Characterizations of near-rings by interval valued $(\alpha, \beta)$-Fuzzy ideals

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ABSTRACT. In this paper, we introduce the concept of interval valued $(\alpha, \beta)$-fuzzy subnear-rings and ideal of near-rings, where $\alpha, \beta$ any two of the $\{\in, q, \in \land q\}$ with $\alpha \neq \in \land q$, by using belongs to relation $\in$ and quasi-coincidence with relation $q$ between interval valued fuzzy points and interval valued fuzzy sets. We also discussed some characterizations of interval valued $(\alpha, \in \lor q)$-fuzzy ideals(subnear-rings), mainly discuss interval valued $(\in, \in \lor q)$-fuzzy ideals(subnear-rings) of near-rings.

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1. Introduction

A near-ring satisfies all axioms of an associative ring, except commutative of addition and one of the two distributive laws. In 1965, the fundamental concept of a fuzzy set was first initiated by Zadeh[24]. Then the fuzzy sets have been used in the reconsideration of classical mathematics. Ten years later Zadeh[25] introduced the concept of interval valued fuzzy subsets, where the values of the membership functions are the intervals instead of numbers. Rosenfeld[19] introduced the concept of fuzzy subgroup and give some of its properties. The concept of the interval valued fuzzy subgroup was first discussed by Biswas[6] in 1994. Abou-zaid [1] proposed the notion of fuzzy subnear-rings and ideals of near-rings. A new type of fuzzy subgroup, namely, $(\alpha, \beta)$-fuzzy subgroup was introduced by Bhakat and Das[3, 4, 5] using the relation “belongs to” $(\in)$ and “quasi-coincidence” $(q)$ of fuzzy points and fuzzy sets initiated by Pu Pao-Ming and Liu-Ming[18].The $(\in, \in \lor q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld’s fuzzy subgroup. In [12], Dudek et al. introduced the concept of $(\alpha, \beta)$-fuzzy ideals and $(\alpha, \beta)$-fuzzy h-ideals.
in hemirings. Davvaz\cite{7,8} used this concept in the theory of near-rings and introduced $(\in, \in \cap q)$-fuzzy subnear-rings (ideals, $R$-subgroups) of near-rings. Young Bae Jun\cite{22,23}, gave some results on $(\alpha, \beta)$-fuzzy $h$-ideals in hemirings and discussed some properties of $(\in, \in \cap q_k)$-fuzzy subalgebras in BCK/BCI-algebras. Narayanan and Manikandan\cite{16} introduced the notion of an $(\in, \in \cap q)$-fuzzy quasi-ideals in near-rings. Asger Khan\cite{2} introduced the notion of generalized fuzzy ideals of ordered semigroups. Muhammad Shabir\cite{15}, initiated the concept of interval valued generalized fuzzy ideals of regular LA-semigroups. Deena and Coumaressane\cite{11} proposed the notion of $(\in, \in \cap q)$-fuzzy subnear-rings and ideals of near-rings which is a generalization of $(\in, \in \cap q_k)$-fuzzy subnear-rings and ideals. In \cite{13,14}, Zhan et al. have considered the idea of interval valued $(\alpha, \beta)$-fuzzy hyperideals of hypernear-rings and a new view of fuzzy hypernear-rings. Davvaz\cite{9,10}, discussed few concepts of fuzzy ideals of near-rings and generalized fuzzy $H_v$-submodules endowed with interval valued membership functions.

2. Preliminaries

In this section, we present some elementary definitions that we use in the sequel.

**Definition 2.1** (\cite{8,17}). A near-ring is an algebraic system $(R, +, \cdot)$ consisting of a non empty set $R$ together with two binary operations called $+$ and $\cdot$ such that $(R, +)$ is a group not necessarily abelian and $(R, \cdot)$ is a semigroup connected by the following distributive law: $x \cdot (y + z) = x \cdot y + x \cdot z$ valid for all $x, y, z \in R$. We will use the word ‘near-ring’ to mean ‘left near-ring’. We denote $x y$ instead of $x \cdot y$. An ideal $I$ of a near-ring $R$ is the subset of $R$ such that (i) $(I, +)$ is a normal subgroup of $(R, +)$, (ii) $RI \subseteq I$, (iii) $(x + a)y - xy \in I$, for any $a \in I$ and $x, y \in R$.

Note that $I$ is a left ideal of $R$ if $I$ satisfies (i) and (ii), and right ideal of $R$ if $I$ satisfies (i) and (iii).

**Definition 2.2** (\cite{7}). A fuzzy subset $\mu$ of $R$ is said to be an $(\in, \in \cap q)$-fuzzy subnear-ring of $R$ if for all $x, y \in R$ and $t, r \in (0, 1]$:

1. $x_t, y_r \in \mu$ implies $(x + y)_{\min(t, r)} \in \forall q \mu$,
2. $x_t \in \mu$ implies $(-x)_t \in \forall q \mu$,
3. $x_t, y_r \in \mu$ implies $(xy)_{\min(t, r)} \in \forall q \mu$.

$\mu$ is called an $(\in, \in \cap q)$-fuzzy ideal of $R$ if $\mu$ is a $(\in, \in \cap q)$-fuzzy subnear-ring of $R$ and

4. $x_t \in \mu$ implies $(y - x)_t \in \forall q \mu$,
5. $y_r \in \mu$ and $x \in R$ implies $(xy)_r \in \forall q \mu$,
6. $a_t \in \mu$ and $x, y \in R$ implies $((x + a)y - xy)_t \in \forall q \mu$, for any $x, y, a \in R$.

**Definition 2.3.** A fuzzy subset $\mu$ of $R$ is a map $\mu : R \to [0, 1]$. A fuzzy subset of the form

$$\mu(y) = \begin{cases} t \in (0, 1], & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

is called a fuzzy point with support $x$ and value $t$ and is denoted by $x_t$.

For a fuzzy point $x_t$ and a fuzzy subset $\mu$ of the same set $R$, Pu Ming and Liu Ming\cite{18} introduced the symbol $x_t \mu$, where $\mu \in \{q, q \cap q, q \cap q \}$ and $q \in \forall q \mu$. A fuzzy point $x_t$ is said to be an element of (resp. quasi-coincident with) a fuzzy subset $\mu$, written
as \( x_t \in \mu \) (resp. \( x_t \in \nu \)) if \( \mu(x) \geq t \) (resp. \( \mu(x) + t > 1 \)). The symbol \( x_t \in \mu \) or \( x_t \in \nu \) means that \( x_t \in \mu \) or \( x_t \in \nu \) and \( x_t \in \mu \) and \( x_t \in \nu \). Similarly, \( x_t \in \wedge \nu \mu \) denotes that \( x_t \in \mu \) and \( x_t \in \nu \). Let \( x_t \in \mu \) and \( x_t \in \nu \) do not hold, respectively.

**Notation 2.4** ([10, 20]). By an interval number \( \tilde{a} \), we mean an interval \([a^-, a^+]\) such that \( 0 \leq a^- \leq a^+ \leq 1 \) where \( a^- \) and \( a^+ \) are the lower and upper limits of \( \tilde{a} \) respectively. The set of all closed subintervals of \([0, 1]\) is denoted by \( D[0, 1] \). We also identify the interval \([a, a]\) by the number \( a \in [0, 1] \). For any interval numbers \( \tilde{a}_i = [a^-_i, a^+_i], \tilde{b}_i = [b^-_i, b^+_i] \in D[0, 1], i \in I \) we define

\[
\begin{align*}
\max \{\tilde{a}_i, \tilde{b}_i\} &= [\max\{a^-_i, b^-_i\}, \max\{a^+_i, b^+_i\}], \\
\min \{\tilde{a}_i, \tilde{b}_i\} &= [\min\{a^-_i, b^-_i\}, \min\{a^+_i, b^+_i\}], \\
\inf \tilde{a}_i &= \left[ \bigcap_{i \in I} a^-_i, \bigcap_{i \in I} a^+_i \right], \\
\sup \tilde{a}_i &= \left[ \bigcup_{i \in I} a^-_i, \bigcup_{i \in I} a^+_i \right]
\end{align*}
\]

and let

1. \( \tilde{a} \leq \tilde{b} \iff a^- \leq b^- \) and \( a^+ \leq b^+ \),
2. \( \tilde{a} \equiv \tilde{b} \iff a^- = b^- \) and \( a^+ = b^+ \),
3. \( \tilde{a} < \tilde{b} \iff \tilde{a} \leq \tilde{b} \) and \( \tilde{a} \neq \tilde{b} \),
4. \( \tilde{a} + k = [ka^-, ka^+] \), whenever \( 0 \leq k \leq 1 \).

**Definition 2.5** ([20]). Let \( X \) be a non-empty set. A mapping \( \tilde{\mu} : X \to D[0, 1] \) is called an interval valued fuzzy subset of \( X \). For any \( x \in X \), \( \tilde{\mu}(x) = [\tilde{\mu}^-(x), \tilde{\mu}^+(x)] \), where \( \tilde{\mu}^- \) and \( \tilde{\mu}^+ \) are fuzzy subsets of \( X \) such that \( \tilde{\mu}^-(x) \leq \tilde{\mu}^+(x) \). Thus \( \tilde{\mu}(x) \) is an interval (a closed subset of \([0, 1]\)) and not a number from the interval \([0, 1]\) as in the case of a fuzzy set.

Let \( \tilde{\mu}, \tilde{\nu} \) be interval valued fuzzy subsets of \( X \). The following are defined by

1. \( \tilde{\mu} \leq \tilde{\nu} \iff \tilde{\mu}(x) \leq \tilde{\nu}(x) \).
2. \( \tilde{\mu} = \tilde{\nu} \iff \tilde{\mu}(x) = \tilde{\nu}(x) \).
3. \( \tilde{\mu} \cup \tilde{\nu} = \max \{\tilde{\mu}(x), \tilde{\nu}(x)\} \).
4. \( \tilde{\mu} \cap \tilde{\nu} = \min \{\tilde{\mu}(x), \tilde{\nu}(x)\} \).

**Definition 2.6** ([20]). Let \( \tilde{\mu} \) be an interval valued fuzzy subset of \( X \) and \( [t_1, t_2] \in D[0, 1] \). Then the set \( \tilde{U}(\tilde{\mu} : [t_1, t_2]) = \{x \in X | \tilde{\mu}(x) \geq [t_1, t_2]\} \) is called the upper level set of \( \tilde{\mu} \).

