Soft strongly generalized closed sets with respect to soft ideals

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Abstract. The concept of soft ideal was first introduced by Kandil et al. [16]. The notion of generalized closed soft sets in soft topological spaces was introduced by Kannan [23] in 2012, which is generalized in [22, 24, 31]. In this paper, we generalize the notions of soft strongly $g$-closed sets and soft strongly $g$-open sets [24] by using the soft ideals notion [16] in soft topological spaces and study their basic properties. Here we used the concept of soft ideals, soft closure, soft interior and soft open sets to define soft strongly $\tilde{I}g$-closed sets. The relationship between soft strongly $\tilde{I}g$-closed sets and other existing soft sets have been investigated. Furthermore, the union and intersection of two soft strongly $\tilde{I}g$-closed (resp. open) sets have been obtained.

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1. Introduction

In real life situation, the problems in economics, engineering, social sciences, medical science etc. do not always involve crisp data. So, we cannot successfully use the traditional classical methods because of various types of uncertainties presented in these problems. To exceed these uncertainties, some kinds of theories were given like theory of fuzzy set, intuitionistic fuzzy set, rough set, bipolar fuzzy set, i.e. which we can use as mathematical tools for dealings with uncertainties. But, all these theories have their inherent difficulties. The reason for these difficulties Molodtsov [30] initiated the concept of soft set theory as a new mathematical tool for dealing with uncertainties which is free from the above difficulties. In [30, 29],
Molodtsov successfully applied the soft theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement, and so on. After presentation of the operations of soft sets [28], the properties and applications of soft set theory have been studied increasingly [5, 25, 29]. In [13], Kandil et al. introduced some soft operations such as semi open soft, pre open soft, α-open soft and β-open soft and investigated their properties in detail. Kandil et al. [26] introduced the notion of soft semi separation axioms, which is extended by Abd-El-latif et al. in [3]. Maji et al. [27] initiated the study involving both fuzzy sets and soft sets. In [4], Karal and Ahmed defined the notion of a mapping on classes of fuzzy soft sets, which is a fundamental important in fuzzy soft set theory, to improve this work and they studied properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets. Some fuzzy soft topological properties were introduced in [1, 2, 11, 12, 21].

In 1970, Levine [26] introduced the notion of g-closed sets in topological spaces as a generalization of closed sets. Indeed ideals are very important tools in general topology. It was the works of Newcomb [32], Rancin [33], Samuels [34] and Hanlet Jankovic [8, 9] which motivated the research in applying topological ideals to generalize the most basic properties in general Topology. S. Jafari and N. Rajesh introduced the concept of g-closed sets with respect to an ideal which is a extension of the concept of g-closed sets. The notion of soft ideal was initiated for the first time by Kandil et al. [16]. They also introduced the concept of soft local function. These concepts are discussed with a view to find new soft topologies from the original one, called soft topological spaces with soft ideal (X,τ,E,⪯). Applications to various fields were further investigated in [10, 14, 15, 17, 18, 19, 22]. Recently, K. Kannan [23] introduced the concept of g-closed soft sets in a soft topological spaces, which is generalized in [22, 24, 31].

In this paper, we introduce and study the concept of soft strongly g-closed sets with respect to an ideal, which is the extension of the concept of soft strongly g-closed set. Also, we study the relationship between soft strongly Ig-closed sets and other existing soft sets have been investigated.

2. Preliminaries

In this section, we present the basic definitions and results of soft set theory.

Definition 2.1 ([30]). Let X be an initial universe and E be a set of parameters. Let P(X) denote the power set of X and A be a non-empty subset of E. A pair (F,A) denoted by FA is called a soft set over X, where F is a mapping given by F : A → P(X). In other words, a soft set over X is a parametrized family of subsets of the universe X. For a particular e ∈ A, F(e) may be considered the set of e-approximate elements of the soft set FA and if e /∈ A, then F(e) = φ, i.e.

FA = {F(e) : e ∈ A ⊆ E, F : A → P(X)}. The family of all these soft sets denoted by SS(X)A.

Definition 2.2 ([28]). Let FA, GB ∈ SS(X)E. Then, FA is soft subset of GB, denoted by FA ⊆GB, if

1. A ⊆ B and
In this case, $F_A$ is said to be a soft subset of $G_B$, and $G_B$ is said to be a soft superset of $F_A$, $G_B \supseteq F_A$.

