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A new approach to combine generalized interval valued fuzzy numbers based on average width of fuzzy set concept

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Abstract. The objective of this paper is to devise a new technique to combine generalized interval valued fuzzy numbers (GIVFNs)based on average width of fuzzy set concept. The effectiveness and applicability of the proposed technique are illustrated with an example.

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1. Introduction

In general, real world problems are ill defined, i.e., objectives and parameters are not precisely known. Initially, the obstacles of lack of precision have been dealt with using the well established and age old classical probability approach. But due to the fact that the requirement on the data and on the environment are very high and that many real word problems are fuzzy nature and not random, the probability applications are inappropriate in lots of cases. Also, the applications of fuzzy set theory in real world problems give better results. More often, type-I fuzzy set theory[40] is used to deal with imprecision, vagueness, lack of data etc. However, in some situations it is not always possible for a membership function of the type to precisely assign one point from [0,1] so it is more realistic to assign interval value. According to Gehrke et al.[15] many people believe that assigning an exact number to expert's opinion is too restrictive and the assignment of an interval valued is more realistic. In such situations interval valued fuzzy set (IVFS) comes into picture. In May, 1975 Sambuc[30] presented in his doctoral research (thesis) the concept of IVFS named as Φ-fuzzy set. After development of IVFVs, different researchers have been studied this issue and applied in different areas.

Sambuc [30] and Grattan [18] noted that the presentation of a linguistic expression in the form of fuzzy sets is not enough. Interval valued fuzzy sets were suggested by Gorzlczany [16, 17] and Turksen[32]. Wnag and Li[36] defined interval valued fuzzy numbers (IVFN) and gave their extended operations. Turksen[33] studied interval valued logic and applied in preference modeling. Wang and Li[35] presented the correlation coefficient of interval-valued fuzzy numbers and some of their properties. Lin[24] used interval-valued fuzzy numbers to represent vague processing time in job-shop scheduling problems. Yao and Lin[38] used interval-valued fuzzy numbers to represent unknown job processing time for constructing a fuzzy flow-shop sequencing model.

Chen et al.[7] proposed a fuzzy risk analysis method based on similarity measure of interval valued fuzzy number. Chen[8], Chen et al.[6], Wei et al[37] have developed a fuzzy risk analysis based on the similarity measure of interval-valued fuzzy number. Carlsson et al.[5] introduced a possibilistic mean value for interval-valued fuzzy numbers as the arithmetic mean of the possibilistic mean values of its upper and lower fuzzy numbers. Carlsson et al.[4] introduced the concept that means value and variance can be utilized as a ranking method for interval-valued fuzzy numbers. Tomas et al.[31] discussed Group Multi-Criteria decision making based upon Interval-Valued Fuzzy numbers and extended the MULTIMOORA Method, Jozsef[20] combined interval-valued fuzzy sets and OWA operators to create new aggregation methods and proved that the new operators satisfy some important properties.

Some researchers [3, 19, 23, 27, 39, 41] investigated and suggested some methods for measuring distance between interval valued fuzzy sets (IVFS). Application of IVFSs in medical diagnosis can be found in [1, 2, 10, 14, 22, 25, 26, 28].

An IVFS is a set in which every element has degree of membership in the form of an interval. One can say, IVFS consist of two membership function, one is upper membership function (UMF) and other is lower membership function (LMF). In this study we consider these UMF and LMF as generalized fuzzy set. In 1985, Chen [9]developed the concept of generalized fuzzy number. A generalized fuzzy number is a fuzzy number in which height of the fuzzy number is less than unit. Dutta[12] proposed new method to study all basic arithmetic operations between Generalized triangular Fuzzy Numbers using α-cut technique apart from Chen's approach. Also proposed an alternative approach to perform arithmetic operations between GFNs and dubbed as Normalized approach. An uncertainty measure technique called average width is proposed too for GFNs.

Rashid et al.,[29]proposed a new method to aggregate opinion of several decision makers using generalized interval valued fuzzy numbers. Das et al.,[11] discussed permanent of interval valued fuzzy matrices.

In this paper, an attempt has been made to combine generalized IVFNs based on average width of fuzzy set concept in which LFMs of IVFNs are treated as generalized fuzzy numbers. In literature, in the operation of triangular generalized IVFNs, it is observed that resulting fuzzy number is obtained in the form of generalized triangular IVFN, but in the present study it is found that the resulting fuzzy number is obtained in the form of generalized IVFN in which UMF is type-I normal fuzzy number while LMF is trapezoidal type fuzzy number. Furthermore, the resultant IVFN will be generalized trapezoidal type IVFN if LMF and UMF are considered to be generalized triangular IVFNs.

2. Preliminaries

Fuzzy set theory provides a way to characterize the imprecisely defined variables, define relationships between variables based on expert human knowledge and use them to compute results. In this section, some necessary backgrounds and notions of type-I fuzzy set theory [9, 13, 40] and IVFS theory [16, 17, 32, 33, 34] that will be required in the sequel are reviewed.

Definition 2.1. Let X be a universal set. Then the fuzzy subset A of X is defined by its membership function

$$\mu_A : X \to [0,1]$$

which assign a real number $\mu_A(x)$in the interval [0, 1], to each element $x \in A$, where the value of $\mu_A(x)$ at x shows the grade of membership of x in A.

Definition 2.2. Given a fuzzy set A in X and any real number $\alpha \in [0,1]$.

(i) The α -cut of A, denoted by ${}^{\alpha}A$ is the crisp set

$${}^{\alpha}A = \{x \in X : \mu_A(x) \geq \alpha\}.$$

(ii) The strong α-cut, denoted by ${}^{\alpha+}A$ is the crisp set

$${}^{\alpha+}A = \{x \in X : \mu_A(x) > \alpha\}.$$

Definition 2.3. The support of a fuzzy set A on X is a crisp set defined as

$$Supp(A) = \{x \in X : \mu_A(x) > 0\}.$$

Definition 2.4. The height of a fuzzy set A, denoted by $h(A)$ is the largest membership grade obtain by any element in the set and it is denoted as

$$h(A) = \sup_{x \in X} \mu_A(x).$$

Definition 2.5. Generalized Fuzzy Numbers (GFN): The membership function of GFN $A = [a, b, c, d; w]$, where $a \leq b \leq c \leq d, 0 < w \leq 1$ is defined as

$$\mu_A(x) = \begin{cases} 0, & x < a \\ w, & \frac{x-a}{b-a}, a \leq x \leq b \\ w, & b \leq x \leq c \\ w, & \frac{x-c}{d-c}, c \leq x \leq d \\ 0, & x > d. \end{cases}$$

If $w = 1$, then GFN A is a normal trapezoidal fuzzy number $A = [a, b, c, d]$. If $a = b$ and $c = d$, then A is a crisp interval .If $b = c$ then A is a generalized triangular fuzzy number. If $a = b = c = d$ and $w = 1$ then A is a real number. Compared to normal fuzzy number the GFN can deal with uncertain information in a more flexible manner because of the parameter w that represent the degree of confidence of opinions of decision maker's.

Definition 2.6. An interval valued fuzzy set A defined in the universe of discourse X is represented by

$$A = \left\{(x, [\mu_A^L(x), \mu_A^U(x)] : x \in X)\right\},$$

where $0 \leq \mu_A^L(x) \leq \mu_A^U(x) \leq 1$ and the membership grade $\overline{\mu}_A(x)$ of elements of x to the interval valued fuzzy set A is represented by an interval $[\mu_A^L(x), \mu_A^U(x)]$ $\left(i.e., \overline{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]\right)$.