**Definition 2.7** ([20]). Let \( I \) be a subset of a near-ring \( R \). Define a function \( \tilde{f}_I : R \to D[0, 1] \) by

\[
\tilde{f}_I(x) = \begin{cases} 
1 & \text{if } x \in I \\
0 & \text{otherwise}
\end{cases}
\]

for all \( x \in R \). Clearly \( \tilde{f}_I \) is an interval valued fuzzy subset of \( R \) and \( \tilde{f}_I \) is called the interval valued characteristic function of \( I \).

3. INTERVAL VALUED \((\alpha, \beta)\)-FUZZY IDEALS

We now extend the idea of quasi-coincident fuzzy point with a fuzzy set to the concept of quasi-coincidence of a interval value fuzzy point with an interval valued fuzzy set as follows.

**Definition 3.1.** An interval valued fuzzy set \( \tilde{\mu} \) of a near-ring \( R \) of the form
is said to be an interval value fuzzy point with support $x$ and interval value $\bar{t}$ and is denoted by $x_{\bar{t}}$. An interval value fuzzy point $x_{\bar{t}}$ is said to belong to (resp. be quasi-coincidence with) an interval valued fuzzy set $\tilde{\mu}$, written as $x_{\bar{t}} \subseteq \tilde{\mu}$ (resp. $x_{\bar{t}} \approx \tilde{\mu}$) if $\tilde{\mu}(x) \geq \bar{t}$ (resp. $\tilde{\mu}(x) + \bar{t} > [1,1]$). If $x_{\bar{t}} \subseteq \tilde{\mu}$ or $x_{\bar{t}} \approx q\tilde{\mu}$, then we write $x_{\bar{t}} \in \sqcup q\tilde{\mu}$ and if $x_{\bar{t}} \subseteq \tilde{\mu}$ and $x_{\bar{t}} \approx q\tilde{\mu}$, then we write $x_{\bar{t}} \in \wedge q\tilde{\mu}$. The symbol $\exists x \in \forall q \text{ means } \forall q \text{ does not hold.}$

Throughout this paper $R$ will denote a left near-ring and $\alpha$ and $\beta$ denote any one of $\{\varepsilon, q, \in \forall q, \in \wedge q\}$ unless otherwise specified. Also $\tilde{\mu}(x) = [\mu^-(x), \mu^+(x)]$ satisfies the following conditions:

1. Any two elements of $D[0,1]$ are comparable.
2. $[\mu^-(x), \mu^+(x)] \geq [0.5,0.5]$ or $[\mu^-(x), \mu^+(x)] \leq [0.5,0.5]$, for all $x \in R$.

In this section, we present some fundamental concepts and characterizations of interval valued $(\alpha, \beta)$-fuzzy ideals in which the central role is played by $(\alpha, \in \forall q)$-fuzzy ideals.

We first extend the idea of fuzzy ideals to interval valued $(\alpha, \beta)$-fuzzy ideals of near-rings.

**Definition 3.2.** An interval valued fuzzy set $\tilde{\mu}$ of $R$ is said to be an interval valued $(\alpha, \beta)$-fuzzy subnear-ring of $R$ with $\alpha \neq \in \wedge q$ if it satisfies the following conditions:

1. $x_{\bar{t}} \alpha \tilde{\mu}$ and $y_{\bar{r}} \alpha \tilde{\mu}$ implies $(x + y)_{\min\{t,r\}} \beta \tilde{\mu}$,
2. $x_{\bar{t}} \alpha \tilde{\mu}$ implies $(-x)_{\bar{t}} \beta \tilde{\mu}$,
3. $x_{\bar{t}} \alpha \tilde{\mu}$ and $y_{\bar{r}} \alpha \tilde{\mu}$ implies $(xy)_{\min\{t,r\}} \beta \tilde{\mu}$, for all $t, r \in (0,1]$ and $x, y \in R$.

**Definition 3.3.** An interval valued fuzzy set $\tilde{\mu}$ of $R$ is said to be an interval valued $(\alpha, \beta)$-fuzzy ideals of $R$ with $\alpha \neq \in \wedge q$ if the following conditions hold:

1. $\tilde{\mu}$ is an interval valued $(\alpha, \beta)$-fuzzy subnear-ring of $R$,
2. $x_{\bar{t}} \alpha \tilde{\mu}$ and $y \in R$ implies $(y + x - y)_{\bar{t}} \beta \tilde{\mu}$,
3. $y_{\bar{t}} \alpha \tilde{\mu}$ and $x \in R$ implies $(xy)_{\bar{t}} \beta \tilde{\mu}$,
4. $x_{\bar{t}} \alpha \tilde{\mu}$ and $x, y \in R$ implies $\alpha \tilde{\mu}$,
5. $z_{\bar{t}} \alpha \tilde{\mu}$ and $x, y \in R$ implies $(x + z)_{\bar{t}} \beta \tilde{\mu}$, for all $t, r \in (0,1]$ and $x, y, z \in R$.

The conditions (1) and (2) in Definition 3.2 is equivalent to the following condition:

1. $x_{\bar{t}} \alpha \tilde{\mu}$, and $y_{\bar{r}} \alpha \tilde{\mu}$ implies $(x - y)_{\min\{t,r\}} \beta \tilde{\mu}$.

Let $\tilde{\mu}$ be an interval valued fuzzy subset of $R$ such that $\tilde{\mu}(x) \leq [0.5,0.5]$ for all $x \in R$. Suppose that $x \in R$ and $t \in (0,1]$ such that $x_{\bar{t}} \in \sqcup q\tilde{\mu}$. Then $\tilde{\mu}(x) \geq \bar{t}$ and $\tilde{\mu}(x) + \bar{t} > [1,1]$. It follows that $[1,1] \leq \tilde{\mu}(x) + \bar{t} \leq \tilde{\mu}(x) + \bar{t} = 2\tilde{\mu}(x)$. This means that $\tilde{\mu}(x) > [0.5,0.5]$, and so $\{x_{\bar{t}} \mid x_{\bar{t}} \in \sqcup q\tilde{\mu}\} = \emptyset$. Therefore the case $\alpha = \in \wedge q$ in Definitions 3.2 and 3.3 are omitted.

**Example 3.4.** Let $R = \{a, b, c, d\}$ be a set with two binary operations defined as follows:

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Then \((R, +, \cdot)\) is a near-ring and \(I = \{a, b\}\) is its ideal. Let \(\bar{\mu} : R \rightarrow D[0, 1]\) be an interval valued fuzzy subset of \(R\) defined by \(\bar{\mu}(a) = [0.8, 0.9], \bar{\mu}(b) = [0.6, 0.7]\) and \(\bar{\mu}(c) = [0.5, 0.5] = \bar{\mu}(d)\). Then, clearly, \(\bar{\mu}\) is an interval valued \((\in, \in \setminus \emptyset)\)-fuzzy ideal of \(R\). But

\(1\) \(\bar{\mu}\) is not an interval valued \((\in, \in)\)-fuzzy ideal of \(R\), since
\[
b_{[0.58, 0.68]}(a, b, c, d) \in \bar{\mu}\) but \((a + b - c - d) [0.58, 0.68] = d_{[0.58, 0.68]} f \bar{\mu}.
\]

\(2\) \(\bar{\mu}\) is not an interval valued \((q, q)\)-fuzzy ideal of \(R\), since
\[
a_{[0.2, 0.3]} q \bar{\mu} and b_{[0.48, 0.58]} q \bar{\mu} but \((a - b) [0.2, 0.3] = b_{[0.2, 0.3]} q \bar{\mu}.
\]

\(3\) \(\bar{\mu}\) is not an interval valued \((q, \in \setminus \emptyset)\)-fuzzy ideal of \(R\), since
\[
a_{[0.2, 0.3]} q \bar{\mu} and c_{[0.58, 0.59]} q \bar{\mu} but \((a - c) [0.2, 0.3] = d_{[0.2, 0.3]} q \bar{\mu}.
\]

\(4\) \(\bar{\mu}\) is not an interval valued \((\in, \in \setminus \emptyset)\)-fuzzy ideal of \(R\), since
\[
b_{[0.58, 0.68]} \in \bar{\mu} and c_{[0.48, 0.49]} \in \bar{\mu} but \((b - c) [0.48, 0.49] = c_{[0.48, 0.49]} \in \emptyset \bar{\mu}.
\]

\(5\) \(\bar{\mu}\) is not an interval valued \((\in, \in)\)-fuzzy ideal of \(R\), since
\[
b_{[0.58, 0.68]} \in \bar{\mu} and c_{[0.48, 0.49]} \in \bar{\mu} but \((b - c) [0.48, 0.49] = c_{[0.48, 0.49]} \in \emptyset \bar{\mu}.
\]

\(6\) \(\bar{\mu}\) is not an interval valued \((\in, \in \setminus \emptyset)\)-fuzzy ideal of \(R\), since
\[
b_{[0.58, 0.68]} \in \bar{\mu} and c_{[0.48, 0.49]} \in \bar{\mu} but \((b - c) [0.48, 0.49] = c_{[0.48, 0.49]} \in \emptyset \bar{\mu}.
\]

\(7\) \(\bar{\mu}\) is not an interval valued \((\in, \in)\)-fuzzy ideal of \(R\), since
\[
b_{[0.58, 0.68]} q \bar{\mu} and c_{[0.52, 0.54]} q \bar{\mu} but \((b - c) [0.52, 0.54] = c_{[0.52, 0.54]} \in \emptyset \bar{\mu}.
\]

\(8\) \(\bar{\mu}\) is not an interval valued \((\in, \in \setminus \emptyset)\)-fuzzy ideal of \(R\), since
\[
b_{[0.58, 0.68]} \in \bar{\mu} and c_{[0.52, 0.54]} \in \bar{\mu} but \((b - c) [0.52, 0.54] = c_{[0.52, 0.54]} \in \emptyset \bar{\mu}.
\]

\(9\) \(\bar{\mu}\) is not an interval valued \((\in, \in)\)-fuzzy ideal of \(R\), since
\[
a_{[0.2, 0.2]} \in \bar{\mu} and b_{[0.3, 0.4]} \in \bar{\mu} but \(a - b) [0.2, 0.2] = b_{[0.2, 0.2]} \in \emptyset \bar{\mu}.
\]

In the next theorem, using an interval valued \((\alpha, \beta)\)-fuzzy ideal of \(R\), we present a method of constructing an ideal of \(R\).

**Theorem 3.5.** Let \(\bar{\mu}\) be an interval valued \((\alpha, \beta)\)-fuzzy ideal of \(R\). Then the set \(S_{\bar{\mu}} = \{x \in R \mid \bar{\mu}(x) > [0, 0]\}\) is an ideal of \(R\).

