functor from $\mathbf{FBS_n}$ to $\mathbf{FBSys_n}$ defined thus.
J acts on an object (X, τ) as $J(X, \tau) = (X, \in, Cont(X, \tau))$ where $gr(x \in t) = t(x)$ and on a morphism f as $J(f) = (f, \_ \circ f)$.

Lemmas 5.30, 5.31 show that J is a functor.

Functor Lag.

Definition 5.33. Lag is a functor from $\mathbf{FBSys_n}$ to $(\mathbf{Ł_n^c\text{-}Alg})^{op}$ defined to act on an object $(X, \models, A)$ as $Lag(X, \models, A) = A$ and on a morphism (f_1, f_2) as $Lag(f_1, f_2) = f_2$.

Functor S.

Definition 5.34. Let A be an $Ł_n^c$-algebra, $Hom(A, \bar{n}) = \{Ł_n^c \ hom \ v : A \longrightarrow \bar{n}\}$.

Lemma 5.35. *Let A be an $Ł_n^c$-algebra. Then $(Hom(A, \bar{n}), \models_*, A)$ is an $\bar{n}$-fuzzy Boolean system.*

Proof. Clearly $Hom(A, \bar{n})$ is a set, A is a frame. It suffices to show that
(i) $gr(v \models a * b) = max(0, gr(v \models_* a) + gr(v \models_* b) - 1)$.
(ii) $gr(v \models_* a^{\perp}) = 1 - gr(v \models_* a)$.
(iii) $gr(v \models_* r) = v(r) = r$ for all $r \in \bar{n}$.
(iv) $v_1 \neq v_2 \Rightarrow gr(v_1 \models a) \neq gr(v_2 \models a)$ for some $a \in A$.
(i) $gr(v \models_* a * b) = v(a * b) = v(a) * v(b) = max(0, v(a) + v(b) - 1) = max(0, gr(v \models_* a) + gr(v \models_* b) - 1)$.
Similarly we can show (ii) and (iii).
(iv) As, $v_1 \neq v_2$ so we have, $v_1(a) \neq v_2(a)$ for some $a \in A$.
Hence $v_1 \neq v_2 \Rightarrow gr(v_1 \models a) \neq gr(v_2 \models a)$ for some $a \in A$. □

It is easy to see that $(\_ \circ f, f)$ is continuous provided f is an $Ł_n^c$-hom.

Definition 5.36. S is a functor from $(\mathbf{Ł_n^c\text{-}Alg})^{op}$ to $\mathbf{FBSys_n}$ defined as follows.
S acts on an object A as $S(A) = (Hom(A, \bar{n}), \models_*, A)$ and on a morphism f as $S(f) = (\_ \circ f, f)$, where $gr(v \models_* a) = v(a)$.

The above fact and Lemma 5.35 show that S is a functor.

Theorem 5.37. *Ext is the right adjoint to the functor J.*

Proof. We will prove the theorem presenting the co-unit of the adjunction.
Let us draw the diagram of co-unit

$\mathbf{FBSys_n}$	$\mathbf{FBS_n}$
$$\begin{tikzcd} J(Ext(X,\models,A)) \arrow[r, "\xi_X"] & (X,\models,A) \\ J(Y,\tau') \arrow[u, "J(f)(\equiv (f_1,f_1^{-1}))"] \arrow[ur, "\hat{f}(\equiv (f_1,f_2))"'] & \end{tikzcd}$$	$$\begin{tikzcd} Ext(X,\models,A) \\ (Y,\tau') \arrow[u, "f(\equiv f_1)"'] \end{tikzcd}$$

Recall that $J(X,\tau) = (S, \in, Cont(X,\tau))$ and $Ext(X,\models,A) = (X, ext(A))$.
So, $J(Ext(X,\models,A)) = (X, \in, Cont(X, ext(A)))$. Co-unit is defined by,

$$(X, \in, Cont(X, ext(A))) \xrightarrow[(id_X,\, ext^*)]{\xi_X} (X,\models,A)$$

where $ext^*(a) = ext(a)$.