Definition 2.7. If an interval valued fuzzy set A satisfies the following properties:
(i) A is normal,
(ii) A is defined in a closed bounded interval,
(iii) A is convex set,
then A is called an interval valued fuzzy number.

Definition 2.8 ([21], $\alpha-$cut of IVFN)**.** Let A be a continuous and convex IVFN with UMF $\mu_A^U(or\overline{A})$ and LMF $\mu_A^L(or\underline{A})$. Let α-cut of UMF $(\overline{A})$ be ${}^\alpha\overline{A} = [L_{\overline{x}_\alpha}, R_{\overline{x}_\alpha}]$ and of LMF $(\underline{A})$ be ${}^\alpha\underline{A} = [L_{\underline{x}_\alpha}, R_{\underline{x}_\alpha}]$. Then, $\alpha-$cut of IVFS A can be calculated by the following formula:

$${}^\alpha A = \begin{cases} ([L_{\overline{x}_\alpha}, R_{\overline{x}_\alpha}], [L_{\underline{x}_\alpha}, R_{\underline{x}_\alpha}]), \alpha \leq h(\overline{A}) \ \ \& \ \ \alpha \leq h(\underline{A}) \\ ([L_{\overline{x}_\alpha}, R_{\overline{x}_\alpha}], \Phi), \alpha \leq h(\overline{A}) \ \ \& \ \ \alpha > h(\underline{A}) \\ (\Phi, \Phi), \alpha > h(\overline{A}), \end{cases}$$

where $\forall\alpha : L_{\overline{x}_\alpha} \leq L_{\underline{x}_\alpha} \leq R_{\underline{x}_\alpha} \leq R_{\overline{x}_\alpha}$, $h(\underline{A})$ is the height of LMF, $h(\overline{A})$ is the height of UMF and Φ is an empty set.

Definition 2.9. $\alpha-$cut of Generalized IVFN: Let A be a generalized IVFN and its membership function is given as

$$\overline{\mu}_A(x) = \begin{cases} \left.\begin{array}{ll} w_A^U \ \frac{x-a_1}{c_1-a_1}, & x \in [a_1, c_1] \end{array}\right\} UMF \\ \left.\begin{array}{ll} w_A^U \ \frac{e_1-x}{e_1-c_1}, & x \in [c_1, e_1] \end{array}\right\} UMF \\ \\ \left.\begin{array}{ll} w_A^L \ \frac{x-b_1}{c_1-b_1}, & x \in [b_1, c_1] \end{array}\right\} LMF \\ \left.\begin{array}{ll} w_A^L \ \frac{d_1-x}{d_1-c_1}, & x \in [c_1, d_1] \end{array}\right\} LMF \end{cases}$$

Then, $\alpha-$cut of the generalized IVFN A is

$${}^\alpha A = \begin{cases} ([\frac{\alpha}{w_A^U}(c_1-a_1)+a_1, e_1-\frac{\alpha}{w_A^U}(e_1-c_1)], [\frac{\alpha}{w_A^L}(c_1-b_1)+b_1, d_1-\frac{\alpha}{w_A^L}(d_1-c_1)]), \\ \qquad\qquad\qquad\qquad\qquad\qquad \alpha \leq w_A^U; \alpha \leq w_A^L \\ ([\frac{\alpha}{w_A^U}(c_1-a_1)+a_1, e_1-\frac{\alpha}{w_A^U}(e_1-c_1)], \Phi), \alpha \leq w_A^U; \alpha > w_A^L \\ (\Phi, \Phi), \alpha > w_A^U, \end{cases}$$

where $w_A^U \geq w_A^L$.

3. Average Width of GFN

Let A be a GFN whose membership function is

$$\mu_A(x) = \begin{cases} 0, & x < a \\ w, & \frac{x-a}{b-a}, a \le x \le b \\ w, & \frac{c-x}{c-b}, b \le x \le c \\ 0, & x > c. \end{cases}$$

Then the α-cut of A is

$$^{\alpha}A = \left[\frac{\alpha}{w}(b-a) + a,\ c - \frac{\alpha}{w}(c-b),\ \alpha \in [0,\ w]\right].$$

We calculate average width of GFN A following the steps below:
Step 1: Consider consecutive N numbers of values from [0, w].
Step 2: Then find α-cuts for each α-values.
Step 3: Calculate width of each α-cuts.
That is, $c - \frac{\alpha}{w}(c-b) - \left\{\frac{\alpha}{w}(b-a) + a\right\} = (c-a)(1-\frac{\alpha}{w})$.
Step 4: Sum up all the widths and divide by N.
That is, $\frac{\sum (c-a)(1-\frac{\alpha}{w})}{N}$, which will give the average width of the generalized fuzzy number A.

4. Arithmetic operations on Generalized IVFNs

In this section, we perform arithmetic operations between generalized interval valued fuzzy numbers (GIVFNs) based on width of interval concept where LMFs are considered as generalized triangular fuzzy sets.

Let A and B be two generalized IVFNs and their membership functions are given as

$$\overline{\mu}_A(x) = \begin{cases} \left.\begin{matrix} w_A^U\ \frac{x-a_1}{c_1-a_1}, & x\ \in [a_1, c_1] \end{matrix}\right\} UMF \\ \left.\begin{matrix} w_A^U\ \frac{e_1-x}{e_1-c_1}, & x\ \in [c_1, e_1] \end{matrix}\right\} UMF \\ \\ \left.\begin{matrix} w_A^L\ \frac{x-b_1}{c_1-b_1}, & x\ \in [b_1, c_1] \end{matrix}\right\} LMF \\ \left.\begin{matrix} w_A^L\ \frac{d_1-x}{d_1-c_1}, & x\ \in [c_1, d_1] \end{matrix}\right\} LMF \end{cases}$$

$$\overline{\mu}_B(x) = \begin{cases} \left.w_B^U \; \frac{x-a_2}{c_2-a_2}, \quad x \in [a_2, c_2]\right\} UMF \\ \left.w_B^U \; \frac{e_2-x}{e_2-c_2}, \quad x \in [c_2, e_2]\right\} UMF \\ \\ \left.w_B^L \; \frac{x-b_2}{c_2-b_2}, \quad x \in [b_2, c_2]\right\} LMF \\ \left.w_B^L \; \frac{d_2-x}{d_2-c_2}, \quad x \in [c_2, d_2]\right\} LMF \end{cases}$$

where $w_A^U = 1 = w_B^U$ and w_A^L,$w_B^L < 1$.
Then, $\alpha-$cut of GIVFNs A and B are

$$^{\alpha}A = \begin{cases} ([\frac{\alpha}{w_A^U}(c_1-a_1)+a_1, e_1-\frac{\alpha}{w_A^U}(e_1-c_1)], [\frac{\alpha}{w_A^L}(c_1-b_1)+b_1, d_1-\frac{\alpha}{w_A^L}(d_1-c_1)]), \\ \text{when} \quad \alpha \le w_A^U \;\; \& \;\; \alpha \le w_A^L. \\ ([\frac{\alpha}{w_A^U}(c_1-a_1)+a_1, e_1-\frac{\alpha}{w_A^U}(e_1-c_1)], \Phi), \alpha \le w_A^U \;\; \& \;\; \alpha > w_A^L. \\ (\Phi, \Phi), \alpha > w_A^U. \end{cases}$$

and

$$^{\alpha}B = \begin{cases} ([\frac{\alpha}{w_A^U}(c_2-a_2)+a_2, e_2-\frac{\alpha}{w_B^U}(e_2-c_2)], [\frac{\alpha}{w_B^L}(c_2-b_2)+b_2, d_2-\frac{\alpha}{w_B^L}(d_2-c_2)]), \\ \text{when} \quad \alpha \le w_B^U \;\; \& \;\; \alpha \le w_B^L \\ ([\frac{\alpha}{w_B^U}(c_2-a_2)+a_2, e_2-\frac{\alpha}{w_B^U}(e_2-c_2)], \Phi), \alpha \le w_B^U \;\; \& \;\; \alpha > w_B^L \\ (\Phi, \Phi), \alpha > w_B^U \end{cases}$$

respectively.