**Proof.** \(S_{\bar{\mu}} = \{x \in R \mid \bar{\mu}(x) > [0, 0]\}\). Let \(x, y \in S_{\bar{\mu}}\) be such that \(\bar{\mu}(x) > [0, 0]\) and \(\bar{\mu}(y) > [0, 0]\). Let \(\bar{\mu}(x - y) = [0, 0]\). If \(\alpha \in \in \setminus \emptyset\), then \(x \in \bar{\mu}(x) \in \bar{\mu}(y) \in \bar{\mu}\) but \(\bar{\mu}(x - y) = [0, 0]\), so \(\bar{\mu}(x) \in \bar{\mu}(y) \in \bar{\mu}\) but \(\bar{\mu}(x - y) = [0, 0]\). Hence, \(\bar{\mu}(x - y) > [0, 0]\), that is, \(x - y \in S_{\bar{\mu}}\). Also, \(x_{[1, 1]} \in \bar{\mu}\) and \(y_{[1, 1]} \in \bar{\mu}\) but \((x - y)_{[1, 1]} \in \bar{\mu}\) for every \(\beta \in \in \setminus \emptyset\), a contradiction. Hence \(\bar{\mu}(x - y) > [0, 0]\), that is, \(x - y \in S_{\bar{\mu}}\). Now, let \(x \in S_{\bar{\mu}}\). Then \(x = (x - y) + y \in R \mid \bar{\mu}(x) > [0, 0]\) and \(\bar{\mu}(y) > [0, 0]\). If \(\alpha \in \in \setminus \emptyset\), then \(x \in \bar{\mu}(x) \in \bar{\mu}(y) \in \bar{\mu}\) but \((x - y)_{[1, 1]} \in \bar{\mu}\) for every \(\beta \in \in \setminus \emptyset\), a contradiction, this implies that \(xy \in S_{\bar{\mu}}\). Also, \(y_{[1, 1]} \in \bar{\mu}\) but \((x - y)_{[1, 1]} \in \bar{\mu}\) for every \(\beta \in \in \setminus \emptyset\). This leads to a contradiction and so \(\bar{\mu}(x + y - x) > [0, 0]\), that is, \(y + x - y \in S_{\bar{\mu}}\). Again, let \(y \in S_{\bar{\mu}}\), \(x \in R \mid \bar{\mu}(x) > [0, 0]\). Let \(\bar{\mu}(x) = [0, 0]\). If \(\alpha \in \in \setminus \emptyset\), then \(y_{[1, 1]} \in \bar{\mu}\) but \((x - y)_{[1, 1]} \in \bar{\mu}\) for every \(\beta \in \in \setminus \emptyset\), a contradiction, this implies that \(xy \in S_{\bar{\mu}}\). Also, \(y_{[1, 1]} \in \bar{\mu}\) but \((x - y)_{[1, 1]} \in \bar{\mu}\) for every \(\beta \in \in \setminus \emptyset\). This leads to a contradiction and so \(x \in \bar{\mu}(x) > [0, 0]\), that is, \(x \in S_{\bar{\mu}}\). Let \(z \in S_{\bar{\mu}}\) and \(x, y \in R\). Then \(\bar{\mu}(z) > [0, 0]\). Suppose that \(\bar{\mu}((x + z) - (y + z)) = [0, 0]\). If \(\alpha \in \in \setminus \emptyset\), then \(z_{[1, 1]} \in \bar{\mu}\) but \((x + z) - (y + z)_{[1, 1]} \in \bar{\mu}\) for every \(\beta \in \in \setminus \emptyset\). This leads to a contradiction and so \(x \in \bar{\mu}(x) > [0, 0]\), that is, \((x + z) - (y + z) \in S_{\bar{\mu}}\). Also,
If \((x + z)y - xy\) \(\in [1, 1]\) \(\bar{q}\mu\) but \((x + z)y - xy\) \(\in [1, 1]\) \(\bar{\mu}\) for every \(\beta \in \{\epsilon, q, \varphi, \in \wedge\}\), a contradiction. Thus \(\bar{\mu}((x + z)y - xy) > [0, 0]\), implies, \((x + z)y - xy \in S_\mu\). This shows that \(S_\mu\) is an ideal of \(R\). □

**Theorem 3.6.** If \(I\) is an ideal of \(R\), then an interval valued fuzzy subset \(\bar{\mu}\) of \(R\) such that

\[
\bar{\mu}(x) = \begin{cases} 
\geq [0.5, 0.5] & \text{if } x \in I \\
[0, 0] & \text{otherwise}
\end{cases}
\]

is an interval valued \((\alpha, \in \varphi)\)-fuzzy ideal of \(R\).

**Proof.** (a) Let \(x, y \in R\) and \(\bar{t}, \bar{r} \in D[0, 1]\) with \(\bar{t}, \bar{r} \neq [0, 0]\) be such that \(x_\bar{t} \in \bar{\mu}\) and \(y_{\bar{r}} \in \bar{\mu}\). Then \(\bar{\mu}(x) \geq \bar{t}\) and \(\bar{\mu}(y) \geq \bar{r}\). Thus \(x, y \in I\) and so \(x - y \in I\), that is, \(\bar{\mu}(x - y) \geq [0.5, 0.5]\). If \(\min \{\bar{t}, \bar{r}\} \leq [0.5, 0.5],\) then \(\bar{\mu}(x - y) \geq [0.5, 0.5] \geq \min \{\bar{t}, \bar{r}\}\). Hence \((x - y)_{\min \{\bar{t}, \bar{r}\}} \in \bar{\mu}\). If \(\min \{\bar{t}, \bar{r}\} > [0.5, 0.5],\) then \(\bar{\mu}(x - y) + \bar{t} > [0.5, 0.5] + [0.5, 0.5] = [1, 1]\) and so \((xy)_{\bar{t}} \in \bar{\mu}\). Thus \((x - y)_{\bar{t}} \in \bar{\nu}\). Similarly, we can prove that \((xy)_{\bar{t}} \in \bar{\nu}\).

(b) Let \(x, y \in R\) and \(\bar{t}, \bar{r} \in D[0, 1]\) with \(\bar{t}, \bar{r} \neq [0, 0]\) be such that \(x_\bar{t} \bar{q}\mu\) and \(y_{\bar{r}} \bar{q}\mu\).

Then \(x, y \in I\), \(\bar{\mu}(x) + \bar{t} > [1, 1]\) and \(\bar{\mu}(y) + \bar{r} > [1, 1]\). Since \(x - y \in I\), we have \(\bar{\mu}(x - y) \geq [0.5, 0.5]\). If \(\min \{\bar{t}, \bar{r}\} \leq [0.5, 0.5],\) then \(\bar{\mu}(x - y) \geq [0.5, 0.5] \geq \min \{\bar{t}, \bar{r}\}\). Hence \((x - y)_{\min \{\bar{t}, \bar{r}\}} \in \bar{\mu}\). If \(\min \{\bar{t}, \bar{r}\} > [0.5, 0.5],\) then \(\bar{\mu}(x - y) + \min \{\bar{t}, \bar{r}\} > [0.5, 0.5] + [0.5, 0.5] = [1, 1]\) and so \((xy)_{\bar{t}} \in \bar{\nu}\). Thus \((x - y)_{\bar{t}} \in \bar{\nu}\).
(c) Similar consequence of (a) and (b), we have to prove that $\tilde{\mu}$ is an interval valued $(\in \vee q, \in \vee q)$-fuzzy ideal of $R$. □

Remark 3.7. The following example proves that every interval valued fuzzy set $\tilde{\mu}$ defined in Theorem 3.6 is an interval valued $(\alpha, \in \vee q)$-fuzzy ideal of $R$ but $\tilde{\mu}$ is not an interval valued $(\alpha, \beta)$-fuzzy ideal of $R$, for every $\beta \in \{\in, \vee q, \in \wedge q\}$.

Example 3.8. Let $R = \{a, b, c, d\}$ be a set with two binary operations defined as follows:

<table>
<thead>
<tr>
<th>+</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
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<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
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</table>

Then $(R, +, \cdot)$ is a near-ring and $I = \{a, b\}$ is its ideal. Let $\tilde{\mu}: R \rightarrow D[0, 1]$ be an interval valued fuzzy subset of $R$ defined by $\tilde{\mu}(a) = [0.6, 0.!1, \tilde{\mu}(b) = [0.5, 0.6]$ and $\tilde{\mu}(c) = [0, 0] = \tilde{\mu}(d)$. Then, clearly, $\tilde{\mu}$ is an interval valued $(\in, \in \vee q)$-fuzzy ideal of $R$. Since, $a_{[0.26,0.28]} \in \tilde{\mu}$. Then, $(a-a)_{[0.26,0.28]} = a_{[0.26,0.28]} \in \vee q \tilde{\mu}$ but $(a-a)_{[0.26,0.28]} = a_{[0.26,0.28]} \tilde{\mu}$, which implies that $(a-a)_{[0.26,0.28]} = a_{[0.26,0.28]} \in \vee q \tilde{\mu}$.

4. Interval valued $(\in, \in \vee q)$-fuzzy ideal of near-rings

In this section, we introduce the notion of interval valued $(\in, \in \vee q)$-fuzzy ideal of near-ring and investigate some of its properties.

Definition 4.1 ([21]). An interval valued fuzzy subset $\mu$ of a near-ring $R$ is said to be an $i$-v $(\in, \in \vee q)$-fuzzy near-ring of $R$ if for all $x, y \in R$ and $t, r \in (0, 1)$:

1. $x_t \in \tilde{\mu}$ and $y_r \in \tilde{\mu}$ implies $(x + y)_{\min(t,r)} \in \vee q \tilde{\mu}$,
2. $x_t \in \tilde{\mu}$ implies $(-x)_t \in \vee q \tilde{\mu}$,
3. $x_t \in \tilde{\mu}$ and $y_r \in \tilde{\mu}$ implies $(xy)_{\min(t,r)} \in \vee q \tilde{\mu}$,

The conditions (1) and (2) in Definition 4.1 is equivalent to

(1') $x_t, y_r \in \tilde{\mu}$ implies $(x - y)_{\min(t,r)} \in \vee q \tilde{\mu}$.

Definition 4.2. An interval valued fuzzy subset $\tilde{\mu}$ of $R$ is said to be an interval valued $(\in, \in \vee q)$-fuzzy ideal of $R$ if it satisfies the following conditions for all $t, r \in (0, 1]$ and $x, y, z \in R$,

1. $\tilde{\mu}$ is an interval valued $(\in, \in \vee q)$-fuzzy near-ring of $R$,
2. $x_t \in \tilde{\mu}$ and $y \in R$ implies $(y + x - y)_t \in \vee q \tilde{\mu}$,
3. $y_r \in \tilde{\mu}$ and $x \in R$ implies $(xy)_r \in \vee q \tilde{\mu}$.
4. $z_t \in \tilde{\mu}$ and $x, y \in R$ implies $(x + z)y - xy)_t \in \vee q \tilde{\mu}$.