**Definition 2.3** ([28]). Two soft subset $F_A$ and $G_B$ over a common universe set $X$ are said to be soft equal if $F_A$ is soft subset of $G_B$ and $G_B$ is soft subset of $F_A$.

**Definition 2.4** ([5]). The complement of a soft set $F_A$, denoted by $F_A^c$, is defined by $F_A^c = (F^c, A)$, $F^c : A \to P(X)$ is mapping given by $F^c(e) = X - F(e)$, $\forall e \in A$ and $F^c$ is called the soft complement function of $F$.

Clearly $(F^c)^c$ is the same as $F$ and $(F_A^c)^c = F_A$.

**Definition 2.5** ([35]). The difference of two soft sets $F_E$ and $G_E$ over the common universe $X$, denoted by $F_E - G_E$ is the soft set $H_E$ where for all $e \in E$, $H(e) = F(e) - G(e)$.

**Definition 2.6** ([35]). Let $F_E$ be a soft set over $X$ and $x \in X$. We say that $x \in F_E$ read as $x$ belongs to the soft set $F_E$ whenever $x \in F(e)$ for all $e \in E$.

**Definition 2.7** ([28]). A soft set $F_A$ over $X$ is said to be a null soft set denoted by $\phi$ or $\phi_A$ if for all $e \in A$, $F(e) = \phi$ (null set).

**Definition 2.8** ([28]). A soft set $F_A$ over $X$ is said to be an absolute soft set denoted by $\hat{A}$ or $X_A$ if for all $e \in A$, $F(e) = X$. Clearly, we have $X_A^c = \phi_A$ and $\phi_A^c = X_A$.

**Definition 2.9** ([28]). The union of two soft sets $F_A$ and $G_B$ over the common universe $X$ is the soft set $(H, C)$, where $C = A \cup B$ and for all $e \in C$, $H(e) = \begin{cases} F(e), & e \in A - B, \\ G(e), & e \in B - A, \\ F(e) \cup G(e), & e \in A \cap B \end{cases}$.

**Definition 2.10** ([28]). The intersection of two soft sets $F_A$ and $G_B$ over the common universe $X$ is the soft set $(H, C)$, where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$. Note that, in order to efficiently discuss, we consider only soft sets $F_E$ over a universe $X$ in which all the parameter set $E$ are same. We denote the family of these soft sets by $SS(X)_E$.

**Definition 2.11** ([35]). Let $\tau$ be a collection of soft sets over a universe $X$ with a fixed set of parameters $E$, then $\tau \subseteq SS(X)_E$ is called a soft topology on $X$ if

1. $\hat{X}, \phi \in \tau$, where $\phi(e) = \phi$ and $\hat{X}(e) = X$, $\forall e \in E$,

2. the union of any number of soft sets in $\tau$ belongs to $\tau$,

3. the intersection of any two soft sets in $\tau$ belongs to $\tau$.

The triplet $(X, \tau, E)$ is called a soft topological space over $X$.

**Definition 2.12** ([35]). Let $(X, \tau, E)$ be a soft topological space. A soft set $F_A$ over $X$ is said to be closed soft set in $X$, if its relative complement $F_A^c$ is open soft set.

**Definition 2.13** ([35]). Let $(X, \tau, E)$ be a soft topological space. The members of $\tau$ are said to be open soft sets in $X$. We denote the set of all open soft sets over $X$ by $OS(X, \tau, E)$, or when there can be no confusion by $OS(X)$ and the set of all closed soft sets by $CS(X, \tau, E)$, or $CS(X)$.
Definition 2.14 ([6]). Let \((X, \tau, E)\) be a soft topological space and \(x, y \in X\) such that \(x \neq y\). Then, \((X, \tau, E)\) is called a soft Hausdorff space or soft \(T_2\) space if there exist open soft sets \(F_A\) and \(G_B\) such that \(x \in F_A, y \in G_B\) and \(F_A \cap G_B = \emptyset\).

Definition 2.15 ([35]). Let \((X, \tau, E)\) be a soft topological space, \(F_E \in SS(X)_E\) and \(Y\) be a non-null subset of \(X\). Then, the sub soft set of \(F_E\) over \(Y\) denoted by \((F_Y, E)\), is defined as follows:
\[
F_Y(e) = Y \cap F(e) \quad \forall e \in E.
\]
In other words \((F_Y, E) = \hat{Y} \cap F_E\).