It is well known that $\alpha-$cut of GIVFN gives closed intervals, to perform arithmetic operation $\otimes$ between A and B it is necessary to consider all the four combinations (4.1) (i.e., $^{\alpha}\underline{A} \otimes ^{\alpha} \underline{B}$). In the first combination, it is taken as $\alpha \le min(w_A^U, w_B^U)$ (i.e., $\alpha \le h(\overline{A})$ & $\alpha \le h(\overline{B})$. Similarly, for the other combinations $\alpha \le min(w_A^U, w_B^L)$, $\alpha \le min(w_A^L, w_B^U)$ and $\alpha \le min(w_A^L, w_B^L)$. Then

(4.1)

$$^{\alpha}A \otimes ^{\alpha}B = \begin{cases} [\frac{\alpha}{w_A^U}(c_1-a_1)+a_1, e_1-\frac{\alpha}{w_A^U}(e_1-c_1)] \otimes [\frac{\alpha}{w_B^U}(c_2-a_2)+a_2, e_2-\frac{\alpha}{w_B^U}(e_2-c_2)], \\ \text{when} \;\; \alpha \in [0, min(w_A^U, w_B^U)] \\ [\frac{\alpha}{w_A^U}(c_1-a_1)+a_1, e_1-\frac{\alpha}{w_A^U}(e_1-c_1)] \otimes [\frac{\alpha}{w_B^L}(c_2-b_2)+b_2, d_2-\frac{\alpha}{w_B^L}(d_2-c_2)], \\ \text{when} \;\; \alpha \in [0, min(w_A^U, w_B^L)] \\ [\frac{\alpha}{w_A^L}(c_1-b_1)+b_1, d_1-\frac{\alpha}{w_A^L}(d_1-c_1)] \otimes [\frac{\alpha}{w_B^U}(c_2-a_2)+a_2, e_2-\frac{\alpha}{w_B^U}(e_2-c_2)], \\ \text{when} \;\; \alpha \in [0, min(w_A^L, w_B^U)] \\ [\frac{\alpha}{w_A^L}(c_1-b_1)+b_1, d_1-\frac{\alpha}{w_A^L}(d_1-c_1)] \otimes [\frac{\alpha}{w_B^L}(c_2-b_2)+b_2, d_2-\frac{\alpha}{w_B^L}(d_2-c_2)], \\ \text{when} \;\; \alpha \in [0, min(w_A^L, w_B^L)] \end{cases}$$

Where $\otimes$ indicates the basic four arithmetic operations $(+, -, \times, \div)$. Using Decomposition theorem four fuzzy sets will be obtained and among of these fuzzy sets UMF and LMF of the resulting IVFS can be evaluated using width of generalized fuzzy number. Maximum and minimum average width will represent UMF and LMF of the resultant IVFS respectively.

Theorem 4.1. *Addition of two generalized interval valued triangular fuzzy numbers produces a generalized interval valued fuzzy number in which UMF is triangular fuzzy set and LMF is trapezoidal fuzzy set.*

Proof. Consider the first combination of ${}^{\alpha}A +^{\alpha} B$ of (4.1). That is,

$$\left[\tfrac{\alpha}{w_A^U}(c_1-a_1)+a_1, e_1-\tfrac{\alpha}{w_A^U}(e_1-c_1)\right] + \left[\tfrac{\alpha}{w_B^U}(c_2-a_2)+a_2, e_2-\tfrac{\alpha}{w_B^U}(e_2-c_2)\right],$$

where $\alpha \in [0, min(w_A^U, w_B^U)]$.

$$\left[\frac{\alpha}{w_A^U}(c_1-a_1)+a_1+\frac{\alpha}{w_B^U}(c_2-a_2)+a_2, e_1-\frac{\alpha}{w_A^U}(e_1-c_1)+e_2-\frac{\alpha}{w_B^U}(e_2-c_2)\right] \tag{4.2}$$

$$= \left[(a_1+a_2)+\alpha\left(\tfrac{c_1-a_1}{w_A^U}+\tfrac{c_2-a_2}{w_B^U}\right), (e_1+e_2)+\alpha\left(\tfrac{e_1-c_1}{w_A^U}+\tfrac{e_2-c_2}{w_B^U}\right)\right].$$

To find the membership function $\mu^1_{A+B}(x)$, we equate both the first and second component of (4.2) to x, which gives

$$x = (a_1+a_2)+\alpha\left(\tfrac{c_1-a_1}{w_A^U}+\tfrac{c_2-a_2}{w_B^U}\right) \text{ and } x = (e_1+e_2)+\alpha\left(\tfrac{e_1-c_1}{w_A^U}+\tfrac{e_2-c_2}{w_B^U}\right).$$

Now, expressing α in terms of x, and setting $\alpha = 0$ and $\alpha = 1$ in (4.2), we get α together with the domain of x, that is

$$\alpha = \frac{x-(a_1+a_2)}{\left\{\tfrac{c_1-a_1}{w_A^U}+\tfrac{c_2-a_2}{w_B^U}\right\}}, x\epsilon\left[(a_1+a_2), (a_1+a_2)+w\left(\frac{c_1-a_1}{w_A^U}+\frac{c_2-a_2}{w_B^U}\right)\right] \tag{4.3}$$

and

$$\alpha = \frac{(e_1+e_2)-x}{\left\{\tfrac{(e_1-c_1)}{w_A^U}+\tfrac{(e_2-c_2)x}{w_B^U}\right\}}, x\epsilon\left[(e_1+e_2)-w\left(\frac{e_1-c_1}{w_A^U}+\frac{e_2-c_2}{w_B^U}\right), (e_1+e_2)\right]. \tag{4.4}$$

which gives the membership function of the fuzzy number $A+B$

$$\mu^1_{A+B}(x)=\begin{cases}\dfrac{x-(a_1+a_2)}{\left\{\frac{c_1-a_1}{w_A^U}+\frac{c_2-a_2}{w_B^U}\right\}}, x\in\left[(a_1+a_2),(a_1+a_2)+w\left(\frac{c_1-a_1}{w_A^U}+\frac{c_2-a_2}{w_B^U}\right)\right]\\ w, x\in\left[(a_1+a_2)+w\left(\frac{c_1-a_1}{w_A^U}+\frac{c_2-a_2}{w_B^U}\right),\right.\\ \qquad\qquad\left.(e_1+e_2)-w\left(\frac{e_1-c_1}{w_A^U}+\frac{e_2-c_2}{w_B^U}\right)\right]\\ \dfrac{(e_1+e_2)-x}{\left\{\frac{e_1-c_1}{w_A^U}+\frac{e_2-c_2}{w_B^U}\right\}}, x\in\left[(e_1+e_2)-w\left(\frac{e_1-c_1}{w_A^U}+\frac{e_2-c_2}{w_B^U}\right),(e_1+e_2)\right].\end{cases}$$