Theorem 4.3 ([21]). An interval valued fuzzy subset $\tilde{\mu}$ of $R$ is an interval valued $(\in, \in \vee q)$-fuzzy near-ring of $R$ if and only if

1. $\tilde{\mu}(x - y) \geq \min^i\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\}$,
2. $\tilde{\mu}(xy) \geq \min^i\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\}$, for all $x, y \in R$.

Lemma 4.4. Let $\tilde{\mu}$ be an interval valued fuzzy subset of $R$ and $\overline{\tilde{\mu}}, \overline{\tilde{\mu}} \in D[0, 1]$ with $\overline{\tilde{\mu}}, \overline{\tilde{\mu}} \neq [0, 0]$. Then
(1) \( \mu \) is an \((\in, \in \lor q)\)-fuzzy subnear-ring of \( R \) and
\[
(\bar{\mu}(x-y) \geq \min^1\{\bar{\mu}(x), \bar{\mu}(y), [0.5, 0.5]\}, \bar{\mu}(xy) \geq \min^1\{\bar{\mu}(x), \bar{\mu}(y), [0.5, 0.5]\}
\]
for all \( x, y \in R \) are equivalent.

(2) \( c \) \( x_1 \in \bar{\mu} \) and \( y \in R \) implies \((y + x - y)_i \in \forall q \bar{\mu}\), and
\[
(\bar{\mu}(y + x - y) \geq \min^1\{\bar{\mu}(x), [0.5, 0.5]\}, \text{ for all } x, y \in R \text{ are equivalent.}
\]

(3) \( e \) \( y_1 \in \bar{\mu} \) and \( x \in R \) implies \((xy)_i \in \forall q \bar{\mu}\) and
\[
(\bar{\mu}(xy) \geq \min^1\{\bar{\mu}(y), [0.5, 0.5]\}, \text{ for all } x, y \in R \text{ are equivalent.}
\]

(4) \( f \) \( z_1 \in \bar{\mu} \) and \( x, y \in R \) implies \((x + y) \in \forall q \bar{\mu}\) and
\[
(\bar{\mu}((x+z) - (y-x)) \geq \min^1\{\bar{\mu}(z), [0.5, 0.5]\}, \text{ for all } x, y, z \in R \text{ are equivalent.}
\]

Proof. Let \( \bar{\mu} \) be an interval-valued fuzzy subset of \( R \).

(1) \( a \) \( b \), Theorem 4.3.

(2) \( e \) \( d \). Suppose that \( d \) is not valid, then there exists \( x, y \in R \) such that
\[
\bar{\mu}(y + x - y) < \min^1\{\bar{\mu}(x), [0.5, 0.5]\}.
\]
Now, we aries the following two cases:

(i) \( \bar{\mu}(x) \leq [0.5, 0.5] \)
(ii) \( \bar{\mu}(x) > [0.5, 0.5] \).

Case (i): We have \( \bar{\mu}(y + x - y) < \bar{\mu}(x) \). Choose an interval \( \bar{\mu} \) such that \( \bar{\mu}(y + x - y) < \bar{\mu}(x) \). This implies \( x_1 \in \bar{\mu} \) and \( (y + x - y)_i \in \forall q \bar{\mu} \), which contradicts (c). So, \( \bar{\mu}(y + x - y) \geq \bar{\mu}(x) = \min^1\{\bar{\mu}(x), [0.5, 0.5]\} \).

Case (ii): We have \( \bar{\mu}(y + x - y) \leq [0.5, 0.5] \). Then \( x_1 \in \bar{\mu} \) and \( (y + x - y)_i \in \forall q \bar{\mu} \), which is a contradiction to (c). Hence \( \bar{\mu}(y + x - y) \geq [0.5, 0.5] = \min^1\{\bar{\mu}(x), [0.5, 0.5]\} \).

(3) \( e \) \( f \). Let us assume that \( f \) is not valid. Then \( x, y \in R \), we can write \( \bar{\mu}(xy) < \min^1\{\bar{\mu}(y), [0.5, 0.5]\} \). We consider the following two cases:

(i) \( \bar{\mu}(y) \leq [0.5, 0.5] \)
(ii) \( \bar{\mu}(y) > [0.5, 0.5] \).

Case (i): We have \( \bar{\mu}(xy) < \bar{\mu}(y) \). Choose \( \bar{\mu} \) such that \( \bar{\mu}(xy) < \bar{\mu}(y) \). Then \( y_1 \in \bar{\mu} \), but \( (xy)_i \in \forall q \bar{\mu} \), which contradicts (e).

Case (ii): We have \( \bar{\mu}(xy) < [0.5, 0.5] \leq \bar{\mu}(y) \). This implies that \( y_1 \in \bar{\mu} \), but \( (xy)_i \in \forall q \bar{\mu} \), which contradicts (e). Therefore \( (xy)_i \in \forall q \bar{\mu} \).

(4) \( f \) \( e \) : Let \( y_1 \in \bar{\mu} \) and \( x \in R \) be such that \( \bar{\mu}(y) \geq \bar{\mu} \). We have \( \bar{\mu}(xy) \geq \min^1\{\bar{\mu}(y), [0.5, 0.5]\} \geq \min^1\{\bar{\mu}(x), [0.5, 0.5]\} \), which implies that \( \bar{\mu}(xy) \geq \bar{\mu}(x) \) and \( \bar{\mu}(xy) \geq \bar{\mu}(y) \) according to \( \bar{\mu} \leq [0.5, 0.5] \) or \( \bar{\mu} > [0.5, 0.5] \). Therefore \( (xy)_i \in \forall q \bar{\mu} \).

Similarly, we can prove \( (4)(g) \Rightarrow (h) \) and \( (h) \Rightarrow (g) \). This completes the proof.

By Definition 4.2 and Lemma 4.4, we obtain the following theorem.

**Theorem 4.5.** An interval-valued fuzzy subset \( \bar{\mu} \) of \( R \) is an interval-valued \((\in, \in \lor q)\)-fuzzy ideal of \( R \) if and only if

(1) \( \bar{\mu} \) is an interval-valued \((\in, \in \lor q)\)-fuzzy subnear-ring of \( R \),
(2) \( \bar{\mu}(x+y-x) \geq \min^1\{\mu(x), [0.5, 0.5]\} \),
(3) \( \bar{\mu}(xy) \geq \min^1\{\mu(y), [0.5, 0.5]\} \),
(4) \( \bar{\mu}((x+z) - (y-x)) \geq \min^1\{\mu(z), [0.5, 0.5]\} \), for all \( x, y, z \in R \).
In the following theorem, we explain the construction of an interval valued generalized fuzzy ideal form an ideal.

**Theorem 4.6.** Let $I$ be an ideal of $R$. For every $\bar{t} \in D[0,0.5]$ with $\bar{t} \neq [0,0]$ there exists an interval valued $(\varepsilon, \in \vee q)$-fuzzy ideal $\tilde{\mu}$ of $R$ such that $\tilde{U}(\tilde{\mu} : \bar{t}) = I$.

**Proof.** Let $\tilde{\mu}$ be an interval valued fuzzy subset in $R$ defined by

$$\tilde{\mu}(x) = \begin{cases} \bar{t} & \text{if } x \in I \\ [0,0] & \text{otherwise} \end{cases}$$

for all $x \in R$, where $\bar{t} \in D[0,0.5]$ with $\bar{t} \neq [0,0]$. Obviously, $\tilde{U}(\tilde{\mu} : \bar{t}) = I$. Assume that $\mu(x-y) < \min\{\mu(x), \mu(y), [0.5,0.5]\}$, for some $x, y \in R$. Since $|\text{Im}(\mu)| = 2$, it follows that $\mu(x-y) = [0,0]$ and $\min\{\mu(x), \mu(y), [0.5,0.5]\} = \bar{t}$. Hence $\mu(x) = \bar{t} = \mu(y)$ and so $x, y \in I$. Thus $x - y \in I$, since $I$ is an ideal of $R$ and so $\mu(x-y) = \bar{t}$, which is a contradiction. Therefore $\mu(x-y) \geq \min\{\mu(x), \mu(y), [0.5,0.5]\}$. Let us suppose that $\mu(x+y-z) < \min\{\mu(x), [0.5,0.5]\}$, for some $x, y \in R$. It follows that $\mu(x+y-z) = [0,0]$ and $\min\{\mu(x), [0.5,0.5]\} = \bar{t}$. Hence $\mu(x) = \bar{t}$ and so $x \in I$. Since $I$ is an ideal of $R$, then $y + x - y \in I$. Thus $\mu(y+x-y) = \bar{t}$, which is a contradiction and hence $\mu(x+y-z) \geq \min\{\mu(x), [0.5,0.5]\}$. Similarly, the same procedure we have $\mu((x+z)y - xy) \geq \min\{\mu(z), [0.5,0.5]\}$. □

The next theorem brings out the relationship between interval valued $(\varepsilon, \in \vee q)$-fuzzy ideals of $R$ and the crisp ideals of $R$.

**Theorem 4.7.** A nonempty subset $I$ of $R$ is an ideal of $R$ if and only if $\tilde{f}_I$ is an interval valued $(\varepsilon, \in \vee q)$-fuzzy ideal of $R$.

**Proof.** Let $I$ be an ideal of $R$. Then $\tilde{f}_I$ is an interval valued $(\varepsilon, \in \vee q)$-fuzzy ideal of $R$ by Theorem 4.6.

Conversely, assume that $\tilde{f}_I$ is an interval valued $(\varepsilon, \in \vee q)$ fuzzy ideal of $R$. Then clearly, $\tilde{f}_I(x-y) \geq \min\{\tilde{f}_I(x), \tilde{f}_I(y), [0.5,0.5]\} = \min\{1,1,[0.5,0.5]\} = [0.5,0.5] \neq [0,0]$, which implies $\tilde{f}_I(x-y) = [1,1]$ and so $x - y \in I$. Let $x \in I$ and $y \in R$. Then, $\tilde{f}_I(y+x-y) \geq \min\{\tilde{f}_I(x), [0.5,0.5]\} = \min\{1,1,[0.5,0.5]\} = [0.5,0.5] \neq [0,0]$. This implies that $\tilde{f}_I(y+x-y) = [1,1]$ and so $y + x - y \in I$. Let $y \in I$ and $x \in R$ be such that $\tilde{f}_I(y) = [1,1]$. Then, $\tilde{f}_I(xy) \geq \min\{\tilde{f}_I(y), [0.5,0.5]\} = [0.5,0.5] \neq [0,0]$. This implies that $\tilde{f}_I(xy) = [1,1]$ and so $xy \in I$. Similarly, we proceed like this $(x+z)y - xy \in I$. □

Now, we characterize the interval valued $(\varepsilon, \in \vee q)$-fuzzy ideals using their level ideals.

