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Simple and generalized T-fuzzy bi-ideals of Γ-semigroup

S. Coumaressane, V. Chinnadurai, N. Thillaigovindan

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Abstract. In this paper, we introduce the concept of $T-$fuzzy interior ideal, $T-$fuzzy simple, $T-$fuzzy bi-ideal, and generalized $T-$fuzzy bi-ideal of a $\Gamma-$semigroup. We characterize $\Gamma-$semigroups through $T-$fuzzification. We find equivalent conditions on these fuzzy subsets in simple $\Gamma-$semigroups and $\Gamma-$semigroups. Finally, we establish a necessary and sufficient condition for a T-fuzzy bi-ideal to be a generalized T-fuzzy bi-ideal in a $\Gamma-$semigroup.

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Corresponding Author: V. Chinnadurai (kv.chinnadurai@yahoo.com)

1. Introduction

The fundamental concepts of fuzzy set was introduced by Zadeh[12]. Sen [8] has introduced Γ-semigroup in 1981. Sen and Saha [9] have introduced Γ-semigroup different from the first definition of Γ-semigroup in the sense of Sen [8]. T-fuzzy concept is a generalization of fuzzy set theory. In 1960, Schweiger and Sklar [7] introduced the concept of t-norm for generalizing the triangular inequality of metric spaces. Using t-norm, Anthony and Sherwood [1] first redefined Rosenfeld's [6] notion of fuzzy groups. Since then t-norm has played an important role in fuzzy algebraic structure [1, 2, 3, 5, 9, 10, 11]. Coumaressane[3] introduced the concept of $T-$fuzzy subset of semiring by introducing the notions of $T-$fuzzy $k-$ideal and $T-$fuzzy $k-$closure in semirings and he generalized the fuzzy subset through triangular norms in semirings.

In this paper, we introduce the concept of $T-$fuzzy interior ideal, $T-$fuzzy simple, $T-$fuzzy bi-ideal, and generalized $T-$fuzzy bi-ideal of a $\Gamma-$semigroup. We characterize $\Gamma-$semigroups by $T-$fuzzification and exhibit by examples that $T-$fuzzy interior ideal is not fuzzy interior ideal, $T-$fuzzy interior ideal is not $T-$fuzzy ideal and generalized $T-$fuzzy bi-ideal is not generalized fuzzy bi-ideal of a $\Gamma-$semigroup.

We find equivalent conditions on these fuzzy subsets in simple $\Gamma-$semigroups and $\Gamma-$semigroups. Finally, we establish a necessary and sufficient condition for a T-fuzzy bi-ideal to be a generalized T-fuzzy bi-ideal in a $\Gamma-$semigroup.

2. Preliminaries

We recall some definitions and results which will be used in later section of this paper.

Definition 2.1. Let A and B be subsets of semigroup S. The *product* of A and B is defined as $AB = \{ ab \in S \mid a \in A \text{ and } b \in B \}$. A nonempty subset A of S is called a subsemigroup of S if $AA \subseteq A$. A nonempty subset A of S is called a left (resp. right) ideal of S if $SA \subseteq A$ (resp. $AS \subseteq A$). A is called a two-sided ideal (simply, ideal) of S if it is both a left and a right ideals of S. A nonempty subset A of S is called an interior ideal of S if $AA \subseteq A$ and $SAS \subseteq A$. A subsemigroup A of S is called a quasi-ideal of S if $AS \cap SA \subseteq A$ and it is called a bi-ideal of S if $ASA \subseteq A$. A semigroup S is called regular if for each element $a \in S$ there exists $x \in S$ such that $a = axa$. A function μ from a nonempty set A into the unit interval $[0,\ 1]$ is called a fuzzy subset of A.

Definition 2.2 ([4])**.** Let M and Γ be any two nonempty sets. M is called a $\Gamma-$semigroup if, for all $a, b, c \in M$ and $x, y \in \Gamma$,
(1) $M\Gamma M \subseteq M$ and $\Gamma M \Gamma \subseteq \Gamma$;
(2) $(axb)yc = a(xby)c = ax(byc)$.

Notation 2.3 ([4])**.** For subsets A and B of M, let
$A\Gamma B = \{a\gamma b \mid a \in A, b \in B, \gamma \in \Gamma\}$.