In a similar fashion, another three membership functions for the remaining combinations ${}^{\alpha}A +^{\alpha} B$ of (4.1) can be evaluated and respectively given below.

$$\mu^2_{A+B}(x)=\begin{cases}\dfrac{x-(a_1+b_2)}{\left\{\frac{c_1-a_1}{w_A^U}+\frac{c_2-b_2}{w_B^L}\right\}}, x\in\left[(a_1+b_2),(a_1+b_2)+w\left(\frac{c_1-a_1}{w_A^U}+\frac{c_2-b_2}{w_B^L}\right)\right]\\ w, x\in\left[(a_1+b_2)+w\left(\frac{c_1-a_1}{w_A^U}+\frac{c_2-b_2}{w_B^L}\right),\right.\\ \qquad\qquad\left.(e_1+d_2)-w\left(\frac{e_1-c_1}{w_A^U}+\frac{d_2-c_2}{w_B^L}\right)\right]\\ \dfrac{(e_1+d_2)-x}{\left\{\frac{e_1-c_1}{w_A^U}+\frac{d_2-c_2}{w_B^L}\right\}}, x\in\left[(e_1+d_2)-w\left(\frac{e_1-c_1}{w_A^U}+\frac{d_2-c_2}{w_B^L}\right),(e_1+d_2)\right],\end{cases}$$

$$\mu^3_{A+B}(x)=\begin{cases}\dfrac{x-(b_1+a_2)}{\left\{\frac{c_1-b_1}{w_A^L}+\frac{c_2-a_2}{w_B^U}\right\}}, x\in\left[(b_1+a_2),(b_1+a_2)+w\left(\frac{c_1-b_1}{w_A^L}+\frac{c_2-a_2}{w_B^U}\right)\right]\\ w, x\in\left[(b_1+a_2)+w\left(\frac{c_1-b_1}{w_A^L}+\frac{c_2-a_2}{w_B^U}\right),\right.\\ \qquad\qquad\left.(d_1+e_2)-w\left(\frac{d_1-c_1}{w_A^L}+\frac{e_2-c_2}{w_B^U}\right)\right]\\ \dfrac{(d_1+e_2)-x}{\left\{\frac{d_1-c_1}{w_A^L}+\frac{e_2-c_2}{w_B^U}\right\}}, x\in\left[(d_1+e_2)-w\left(\frac{d_1-c_1}{w_A^L}+\frac{e_2-c_2}{w_B^U}\right),(d_1+e_2)\right],\end{cases}$$

$$\mu^4_{A+B}(x)=\begin{cases}\dfrac{x-(b_1+b_2)}{\left\{\frac{c_1-b_1}{w^L_A}+\frac{c_2-b_2}{w^L_B}\right\}}, x\in\left[(b_1+b_2),(b_1+b_2)+w\left(\frac{c_1-b_1}{w^L_A}+\frac{c_2-b_2}{w^L_B}\right)\right]\\ w, x\in\left[(b_1+b_2)+w\left(\frac{c_1-b_1}{w^L_A}+\frac{c_2-b_2}{w^L_B}\right),\right.\\ \qquad\qquad\left.(d_1+d_2)-w\left(\frac{d_1-c_1}{w^L_A}+\frac{d_2-c_2}{w^L_B}\right)\right]\\ \dfrac{(d_1+d_2)-x}{\left\{\frac{d_1-c_1}{w^L_A}+\frac{d_2-c_2}{w^L_B}\right\}}, x\in\left[(d_1+d_2)-w\left(\frac{d_1-c_1}{w^L_A}+\frac{d_2-c_2}{w^L_B}\right),(d_1+d_2)\right].\end{cases}$$

The average width of these four membership function can be evaluated at the common height $min(w^U_A, w^L_A, w^U_B, w^L_B)$. The membership function whose average width is maximum represents UMF while the membership function whose average width is minimum represents LMF of the resultant IVFN. Shape of the UMF is triangular fuzzy set (as $w^U_A = w^U_B = 1$) , on the other hand LMF is generalized trapezoidal fuzzy set with height $min(w^L_A, w^L_B)$. □

Remark 4.2. If w^U_A and $w^U_B < 1$ then shape of the UMF will also be generalized trapezoidal fuzzy set with height $\min(w^U_A, w^U_B)$.

Theorem 4.3. *subtraction of two generalized interval valued triangular fuzzy numbers produces a generalized interval valued fuzzy number in which UMF is triangular fuzzy set and LMF is trapezoidal fuzzy set.*

Proof. Consider the first combination of ${}^\alpha A -{}^\alpha B$ of (4.1). That is,

$$\left[\frac{\alpha}{w^U_A}(c_1-a_1)+a_1, e_1-\frac{\alpha}{w^U_A}(e_1-c_1)\right]-\left[\frac{\alpha}{w^U_B}(c_2-a_2)+a_2, e_2-\frac{\alpha}{w^U_B}(e_2-c_2)\right],$$

where $\alpha\in[0, min(w^U_A, w^U_B)]$

$$=\left[\left\{\frac{\alpha}{w^U_A}(c_1-a_1)+a_1\right\}-\left\{e_2-\frac{\alpha}{w^U_B}(e_2-c_2)\right\},\left\{e_1-\frac{\alpha}{w^U_A}(e_1-c_1)\right\}-\right.$$
$$\left.\left\{\frac{\alpha}{w^U_B}(c_2-a_2)+a_2\right\}\right]$$

which gives

$$\left[(a_1-e_1)+\alpha\left\{\frac{c_1-a_1}{w^U_A}+\frac{c_2-a_2}{w^U_B}\right\},(a_1-e_1)+\alpha\left\{\frac{c_1-a_1}{w^U_A}+\frac{c_2-a_2}{w^U_B}\right\}\right]. \tag{4.5}$$

To find the membership function $\mu^1_{A-B}(x)$, we equate both the first and second component of (4.5) to x, which gives

$$x=(a_1-e_2)+\alpha\left(\frac{c_1-a_1}{w^U_A}+\frac{c_2-a_2}{w^U_B}\right)\text{ and } x=(e_1-a_2)+\alpha\left(\frac{c_1-a_1}{w^U_A}+\frac{c_2-a_2}{w^U_B}\right).$$