Definition 2.16 ([35]). Let \((X, \tau, E)\) be a soft topological space and \(Y\) be a non-null subset of \(X\). Then,
\[
\tau_Y = \{(F_Y, E) : F_E \in \tau\}
\]
is said to be the soft relative topology on \(Y\) and \((Y, \tau_Y, E)\) is called a soft subspace of \((X, \tau, E)\).

Definition 2.17 ([13]). Let \((X, \tau, E)\) be a soft topological space and \(F_E \in SS(X)_E\). Then \(F_E\) is said to be semi open soft set if \(F_E \subseteq cl(int F_E)\). The set of all semi open soft sets is denoted by \(SOS(X)\) and the set of all semi closed soft sets is denoted by \(SCS(X)\).

Definition 2.18 ([36]). Let \((X, \tau_1, A)\) and \((Y, \tau_2, B)\) be soft topological spaces and \(f_{pu} : SS(X)_A \rightarrow SS(Y)_B\) be a function. Then,
1. The function \(f_{pu}\) is called continuous soft (cts-soft) if \(f_{pu}^{-1}(G, B) \in \tau_1 \forall (G, B) \in \tau_2\).
2. The function \(f_{pu}\) is called open soft if \(f_{pu}(G, A) \in \tau_2 \forall (G, A) \in \tau_1\).
3. The function \(f_{pu}\) is called closed soft if \(f_{pu}(G, A) \in \tau_2^c \forall (G, A) \in \tau_1^c\).

Theorem 2.1 ([4]). Let \(SS(X)_A\) and \(SS(Y)_B\) be families of soft sets. For the soft function \(f_{pu} : SS(X)_A \rightarrow SS(Y)_B\), the following statements hold:
(a) \(f_{pu}^{-1}((G, B)^c) = (f_{pu}^{-1}(G, B))^c \forall (G, B) \in SS(Y)_B\).
(b) \(f_{pu}(f_{pu}^{-1}((G, B))) \subseteq (G, B) \forall (G, B) \in SS(Y)_B\). If \(f_{pu}\) is surjective, then the equality holds.
(c) \((F, A) \subseteq f_{pu}^{-1}(f_{pu}(F, A)) \forall (F, A) \in SS(X)_A\). If \(f_{pu}\) is injective, then the equality holds.
(d) \(f_{pu}(X) \subseteq Y\). If \(f_{pu}\) is surjective, then the equality holds.
(e) \(f_{pu}^{-1}(Y) = \bar{X}\) and \(f_{pu}(\hat{A}) = \hat{B}\).
(f) If \((F, A) \subseteq (G, A)\), then \(f_{pu}(F, A) \subseteq f_{pu}(G, A)\).
(g) If \((F, B) \subseteq (G, B)\), then \(f_{pu}^{-1}(F, B) \subseteq f_{pu}^{-1}(G, B) \forall (F, B), (G, B) \in SS(Y)_B\).
(h) \(\forall (F, B), (G, B) \in SS(Y)_B\),
\[
f_{pu}^{-1}((F, B) \cup (G, B)) = f_{pu}^{-1}(F, B) \cup f_{pu}^{-1}(G, B)
\]
and
\[
f_{pu}^{-1}((F, B) \cap (G, B)) = f_{pu}^{-1}(F, B) \cap f_{pu}^{-1}(G, B).
\]
(i) \(\forall (F, A), (G, A) \in SS(X)_A\),
\[
f_{pu}((F, A) \cup (G, A)) = f_{pu}(F, A) \cup f_{pu}(G, A)
\]
and
\[
f_{pu}((F, A) \cap (G, A)) = f_{pu}(F, A) \cap f_{pu}(G, A).
\]
Theorem 3.1. Every soft set \( S \subseteq \mathcal{P}(X) \) is the soft topology over \( X \) if \( f_{pu}((F, A) \cap (G, A)) \subseteq f_{pu}(F, A) \cap f_{pu}(G, A) \).

If \( f_{pu} \) is injective, then the equality holds.

**Definition 2.19** ([23]). A soft set \( F \subseteq SS(X, E) \) is called soft generalized closed in a soft topological space \((X, \tau, E)\) if \( cl(F) \subseteq G_E \) whenever \( F_E \subseteq G_E \) and \( G_E \in \tau \).

**Definition 2.20** ([16]). Let \( I \) be a non-null collection of soft sets over a universe \( X \) with the same set of parameters \( E \). Then, \( I \subseteq SS(X)_E \) is called a soft ideal on \( X \) with the same set \( E \) if

1. \( F_E \in I \) and \( G_E \in I \Rightarrow F_E \cup G_E \in I \),
2. \( F_E \in I \) and \( G_E \subseteq F_E \Rightarrow G_E \in I \),

i.e., \( I \) is closed under finite soft unions and soft subsets.