Definition 2.4 ([4])**.** Let M be a Γ-semigroup and A a nonempty subset of M. A is called a left (resp. right) ideal of M if
$M\Gamma A \subseteq A$ (resp. $A\Gamma M \subseteq A$).
A is a two-sided ideal (simply, ideal) of a Γ-semigroup M if it is both a left ideal and a right ideal of M.

Throughout this paper, M denotes Γ-semigroup and $\mathbf{M}$ denote the characteristic function of M unless otherwise specified.

Definition 2.5 ([4])**.** A subsemigroup B of M is called a bi-ideal of M if
$B\Gamma M\Gamma B \subseteq B$.

Definition 2.6. Let I be a subset of a Γ-semigroup. Define a function $\chi_I(x) : \mu \to [0, 1]$ by

$$\chi_I(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in M$. Then χ_I is a fuzzy subset of μ. If $I = M$ we denote $\chi_M = \mathbf{M}$. Clearly $\mathbf{M}$ is a fuzzy subset of M.

Definition 2.7. A subset A of a $\Gamma-$semigroup M is called an interior ideal if
(1) $A\Gamma A \subseteq A$;
(2) $M\Gamma A\Gamma M \subseteq A$.

Definition 2.8 ([4])**.** Let μ and λ be any two fuzzy subsets of M. Then $\mu \wedge \lambda$ and $\mu * \lambda$ are fuzzy subsets of M defined by

$(\mu \wedge \lambda)(x) = min\{\mu(x),\ \lambda(x)\}$,

$$(\mu * \lambda)(z) = \begin{cases} \sup\limits_{z=x\gamma y} \{min\{\mu(x),\ \lambda(y)\}\}, & \text{if } z \text{ can be expressed as } z = x\gamma y, \\ & \text{where } x,\ y,\ z \in M \text{ and } \gamma \in \Gamma, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.9. A fuzzy subset μ of M is called a fuzzy left (resp. right) ideal of M if for all $a,\ b \in M$ and $\gamma \in \Gamma$,

(1) $\mu(a\gamma b) \geq min\{\mu(a),\ \mu(b)\}$;

(2) $\mu(a\gamma b) \geq \mu(b)$ (resp. $\mu(a\gamma b) \geq \mu(a)$).

μ is called a *fuzzy ideal* of M if it is both a fuzzy left and a fuzzy right ideals of M.

Definition 2.10. A fuzzy subset μ of M is called a fuzzy interior ideal of M if for all $a, b, x \in M$ and $\gamma_1, \gamma_2, \gamma \in \Gamma$,

(1) $\mu(a\,\gamma\,b) \geq min\{\,\mu(a), \mu(b)\}$;

(2) $\mu(a\,\gamma_1\,x\,\gamma_2\,b) \geq \mu(x)$.

Definition 2.11. A fuzzy subset λ of M is called a fuzzy generalized bi-ideal of M if

$\lambda(xyz) \geq min\{\lambda(x), \lambda(z)\}$,

for all $x, y, z \in M$.

Definition 2.12 ([2])**.** A t-norm is a function $T : [0,\ 1] \times [0,\ 1] \to [0,\ 1]$ that satisfies the following conditions for all $x,\ y,\ z \in [0,\ 1]$,

(T1) $T(x,\ 1) = x$;

(T2) $T(x',\ y') \leq T(x,\ y)$ if $x' \leq x$ and $y' \leq y$ (monotonicity);

(T3) $T(x,\ y) = T(y,\ x)$ (commutative);

(T4) $T(x,\ T(y,\ z)) = T(T(x,\ y),\ z)$ (associativity).

In general, for any $t-$norm T, $T(x,\ y) \leq min\{x, y\}$, $T(x,\ 0) = 0$, $T(0,\ 0) = 0$, and $T(1,\ 1) = 1$ are always true.
Definition 2.8 is now generalized using t-norm as follows.