Now expressing α in terms of x, and setting $\alpha = 0$ and $\alpha = 1$ in (4.5), we obtain α together with the domain of x, that is

$$\alpha = \frac{x-(a_1-e_2)}{\frac{c_1-a_1}{w_A^U}+\frac{e_2-c_2}{w_B^U}}, x \in \left[(a_1-e_2), (a_1-e_2)+w\left\{\frac{c_1-a_1}{w_A^U}+\frac{e_2-c_2}{w_B^U}\right\}\right]$$

$$\alpha = \frac{(e_1-a_2)-x}{\frac{e_1-c_1}{w_A^U}+\frac{c_2-a_2}{w_B^U}}, x \in \left[(e_1-a_2)-w\left\{\frac{e_1-c_1}{w_A^U}+\frac{c_2-a_2}{w_B^U}\right\}, (e_1-a_2)\right]$$

where w=min(w_A^U, w_B^U) and $\alpha \in [0, w]$.
Which gives the following membership function

$$\mu^1_{A-B}(x) = \begin{cases} \frac{x-(a_1-e_2)}{\left\{\frac{c_1-a_1}{w_A^U}+\frac{e_2-c_2}{w_B^U}\right\}}, x \in \left[(a_1-e_2), (a_1-e_2)+w\left\{\frac{c_1-a_1}{w_A^U}+\frac{e_2-c_2}{w_B^U}\right\}\right] \\ w, x \in \left[(a_1-e_2)+w\left\{\frac{c_1-a_1}{w_A^U}+\frac{e_2-c_2}{w_B^U}\right\}, \right. \\ \qquad \left. (e_1-a_2)-w\left\{\frac{e_1-c_1}{w_A^U}+\frac{c_2-a_2}{w_B^U}\right\}\right] \\ \frac{(e_1-a_2)-x}{\left\{\frac{e_1-c_1}{w_A^U}+\frac{c_2-a_2}{w_B^U}\right\}}, x \in \left[(e_1-a_2)-w\left\{\frac{e_1-c_1}{w_A^U}+\frac{c_2-a_2}{w_B^U}\right\}, (e_1-a_2)\right]. \end{cases}$$

Similarly, membership functions of other remaining combinations can be evaluated and respectively depicted below.

$$\mu^2_{A-B}(x) = \begin{cases} \frac{x-(a_1-d_2)}{\left\{\frac{c_1-a_1}{w_A^U}+\frac{d_2-c_2}{w_B^L}\right\}}, x \in \left[(a_1-d_2), (a_1-d_2)+w\left\{\frac{c_1-a_1}{w_A^U}+\frac{d_2-c_2}{w_B^L}\right\}\right] \\ w, x \in \left[(a_1-d_2)+w\left\{\frac{c_1-a_1}{w_A^U}+\frac{d_2-c_2}{w_B^L}\right\}, \right. \\ \qquad \left. (e_1-b_2)-w\left\{\frac{e_1-c_1}{w_A^U}+\frac{c_2-b_2}{w_B^L}\right\}\right] \\ \frac{(e_1-b_2)-x}{\left\{\frac{e_1-c_1}{w_A^U}+\frac{c_2-b_2}{w_B^L}\right\}}, x \in \left[(e_1-b_2)-w\left\{\frac{e_1-c_1}{w_A^U}+\frac{c_2-b_2}{w_B^L}\right\}, (e_1-b_2)\right], \end{cases}$$

$$\mu^3_{A-B}(x)=\begin{cases}\dfrac{x-(b_1-e_2)}{\left\{\frac{c_1-b_1}{w^L_A}+\frac{e_2-c_2}{w^U_B}\right\}}, x\in\left[(b_1-e_2),(b_1-e_2)+w\left\{\frac{c_1-b_1}{w^L_A}+\frac{e_2-c_2}{w^U_B}\right\}\right]\\ w, x\in\left[(b_1-e_2)+w\left\{\frac{c_1-b_1}{w^L_A}+\frac{e_2-c_2}{w^U_B}\right\},\right.\\ \qquad\qquad \left.(d_1-a_2)-w\left\{\frac{d_1-c_1}{w^L_A}+\frac{c_2-a_2}{w^U_B}\right\}\right]\\ \dfrac{(d_1-a_2)-x}{\left\{\frac{d_1-c_1}{w^L_A}+\frac{c_2-a_2}{w^U_B}\right\}}, x\in\left[(d_1-a_2)-w\left\{\frac{d_1-c_1}{w^L_A}+\frac{c_2-a_2}{w^U_B}\right\},(d_1-a_2)\right],\end{cases}$$

$$\mu^4_{A-B}(x)=\begin{cases}\dfrac{x-(b_1-d_2)}{\left\{\frac{c_1-b_1}{w^L_A}+\frac{d_2-c_2}{w^L_B}\right\}}, x\in\left[(b_1-d_2),(b_1-d_2)+w\left\{\frac{c_1-b_1}{w^L_A}+\frac{d_2-c_2}{w^L_B}\right\}\right]\\ w, x\in\left[(b_1-d_2)+w\left\{\frac{c_1-b_1}{w^L_A}+\frac{d_2-c_2}{w^L_B}\right\},\right.\\ \qquad\qquad \left.(d_1-b_2)-w\left\{\frac{d_1-c_1}{w^L_A}+\frac{c_2-b_2}{w^L_B}\right\}\right]\\ \dfrac{(d_1-b_2)-x}{\left\{\frac{d_1-c_1}{w^L_A}+\frac{c_2-b_2}{w^L_B}\right\}}, x\in\left[(d_1-b_2)-w\left\{\frac{d_1-c_1}{w^L_A}+\frac{c_2-b_2}{w^L_B}\right\},(d_1-b_2)\right].\end{cases}$$

Similarly, the UMF and LMF of the resultant IVFN can be evaluated using average width concept at the common height $min(w^U_A, w^L_A, w^U_B, w^L_B)$. Here also, maximum and minimum average width represent UMF and LMF of the resultant IVFN respectively. The shape of the UMF is triangular fuzzy set (as $w^U_A = w^U_B = 1$), on the other hand LMF is generalized trapezoidal fuzzy set with height. □

Remark 4.4. Here also, if w^U_A & $w^U_B < 1$, then shape of the UMF will also be generalized trapezoidal fuzzy set with height $\min(w^U_A, w^U_B)$.

Theorem 4.5. *Multiplication of two generalized interval valued triangular fuzzy numbers produces a generalized interval valued fuzzy number in which UMF is triangular type fuzzy set and LMF is trapezoidal type fuzzy set.*

Proof. Consider the first combination of ${}^{\alpha}A^{\alpha}B$ of (4.1). That is,

$$\left[\tfrac{\alpha}{w^U_A}(c_1-a_1)+a_1, e_1-\tfrac{\alpha}{w^U_A}(e_1-c_1)\right]\left[\tfrac{\alpha}{w^U_B}(c_2-a_2)+a_2, e_2-\tfrac{\alpha}{w^U_B}(e_2-c_2)\right],$$

where $\alpha\in[0, min(w^U_A, w^U_B)]$.

(4.6)

$$\left[\left\{\frac{\alpha}{w^U_A}(c_1-a_1)+a_1\right\}\left\{\frac{\alpha}{w^U_B}(c_2-a_2)+a_2\right\},\left\{e_1-\frac{\alpha}{w^U_A}(e_1-c_1)\right\}\left\{e_2-\frac{\alpha}{w^U_B}(e_2-c_2)\right\}\right].$$

To find the membership function $\mu^1_{AB}(x)$, we equate both the first and second component of (4.6) to x, which gives

$$x = \left\{\frac{\alpha}{w_A^U}(c_1 - a_1) + a_1\right\}\left\{\frac{\alpha}{w_B^U}(c_2 - a_2) + a_2\right\}.$$

Then

$$x = \frac{\alpha^2}{w_A^U w_B^U}(c_1 - a_1)(c_2 - a_2) + \frac{\alpha}{w_A^U} a_2(c_1 - a_1) + \frac{\alpha}{w_B^U} a_1(c_2 - a_2) + a_1 a_2.$$