**Definition 2.21** ([31]). A soft set \( F \subseteq SS(X, E) \) is called soft generalized closed set with respect to soft ideal \( I \) (soft \( Ig \)-closed set) in a soft topological space \((X, \tau, E)\) if \( cl(F_E) \setminus G_E \in I \) whenever \( F_E \subseteq G_E \) and \( G_E \in \tau \).

3. **Soft strongly generalized closed sets with respect to soft ideal**

Kannan et al. [24] introduced the notion of soft strongly generalized closed sets in soft topological spaces. In this section, we generalize this notion by using the soft ideal notion.

**Definition 3.1.** A soft set \( F \subseteq SS(X, E) \) is called soft strongly generalized closed set with respect to soft ideal \( I \) (soft strongly \( Ig \)-closed set) in a soft topological space \((X, \tau, E)\) if \( cl(int(F_E)) \setminus G_E \in I \) whenever \( F_E \subseteq G_E \) and \( G_E \in \tau \).

**Example 3.1.** Let \( X = \{a, b\} \) be the set of two phones under consideration and, \( E = \{e_1(\text{costly}), e_2(\text{Luxurious})\} \). Let \( A_E, B_E, C_E \) be three soft sets representing the attractiveness of the phone which Mr. X, Mr. Y and M. Z are going to buy, where

- \( A(e_1) = \phi \quad A(e_2) = \{a\} \),
- \( B(e_1) = \{b\} \quad B(e_2) = \phi \),
- \( C(e_1) = \{b\} \quad C(e_2) = \{a\} \).

Then \( A_E, B_E \) and \( C_E \) are soft sets over \( X \) and

\[ \tau = \{X, \phi, A_E, B_E, C_E\} \]

is the soft topology over \( X \). Let \( I = \{\phi, F_E, G_E, H_E\} \) be a soft ideal on \( X \), where

- \( F(e_1) = \{b\} \quad F(e_2) = \phi \),
- \( G(e_1) = \{b\} \quad G(e_2) = \{a\} \),
- \( H(e_1) = \phi \quad H(e_2) = \{a\} \).

The soft sets \( M_E, N_E, L_E \) are soft strongly \( Ig \)-closed, where

- \( M(e_1) = \phi \quad M(e_2) = X \),
- \( N(e_1) = X \quad N(e_2) = \{a\} \),
- \( L(e_1) = \{b\} \quad L(e_2) = \{b\} \).

On the other hand, the soft set \( S_E \) is not soft strongly \( Ig \)-closed, where \( S(e_1) = \phi \quad S(e_2) = \{a\} \).

**Theorem 3.1.** Every soft \( g \)-closed set is soft strongly \( Ig \)-closed.
Proof. Let $F_E \subseteq G_E$ and $G_E \in \tau$. Since $F_E$ is soft $g$-closed, then $cl(F_E) \subseteq G_E$ and $cl(int(F_E)) \subseteq cl(F_E)$. Hence, $cl(int(F_E)) \setminus G_E = \emptyset \in I$. Therefore, $F_E$ is soft strongly $\tilde{I}g$-closed. □

**Remark 3.1.** The converse of the above theorem is not true in general. The following example supports our claim.

**Example 3.2.** Suppose that there are two dresses in the universe $X$ given by $X = \{a, b\}$. Let $E = \{e_1(\text{cotton}), e_2(\text{woollen})\}$ be the set of parameters showing the material of the dresses. Let $A_E, B_E, C_E, D_E$ be four soft sets over the common universe $X$, which describe the composition of the dresses, where
\[
\begin{align*}
A(e_1) &= \{a\}, & A(e_2) &= X, \\
B(e_1) &= \{a\}, & B(e_2) &= \emptyset, \\
C(e_1) &= \{a\}, & C(e_2) &= \{b\}, \\
D(e_1) &= \{a\}, & D(e_2) &= \{a\}.
\end{align*}
\]
Thus, $\tau = \{\emptyset, \tilde{X}, A_E, B_E, C_E, D_E\}$ is the soft topology over $X$. Let $I = \{\emptyset, F_E, G_E, H_E\}$ be a soft ideal on $X$, where
\[
\begin{align*}
F(e_1) &= \{a\}, & F(e_2) &= \emptyset, \\
G(e_1) &= \{a\}, & G(e_2) &= \{a\}, \\
H(e_1) &= \emptyset, & H(e_2) &= \{a\}.
\end{align*}
\]
The soft set $M_E$ is soft strongly $\tilde{I}g$-closed but not soft $g$-closed, where $M(e_1) = \emptyset$ and $M(e_2) = \{b\}$.