Definition 2.13. Let λ and μ be any two fuzzy subsets of M and T be any t-norm. Then $\mu \cap \lambda$ and $\mu *_T \lambda$ are fuzzy subsets of M defined by

$(\mu \cap \lambda)(x) = T(\mu(x),\ \lambda(x))$.

$$(\mu *_T \lambda)(x) = \begin{cases} \sup\limits_{x=y\gamma z} \{T(\mu(y),\ \lambda(z)\}, & \text{if } x \text{ can be expressed as } x = y\gamma z, \\ & \text{where } x,\ y,\ z \in M \text{ and } \gamma \in \Gamma, \\ 0, & \text{otherwise.} \end{cases}$$

For any fuzzy sets $\mu,\ \lambda$ and θ under any $t-$norm T,

$$\begin{aligned} \mu *_T \lambda *_T \theta &= \mu *_T (\lambda *_T \theta) \\ &= (\mu *_T \lambda) *_T \theta \end{aligned}$$

is always true. For any $x,\ y,\ z \in M$ and $\gamma \in \Gamma$ such that $x = y\gamma z$, we have

$$\begin{aligned}(\mu *_T \lambda *_T \theta)(x) &= \sup_{x=y\gamma z}\{T((\mu *_T \lambda)(y),\, \theta(z))\}\\ &= \sup_{x=y\gamma z}\{T[\sup_{y=p\gamma_1 q} T(\mu(p),\, \lambda(q)), \theta(z)]\}\\ &= \sup_{x=p\gamma_1 q\gamma z}\{T[T(\mu(p),\, \lambda(q)),\, \theta(z)]\}\\ &= \sup_{x=p\gamma_1 q\gamma z}\{T[\mu(p),\, T(\lambda(q),\, \theta(z))]\}.\end{aligned}$$

Theorem 2.14. *Let I be a nonempty subset of M. I is left (resp. right) ideal of M if and only if χ_I is a T-fuzzy left (resp. right) ideal of M.*

Proof. Proof is very similar to ([3], Lemma 3.2), hence it is omitted. □

Lemma 2.15. *For any nonempty subsets A and B of M,*
(1) $\chi_A \cap \chi_B = \chi_{A\cap B}$;
(2) $\chi_A * \chi_B = \chi_{A\Gamma B}$;
(3) $\chi_A *_T \chi_B = \chi_{A\Gamma B}$.

Proof. Proof of (1) and (2) followed from ([4], Lemma 3.9).
(3). Let $x \in M$. Suppose $x \in A\Gamma B$. Then there exists $a \in A$, $\gamma \in \Gamma$ and $b \in B$ such that $x = a\gamma b$. Thus, for $x = a\gamma b$,

$$\begin{aligned}(\chi_A *_T \chi_B)(x) &= \sup_{x=u\gamma_1 v}\{T(\chi_A(u),\, \chi_B(v))\}\\ &\geq T(\chi_A(a),\, \chi_B(b))\\ &= T(1,\, 1)\\ &= 1.\end{aligned}$$

For $0 \leq (\chi_A *_T \chi_B)(x) \leq 1$, we have

$$\begin{aligned}(\chi_A *_T \chi_B)(x) &= 1\\ &= \chi_{A\Gamma B}(x).\end{aligned}$$

In the case when $x \notin A\Gamma B$, then x cannot be expressed as $x = a\gamma b$ for any $a \in A$, $\gamma \in \Gamma$ and $b \in B$. Now,

$$\begin{aligned}(\chi_A *_T \chi_B)(x) &= 0\\ &= \chi_{A\Gamma B}(x).\end{aligned}$$

Hence $(\chi_A *_T \chi_B)(x) = \chi_{A\Gamma B}(x)$ for all $x \in M$. Thus $\chi_A *_T \chi_B = \chi_{A\Gamma B}$. □

3. Simple and generalized T-fuzzy bi-ideals of Γ-semigroup

In this section we define the concepts of T-fuzzy interior-ideal, generalized $T-$fuzzy bi-ideal and simple Γ-semigroup in M. We establish the conditions under which these are equivalent.

Definition 3.1. A fuzzy subset μ of M is called a fuzzy interior ideal of M with respect to t-norm T (in short, T-fuzzy interior ideal) if
(1) $\mu(a\,\gamma\,b) \geq T(\mu(a), \mu(b))$;
(2) $\mu(a\,\gamma_1\,x\,\gamma_2\,b) \geq \mu(x)$.

The following lemma gives the one-to-one correspondence between interior ideal and T-fuzzy interior ideal of a $\Gamma-$semigroup.