Thus

$$\alpha^2 \frac{(c_1-a_1)(c_2-a_2)}{w_A^U w_B^U} + \alpha\left\{\frac{a_2(c_1-a_1)}{w_A^U} + \frac{a_1(c_2-a_2)}{w_B^U}\right\} + (a_1 a_2 - x) = 0.$$

This is a quadratic equation and solving. So we get

$$\alpha = \Bigg[-\{a_2(c_1 - a_1)w_B^U + a_1(c_2 - a_2)w_A^U\} + \sqrt{\{a_2(c_1 - a_1){w_B}^U + a_1(c_2 - a_2){w_A}^U\}^2 - 4{w_A}^U {w_B}^U (c_1 - a_1)(c_2 - a_2)(a_1 a_2 - x)}\Bigg] \Big/ \Bigg[2(c_1 - a_1)(c_2 - a_2\Bigg].$$

Similarly, $x = \left\{e_1 - \frac{\alpha}{w_A^U}(e_1 - c_1)\right\}\left\{e_2 - \frac{\alpha}{w_B^U}(e_2 - c_2)\right\}$ gives

$$\alpha = \Bigg[\{e_2(e_1 - c_1)w_B^U + e_1(e_2 - c_2)w_A^U\} - \sqrt{\{e_2(e_1 - c_1)w_B^U + e_1(e_2 - c_2)w_A^U\}^2 - 4w_A^U w_B^U (e_1 - c_1)(e_2 - c_2)(e_1 e_2 - x)}\Bigg] \Big/ \Bigg[2(e_1 - c_1)(e_2 - c_2)\Bigg].$$

Now, setting $\alpha = 0$ and $\alpha = 1$ in (4.6), we have the membership function $\mu^1_{AB}(x)$

$$\mu^1_{AB}(x)=\begin{cases}\frac{-\{a_2(c_1-a_1)w^U_B+a_1(c_2-a_2)w^U_A\}+\sqrt{\{a_2(c_1-a_1){w_B}^U+a_1(c_2-a_2){w_A}^U\}^2-4{w_A}^U{w_B}^U(c_1-a_1)(c_2-a_2)(a_1a_2-x)}}{2(c_1-a_1)(c_2-a_2)} \\ \text{if } x\ \in\left[a_1a_2, a_1a_2+\frac{w^2}{w^U_Aw^U_B}(c_1-a_1)(c_2-a_2)+\frac{w}{w^U_A}a_2(c_1-a_1)+\frac{w}{w^U_B}a_1(c_2-a_2)\right] \\ w,\ x\in\left[\begin{array}{c} a_1a_2+\frac{w^2}{w^U_Aw^U_B}(c_1-a_1)(c_2-a_2)+\frac{w}{w^U_A}a_2(c_1-a_1)+\frac{w}{w^U_B}a_1(c_2-a_2), \\ e_1e_2+\frac{w^2}{w^U_Aw^U_B}(c_1-e_1)(c_2-e_2)-\frac{w}{w^U_A}e_2(c_1-e_1)-\frac{w}{w^U_B}e_1(c_2-e_2)\end{array}\right] \\ \frac{\{e_2(e_1-c_1)w^U_B+e_1(e_2-c_2)w^U_A\}-\sqrt{\{e_2(e_1-c_1)w^U_B+e_1(e_2-c_2)w^U_A\}^2-4w^U_Aw^U_B(e_1-c_1)(e_2-c_2)(e_1e_2-x)}}{2(e_1-c_1)(e_2-c_2)} \\ \text{if } x\ \in\left[e_1e_2+\frac{w^2}{w^U_Aw^U_B}(e_1-c_1)(e_2-c_2)-\frac{w}{w^U_A}e_2(e_1-c_1)-\frac{w}{w^U_B}e_1(e_2-c_2), e_1e_2\right]\end{cases}$$

Similarly, membership functions for remaining combinations can be calculated and respectively listed below.

$$\mu^2_{AB}(x)=\begin{cases}\frac{-\{b_2(c_1-a_1)w^L_B+a_1(c_2-b_2)w^U_A\}+\sqrt{\{b_2(c_1-a_1){w_B}^L+a_1(c_2-b_2){w_A}^U\}^2-4{w_A}^U{w_B}^L(c_1-a_1)(c_2-b_2)(a_1b_2-x)}}{2(c_1-a_1)(c_2-b_2)} \\ \text{if } x\ \in\left[a_1b_2, a_1b_2+\frac{w^2}{w^U_Aw^L_B}(c_1-a_1)(c_2-b_2)+\frac{w}{w^U_A}b_2(c_1-a_1)+\frac{w}{w^L_B}a_1(c_2-b_2)\right] \\ w,\ x\in\left[\begin{array}{c} a_1b_2+\frac{w^2}{w^U_Aw^L_B}(c_1-a_1)(c_2-b_2)+\frac{w}{w^U_A}b_2(c_1-a_1)+\frac{w}{w^L_B}a_1(c_2-b_2), \\ e_1d_2+\frac{w^2}{w^U_Aw^L_B}(d_1-c_1)(d_2-c_2)-\frac{w}{w^U_A}d_2(e_1-c_1)-\frac{w}{w^L_B}e_1(d_2-c_2)\end{array}\right] \\ \frac{\{d_2(e_1-c_1)w^L_B+e_1(d_2-c_2)w^U_A\}-\sqrt{\{d_2(e_1-c_1)w^L_B+e_1(d_2-c_2)w^U_A\}^2-4w^U_Aw^L_B(e_1-c_1)(d_2-c_2)(e_1d_2-x)}}{2(e_1-c_1)(d_2-c_2)} \\ \text{if } x\ \in\left[e_1d_2+\frac{w^2}{w^U_Aw^L_B}(d_1-c_1)(d_2-c_2)-\frac{w}{w^U_A}d_2(e_1-c_1)-\frac{w}{w^L_B}e_1(d_2-c_2), e_1d_2\right]\end{cases}$$

$$\mu^3_{AB}(x) = \begin{cases} \frac{-\{a_2(c_1-b_1)w_B^U+b_1(c_2-a_2)w_A^L\}+\sqrt{\{a_2(c_1-b_1)w_B{}^U+b_1(c_2-a_2)w_A{}^L\}^2-4w_A{}^L w_B{}^U(c_1-b_1)(c_2-a_2)(b_1a_2-x)}}{2(c_1-b_1)(c_2-a_2)} \\ \text{if } x \ \in \left[b_1a_2, b_1a_2+\frac{w^2}{w_A^L w_B^U}(c_1-b_1)(c_2-a_2)+\frac{w}{w_A^L}a_2(c_1-b_1)+\frac{w}{w_B^U}b_1(c_2-a_2)\right] \\ w, \ x \in \begin{bmatrix} b_1a_2+\frac{w^2}{w_A^L w_B^U}(c_1-b_1)(c_2-a_2)+\frac{w}{w_A^L}a_2(c_1-b_1)+\frac{w}{w_B^U}b_1(c_2-a_2), \\ d_1e_2+\frac{w^2}{w_A^L w_B^U}(d_1-c_1)(e_2-c_2)-\frac{w}{w_A^L}e_2(d_1-c_1)-\frac{w}{w_B^U}d_1(e_2-c_2) \end{bmatrix} \\ \frac{\{e_2(d_1-c_1)w_B^U+d_1(e_2-c_2)w_A^L\}-\sqrt{\{e_2(d_1-c_1)w_B^U+d_1(e_2-c_2)w_A^L\}^2-4w_A^L w_B^U(d_1-c_1)(e_2-c_2)(d_1e_2-x)}}{2(d_1-c_1)(e_2-c_2)} \\ \text{if } x \ \in \left[d_1e_2+\frac{w^2}{w_A^L w_B^U}(d_1-c_1)(e_2-c_2)-\frac{w}{w_A^L}e_2(d_1-c_1)-\frac{w}{w_B^U}d_1(e_2-c_2), d_1e_2\right] \end{cases}$$