**Theorem 3.2.** Every closed soft set is soft strongly $\tilde{I}g$-closed.

Proof. Let $F_E \subseteq G_E$ and $G_E \in \tau$. Since $F_E$ is closed soft, then $cl(int(F_E)) \subseteq cl(F_E) = F_E \subseteq G_E$. Hence, $cl(int(F_E)) \setminus G_E = \emptyset \in I$. Therefore, $F_E$ is soft strongly $\tilde{I}g$-closed. □

**Remark 3.2.** The converse of the above theorem is not true in general as shall shown in the following example.

**Example 3.3.** In Example 3.1, the soft set $N_E$ is soft strongly $\tilde{I}g$-closed but not closed soft set, where $N(e_1) = X$ and $N(e_2) = \{a\}$.

**Theorem 3.3.** Every soft $\tilde{I}g$-closed set is soft strongly $\tilde{I}g$-closed.

Proof. Let $F_E \subseteq H_E$ and $H_E \in \tau$. Then, $cl(int(F_E)) \setminus I_E \subseteq cl(F_E) \setminus I_E \in \tilde{I}$ for some $I_E \in \tilde{I}$. Therefore, $F_E$ is strongly $\tilde{I}g$-closed. □

**Remark 3.3.** The converse of the above theorem is not true in general, as shown in the following example.

**Example 3.4.** In Example 3.2, the soft set $M_E$ is soft strongly $\tilde{I}g$-closed but not closed soft, where $M(e_1) = \emptyset$ and $M(e_2) = \{b\}$.

**Theorem 3.4.** If a soft subset $F_E$ of a soft topological space $(X, \tau, E)$ is open soft, then it is soft strongly $\tilde{I}g$-closed if and only if it is soft $\tilde{I}g$-closed.
Proof. It is clear. □

**Theorem 3.5.** A soft set $A_E$ is soft strongly $\hat{I}_g$-closed in a soft topological space $(X, \tau, E)$ if and only if $F_E \subseteq \text{cl}(\text{int}(A_E)) \setminus A_E$ and $F_E$ is closed soft implies $F_E \in \hat{I}$.

Proof. (⇒) Let $F_E \subseteq \text{cl}(\text{int}(A_E)) \setminus A_E$ and $F_E$ is closed soft. Then, $A_E \subseteq F_E^c$. By hypothesis, $\text{cl}(\text{int}(A_E)) \setminus F_E^c \in \hat{I}$. But, $F_E \subseteq \text{cl}(\text{int}(A_E)) \cap F_E = \text{cl}(\text{int}(A_E)) \setminus F_E^c$. Thus, $F_E \in \hat{I}$ from Definition 2.20.

(⇐) Assume that $A_E \subseteq G_E$ and $G_E \in \tau$. Then, $\text{cl}(\text{int}(A_E)) \setminus G_E = \text{cl}(\text{int}(A_E)) \cap G_E^c$ is a closed soft set and $\text{cl}(\text{int}(A_E)) \setminus G_E \subseteq \text{cl}(\text{int}(A_E)) \setminus G_E$. By assumption, $\text{cl}(\text{int}(A_E)) \setminus G_E \in \hat{I}$. So, $A_E$ is soft strongly $\hat{I}_g$-closed. □

**Theorem 3.6.** If $F_E$ is soft strongly $\hat{I}_g$-closed in a soft topological space $(X, \tau, E)$ and $F_E \subseteq G_E \subseteq \text{cl}(\text{int}(F_E))$, then $G_E$ is soft strongly $\hat{I}_g$-closed.

Proof. Let $G_E \subseteq H_E$ and $H_E \in \tau$. Then, $F_E \subseteq H_E$. Since $F_E$ is soft strongly $\hat{I}_g$-closed, then $\text{cl}(\text{int}(F_E)) \setminus H_E \in \hat{I}$. Now, $G_E \subseteq \text{cl}(\text{int}(F_E))$ implies that $\text{cl}(G_E) \subseteq \text{cl}(\text{int}(F_E))$. Thus, $\text{cl}(\text{int}(G_E)) \setminus H_E \subseteq \text{cl}(\text{int}(F_E)) \setminus H_E$. So, $\text{cl}(\text{int}(G_E)) \setminus H_E \in \hat{I}$ from Definition 2.20. Hence, $G_E$ is soft strongly $\hat{I}_g$-closed. □

**Remark 3.4.** The soft intersection of two soft strongly $\hat{I}_g$-closed sets need not be a soft strongly $\hat{I}_g$-closed as shown by the following example.