Lemma 3.2. *Let A be a nonempty subset of a Γ-semigroup M. A is an interior ideal of M if and only if χ_A is a T-fuzzy interior ideal of M.*

Proof. Let A be an interior ideal of M. Suppose that $\chi_A(x\,\gamma_1\,a\,\gamma_2\,y) < \chi_A(a)$ for some $x, a, y \in M$ and $\gamma_1, \gamma_2 \in \Gamma$. Then $\chi_A(a) = 1$ implies that $a \in A$. Since A is an interior ideal of M, $x\gamma_1 a\gamma_2 y \in M\Gamma A\Gamma M \subseteq A$. This implies that $\chi_A(x\gamma_1 a\gamma_2 y) = 1$, a contradiction. Thus $\chi_A(x\,\gamma_1\,a\,\gamma_2\,y) \geq \chi_A(a)\ \forall\, x, a, y \in M$ and $\gamma_1, \gamma_2 \in \Gamma$. Again suppose that $\chi_A(a\,\gamma\,b) < T(\chi_A(a), \chi_A(b))$ for some $a, b \in M$ and $\gamma \in \Gamma$. Then $\chi_A(a) = 1$ and $\chi_A(b) = 1$. Hence $a, b \in A$. Since A is an interior ideal of M, $a\,\gamma\,b \in A$ and $\chi_A(a\,\gamma\,b) = 1$, a contradiction. Hence $\chi_A(a\,\gamma\,b) \geq T(\chi_A(a), \chi_A(b))$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Conversely, assume that χ_A is a T-fuzzy interior ideal of M. Let $z = x\,\gamma_1\,a\,\gamma_2\,y \in M\Gamma A\Gamma M$ where $x, y \in M, \gamma_1, \gamma_2 \in \Gamma$ and $a \in A$. Then, $\chi_A(x\gamma_1 a\gamma_2 y) \geq \chi_A(a) = 1$. Thus $\chi_A(x\gamma_1 a\gamma_2 y) = 1$ which implies that $x\gamma_1 a\gamma_2 y \in A$. Thus $M\Gamma A\Gamma M \subseteq A$. Let $x, y \in A$, and $\gamma \in \Gamma$. Since χ_A is a $T-$fuzzy interior ideal of M, $\chi_A(x\,\gamma\,y) \geq T(\chi_A(x), \chi_A(y)) = T(1, 1) = 1$.

Hence, $x\gamma y \in A$, which implies that $A\Gamma A \subseteq A$. This shows that A is an interior ideal of M. □

Remark 3.3. Every fuzzy interior ideal of a Γ-semigroup M is a T-fuzzy interior ideal of M. However the converse is not true in general, which is shown in the following example.

Example 3.4. Let $M = \{0, 1, 2, 3\}$ and $\Gamma = \{0, 1, 2\}$. Define '$\bullet$' in M as follows:

$\bullet$	0	1	2	3
0	0	0	0	0
1	0	1	2	0
2	0	2	1	0
3	0	3	0	0

Then $(M, \bullet)$ is a Γ-semigroup. Define fuzzy subset μ of M as

$$\mu(x) = \begin{cases} 0.9 & \text{if } x = 0, \\ 0.3 & \text{if } x = 1, \\ 0.25 & \text{if } x = 2, \\ 0.2 & \text{if } x = 3. \end{cases}$$

Let $T : [0, 1] \times [0, 1] \to [0, 1]$ be a function defined by $T(a, b) = a.b$. Then T is a t-norm. Now $\mu(x\gamma y) \geq T(\mu(x), \mu(y))$ for all $x, y \in M$ and $\gamma \in \Gamma$ and $\mu(a\gamma_1 x\,\gamma_2 b) \geq \mu(x)$ for all $a, x, b \in M$ and $\gamma_1, \gamma_2 \in \Gamma$. Thus μ is a T-fuzzy interior ideal of M. However $\mu(1.2.1) \ngeq \min\{\mu(1), \mu(1)\}$. Hence μ is not a fuzzy interior ideal of M.

Now we find the equivalence of $T-$fuzzy interior ideal in terms of product of $T-$fuzzy sets.