**Theorem 4.8.** An interval valued fuzzy subset $\tilde{\mu}$ of $R$ is an interval valued $(\varepsilon, \in \vee q)$-fuzzy ideal of $R$ if and only if the level subset $\tilde{U}(\tilde{\mu} : \bar{t})$ is an ideal of $R$ for all $[0,0] < \bar{t} \leq [0.5,0.5]$. 43
Proof. Let \( \tilde{\mu} \) be an interval valued \((e, \in \forall q)\)-fuzzy ideal of \( R \) and \([0,0] < \tilde{t} \leq [0.5,0.5] \). Let \( x, y \in \tilde{U}(\tilde{\mu} : \tilde{t}) \) then \( \tilde{\mu}(x) \geq \tilde{t} \) and \( \tilde{\mu}(y) \geq \tilde{t} \). Now by Theorem 4.5, we have \( \tilde{\mu}(x - y) \geq \min^t\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5,0.5]\} \geq \min^t\{\tilde{t}, [0.5,0.5]\} = \tilde{t} \). So \( x - y \in \tilde{U}(\tilde{\mu} : \tilde{t}) \). If \( x \in \tilde{U}(\tilde{\mu} : \tilde{t}) \) and \( y \in R \). Then \( \tilde{\mu}(x) \geq \tilde{t} \). Consequently by Theorem 4.5, we have \( \tilde{\mu}(y + x - y) \geq \min^t\{\tilde{\mu}(x), [0.5,0.5]\} \geq \min^t\{\tilde{t}, [0.5,0.5]\} = \tilde{t} \). So \( y + x - y \in \tilde{U}(\tilde{\mu} : \tilde{t}) \). Let \( y \in \tilde{U}(\tilde{\mu} : \tilde{t}) \) and \( x \in R \). Then \( \tilde{\mu}(y) \geq \tilde{t} \). Since \( \tilde{\mu} \) is an interval valued \((e, \in \forall q)\)-fuzzy ideal of \( R \), we have \( \tilde{\mu}(xy) \geq \min^t\{\tilde{\mu}(y), [0.5,0.5]\} \geq \min^t\{\tilde{t}, [0.5,0.5]\} = \tilde{t} \). Thus \( xy \in \tilde{U}(\tilde{\mu}, \tilde{t}) \) and so \( \tilde{U}(\tilde{\mu} : \tilde{t}) \) is a left ideal of \( R \). Also, for every \( z \in \tilde{U}(\tilde{\mu} : \tilde{t}) \) and \( x, y \in R \) such that \( \tilde{\mu}(z) \geq \tilde{t} \). Then \( \tilde{\mu}((x + z) y - x y) \geq \min^t\{\tilde{\mu}(z), [0.5,0.5]\} \geq \min^t\{\tilde{t}, [0.5,0.5]\} = \tilde{t} \) and so \((x + z) y - x y \in \tilde{U}(\tilde{\mu} : \tilde{t}) \). Therefore \( \tilde{U}(\tilde{\mu} : \tilde{t}) \) is an ideal of \( R \).

Conversely, assume that \( \tilde{\mu} \) is an interval valued fuzzy subset of \( R \) such that \( \tilde{U}(\tilde{\mu} : \tilde{t})(\neq \emptyset) \) become an ideal of \( R \), for all \([0,0] < \tilde{t} \leq [0.5,0.5] \). Let \( x, y \in R \). Suppose that \( \tilde{\mu}(x - y) < \min^t\{\tilde{\mu}(x), [0.5,0.5]\} \). Then we can choose \( t \) such that \( \tilde{\mu}(x - y) < \tilde{t} < \min^t\{\tilde{\mu}(x), [0.5,0.5]\} \). This implies that \( x, y \in \tilde{U}(\tilde{\mu} : \tilde{t}) \). Since \( \tilde{U}(\tilde{\mu} : \tilde{t}) \) is an additive subgroup of \( R \), then \((x - y) \in \tilde{U}(\tilde{\mu} : \tilde{t}) \) and so \( \tilde{\mu}(x - y) \geq \tilde{t} \), which is a contradiction. Thus \( \tilde{\mu}(x - y) \geq \min^t\{\tilde{\mu}(x), [0.5,0.5]\} \). Let us assume that \( \tilde{\mu}(y + x - y) < \min^t\{\tilde{\mu}(x), [0.5,0.5]\} \). Choose \( \tilde{t} \) such that \( \tilde{\mu}(y + x - y) < \tilde{t} < \min^t\{\tilde{\mu}(x), [0.5,0.5]\} \). Then \( x \in \tilde{U}(\tilde{\mu} : \tilde{t}) \) and \( y + x - y \in \tilde{U}(\tilde{\mu} : \tilde{t}) \), since \( \tilde{U}(\tilde{\mu} : \tilde{t}) \) is an ideal of \( R \). This implies that \( \tilde{\mu}(y + x - y) \geq \tilde{t} \), which contradicts to our hypothesis. Hence \( \tilde{\mu}(x + y - x) \geq \min^t\{\tilde{\mu}(x), [0.5,0.5]\} \). Suppose that \( \tilde{\mu}(xy) < \min^t\{\tilde{\mu}(y), [0.5,0.5]\} \), for all \( x, y \in R \). Then there exist \( \tilde{t} \) such that \( \tilde{\mu}(xy) < \tilde{t} < \min^t\{\tilde{\mu}(y), [0.5,0.5]\} \). Thus \( y \in \tilde{U}(\tilde{\mu} : \tilde{t}) \) and so \( xy \in \tilde{U}(\tilde{\mu} : \tilde{t}) \), since \( \tilde{U}(\tilde{\mu} : \tilde{t}) \) is an ideal of \( R \). Hence \( \tilde{\mu}(xy) \geq \tilde{t} \), which contradicts to our hypothesis. Hence \( \tilde{\mu}(xy) \geq \min^t\{\tilde{\mu}(y), [0.5,0.5]\} \). Similarly, we can prove that \( \tilde{\mu}((x + z) y - x y) \geq \min^t\{\tilde{\mu}(z), [0.5,0.5]\} \). Therefore \( \tilde{\mu} \) is an interval valued \((e, \in \forall q)\)-fuzzy ideal of \( R \).

Next, we discuss the relationship between these generalized interval valued fuzzy ideals.

**Theorem 4.9.** Every interval valued \((e, \in \forall q)\)-fuzzy ideal of \( R \) is an interval valued \((e, \in \forall q)\)-fuzzy ideal of \( R \).

**Proof.** Let \( \tilde{\mu} \) be an interval valued \((e, \in \forall q)\)-fuzzy ideal of \( R \). Suppose that \( x, y \in R \) and \( \tilde{t}, \tilde{t} \in D[0,1] \) with \( \tilde{t}, \tilde{t} \neq [0,0] \) such that \( x_\tilde{t} \in \tilde{\mu} \) and \( y_\tilde{t} \in \tilde{\mu} \). Then \( x_\tilde{t} \in \forall q \tilde{\mu} \) and \( y_\tilde{t} \in \forall q \tilde{\mu} \). By the hypothesis \( (x - y)_{\min^t(\tilde{t}, \tilde{t})} \in \forall q \tilde{\mu} \). Now \( x, y \in R \) and \( \tilde{t}, \tilde{t} \in D[0,1] \) with \( \tilde{t}, \tilde{t} \neq [0,0] \) such that \( x_\tilde{t} \in \tilde{\mu} \). Then \( x_\tilde{t} \in \forall q \tilde{\mu} \), so by hypothesis \( (y + x - y)_{\tilde{t}} \in \forall q \tilde{\mu} \). Similarly, we prove \( (xy)_{\tilde{t}} \in \forall q \tilde{\mu} \) and \( ((x + z) y - x y)_{\tilde{t}} \in \forall q \tilde{\mu} \). Therefore \( \tilde{\mu} \) is an interval valued \((e, \in \forall q)\)-fuzzy ideal of \( R \).

The following theorem gives the connection between interval valued \((e, \in)\)-fuzzy ideal and interval valued fuzzy ideal.

**Theorem 4.10.** An interval valued fuzzy subset \( \tilde{\mu} \) of \( R \) is an interval valued \((e, \in)\)-fuzzy ideal of \( R \) if and only if it is an interval valued fuzzy ideal of \( R \).

**Proof.** Assume that \( \tilde{\mu} \) is an interval valued fuzzy ideal of \( R \). Let \( x, y \in R \) and \( \tilde{t}, \tilde{t} \in D[0,1] \) with \( \tilde{t}, \tilde{t} \neq [0,0] \) be such that \( x_\tilde{t}, y_\tilde{t} \in \tilde{\mu} \). Then \( \tilde{\mu}(x) \geq \tilde{t} \) and \( \tilde{\mu}(y) \geq \tilde{t} \).
Since $\tilde{\mu}$ is an interval valued fuzzy ideal of $R$, we have $\tilde{\mu}(x - y) \geq \min^1\{\tilde{\mu}(x), \tilde{\mu}(y)\} \geq \min^1\{\tilde{t}, \tilde{r}\}$, it follows that $(x - y)_{\min^1\{\tilde{t}, \tilde{r}\}} \leq \tilde{\mu}$. Now let $x, y \in R$ and $\tilde{t} \in \mathbb{D}[0, 1]$ with $\tilde{t} \neq [0, 0]$. Then $x\tilde{t} \in \tilde{\mu}$ and so $\tilde{\mu}(x) \geq \tilde{t}$. Since $\tilde{\mu}$ is an interval valued fuzzy ideal of $R$, we have $\tilde{\mu}(y + x - y) \geq \tilde{\mu}(x) \geq \tilde{t}$. Hence $(y + x - y)_{\tilde{t}} \in \tilde{\mu}$. Let $x, y \in R$ and $\tilde{t} \in \mathbb{D}[0, 1]$ with $\tilde{t} \neq [0, 0]$. Then $y\tilde{t} \in \tilde{\mu}$ and so $\tilde{\mu}(y) \geq \tilde{t}$. Hence $\tilde{\mu}(xy) \geq \tilde{\mu}(y) \geq \tilde{t}$, because $\tilde{\mu}$ is an interval valued fuzzy ideal of $R$. Thus $(xy)_{\tilde{t}} \in \tilde{\mu}$. Again let $x, z, y \in R$ and $\tilde{t} \in \mathbb{D}[0, 1]$ with $\tilde{t} \neq [0, 0]$ be such that $\tilde{\mu}(z) \geq \tilde{t}$. Since $\tilde{\mu}$ is an interval valued fuzzy ideal of $R$, then $\tilde{\mu}((x + z)y - xy) \geq \tilde{\mu}(z) \geq \tilde{t}$. Thus $((x + z)y - xy)\tilde{t} \in \tilde{\mu}$ and therefore $\tilde{\mu}$ is an interval valued $(\varepsilon, \varepsilon)$-fuzzy ideal of $R$.