**Example 3.5.** In Example 3.1, the soft sets $M_E, N_E$ are soft strongly $\hat{I}_g$-closed. But, $P_E = M_E \cap N_E$ is not soft strongly $\hat{I}_g$-closed, where $P(e_1) = \phi$ and $P(e_2) = \{a\}$.

**Theorem 3.7.** If $A_E$ is soft strongly $\hat{I}_g$-closed and $F_E$ is closed soft in a soft topological space $(X, \tau, E)$. Then, $A_E \cap F_E$ is soft strongly $\hat{I}_g$-closed.

Proof. Assume that $A_E \cap F_E \subseteq G_E \in \tau$. Then, $A_E \subseteq G_E \cup F_E^c$. Since $A_E$ is soft strongly $\hat{I}_g$-closed, so $\text{cl}(\text{int}(A_E)) \setminus (G_E \cup F_E^c) \in \hat{I}$. Now,

$$
\text{cl}(\text{int}((A_E \cap F_E))) \subseteq \text{cl}(\text{int}(A_E)) \cap \text{cl}(\text{int}(F_E))
$$

$$
= \text{cl}(\text{int}(A_E)) \cap \text{cl}(F_E)
$$

$$
= \text{cl}(\text{int}(A_E)) \cap F_E^c.
$$

Thus,

$$
\text{cl}(\text{int}((A_E \cap F_E))) \setminus G_E \subseteq \text{cl}(\text{int}(A_E)) \cap F_E^c \setminus [F_E^c \cup G_E]
$$

$$
\subseteq \text{cl}(\text{int}(A_E)) \setminus [G_E \cup F_E^c] \in \hat{I}.
$$

So, $A_E \cap F_E$ is soft strongly $\hat{I}_g$-closed. □

**Theorem 3.8.** Let $(Y, \tau_Y, E)$ be a soft subspace of a soft topological space $(X, \tau, E)$, $F_E \subseteq Y_E$ and $F_E$ is strongly $\hat{I}_g$-closed in $(X, \tau, E)$. Then, $F_E$ is strongly $\hat{I}_Y$-closed in $(Y, \tau_Y, E)$.

Proof. Assume that $F_E \subseteq B_{E} \subseteq Y_E$ and $B_E \in \tau_Y$. Then, $B_E \subseteq Y_E \subseteq \tau_Y$ and $F_E \subseteq B_E$. Since $F_E$ is soft strongly $\hat{I}_g$-closed in $(X, \tau, E)$, then $\text{cl}(\text{int}(F_E)) \setminus B_E \in \hat{I}$. Now,

$$
[\text{cl}(\text{int}(F_E)) \cap Y_E] \setminus [B_E \cap Y_E] = [\text{cl}(\text{int}(F_E)) \setminus B_E] \cap Y_E \in \hat{I}_Y.
$$
Thus, $F_E$ is soft strongly $Ig$-closed in $(Y, \tau_Y, E)$. \hfill \Box

**Theorem 3.9.** Let $(X_1, \tau_1, E), (X_2, \tau_2, K)$ be soft topological spaces. Let $f_{pu} : SS(X_1)_K \rightarrow SS(X_2)_K$ be a closed and continuous soft function. If $A_E \in SS(X, E)$ is both a soft strongly $Ig$-closed and semi open soft in $(X_1, \tau_1, E)$, then $f_{pu}(A_E)$ is a soft strongly $f_{pu}(Ig)$-closed in $(X_2, \tau_2, K)$.