Theorem 3.5. *Let μ be a T-fuzzy subsemigroup of M. μ is a T-fuzzy interior ideal of M if and only if $\mathbf{M} *_T \mu *_T \mathbf{M} \leq \mu$.*

Proof. Assume that μ is a T-fuzzy interior ideal of M. Let $x \in M$ and $a, b, p, q \in M$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $x = a\,\gamma_1 b$ and $b = p\,\gamma_2\,q$. Since $\mu(a\,\gamma_1\,p\,\gamma_2\,q) \geq \mu(p)$, we have

$$\begin{aligned}(\mathbf{M} *_T \mu *_T \mathbf{M})(x) &= \sup_{x=a\gamma_1 p\,\gamma_2 q} \{T(\mathbf{M}(a), T(\mu(p), \mathbf{M}(q))\} \\ &\leq \sup_{x=a\gamma_1 p\,\gamma_2 q} \{\mu(p)\} \\ &\leq \mu(a\gamma_1 p\,\gamma_2 q) = \mu(x).\end{aligned}$$

If x cannot be expressed as $x = a\,\gamma_1\,p\,\gamma_2\,q$, then $(\mathbf{M} *_T \mu *_T \mathbf{M})(x) = 0 \leq \mu(x)$. Thus $(\mathbf{M} *_T \mu *_T \mathbf{M})(x) \leq \mu(x)$,for all $x \in M$.

Conversely, assume that $\mathbf{M} *_T \mu *_T \mathbf{M} \leq \mu$ for any T-fuzzy subsemigroup μ of M.

$$\begin{aligned}\mu(a\,\gamma_1\,x\,\gamma_2\,b) &\geq (\mathbf{M} *_T \mu *_T \mathbf{M})(a\,\gamma_1\,x\,\gamma_2\,b) \\ &= \sup_{a\,\gamma_1\,x\,\gamma_2\,b=p\,\gamma_3\,q\,\gamma_4\,r} \{T(\mathbf{M}(p), T(\mu(q), \mathbf{M}(r))\} \\ &\geq T(\mathbf{M}(a), T(\mu(x), \mathbf{M}(b))) \\ &= \mu(x).\end{aligned}$$

Thus μ is a T-fuzzy interior ideal of M. □

Remark 3.6. Every T-fuzzy ideal of M is a T-fuzzy interior ideal of M. But the converse is not true in general, which is shown in the following example.

Example 3.7. Consider the $\Gamma-$semigroup $(M, \bullet)$, fuzzy subset μ and t-norm T as in Example 3.4. $\mu(x\gamma y) \geq T(\mu(x), \mu(y))$ for all $x, y \in M$ and $\gamma \in \Gamma$ and $\mu(a\gamma_1 x\,\gamma_2 b) \geq \mu(x)$ for all $a, x, b \in M$ and $\gamma_1, \gamma_2 \in \Gamma$. $\mu(x\gamma y) \geq T(\mu(x), \mu(y))$ and $\mu(a\,\gamma_1\,x\,\gamma_2\,b) \geq \mu(x)$ for all $x, y, a, b \in M$, $\gamma_1, \gamma_2 \in \Gamma$. Thus μ is a $T-$fuzzy interior ideal of M. Since $\mu(1.2.1) \ngeq \mu(1)$, μ is not a T-fuzzy ideal of M.

Definition 3.8. A Γ-semigroup M is called left (right) simple if it contains no proper left (right) ideal. A Γ-semigroup M is called two-sided (simply, simple) simple if it contains no proper two sided ideal.

Definition 3.9. A Γ-semigroup M is called T-fuzzy left (right) simple if every T-fuzzy left (right) ideal of M is a constant function.

A Γ-semigroup M is called T-fuzzy two-sided simple (simply, T-fuzzy simple) if every T-fuzzy two-sided ideal of M is a constant function.

Note 3.10. If M is simple, then every left, right and two-sided ideals of M are M itself. This implies, if M is simple, then any interior ideal of M is M itself.

For, $a \in M$ clearly $M\Gamma a, a\Gamma M$ and $M\Gamma M$ are respectively left, right and two-sided ideals of M. Since M is simple, $M\Gamma a = M$, $a\Gamma M = M$ and $M\Gamma M = M$. Thus M is an interior ideal of M.

The following theorem gives relation of simple $\Gamma-$semigroup and $T-$fuzzy set in M.