Conversely, assume that $\tilde{\mu}$ is an interval valued $(\varepsilon, \varepsilon)$-fuzzy ideal of $R$. On the contrary assume that there exist $x, y \in R$ such that $\tilde{\mu}(x - y) < \min^1\{\tilde{\mu}(x), \tilde{\mu}(y)\}$. Choose $\tilde{t}$ such that $\tilde{\mu}(x - y) < \tilde{t} < \min^1\{\tilde{\mu}(x), \tilde{\mu}(y)\}$. Then $x\tilde{t}, y\tilde{t} \in \tilde{\mu}$ and $(x - y)\tilde{t} \notin \tilde{\mu}$. This is a contradiction to our assumption that $\tilde{\mu}$ is an interval valued $(\varepsilon, \varepsilon)$-fuzzy ideal of $R$. Thus $\tilde{\mu}(x - y) \geq \min^1\{\tilde{\mu}(x), \tilde{\mu}(y)\}$. Suppose that $\tilde{\mu}(y + x - y) < \tilde{\mu}(x)$, for some $x, y \in R$. Choose $\tilde{t}$ such that $\tilde{\mu}(y + x - y) < \tilde{t} < \tilde{\mu}(x)$. Then $x\tilde{t} \in \tilde{\mu}$ and $(y + x - y)\tilde{t} \notin \tilde{\mu}$, which is a contradiction and hence $\tilde{\mu}(y + x - y) \geq \tilde{\mu}(x)$. Let us assume that $\tilde{\mu}(xy) < \tilde{\mu}(y)$, for some $x, y \in R$. Then there exist $\tilde{t}$ such that $\tilde{\mu}(xy) < \tilde{t} < \tilde{\mu}(y)$. This implies that $y\tilde{t} \in \tilde{\mu}$ but $(xy)\tilde{t} \notin \tilde{\mu}$. This contradicts our hypothesis. Hence $\tilde{\mu}(xy) \geq \tilde{\mu}(y)$. Again the contrary assume that there exist $x, y, z \in R$ such that $\tilde{\mu}((x + z)y - xy) < \tilde{\mu}(z)$. Let $\tilde{t}$ be such that $\tilde{\mu}((x + z)y - xy) < \tilde{t} < \tilde{\mu}(z)$. Then $z\tilde{t} \in \tilde{\mu}$ but $((x + z)y - xy)\tilde{t} \notin \tilde{\mu}$, which is a contradiction and so $\tilde{\mu}((x + z)y - xy) \geq \tilde{\mu}(z)$. Therefore $\tilde{\mu}$ is an interval valued fuzzy ideal of $R$. 

**Theorem 4.11.** Every interval valued $(\varepsilon, q)$-fuzzy ideal of $R$ is an interval valued $(\varepsilon, \sqrt{q})$-fuzzy ideal of $R$.

**Proof.** The proof is straightforward. □

The converse part of the above Theorem 4.11 is not true as general as shown in Example 3.4(6).

**Theorem 4.12.** Every interval valued $(\varepsilon, \varepsilon)$-fuzzy ideal of $R$ is an interval valued $(\varepsilon, \sqrt{q})$-fuzzy ideal of $R$.

**Proof.** The proof is straightforward. □

The converse part of the above Theorem 4.12 is not true as general as shown in Example 3.4(1).

In the following theorem, we give a condition for an interval valued $(\varepsilon, \varepsilon)$-fuzzy ideal of $R$ to be an interval valued $(\varepsilon, \varepsilon)$-fuzzy ideal of $R$.

**Theorem 4.13.** Let $\tilde{\mu}$ be an interval valued $(\varepsilon, \sqrt{q})$-fuzzy ideal of $R$ such that $\tilde{\mu}(x) < [0.5, 0.5]$ for all $x \in R$. Then $\tilde{\mu}$ is an interval valued $(\varepsilon, \varepsilon)$-fuzzy ideal of $R$.

**Proof.** Let $x, y \in R$ and $\tilde{t}, \tilde{r} \in \mathbb{D}[0, 1]$ with $\tilde{t}, \tilde{r} \neq [0, 0]$ be such that $x\tilde{t}, y\tilde{r} \in \tilde{\mu}$. Then $\tilde{\mu}(x) \geq \tilde{t}, \tilde{\mu}(y) \geq \tilde{r}$. Since $\tilde{\mu}$ is an interval valued $(\varepsilon, \sqrt{q})$-fuzzy ideal of $R$, then $\tilde{\mu}(x - y) \geq \min^1\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \geq \min^1\{\tilde{t}, \tilde{r}, [0.5, 0.5]\} = \min^1\{\tilde{t}, \tilde{r}\}$ and so $(x - y)_{\min^1\{\tilde{t}, \tilde{r}\}} \leq \tilde{\mu}$. Let $x, y \in R$ and $\tilde{t} \in \mathbb{D}[0, 1]$ with $\tilde{t} \neq [0, 0]$ be such that $x\tilde{t} \in \tilde{\mu}$. Then $\tilde{\mu}(x) \geq \tilde{t}$. Thus $\tilde{\mu}(y + x - y) \geq \min^1\{\tilde{\mu}(x), [0.5, 0.5]\} \geq \tilde{t}$, since $\tilde{\mu}$ is an interval valued $(\varepsilon, \varepsilon)$-fuzzy ideal of $R$. □
valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \(R\). Hence \((y + z) - y) \in \tilde{\mu}\). Let \(x, y \in R\) and \(t \in D[0, 1] \) with \(t \neq [0, 0]\). Then \(y \in \tilde{\mu}\) implies \(\mu(y) \geq t\). So \(\tilde{\mu}(xy) = \min \{\tilde{\mu}(y), [0.5, 0.5]\} \geq t\), since \(\tilde{\mu}\) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \(R\). Thus \((xy) \in \tilde{\mu}\). Similarly, we can prove that \((x + z)y - xy) \in \tilde{\mu}\). Therefore \(\tilde{\mu}\) is an interval valued \((\varepsilon, \in)\)-fuzzy ideal of \(R\). \(\square\)

**Theorem 4.14** ([21]). If \(\{\tilde{\mu}_i\}_{i \in \Omega}\) is a family of interval valued \((\varepsilon, \in \vee q)\)-fuzzy subnear-ring of a near-ring \(R\), then \(\tilde{\mu} = \bigcap_{i \in \Omega} \tilde{\mu}_i\) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy subnear-ring of a near-ring \(R\), where \(\Omega\) is any index set.

**Theorem 4.15.** If \(\{\tilde{\mu}_i\}_{i \in \Omega}\) is a family of interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \(R\), then \(\tilde{\mu} = \bigcap_{i \in \Omega} \tilde{\mu}_i\) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \(R\), where \(\Omega\) is any index set.

**Proof.** Let \(x, y, z \in R\). Then, clearly, \(\tilde{\mu} = \bigcap_{i \in \Omega} \tilde{\mu}_i\) is an interval valued \((\varepsilon, \in \vee q)\) fuzzy subnear-ring of \(R\) from Theorem 4.15. Then,

\[
\tilde{\mu}(y + x - y) = \bigcap_{i \in \Omega} \tilde{\mu}_i(y + x - y) = \inf \{\tilde{\mu}_i(y + x - y) : i \in \Omega\}
\]

\[
\geq \inf \{\min \{\tilde{\mu}_i(x), [0.5, 0.5]\} : i \in \Omega\}
\]

\[
= \min \{\inf \{\tilde{\mu}_i(x), [0.5, 0.5]\} : i \in \Omega\}
\]

\[
= \min \{\bigcap_{i \in \Omega} \tilde{\mu}_i(x), [0.5, 0.5]\}
\]

Similarly, \(\tilde{\mu}(x + z)\) \(y - xy) \geq \min \{\tilde{\mu}(z), [0.5, 0.5]\}\). Therefore \(\tilde{\mu} = \bigcap_{i \in \Omega} \tilde{\mu}_i\) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \(R\). \(\square\)

**Theorem 4.16** ([21]). Let \(\tilde{\mu}\) be an interval valued fuzzy subset of \(R\). \(\tilde{\mu} = [\mu^-, \mu^+]\) \(\text{is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy subnear-ring of } R\) if and only if \(\mu^-, \mu^+\) are \((\varepsilon, \in \vee q)\)-fuzzy subnear-ring of \(R\).

The following theorem establishes the connection between interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \(R\) and \((\in, \varepsilon, \vee q)\)-fuzzy ideal of \(R\).
Theorem 4.17. Let \( \bar{\mu} \) be an interval valued fuzzy subset of \( R \). \( \bar{\mu} = [\mu^-, \mu^+] \) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \) if and only if \( \mu^-, \mu^+ \) are \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \).

Proof. Let \( \bar{\mu} \) be an interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \). For any \( x, y, z \in R \).

\[
\begin{align*}
[\mu^-(x - y), \mu^+(x - y)] &= \bar{\mu}(x - y) \\
&\geq \min^i\{\bar{\mu}(x), \bar{\mu}(y), [0, 0.5]\} \\
&= \min^i\{[\mu^-(x), \mu^+(x)], [\mu^-(y), \mu^+(y)], [0, 0.5]\} \\
&= [\min\{\mu^-(x), \mu^-(y), 0.5\}, \min\{\mu^+(x), \mu^+(y), 0.5\}].
\end{align*}
\]

It follows that \( \mu^-(x - y) \geq \min\{\mu^-(x), \mu^-(y), 0.5\} \) and \( \mu^+(x - y) \geq \min\{\mu^+(x), \mu^+(y), 0.5\} \). And

\[
\begin{align*}
[\mu^-(y + x - y), \mu^+(y + x - y)] &= \bar{\mu}(y + x - y) \\
&\geq \min^i\{\bar{\mu}(x), [0, 0.5]\} \\
&= \min^i\{[\mu^-(x), \mu^+(x)], [0, 0.5]\} \\
&= [\min\{\mu^-(x), 0.5\}, \min\{\mu^+(x), 0.5\}].
\end{align*}
\]

It follows that \( \mu^-(y + x - y) \geq \min\{\mu^-(x), 0.5\} \) and \( \mu^+(y + x - y) \geq \min\{\mu^+(x), 0.5\} \). Further,

\[
\begin{align*}
[\mu^-(xy), \mu^+(xy)] &= \bar{\mu}(xy) \\
&\geq \min^i\{\bar{\mu}(y), [0, 0.5]\} \\
&= \min^i\{[\mu^-(y), \mu^+(y)], [0, 0.5]\} \\
&= [\min\{\mu^-(y), 0.5\}, \min\{\mu^+(y), 0.5\}].
\end{align*}
\]

It follows that \( \mu^-(xy) \geq \min\{\mu^-(y), 0.5\} \) and \( \mu^+(xy) \geq \min\{\mu^+(y), 0.5\} \). Similarly, \( \mu^-(y + x - y) \geq \min\{\mu^-(z), 0.5\} \), \( \mu^+(y + x - y) \geq \min\{\mu^+(z), 0.5\} \). Therefore \( \mu^+ \) and \( \mu^- \) are \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \).