**Proof.** Let $A_E \in SS(X)_E$ be a soft strongly $Ig$-closed in $(X_1, \tau_1, E)$ and $f_{pu}(A_E) \subseteq G_K$ for some $G_K \in \tau_2$. Then, $A_E \subseteq f_{pu}^{-1}(G_K)$. It follows that, $cl_{\tau_1}(int_{\tau_1}(A_E)) \subseteq f_{pu}^{-1}(G_K) \in I$ from Definition 3.1. Thus, $f_{pu}(cl_{\tau_1}(int_{\tau_1}(A_E))) \in \tau_2$ from Theorem 2.1. Since $f_{pu}$ is a closed soft function, then $f_{pu}(cl_{\tau_1}(int_{\tau_1}(A_E)))$ is a soft closed in $\tau_2$ from Definition 2.18. Since $A_E$ is semi open soft in $(X_1, \tau_1, E)$, then

$$cl_{\tau_2}(int_{\tau_2}(f_{pu}(A_E))) \subseteq cl_{\tau_2}[f_{pu}(cl_{\tau_1}(int_{\tau_1}(A_E)))],$$

This implies that, $[cl_{\tau_2}(int_{\tau_2}(f_{pu}(A_E))) \subseteq cl_{\tau_2}[f_{pu}(cl_{\tau_1}(int_{\tau_1}(A_E)))]] \subseteq G_K \subseteq f_{pu}(Ig)$.

So, $f_{pu}(A_E)$ is a soft strongly $f_{pu}(Ig)$-closed in $(X_2, \tau_2, K)$. \hfill \Box

4. Soft generalized open sets with respect to soft ideal

**Definition 4.1.** A soft set $F_E \in SS(X, E)$ is called soft strongly generalized open set with respect to soft ideal $I$ (soft strongly $Ig$-open) in a soft topological space $(X, \tau, E)$ if and only if its relative complements $F^c_E$ is soft strongly $Ig$-closed in $(X, \tau, E)$.

**Example 4.1.** In Example 3.1. The soft sets $M^c_E, N^c_E, L^c_E$ are soft strongly $Ig$-open where $M^c_E, N^c_E, L^c_E$ are defined by

$M^c(e_1) = X$, $M^c(e_2) = \phi$,

$N^c(e_1) = \phi$, $N^c(e_2) = \{b\}$,

$L^c(e_1) = \{a\}$, $L^c(e_2) = \{a\}$.

**Theorem 4.1.** A soft set $A_E$ is soft strongly $Ig$-open in a soft topological space $(X, \tau, E)$ if and only if $F_E \subseteq A_E$ for some $B_E \in I$, whenever $F_E \subseteq A_E$ and $F_E$ is closed soft in $(X, \tau, E)$.

**Proof.** ($\Rightarrow$) Suppose that $F_E \subseteq A_E$ and $F_E$ is closed soft. We have $A_E \subseteq F_E \subseteq A_E$ is soft strongly $Ig$-closed and $F_E$ is soft strongly $Ig$-open in $\tau$. By assumption, $cl(int((A_E))) \subseteq F_E$. Then, $cl(int((A_E))) \subseteq F_E = B_E$ for some $B_E \in I$. Thus,

$$cl(int((A_E))) \subseteq F_E = cl(int((A_E))) \cap F_E = B_E \subseteq I.$$  

So, $[cl(int((A_E))) \subseteq F_E = cl(int((A_E))) \cap F_E = B_E \subseteq I]$. This implies that,

$$cl(int((A_E))) \subseteq cl(int((A_E))) \subseteq cl(int((A_E))) \subseteq cl(int((A_E))) \subseteq cl(int((A_E))) \subseteq cl(int((A_E))).$$

Hence, $cl(int((A_E))) \subseteq F_E \subseteq B_E$ for some $B_E \in I$. Furthermore,

$$(F_E \subseteq B_E) \subseteq cl(int((A_E))).$$

Therefore, $F_E \setminus B_E = F_E \setminus cl(int((A_E))).$

($\Leftarrow$) We want to prove that $A_E$ is soft strongly $Ig$-open. It is sufficient to prove that, $A_E$ is soft strongly $Ig$-closed.

Let $A_E \subseteq G_E$ such that $G_E \subseteq \tau$. Then, $G_E \subseteq A_E$. By assumption,
Thus, every open soft set is soft strongly \( \tilde{I} \)-open.

**Theorem 4.2.** Every open soft set is soft strongly \( \tilde{I} \)-open.

**Proof.** Immediate from Theorem 3.2.

**Remark 4.1.** The converse of the above theorem is not true in general as shall shown in the following example.

**Example 4.2.** In Example 3.1, the soft set \( S_E \) is soft strongly \( \tilde{I} \)-open but not open soft set, where \( S(e_1) = \emptyset \) and \( S(e_2) = \{ b \} \).