Theorem 3.11. *For a Γ-semigroup M, the following conditions are equivalent:*
(1) *M is left (right) simple,*
(2) *M is T-fuzzy left (right) simple.*

Proof. (1) ⇒ (2). Let M be left simple. Let μ be any T-fuzzy left ideal of M and $a, b \in M$. Since $M\Gamma a$ and $M\Gamma b$ are left ideals of Γ-semigroup M and M is left simple, we have $M\Gamma a = M$ and $M\Gamma b = M$. Therefore there exist $x, y \in M$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $x\,\gamma_1\,a = b$ and $y\,\gamma_2\,b = a$. Now μ being a T-fuzzy ideal,

$\mu(a) = \mu(y\,\gamma_2\,b) \geq \mu(b) = \mu(x\,\gamma_1\,a) \geq \mu(a)$
and so $\mu(a) = \mu(b)$. Since a and b are arbitrary elements of M, μ is a constant function and so M is T-fuzzy left simple. Thus (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Assume that (2) holds. Let A be any left ideal of M. Then by Theorem 2.14, χ_A is a T-fuzzy left ideal of M. By assumption, χ_A is a constant function. Since χ_A takes only the values 1 or 0, $\chi_A(x) = 1$ otherwise χ_A is an empty fuzzy subset of M, and so $x \in A$. This implies that $M \subseteq A$. Therefore $M = A$, which implies that M is left simple and we have (2) $\Rightarrow$ (1). □

The following theorem gives equivalent condition on a simple Γ-semigroup M in terms of its T-fuzzy simple ideals and T-fuzzy interior ideals.

Theorem 3.12. *For a Γ-semigroup M, the following conditions are equivalent:*
(1) *M is simple;*
(2) *M is T-fuzzy simple;*
(3) *Every T-fuzzy interior ideal of M is a constant.*

Proof. (1) and (2) are equivalent by Theorem 3.11.

Assume that (2) holds. Let μ be any T-fuzzy interior ideal of M and $a, b \in M$. Clearly, $M\Gamma b\Gamma M$ is an interior ideal of M. Since M is simple by Note 3.10 $M\Gamma b\Gamma M = M$. Thus there exist $x, y \in M$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $x\,\gamma_1\,b\,\gamma_2\,y = a$. Since μ is a T-fuzzy interior ideal of M, we have $\mu(a) = \mu(x\,\gamma_1\,b\,\gamma_2\,y) \geq \mu(b)$. In a similar way, one can show that $\mu(b) \geq \mu(a)$. Therefore $\mu(a) = \mu(b)$. Thus μ is constant and (2) $\Rightarrow$ (3) holds.

(3) $\Rightarrow$ (1). Every T-fuzzy ideal of M is a T-fuzzy interior ideal of M. By Note 3.10 and Theorem 3.11, M is simple. □

Definition 3.13. A fuzzy subset μ of M is called a *$T-$fuzzy bi-ideal* of M if for all $x,\, y \in M$ and $\gamma \in \Gamma$,
(1) $\mu(x\gamma y) \geq T(\mu(x),\ \mu(y))$;
(2) $\mu *_T \mathbf{M} *_T \mu \leq \mu$.

Definition 3.14. A fuzzy subset μ of M is called a generalized T-fuzzy bi-ideal of M if $\mu *_T \mathbf{M} *_T \mu \leq \mu$.

It is clear that every T-fuzzy bi-ideal of M is a generalized T-fuzzy bi-ideal of M. We strongly believe that the converse of this statement need not hold in general. That is, a generalized T-fuzzy bi-ideal of M need not be a T-fuzzy bi-ideal of M.

Lemma 3.15. *Every generalized fuzzy bi-ideal is a generalized T-fuzzy bi-ideal of M.*

Proof. Let μ be a generalized fuzzy bi-ideal of M. Then $\mu * \mathbf{M} * \mu \leq \mu$ where $(\mu * \mathbf{M})(x) = \sup\limits_{x = a\,\gamma\,b}\{min\{\mu(a), \mathbf{M}(b)\}\}$. Ascertain that μ satisfies $\mu *_T \mathbf{M} *_T \mu \leq \mu$ under t-norm T. Now

$$\begin{aligned} \mu(x) &\geq (\mu * \mathbf{M} * \mu)(x) \\ &= \sup_{x = a\,\gamma\,b}\{min\{(\mu * \mathbf{M})(a), \mu(b)\}\} \\ &\geq \sup_{x = a\,\gamma\,b}\{T((\mu * \mathbf{M})(a), \mu(b))\} \end{aligned}$$