Lemma 5.38. $(id_X, ext^*) : J(Ext(X,\models,A)) \longrightarrow (X,\models,A)$ *is a continuous map of $\bar{n}$-fuzzy Boolean system.*

Proof. To establish the Lemma it is enough to show that
$gr(x \in ext^*(a)) = gr(id_X(x) \models a)$. i.e. $gr(x \in ext(a)) = gr(x \models a)$.
Let us define f as follows:
Given $(f_1, f_2) : J(T) \longrightarrow (X,\models,A)$, then $f = f_1$.
Now we will prove that the diagram on the left commutes. In other words,

$$\begin{aligned} (f_1,f_2) &= \xi_X \circ J(f) \\ &= (id_X, ext^*) \circ (f_1, f_1^{-1}) && (\text{as } J(f) = (f_1, f_1^{-1})) \\ &= (id_X \circ f_1, f_1^{-1} \circ ext^*) \end{aligned}$$

Clearly $id_X \circ f_1 = f_1$.
It is left to show that $f_2 = f_1^{-1} \circ ext^*$
Now as (id_X, ext^*) is continuous, $ext^*(a)(x) = gr(x \models a)$, i.e., $ext^*(a) = a$,

$$\begin{aligned} f_1^{-1} ext^*(a) &= f_1^{-1}(a) \\ &= f_2(a) && (\text{as } (f_1,f_2) \text{ is continuous}) \end{aligned}$$

Hence $\xi_X(\equiv (id_X, ext^*)) : J(Ext(X,\models,A)) \longrightarrow (X,\models,A)$ is the co-unit, consequently Ext is the right adjoint to the functor J. □

Diagram of the unit of the above adjunction is as follows

$\mathbf{FBS_n}$	$\mathbf{FBSys_n}$
$$\begin{tikzcd} (X,\tau) \arrow[r, "\eta_X"] \arrow[dr, "\hat{f}(\equiv f_1)"'] & Ext(J(X,\tau)) \arrow[d, "ext(f)(\equiv f_1)"] \\ & Ext(Y,\models,B) \end{tikzcd}$$	$$\begin{tikzcd} J(X,\tau) \arrow[d, "f(\equiv (f_1,f_1^{-1}))"] \\ (Y,\models,B) \end{tikzcd}$$

Theorem 5.39. *The category* $\mathbf{FBSys_n}$ *is equivalent to the category* $\mathbf{FBS_n}$.

Proof. We have two natural transformations ξ, η such that for an $\bar{n}$-fuzzy Boolean system $(X, \models, A)$, $\xi_X : J(Ext(X, \models, A)) \longrightarrow (X, \models, A)$ and for an $\bar{n}$-fuzzy Boolean space (X, τ), $\eta_X : (X, \tau) \longrightarrow Ext(J(X, \tau))$. Now it suffices to show that ξ, η are natural isomorphisms. First we will show that ξ_X is an isomorphism. We have, $\xi_X : J(Ext(X, \models, A)) \longrightarrow (X, \models, A)$, i.e., $\xi_X : (X, \in, ext(A)) \longrightarrow (X, \models, A)$ Now ξ_X is a natural transformation between two $\bar{n}$-fuzzy Boolean systems. Hence we have to show that ξ_X is a homeomorphism. Here $\xi_X = (id_X, ext^*)$.