**Theorem 4.3.** Every soft \( \tilde{I} \)-open set is soft strongly \( \tilde{I} \)-open.

**Proof.** Immediate from Theorem 3.3.

**Remark 4.2.** The converse of the above theorem is not true in general, as shown in the following example.

**Example 4.3.** In Example 3.2, the soft set \( Z_E \) is soft strongly \( \tilde{I} \)-open but not soft \( \tilde{I} \)-open, where \( Z(e_1) = X \) and \( Z(e_2) = \{ a \} \).

**Remark 4.3.** The soft intersection (resp. union) of two soft strongly \( \tilde{I} \)-open sets need not be a soft strongly \( \tilde{I} \)-open as shown by the following example.

**Example 4.4.** In Example 3.1, the soft sets \( M_E, N_E, L_E \) are soft strongly \( \tilde{I} \)-open. But, \( H_E = M_E \cup N_E \) is not soft strongly \( \tilde{I} \)-open, where \( H(e_1) = X \) and \( H(e_2) = \{ b \} \). Also, \( K_E = U_E \cap V_E \) is not soft strongly \( \tilde{I} \)-open, where \( K(e_1) = \{ b \} \) and \( K(e_2) = \{ a \} \).

**Theorem 4.4.** If \( F_E \) is soft strongly \( \tilde{I} \)-open in a soft topological space \( (X, \tau, E) \) and \( \text{int}(\text{cl}(F_E)) \subseteq G_E \subseteq F_E \), then \( G_E \) is soft strongly \( \tilde{I} \)-open.

**Proof.** Let \( H_E \subseteq G_E \) and \( H_E \in \tau \). Then, \( H_E \subseteq F_E \). Since \( F_E \) is soft strongly \( \tilde{I} \)-open, then \( G_E \setminus \text{int}(\text{cl}(H_E)) \subseteq F_E \setminus \text{int}(\text{cl}(H_E)) \in \tilde{I} \). It follows that, \( G_E \setminus \text{int}(\text{cl}(H_E)) \in \tilde{I} \). Thus \( G_E \) is soft strongly \( \tilde{I} \)-open.

**Theorem 4.5.** If a soft subset \( F_E \) of a soft topological space \( (X, \tau, E) \) is closed soft, then it is soft strongly \( \tilde{I} \)-open if and only if it is soft \( \tilde{I} \)-open.

**Proof.** It is clear.

**Theorem 4.6.** A soft set \( A_E \) is soft strongly \( \tilde{I} \)-closed in a soft topological space \( (X, \tau, E) \) if and only if \( \text{cl}(\text{int}(A_E)) \setminus A_E \) is soft strongly \( \tilde{I} \)-open.
Proof. \((\Rightarrow)\) Let \(F_E \subseteq \overline{\text{cl}(\text{int}(A_E))} \setminus A_E\) and \(F_E\) be a closed soft set. Then, \(F_E \in \hat{I}\) from Theorem 3.5. Thus, there exists \(I_E \in \hat{I}\) such that \(F_E \setminus I_E = \hat{\phi}\). So, \(F_E \setminus I_E = \hat{\phi} \subseteq \text{int}(\text{cl}(\text{int}(A_E)) \setminus A_E)\). Hence, \(\text{cl}(\text{int}(A_E)) \setminus A_E\) is a soft strongly \(Ig\)-open from Theorem 4.1.

\((\Leftarrow)\) Let \(A_E \subseteq G_E\) such that \(G_E \in \tau\). Then, \(\text{cl}(\text{int}(A_E)) \setminus G_E \subseteq \text{cl}(\text{int}(A_E)) \setminus A_E = \text{cl}(\text{int}(A_E)) \setminus A_E\). By hypothesis, \(\text{cl}(\text{int}(A_E)) \setminus A_E \subseteq \text{int}(\text{cl}(\text{int}(A_E)) \setminus A_E)\) = \(\hat{\phi}\), for some \(I_E \in \hat{I}\) from Theorem 4.1. This implies that, \(\text{cl}(\text{int}(A_E)) \setminus G_E \subseteq I_E \in \hat{I}\). Thus, \(\text{cl}(\text{int}(A_E)) \setminus G_E \in \hat{I}\). So, \(A_E\) is a soft strongly \(Ig\)-closed. \(\square\)

5. Conclusion

Therefore, the notions of soft strongly \(Ig\)-closed sets and soft strongly \(Ig\)-open sets have been introduced and investigated. In future, the generalization of these concepts to supra soft topological spaces [7] will be introduced and the future research will be undertaken in this direction.

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References


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