$$\mu^4_{AB}(x) = \begin{cases} \frac{-\{b_2(c_1-b_1)w_B^L+b_1(c_2-b_2)w_A^L\}+\sqrt{\{b_2(c_1-b_1)w_B{}^L+b_1(c_2-b_2)w_A{}^L\}^2-4w_A{}^L w_B{}^L(c_1-b_1)(c_2-b_2)(b_1b_2-x)}}{2(c_1-b_1)(c_2-b_2)} \\ \text{if } x \ \in \left[b_1b_2, b_1b_2+\frac{w^2}{w_A^L w_B^L}(c_1-b_1)(c_2-b_2)+\frac{w}{w_A^L}b_2(c_1-b_1)+\frac{w}{w_B^L}b_1(c_2-b_2)\right] \\ w, \ x \in \begin{bmatrix} b_1b_2+\frac{w^2}{w_A^L w_B^L}(c_1-b_1)(c_2-b_2)+\frac{w}{w_A^L}b_2(c_1-b_1)+\frac{w}{w_B^L}b_1(c_2-b_2), \\ d_1d_2+\frac{w^2}{w_A^L w_B^L}(d_1-c_1)(d_2-c_2)-\frac{w}{w_A^L}d_2(d_1-c_1)-\frac{w}{w_B^L}d_1(d_2-c_2) \end{bmatrix} \\ \frac{\{d_2(d_1-c_1)w_B^L+d_1(d_2-c_2)w_A^L\}-\sqrt{\{d_2(d_1-c_1)w_B^L+d_1(d_2-c_2)w_A^L\}^2-4w_A^L w_B^L(d_1-c_1)(d_2-c_2)(d_1d_2-x)}}{2(d_1-c_1)(d_2-c_2)} \\ \text{if } x \ \in \left[d_1d_2+\frac{w^2}{w_A^L w_B^L}(d_1-c_1)(d_2-c_2)-\frac{w}{w_A^L}d_2(d_1-c_1)-\frac{w}{w_B^L}d_1(d_2-c_2), d_1d_2\right] \end{cases}$$

Here too, the UMF and LMF of the resultant IVFN can be evaluated using average width concept at the common height $min(w_A^U, w_A^L, w_B^U, w_B^L)$. The maximum and minimum average width represent UMF and LMF of the resultant IVFN respectively. The Shape of the UMF is triangular type fuzzy set (as $w_A^U = w_B^U = 1$), on the other hand LMF is generalized trapezoidal type fuzzy set with height $min(w_A^L, w_B^L)$. $\square$

Remark 4.6. If $w_A^U \& w_B^U < 1$ then shape of UMF will also be generalized trapezoidal type fuzzy set with height $min(w_A^U, w_B^U)$.

Theorem 4.7. *Division of two generalized interval valued triangular fuzzy numbers produces a generalized interval valued fuzzy number in which UMF is triangular type fuzzy set and LMF is trapezoidal type fuzzy set.*

Proof. Consider the first combination of ${}^{\alpha}A \div^{\alpha} B$ of (4.1). That is,

$$\left[\frac{\alpha}{w_A^U}(c_1 - a_1) + a_1, e_1 - \frac{\alpha}{w_A^U}(e_1 - c_1)\right] \div \left[\frac{\alpha}{w_B^U}(c_2 - a_2) + a_2, e_2 - \frac{\alpha}{w_B^U}(e_2 - c_2)\right],$$

where $\alpha \in [0, min(W_A^U, W_B^U)]$. That is,

$$\left[\frac{\left\{\frac{\alpha}{w_A^U}(c_1 - a_1) + a_1\right\}}{\left\{e_2 - \frac{\alpha}{w_B^U}(e_2 - c_2)\right\}}, \frac{\left\{e_1 - \frac{\alpha}{w_A^U}(e_1 - c_1)\right\}}{\left\{\frac{\alpha}{w_B^U}(c_2 - a_2) + a_2\right\}}\right]. \tag{4.7}$$

To find the membership function $\mu^1_{A \div B}$ we equate both the first and second component of (4.7) to x , we have

$$x = \frac{\left\{\frac{\alpha}{w_A^U}(c_1 - a_1) + a_1\right\}}{\left\{e_2 - \frac{\alpha}{w_B^U}(e_2 - c_2)\right\}} \text{ and } x = \frac{\left\{e_1 - \frac{\alpha}{w_A^U}(e_1 - c_1)\right\}}{\left\{\frac{\alpha}{w_B^U}(c_2 - a_2) + a_2\right\}}.$$

Now expressing α in terms of x, and setting $\alpha = 0$ and $\alpha = 1$ in (4.7), we get α together with the domain of x,

Now expressing in terms of , and setting and in (4.7), we get together with the domain of , that is,

$$\alpha = \frac{e_2 x - a_1}{\left\{\frac{(c_1 - a_1}{w_A^U} + \frac{(d_2 - c_2)x}{w_B^L}\right\}}, x\epsilon\left[\frac{a_1}{a_2}, \frac{\frac{w}{w_A^U}(c_1 - a_1) + a_1}{d_2 - \frac{w}{w_B^L}(d_2 - c_2)}\right]$$

and

$$\alpha = \frac{e_1 - b_2 x}{\left\{\frac{(e_1 - c_1)}{w_A^U} + \frac{(c_2 - b_2)x}{w_B^L}\right\}}, x\epsilon\left[\frac{e_1 - \frac{w}{w_A^U}(e_1 - c_1)}{b_2 + \frac{w}{w_B^L}(c_2 - b_2)}, \frac{e_1}{b_2}\right].$$

Then the membership function of $\mu^1_{A \div B}(x)$ the resultant INFN is obtained as

$$\mu^1_{A\div B}(x)=\begin{cases}\dfrac{e_2x-a_1}{\left\{\frac{(c_1-a_1)}{w_A^U}+\frac{(e_2-c_2)x}{w_B^U}\right\}}, x\in\left[\frac{a_1}{e_2},\frac{\frac{w}{w_A^U}(c_1-a_1)+a_1}{e_2-\frac{w}{w_B^U}(e_2-c_2)}\right]\\ w \qquad, x\in\left[\frac{\frac{w}{w_A^U}(c_1-a_1)+a_1}{e_2-\frac{w}{w_B^U}(e_2-c_2)},\frac{e_1-\frac{w}{w_A^U}(e_1-c_1)}{a_2+\frac{w}{w_B^U}(c_2-a_2)}\right]\\ \dfrac{e_1-a_2x}{\left\{\frac{(e_1-c_1)}{w_A^U}+\frac{(c_2-a_2)x}{w_B^U}\right\}}, x\in\left[\frac{e_1-\frac{w}{w_A^U}(e_1-c_1)}{a_2+\frac{w}{w_B^U}(c_2-a_2)},\frac{e_1}{a_2}\right].\end{cases}$$

Similarly, membership functions for other combinations can be evaluated and which are given below.