- We have to show that A and $ext(A)$ are isomorphic $Ł_n^c$-algebras, i.e., $ext^* : A \longrightarrow ext(A)$ is an isomorphism.
  We already have ext^* is an $Ł_n^c$-homomorphism as $ext^*(a) = ext(a)$ for all $a \in A$.
  Next we will show that ext^* is injective. Let $ext(a) = ext(b)$ for $a, b \in A$. So, for all $x \in X$ $ext(a)(x) = ext(b)(x)$, i.e., $gr(x \models a) = gr(x \models b)$. Therefore, for any $r \in \bar{n}$ and $x \in X$, $gr(x \models T_r(a)) = gr(x \models T_r(b))$ (follows from Proposition 5.7.2)
  Now using Proposition 5.10.3, we have for any prime filter P of A and any $r \in \bar{n}$, $T_r(a) \in P$ if and only if $T_r(b) \in P$.
  We claim that $T_r(a) = T_r(b)$ for any $r \in \bar{n}$. If possible let $T_r(a) \neq T_r(b)$ for some $r \in \bar{n}$. Take $T_r(a) \nleq T_r(b)$ without loss of generality. Let $F = \{a_1 \in A : T_r(a) \leq a_1\}$. As $T_r(a)$ is idempotent, so F is an $\bar{n}$-filter of A. Clearly $T_r(b) \notin F$. Using Proposition 5.10.1, we have that there is a prime filter P of A such that $F \subset P$ and $T_r(b) \notin P$. Now $F \subset P$, so $T_r(a) \in P$, which is a contradiction.
  Therefore, $T_r(a) = T_r(b)$ for any $r \in \bar{n}$. So, $\bigwedge_{r\in\bar{n}} T_r(a) \longleftrightarrow T_r(b) = 1$, but $\bigwedge_{r\in\bar{n}} T_r(a) \longleftrightarrow T_r(b) \leq a \longleftrightarrow b$ (Proposition 5.7.4) It follows that $a = b$, and therefore, ext^* is injective.
  It is easy to see that ext^* is onto.
  As a consequence, we get that A and $ext(A)$ are isomorphic.
- Lastly, we have $gr(x \in ext(a)) = ext(a)(x) = gr(x \models a)$.

Hence ξ_X is an isomorphism, and, consequently, ξ is a natural isomorphism.
Now, it is left to show that η is a natural isomorphism. We have $\eta_X : (X, \tau) \longrightarrow (X, ext(Cont(X, \tau)))$. Defining η_X as $\eta_X(x)(t) = t(x)$ for $x \in X$ and $t \in Cont(X, \tau)$, it is possible to show that η is a natural isomorphism. $\square$

Theorem 5.40. *Lag is a left adjoint to the functor S.*

Proof. We will prove the theorem presenting the unit of the adjunction.
Let us draw the diagram of unit:

$\mathbf{FBSys_n}$	$(\mathbf{Ł_n^c\text{-}Alg})^{op}$
$\begin{tikzcd} (X,\models,A) \arrow[r, "\eta_A"] \arrow[dr, "f(\equiv (f_1, f_2))"'] & S(Lag(X,\models,A)) \arrow[d, "S\hat{f}"] \\ & S(B) \end{tikzcd}$	$\begin{tikzcd} Lag(X,\models,A) \arrow[d, "\hat{f}(\equiv f_2)"] \\ B \end{tikzcd}$

Recall that $S(B) = (Hom(B, \bar{n}), \models_*, B)$ where $gr(v \models_* b) = v(b)$

Hence $S(Lag(X, \models, A)) = (Hom(A, \bar{n}), \models_*, A)$

then unit is defined by

$$(X, \models, A) \xrightarrow[(p^*, id_A)]{\eta_A} S(Lag(X, \models, A))$$

where

$p^* : X \longrightarrow Hom(A, \bar{n})$

$x \longmapsto p_x : A \longrightarrow \bar{n}$

where $p_x(a) = gr(x \models a)$.

Lemma 5.41. *For each $x \in X$, $p_x : A \longrightarrow \bar{n}$ is an $Ł_n^c$ homomorphism.*

Proof. Proof is straightfroward.