$$\begin{aligned} &= \sup_{x=a\,\gamma\,b}\{T(\sup_{a=p\,\gamma_1\,q}\{min(\mu(p),\mathbf{M}(q)),\mu(b)\})\} \\ &= \sup_{x=p\,\gamma_1\,q\,\gamma\,b}\{T(min\{\mu(p),\mathbf{M}(q)\},\mu(b))\} \\ &\geq \sup_{x=p\,\gamma_1\,q\,\gamma\,b}\{T(T(\mu(p),\mathbf{M}(q)),\mu(b))\} \\ &= (\mu *_T \mathbf{M} *_T \mu)(x), \end{aligned}$$

and hence $\mu \geq (\mu *_T \mathbf{M} *_T \mu)$. □

The converse of Lemma 3.15 is not true in general, which is shown in the following example.

Example 3.16. Let $M = \{0,\ a,\ b,\ c,\ d\}$ and $\Gamma = \{0,\ a,\ b\}$ define $'\bullet'$ in M as follows:

$\bullet$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	c	c	c
d	0	a	d	d	d

One can easily check that $(M,\ \bullet)$ is a $\Gamma-$semigroup. Let μ be a fuzzy subset of M defined by

$$\mu(x) = \begin{cases} 0.7 & \text{if } \ x = 0, \\ 0.5 & \text{if } \ x = a, c, \\ 0.6 & \text{if } \ x = b, \\ 0.45 & \text{if } \ x = d, \end{cases}$$

and let $T : [0,\ 1] \times [0,\ 1] \to [0,\ 1]$ be a function defined by $T(a,\ b) = ab,\ a,\ b \in [0,\ 1]$, a product norm. Then μ is a $T-$fuzzy bi-ideal of M and hence μ is a generalized T-fuzzy bi-ideal of M. However, since $(\mu*\mathbf{M}*\mu)(babab) > \mu(a)$, we have $(\mu*\mathbf{M}*\mu) > \mu$. Thus μ is not a generalized fuzzy bi-ideal of M.

Lemma 3.17. *A non-empty subset I of M is a generalized bi-ideal of M if and only if χ_I is a generalized T-fuzzy bi-ideal of M.*

Proof. Assume that I is a generalized bi-ideal of M. Consider $\chi_I * \mathbf{M} * \chi_I = \chi_{I\Gamma M\Gamma I} \leq \chi_I$ by Lemma 2.15(2). Hence by Lemma 3.15, χ_I is a generalized $T-$fuzzy bi-ideal of M.

Conversely, let us assume that χ_I is a generalized T-fuzzy bi-ideal of M for any subset I of M. Now let $x \in M$ such that $x \in I\Gamma M\Gamma I$. Then since χ_I is a generalized $T-$fuzzy bi-ideal, we have

$$\begin{aligned} \chi_I(x) &\geq (\chi_I *_T \mathbf{M} *_T \chi_I)(x) \\ &= \chi_{(I\Gamma M\Gamma I)}(x) \ \text{(by Lemma 3.15(3))} \\ &= 1 \text{ and so } x \in I. \end{aligned}$$

Thus $I\Gamma M\Gamma I \subseteq I$ implying that I is a generalized bi-ideal of M. □

Corollary 3.18. *Let I be any nonempty subset of M. I is a generalized bi-ideal of M if and only if χ_I is a generalized fuzzy bi-ideal of M.*

Proof. If $T = min$, then the result follows immediately. □

The next theorem gives a necessary and sufficient condition for a fuzzy subset of Γ-semigroup M to be generalized T-fuzzy bi-ideal of M.

Theorem 3.19. *Let μ be a fuzzy subset of M. μ is a generalized T-fuzzy bi-ideal of M if and only if $\mu(x\,\gamma_1\,y\,\gamma_2\,z) \geq T(\mu(x), \mu(z))$.*

Proof. Let μ be a generalized T-fuzzy bi-ideal of M. Then $\mu *_T \mathbf{M} *_T \mu \leq \mu$. Let $a, x, y, z \in M$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $a = x\,\gamma_1\,y\,\gamma_2\,z$. Then

$$\begin{aligned}\mu(x\,\gamma_1\,y\,\gamma_2\,z) &= \mu(a) \geq (\mu *_T \mathbf{M} *_T \mu)(a)\\ &= \sup_{a=p\,\gamma\,q} \{T((\mu *_T \mathbf{M})(p), \mu(q))\}\\ &\geq T((\mu *_T \mathbf{M})(x\gamma_1 y), \mu(z))\\ &\geq T(T(\mu(x), \mathbf{M}(y)), \mu(z))\\ &= T(\mu(x), \mu(z)).\end{aligned}$$

Conversely assume that $\mu(x\,\gamma_1\,y\,\gamma_2\,z) \geq T(\mu(x), \mu(z))$ is true for any fuzzy subset μ, t-norm T, $x, y, z \in M$ and $\gamma_1, \gamma_2 \in \Gamma$.