Conversely, assume that \( \mu^+ \) and \( \mu^- \) are \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \). Let \( x, y, z \in R \). Then,

\[
\begin{align*}
\bar{\mu}(x - y) &= [\mu^-(x - y), \mu^+(x - y)] \\
&\geq [\min\{\mu^-(x), \mu^-(y), 0.5\}, \min\{\mu^+(x), \mu^+(y), 0.5\}] \\
&= \min^i\{[\mu^-(x), \mu^+(x)], [\mu^-(y), \mu^+(y)], [0, 0.5]\} \\
&= \min^i\{\bar{\mu}(x), \bar{\mu}(y), [0, 0.5]\}.
\end{align*}
\]

Further,

\[
\begin{align*}
\bar{\mu}(y + x - y) &= [\mu^-(y + x - y), \mu^+(y + x - y)] \\
&\geq [\min\{\mu^-(x), 0.5\}, \min\{\mu^+(x), 0.5\}] \\
&= \min^i\{[\mu^-(x), \mu^+(x)], [0, 0.5]\} \\
&= \min^i\{\bar{\mu}(x), [0, 0.5]\}.
\end{align*}
\]
And
\[
\tilde{\mu}(xy) = [\mu^-(xy), \mu^+(xy)] \\
\geq [\min\{\mu^-(y), 0.5\}, \min\{\mu^+(y), 0.5\}] \\
= \min'\{[\mu^-(y), \mu^+(y)], [0.5, 0.5]\} \\
= \min'\{\tilde{\mu}(y), [0.5, 0.5]\}.
\]
Similarly, \(\tilde{\mu}((x+z)y-x-y) \geq \min'\{\tilde{\mu}(z), [0.5, 0.5]\}\). □

**Definition 4.18.** For any interval valued fuzzy subset \(\tilde{\mu}\) of \(R\) and \(\tilde{t} \in D[0, 1]\) with \(\tilde{t} \neq [0, 0]\) we consider two subsets: \(\tilde{Q}(\tilde{\mu}; \tilde{t}) = \{x \in R | x \tilde{\mu} \tilde{t}\}\) and \([\tilde{\mu}]_t = \{x \in R | x \tilde{t} \in \lor \tilde{\mu}\}\). Obviously, \([\tilde{\mu}]_t = \tilde{U}(\tilde{\mu} : \tilde{t}) \cup \tilde{Q}(\tilde{\mu}; \tilde{t})\).

We call \([\tilde{\mu}]_t\) as an \(\in \lor \tilde{\mu}\)-level ideal and \(\tilde{Q}(\tilde{\mu}; \tilde{t})\) a \(q\)-level ideal of \(\tilde{\mu}\).

**Lemma 4.19.** Every interval valued fuzzy subset \(\tilde{\mu}\) of \(R\) satisfies the following assertion \(\tilde{t} \in D[0, 0.5]\) with \(\tilde{t} \neq [0, 0]\) implies \([\tilde{\mu}]_t = \tilde{U}(\tilde{\mu} : \tilde{t})\).

**Proof.** Let \(\tilde{t} \in D[0, 0.5]\) with \(\tilde{t} \neq [0, 0]\). Clearly, \(\tilde{U}(\tilde{\mu} : \tilde{t}) \subseteq [\tilde{\mu}]_t\). Let \(x \in [\tilde{\mu}]_t\). If \(x \notin \tilde{U}(\tilde{\mu} : \tilde{t})\), then \(\tilde{\mu}(x) < \tilde{t}\) and so \(\tilde{\mu}(x) + \tilde{t} < \tilde{t} + \tilde{t} = 2\tilde{t} \leq [1, 1]\). This implies that \(x \notin \tilde{Q}(\tilde{\mu}; \tilde{t})\), that is \(x \notin \tilde{Q}(\tilde{\mu}; \tilde{t})\). Thus \(x \notin \tilde{U}(\tilde{\mu} : \tilde{t}) \cup \tilde{Q}(\tilde{\mu}; \tilde{t}) = [\tilde{\mu}]_t\). This leads to a contradiction and so \(x \in \tilde{U}(\tilde{\mu} : \tilde{t})\). Therefore \([\tilde{\mu}]_t \subseteq \tilde{U}(\tilde{\mu} : \tilde{t})\). □

Using the \((\in \lor \tilde{\mu})\)-level ideals of near-rings, we characterize the interval valued \((\in, \in \lor \tilde{\mu})\)-fuzzy ideals of near-rings.

**Theorem 4.20.** An interval valued fuzzy subset \(\tilde{\mu}\) of \(R\) is an interval valued \((\in, \in \lor \tilde{\mu})\)-fuzzy ideal of \(R\) if and only if \([\tilde{\mu}]_t\) is an \(\in \lor \tilde{\mu}\)-level ideal of \(R\).

**Proof.** Assume that \(\tilde{\mu}\) is an interval valued \((\in, \in \lor \tilde{\mu})\)-fuzzy ideal of \(R\) and let \(\tilde{t} \in D[0, 0.5]\) with \(\tilde{t} \neq [0, 0]\) be such that \([\tilde{\mu}]_t\) is an \(\in \lor \tilde{\mu}\)-level ideal of \(R\). Let \(x, y \in [\tilde{\mu}]_t\) such that \(\tilde{\mu}(x) \geq \tilde{t}\) or \(\tilde{\mu}(x) + \tilde{t} > [1, 1]\) and \(\tilde{\mu}(y) \geq \tilde{t}\) or \(\tilde{\mu}(y) + \tilde{t} > [1, 1]\). We can consider four cases:

(i) \(\tilde{\mu}(x) \geq \tilde{t}\) and \(\tilde{\mu}(y) \geq \tilde{t}\),
(ii) \(\tilde{\mu}(x) \geq \tilde{t}\) and \(\tilde{\mu}(y) + \tilde{t} > [1, 1]\),
(iii) \(\tilde{\mu}(x) + \tilde{t} > [1, 1]\) and \(\tilde{\mu}(y) \geq \tilde{t}\),
(iv) \(\tilde{\mu}(x) + \tilde{t} > [1, 1]\) and \(\tilde{\mu}(y) + \tilde{t} > [1, 1]\).

Consider Case (i): \(\tilde{\mu}(x) \geq \tilde{t}\) and \(\tilde{\mu}(y) \geq \tilde{t}\) and \(\tilde{\mu}(x) + \tilde{t} > [1, 1]\). This implies that
\[
\tilde{\mu}(x-y) \geq \min'\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \geq \min'\{\tilde{t}, [0.5, 0.5]\}
\]
\[
= \begin{cases} 
[0.5, 0.5] & \text{if } \tilde{t} > [0.5, 0.5] \\
\tilde{t} & \text{if } \tilde{t} \leq [0.5, 0.5]
\end{cases}
\]
If \(\tilde{t} > [0.5, 0.5]\), then \(\tilde{\mu}(x-y) \geq [0.5, 0.5]\) and so \(\tilde{\mu}(x-y) + \tilde{t} > [0.5, 0.5] + [0.5, 0.5] = [1, 1]\), that is, \((x-y)\in \lor \tilde{\mu}\). If \(\tilde{t} \leq [0.5, 0.5]\), then \(\tilde{\mu}(x-y) \geq \tilde{t}\) and \((x-y)\in \lor \tilde{\mu}\). Therefore, \((x-y)\in \lor \tilde{\mu}\), that is, \((x-y)\in [\tilde{\mu}]_t\). Case (ii): \(\tilde{\mu}(x) \geq \tilde{t}\) and \(\tilde{\mu}(y) + \tilde{t} > [1, 1]\). If \(\tilde{t} > [0.5, 0.5]\), then \(\tilde{\mu}(x-y) \geq \min'\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \geq \min'\{\tilde{t}, [1, 1] - \tilde{t}, [0.5, 0.5]\} = [1, 1] - \tilde{t}\), that is, \(\tilde{\mu}(x-y) + \tilde{t} > [1, 1]\) and \((x-y)\in \lor \tilde{\mu}\). If \(\tilde{t} \leq [0.5, 0.5]\), then \(\tilde{\mu}(x-y) \geq \min'\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \geq \min'\{\tilde{t}, [1, 1] - \tilde{t}, [0.5, 0.5]\} = \tilde{t}\), that is, \((x-y)\in [\tilde{\mu}]_t\). Similarly, we
can prove the result for the case (iii). Next we consider the case (iv): \( \tilde{\mu}(x) + \tilde{t} > [1, 1] \) and \( \tilde{\mu}(y) + \tilde{t} > [1, 1] \). If \( \tilde{t} > [0.5, 0.5] \), then \( \tilde{\mu}(x - y) \geq \min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \geq \min\{[1, 1] - \tilde{t}, [1, 1] - \tilde{t}, [0.5, 0.5]\} = [1, 1] - \tilde{t} \). So, \( \tilde{\mu}(x - y) + \tilde{t} > [1, 1] \), that is, \( (x - y) \otimes \tilde{\mu} \). If \( \tilde{t} \leq [0.5, 0.5] \), then \( \tilde{\mu}(x - y) \geq \min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \geq \min\{[1, 1] - \tilde{t}, [1, 1] - \tilde{t}, [0.5, 0.5]\} = [0.5, 0.5] \), that is, \( (x - y) \otimes \tilde{\mu} \) and hence \( (x - y) \otimes \tilde{\mu} \in \mathcal{V} \tilde{\mu} \). This means that \( x - y \in [\tilde{\mu}]_T \). Consequently, \( [\tilde{\mu}]_T \) is a subnear-ring of \( (R, +) \). Let \( x \in [\tilde{\mu}]_T \) and \( y \in R \) such that \( \tilde{\mu}(x) > \tilde{t} \) and \( \tilde{\mu}(x) + \tilde{t} > [1, 1] \) and we consider two cases:

Case (i): \( \tilde{\mu}(x) > \tilde{t} \). Since \( \tilde{\mu} \) is an \((\varepsilon, \in \mathcal{V} \mu)\)-fuzzy ideal of \( R \), we have \( \tilde{\mu}(y + x - y) \geq \min\{\tilde{\mu}(x), [0.5, 0.5]\} \) \( \geq \min\{[1, 1] - \tilde{t}, [0.5, 0.5]\} \). If \( \tilde{t} > [0.5, 0.5] \), then \( \tilde{\mu}(y + x - y) \geq [0.5, 0.5] \) and so \( \tilde{\mu}(y + x + y + \tilde{t} > [0.5, 0.5] + [0.5, 0.5] = [1, 1] \), that is, \( \tilde{\mu}(y + x - y + \tilde{t} > [1, 1] \). Thus \( (y + x - y) \otimes \tilde{\mu} \). If \( \tilde{t} \leq [0.5, 0.5] \), then \( \tilde{\mu}(y + x - y) \geq \tilde{t} \). Hence \( (y + x - y) \otimes \tilde{\mu} \). Let \( y \in [\tilde{\mu}]_T \) and \( x \in R \). Then \( \tilde{\mu}(y) + \tilde{t} \). Assume that \( \tilde{\mu}(y) > \tilde{t} \). Since \( \tilde{\mu} \) is an \((\varepsilon, \in \mathcal{V} \mu)\)-fuzzy ideal of \( R \), we have \( \tilde{\mu}(y) \geq \min\{\tilde{\mu}(y), [0.5, 0.5]\} \geq \min\{[1, 1] - \tilde{t}, [0.5, 0.5]\} \). If \( \tilde{t} > [0.5, 0.5] \), then \( \tilde{\mu}(y) \geq [0.5, 0.5] \) and \( \tilde{\mu}(y + x - y) \geq [1, 1] - \tilde{t} \). Thus \( (y + x - y) \otimes \tilde{\mu} \). If \( \tilde{t} \leq [0.5, 0.5] \), then \( \tilde{\mu}(y) \geq \tilde{t} \). Hence \( (y + x - y) \otimes \tilde{\mu} \). Let \( y \in [\tilde{\mu}]_T \) and \( x \in R \). Then \( \tilde{\mu}(y) + \tilde{t} \). Let us assume that \( \tilde{\mu}(x) > \tilde{t} \). Since \( \tilde{\mu} \) is an interval valued \((\varepsilon, \in \mathcal{V} \mu)\)-fuzzy ideal of \( R \), we have \( \tilde{\mu}(x) \geq \min\{\tilde{\mu}(x), [0.5, 0.5]\} \geq \min\{[1, 1] - \tilde{t}, [0.5, 0.5]\} \). If \( \tilde{t} > [0.5, 0.5] \), then \( \tilde{\mu}(x) > [0.5, 0.5] \) and \( \tilde{\mu}(x + y - y) > [1, 1] - \tilde{t} \). Thus \( (x + y - y) \otimes \tilde{\mu} \). If \( \tilde{t} \leq [0.5, 0.5] \), then \( \tilde{\mu}(x) \geq [0.5, 0.5] \) and \( \tilde{\mu}(x + y - y) \geq [1, 1] - \tilde{t} \). Thus \( (x + y - y) \otimes \tilde{\mu} \). This means that \( (x + y - y) \otimes \tilde{\mu} \). Let \( z \in [\tilde{\mu}]_T \) and \( \tilde{\mu} \) is a left ideal of \( R \). Again, let \( x, y \in R \) and \( z \in [\tilde{\mu}]_T \) for \( [0, 0] < \tilde{t} \leq [1, 1] \). Then \( z \in [\tilde{\mu}]_T \). For \( [0, 0] < \tilde{t} \leq [0.5, 0.5] \) and \( x \in [\tilde{\mu}]_T \) \( \tilde{\mu}(y + x - y) \geq \min\{\tilde{\mu}(z), [0.5, 0.5]\} \). Similarly, we can prove that \( (x + z) - x - y \in [\tilde{\mu}]_T \). Therefore, \( [\tilde{\mu}]_T \) is a right ideal of \( R \).

Conversely, assume that \( \tilde{\mu} \) be an interval valued fuzzy subset in \( R \) and let \( [0, 0] < \tilde{t} \leq [1, 1] \) such that \( [\tilde{\mu}]_T \) is an ideal of \( R \). Suppose that \( \tilde{\mu}(x - y) < \min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \). Choose \( \tilde{t} \) such that \( \tilde{\mu}(x - y) < \tilde{t} \). Then \( \tilde{\mu}(x - y) < \tilde{t} \) \( \leq [0.5, 0.5] \). Then \( [0, 0] < \tilde{t} \leq [0.5, 0.5] \) and \( x, y \in U(\tilde{\mu} : \tilde{t}) \subseteq [\tilde{\mu}]_T \). Since \( [\tilde{\mu}]_T \) is an ideal of \( R \), we have \( \tilde{\mu}(x - y) \in [\tilde{\mu}]_T \) and \( \tilde{\mu}(y + x - y) \geq \tilde{t} \) for all \( x, y \in R \). Since \( [\tilde{\mu}]_T \) is an ideal of \( R \), we have \( \tilde{\mu}(y + x - y) \in [\tilde{\mu}]_T \) and \( \tilde{\mu}(y + x - y) \geq \tilde{t} \). This is a contradiction to our assumption. Hence \( \tilde{\mu}(y + x - y) \geq \min\{\tilde{\mu}(x), [0.5, 0.5]\} \).
our assumption. Hence $\bar{\mu}(xy) \geq \min^1 \{\bar{\mu}(y), [0.5, 0.5]\}$, for all $x, y \in R$. Similarly we have to prove $\bar{\mu}((x+z)y - xy) \geq \min^1 \{\bar{\mu}(z), [0.5, 0.5]\}$ and therefore $\bar{\mu}$ is an interval valued $(\bar{\varepsilon} \in \bar{\wedge})$-fuzzy ideal of $R$. 

\[\square\]

**Theorem 4.21.** If $\bar{\mu}$ is an interval valued $(\bar{\varepsilon} \in \bar{\wedge})$-fuzzy ideal of $R$, then the set $\bar{Q}(\bar{\mu}; \bar{t}) (\neq \emptyset)$ is an ideal of $R$ for all $[0.5, 0.5] < \bar{t} \leq [1, 1]$.

**Proof.** Assume that $\bar{\mu}$ is an interval valued $(\bar{\varepsilon} \in \bar{\wedge})$-fuzzy ideal of $R$ and let $[0.5, 0.5] < \bar{t} \leq [1, 1]$ be such that $\bar{Q}(\bar{\mu}; \bar{t}) \neq \emptyset$. Let $x, y \in \bar{Q}(\bar{\mu}; \bar{t})$ be such that $\bar{\mu}(x) + t > [1, 1]$ and $\bar{\mu}(y) + t > [1, 1]$ and we have $\bar{\mu}(x - y) \geq \min^1 \{\bar{\mu}(x), \bar{\mu}(y), [0.5, 0.5]\}$. If $\min^1 \{\bar{\mu}(x), \bar{\mu}(y)\} \geq [0.5, 0.5]$, then $\bar{\mu}(x - y) \geq [0.5, 0.5] > [1, 1] - \bar{t}$. If $\min^1 \{\bar{\mu}(x), \bar{\mu}(y)\} < [0.5, 0.5]$, then $\bar{\mu}(x - y) \geq \min^1 \{\bar{\mu}(x), \bar{\mu}(y)\} > [1, 1] - \bar{t}$. This implies that $x - y \in \bar{Q}(\bar{\mu}; \bar{t})$. Now, let $x \in \bar{Q}(\bar{\mu}; \bar{t})$ and $y \in R$ be such that $\bar{\mu}(x) + \bar{t} > [1, 1]$. Since $\bar{\mu}$ is an interval valued $(\bar{\varepsilon} \in \bar{\wedge})$-fuzzy ideal of $R$, then we have $\bar{\mu}(y + x - y) \geq \min^1 \{\bar{\mu}(x), [0.5, 0.5]\}$. If $\bar{\mu}(x) \geq [0.5, 0.5]$, then $\bar{\mu}(y + x - y) \geq [0.5, 0.5] > [1, 1] - \bar{t}$. If $\bar{\mu}(x) < [0.5, 0.5]$, then $\bar{\mu}(y + x - y) \geq \bar{\mu}(x) > [1, 1] - \bar{t}$. Thus $y + x - y \in \bar{Q}(\bar{\mu}; \bar{t})$. Similarly, let $y \in \bar{Q}(\bar{\mu}; \bar{t})$ and $x \in R$, then $xy \in \bar{Q}(\bar{\mu}; \bar{t})$. Again let $x, y \in R$ and $z \in \bar{Q}(\bar{\mu}; \bar{t})$ be such that $\bar{\mu}(z) + \bar{t} > [1, 1]$. Since $\bar{\mu}$ is an interval valued $(\bar{\varepsilon} \in \bar{\wedge})$-fuzzy ideal of $R$, then we have $\bar{\mu}((x + z)y - xy) \geq \min^1 \{\bar{\mu}(z), [0.5, 0.5]\}$. If $\bar{\mu}(z) \geq [0.5, 0.5]$, then $\bar{\mu}((x + z)y - xy) \geq [0.5, 0.5] > [1, 1] - \bar{t}$ and if $\bar{\mu}(z) < [0.5, 0.5]$, then $\bar{\mu}((x + z)y - xy) \geq \bar{\mu}(z) > [1, 1] - \bar{t}$ and thus $(x + z)y - xy \in \bar{Q}(\bar{\mu}; \bar{t})$. Therefore $\bar{Q}(\bar{\mu}; \bar{t})$ is an ideal of $R$. 

\[\square\]

**References**


[16] Al. Narayanan and T. Manikantan, $\langle \pi, \pi \vee q \rangle$-fuzzy subnear-rings and $\langle \pi, \pi \wedge q \rangle$-fuzzy ideals of near-rings, J. Appl. Math. and Computing 18 (2005) 419–430.


[22] Young Bae Jun, Generalizations of $\langle \pi, \pi \vee q \rangle$-fuzzy subalgebras in BCK—BCI-algebras, Computers and Mathematics with Applications 58 (2009) 1383–1390.


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