Lemma 5.42. $(p^*, id_A) : (X, \models, A) \longrightarrow S(Lag(X, \models, A))$ *is a continuous map of $\bar{n}$-fuzzy Boolean system.*

Proof. Here it will be enough to show that $gr(x \models id_A(a)) = gr(p^*(x) \models_* a)$.
i.e. $gr(p^*(x) \models_* a) = gr(x \models a)$.
As we have $gr(p^*(x) \models_* a) = p^*(x)(a)$ and also we have the fact that $p^*(x)(a) = gr(x \models a)$ hence the lemma is proved.
Let us define $\hat{f}$ as follows $(f_1, f_2) : (X, \models, A) \longrightarrow (Hom(B, \bar{n}), \models_*, B)$ then $\hat{f} = f_2$ (as f_2 is the $Ł_n^c$ homomorphism).
Now we have to show that the triangle on the left commute. In other words $(f_1, f_2) = S(\hat{f}) \circ \eta_A = (- \circ f_2, f_2) \circ (p^*, id_A) = ((- \circ f_2)p^*, id_A \circ f_2)$
Clearly $f_2 = id_A \circ f_2$.
It is only left to show $f_1 = (- \circ f_2)p^*$, i.e., for $x \in X$, $f_1(x) = (\_ \circ f_2)p^*(x) = p_x \circ f_2$.
We have for all $b \in B$, $p_x \circ f_2(b) = p_x(f_2(b)) = gr(x \models f_2(b)) = gr(f_1(x) \models_* b) = f_1(x)(b)$.
Hence $p_x \circ f_2 = f_1(x)$, i.e., $(\_ \circ f_2)p^*(x) = f_1(x)$. So, $(\_ \circ f_2)p^* = f_1$ This completes the proof. □

Diagram of the co-unit of the above adjunction is as follows.

FBSys$_\mathbf{n}$ | **FBS$_\mathbf{n}$**

$$\begin{tikzcd} Lag(S(A)) \arrow[r, "\xi_A"] & A \\ Lag(Y, \models, B) \arrow[u, "Lag(f)(\equiv f')"] \arrow[ur, swap, "\hat{f}(\equiv f')"] & \end{tikzcd} \qquad \begin{tikzcd} S(A) \\ (Y, \models, B) \arrow[u, swap, "f(\equiv (\_ \circ f', f'))"] \end{tikzcd}$$

Theorem 5.43. *The category $\boldsymbol{Ł_n^c}$-**Alg** is dually equivalent to the category* **FBSys$_\mathbf{n}$**.

Proof. We have two natural transformations ξ, η such that for an $Ł_n^c - algebra\ A$, $\xi_A : Lag(S(A)) \longrightarrow A$ and for an $\bar{n}$-fuzzy Boolean system $(X, \models, A)$, $\eta_A : (X, \models, A) \longrightarrow S(Lag(X, \models, A))$. It is easy to see that ξ is a natural isomorphism. We will show that $\eta_A : (X, \models, A) \longrightarrow (Hom(A, \bar{n}), \models_*, A)$ is an isomorphism. η_A is a natural transformation between two $\bar{n}$-fuzzy Boolean systems. So we will show that

$\eta_A = (p^*, id_A)$ is a homeomorphism.
We show that $p^* : X \longrightarrow Hom(A, \bar{n})$ is a bijection, i.e., X and $Hom(A, \bar{n})$ are in bijective correspondence. First we show that p^* is injective. Take $x_1, x_2 \in X$ such that $x_1 \neq x_2$.
It follows that $gr(x_1 \models a) \neq gr(x_2 \models a)$ for some $a \in A$ (using Definition 5.16)
or $p_{x_1}(a) \neq p_{x_2}(a)$ for some $a \in A$,
or $p^*(x_1)(a) \neq p^*(x_2)(a)$ for some $a \in A$. As a consequence, p^* is injective.
From the construction of p^*, it is clear that p^* is onto.
So X and $Hom(A, \bar{n})$ are isomorphic.
Lastly we have $gr(x \models a) = p_x(a) = gr(p^*(x) \models_* a)$. Hence, η_A is an isomorphism, and therefore, $(\mathbf{\L_n^c\text{-}Alg})^{op}$ is equivalent to the category $\mathbf{FBSys_n}$. As a consequence, $\mathbf{\L_n^c\text{-}Alg}$ is dually equivalent to the category $\mathbf{FBSys_n}$. □