In the case when $(\mu *_T \mathbf{M} *_T \mu)(a) = 0 \leq \mu(a)$. Otherwise $a = x\,\gamma\,y$ and $x = p\,\gamma_1\,q$. Since $\mu(p\,\gamma_1\,q\,\gamma\,y) \geq T(\mu(p), \mu(q))$ we have

$$\begin{aligned}(\mu *_T \mathbf{M} *_T \mu)(a) &= \sup_{a=x\,\gamma\,y} \{T((\mu *_T \mathbf{M})(x), \mu(y))\}\\ &= \sup_{a=p\,\gamma_1\,q\,\gamma\,y} \{T(T(\mu(p), \mathbf{M}(q)), \mu(y))\}\\ &= \sup_{a=p\,\gamma_1\,q\,\gamma\,y} \{T(\mu(p), \mu(y))\}\\ &= \sup_{a=p\,\gamma_1\,q\,\gamma\,y} \{\mu(p\,\gamma_1\,q\,\gamma\,y)\}\\ &= \mu(a)\end{aligned}$$

and hence we have $(\mu *_T \mathbf{M} *_T \mu)(a) \leq \mu(a)$ for all $a \in M$. Thus $\mu *_T \mathbf{M} *_T \mu \leq \mu$. □

Lemma 3.20. *Let λ and μ be any fuzzy subset and generalized T-fuzzy bi-ideal of M respectively. Then the product $\lambda *_T \mu$ and $\mu *_T \lambda$ are generalized T-fuzzy bi-ideals of M.*

Proof. Let λ and μ be any fuzzy subset and generalized T-fuzzy bi-ideal of M respectively. Now,

$$\begin{aligned}(\lambda *_T \mu) *_T \mathbf{M} *_T (\lambda *_T \mu) &= \lambda *_T (\mu *_T \mathbf{M} *_T \lambda) *_T \mu\\ &\leq \lambda *_T \mu *_T \mathbf{M} *_T \mathbf{M} *_T \mu\\ &\leq \lambda *_T \mu *_T \mathbf{M} *_T \mu\\ &\leq \lambda *_T \mu,\end{aligned}$$

since μ is a generalized T-fuzzy bi-ideal and so we have $\lambda *_T \mu$ is a T-fuzzy bi-ideal of M. In a similar way, we can establish that $\mu *_T \lambda$ is a generalized T-fuzzy bi-ideal of M. □

The following theorem exhibits a necessary and sufficient condition for a T-fuzzy bi-ideal to be a generalized T-fuzzy bi-ideal in M.

Theorem 3.21. *For a regular Γ-semigroup M, μ is a generalized T-fuzzy bi-ideal of M if and only if μ is a T-fuzzy bi-ideal of M.*

Proof. Let μ be a generalized T-fuzzy bi-ideal of M and $a, b, m \in M$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $b = b\,\gamma_1\, m\, \gamma_2\, b$. Then,

$$\begin{aligned} \mu(a\gamma b) &= \mu(a\,\gamma\, b\,\gamma_1\, m\,\gamma_2\, b) \\ &\geq T(\mu(a), \mu(b)). \end{aligned}$$

This means that μ is a T-fuzzy bi-ideal of M. Since every T-fuzzy bi-ideal of M is a generalized T-fuzzy bi-ideal of M, converse part is clear. □

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S. Coumaressane (coumaressane_s@yohoo.com)
Department of Mathematics, Perunthalaivar Kamarajar Government Arts College, Puducherry, postal code 605 107, India

V. Chinnadurai (kv.chinnadurai@yahoo.com)
Department of mathematics, Annamalai University, Annamalainagar, postal code 608 002, India

N. Thillaigovindan (thillaigovindan.natesan@gmail.com)
Veltech Arts college, Avadi, Chennai, postal code 600 062, India