Theorem 5.44. *$Ext \circ S$ is a right adjoint to the functor $Lag \circ J$.*

Proof. Follows from the combination of the adjoint situations in Theorems 5.37, 5.40. □

Theorem 5.45. *$\mathbf{\L_n^c\text{-}Alg}$ is dually equivalent to the category $\mathbf{FBS_n}$.*

Proof. Follows as a combination of the equivalences of Theorems 5.39, 5.43. □

The obtained functorial relationships can be depicted as follows:

$$\begin{tikzcd}
& \mathbf{FBSys_n} \arrow[dl, bend right, "Ext"'] \arrow[dr, bend left, "Lag"] & \\
\mathbf{FBS_n} \arrow[ur, bend left, "J"] \arrow[rr, bend left, "Lag \circ J"] & & (\mathbf{\L_n^c\text{-}Alg})^{op} \arrow[ul, bend left, "S"] \arrow[ll, bend left, "Ext \circ S"]
\end{tikzcd}$$

The duality which is obtained now between $\mathbf{FBS_n}$ and $\mathbf{\L_n^c\text{-}Alg}$ can be proved to be same as the duality established in [19].

Remark 5.46. The equivalence between $\mathbf{FBSys_n}$ and $\mathbf{FBS_n}$ seems to have resemblance to the equivalence between the category of state property systems and closure space [28]. But to see the clear picture, further investigation is necessary.

6. Concluding remarks

In the present paper we study the categorical interconnections among fuzzy topological systems, frames and fuzzy topological spaces.

Secondly, we provide two ways of constructing new fuzzy topological systems from old ones, by defining sums and tensor product of fuzzy topological systems. These constructions are algebraic rather than categorical.

Thirdly, we introduce the notion of $\bar{n}$-fuzzy Boolean system and establish the equivalence with $\bar{n}$-fuzzy Boolean space as well as duality between the system and

$Ł_n^c$-algebra. As a result, the duality of $\bar{n}$-fuzzy Boolean space and $Ł_n^c$- algebra is also established. (In [19] this duality was shown directly).

It may be noted that during the progress of this research, there have been several parallel studies [9, 26, 27, 29, 30] in the same direction of development as that of the present paper. Some of them address more generalized concepts by taking a lattice instead of [0,1], while some are variety-based. However, our intention being focused on logical studies, we prefer to utilize more concrete structures which are algebraic models of various logics. Vicker's intention for developing topological systems and use of categorical language is to present what is called geometric logic or the logic of finite observations. The semantics of this logic requires an interpretation function, mapping reality to observations. Now, this matching may be partial, or fuzzy and hence graded, the grade being measured by numbers in [0,1]. Thus, the logic of finite observation seems to admit a natural extension by using fuzzy matching functions/relations, and thereby a logic of finite observation incorporating fuzziness is called for. An elementary work in this direction is published [31]. Our aim is to develop such a logic and not category theoretic results per se. But a category theoretic study serves as a step in this direction, since this theory is a very powerful linguistic device for investigation in very general terms. Besides, in continuation to our work in Section 5 where $Ł_n^c$-algebra which is the algebraic model of Łukasiewicz n-valued logic, we would pursue similar investigations with algebraic models of other logics e.g. Łukasiewicz $\aleph_0$-valued logic for which the interval [0,1] is fundamental. Thus in our project, specific frames instead of the general ones are likely to play a significant role.

More specifically, we may indicate some of the future directions of work as follows:

- developing fuzzy (many-valued) geometric logic as mentioned in the introduction, and relating the logic with the logic of graded consequence,
- taking an MV-algebra in place of $Ł_n^c$-algebra, and the MV-algebra $[0,1]$ (the algebraic model of Łukasiewicz $\aleph_0$-valued logic) in place of $\bar{n}$ and continue studies in the line of Section 4,
- replacing fuzzy topology (on crisp sets) by fuzzy topology on fuzzy sets [4, 5], defining fuzzy topological systems accordingly, and proceed along the lines of Section 1 (A preliminary version of which is accepted for a publication in proceedings of Asian Logic Conference 2013).

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