$$\mu^2_{A\div B}(x)=\begin{cases}\dfrac{d_2x-a_1}{\left\{\frac{(c_1-a_1)}{w_A^U}+\frac{(d_2-c_2)x}{w_B^L}\right\}}, x\in\left[\frac{a_1}{d_2},\frac{\frac{w}{w_A^U}(c_1-a_1)+a_1}{d_2-\frac{w}{w_B^L}(d_2-c_2)}\right]\\ w \qquad, x\epsilon\left[\frac{\frac{w}{w_A^U}(c_1-a_1)+a_1}{d_2-\frac{w}{w_B^L}(d_2-c_2)},\frac{e_1-\frac{w}{w_A^U}(e_1-c_1)}{b_2+\frac{w}{w_B^L}(c_2-b_2)}\right]\\ \dfrac{e_1-b_2x}{\left\{\frac{(e_1-c_1)}{w_A^U}+\frac{(c_2-b_2)x}{w_B^L}\right\}}, x\in\left[\frac{e_1-\frac{w}{w_A^U}(e_1-c_1)}{b_2+\frac{w}{w_B^L}(c_2-b_2)},\frac{e_1}{b_2}\right],\end{cases}$$

$$\mu^3_{A\div B}(x)=\begin{cases}\dfrac{e_2x-b_1}{\left\{\frac{(c_1-b_1}{w_A^L}+\frac{(e_2-c_2)x}{w_B^U}\right\}}, x\in\left[\frac{b_1}{e_2},\frac{\frac{w}{w_A^L}(c_1-b_1)+b_1}{e_2-\frac{w}{w_B^U}(e_2-c_2)}\right]\\ w \qquad, x\in\left[\frac{\frac{w}{w_A^L}(c_1-b_1)+b_1}{e_2-\frac{w}{w_B^U}(e_2-c_2)},\frac{d_1-\frac{w}{w_A^L}(d_1-c_1)}{a_2+\frac{w}{w_B^U}(c_2-a_2)}\right]\\ \dfrac{d_1-a_2x}{\left\{\frac{(d_1-c_1)}{w_A^L}+\frac{(c_2-a_2)x}{w_B^U}\right\}}, x\in\left[\frac{d_1-\frac{w}{w_A^L}(d_1-c_1)}{a_2+\frac{w}{w_B^U}(c_2-a_2)},\frac{d_1}{a_2}\right],\end{cases}$$

$$\mu^4_{A\div B}(x)=\begin{cases}\dfrac{d_2x-b_1}{\left\{\frac{(c_1-b_1)}{w_A^L}+\frac{(d_2-c_2)x}{w_B^L}\right\}}, x\in\left[\frac{b_1}{d_2},\frac{\frac{w}{w_A^L}(c_1-b_1)+b_1}{d_2-\frac{w}{w_B^L}(d_2-c_2)}\right]\\ w \qquad, x\in\left[\frac{\frac{w}{w_A^L}(c_1-b_1)+b_1}{d_2-\frac{w}{w_B^L}(d_2-c_2)},\frac{d_1-\frac{w}{w_A^L}(d_1-c_1)}{b_2+\frac{w}{w_B^L}(c_2-b_2)}\right]\\ \dfrac{d_1-b_2x}{\left\{\frac{(d_1-c_1)}{w_A^L}+\frac{(c_2-b_2)x}{w_B^L}\right\}}, x\in\left[\frac{d_1-\frac{w}{w_A^L}(d_1-c_1)}{b_2+\frac{w}{w_B^L}(c_2-b_2)},\frac{d_1}{b_2}\right].\end{cases}$$

In this case also, the UMF and LMF of the resultant IVFN can be evaluated using average width concept at the common height $min(w_A^U, w_A^L, w_B^U, w_B^L)$ where maximum and minimum average width represent UMF and LMF of the resultant IVFN respectively. The Shape of the UMF is triangular type fuzzy set (as $w_A^U = w_B^U = 1$), the LMF is generalized trapezoidal type fuzzy set with height $min(w_A^L, w_B^L)$. □

Remark 4.8. In this case also it is observed that if $w_A^U \& w_B^U < 1$ then shape of UMF will also be generalized trapezoidal type fuzzy set with height $min(w_A^U, w_B^U)$.

5. Numerical Examples

Let A and B be two interval valued generalized fuzzy sets whose membership functions are given as

$$\overline{\mu}_A(x) = \begin{cases} \frac{x-10}{15}, x \in [10,25] \\ \frac{40-x}{15}, x \in [25,40] \text{ UMF} \\ 0.8\frac{x-20}{5}, x \in [20,25] \\ 0.8\frac{30-x}{5}, x \in [25,30] LMF \end{cases} \qquad \overline{\mu}_B(x) = \begin{cases} \frac{x-2}{4}, x \in [2,6] \\ \frac{10-x}{4}, x \in [6,10] \text{ UMF} \\ 0.7\frac{x-4}{2}, x \in [4,6] \\ 0.7\frac{8-x}{2}, x \in [6,8] \text{ LMF} \end{cases}$$

Then α-Cut of the fuzzy members A and B are

$$A^\alpha = \left\{[15\alpha + 10, 40 - 15\alpha], \alpha \in [0,1]; \left[\tfrac{5}{0.8}\alpha + 20, 30 - \tfrac{5}{0.8}\alpha\right], \alpha \in [0.0.8]\right\}$$

and

$$B^\alpha = \left\{[4\alpha + 2, 10 - 4\alpha], \alpha \in [0,1]; \left[\tfrac{2}{0.7}\alpha + 4, 8 - \tfrac{2}{0.7}\alpha\right], \alpha \in [0.0.7]\right\},$$

respectively.

Now, to add fuzzy numbers A and B,

$$A^\alpha + B^\alpha = \begin{cases} [19\alpha + 12, 50 - 19\alpha], \alpha \in [0, min(1,1)] \\ \left[\left(15 + \frac{2}{0.7}\right)\alpha + 14, 48 - \left(15 + \frac{2}{0.7}\alpha\right)\right], \alpha \in [0, min(0.7,1)] \\ \left[\left(\frac{5}{0.8} + 4\right)\alpha + 22, 40 - \left(\frac{5}{0.8} + 4\right)\alpha\right], \alpha \in [0, min(0.8,1)] \\ \left[\left(\frac{5}{0.8} + \frac{2}{0.7}\right)\alpha + 24, 38 - \left(\frac{5}{0.8} + \frac{2}{0.7}\right)\alpha\right], \alpha \in [0, min(0.7, 0.8)]. \end{cases}$$

Then the four membership functions are $\mu^1_{A+B}(x) = \begin{cases} \frac{x-12}{19}, x \in [12,31] \\ \frac{50-x}{19}, x \in [31,50], \end{cases}$

$$\mu^2_{A+B}(x) = \begin{cases} \frac{0.7(x-14)}{12.5}, x \in [14, 26.5] \\ \frac{0.7(48-x)}{12.5}, x \in [35.5, 48], \end{cases}$$

$$\mu^3_{A+B}(x) = \begin{cases} \frac{0.8(x-22)}{8.2}, x \in [22, 30.2] \\ \frac{0.8(40-x)}{8.2}, x \in [31.8, 40], \end{cases}$$

$$\mu^4_{A+B}(x) = \begin{cases} \frac{0.7(x-24)}{6.375}, x \in [24, 30.375] \\ \frac{0.7(38-x)}{6.375}, x \in [31.625, 38]. \end{cases}$$

Thus the average width of the four membership functions are 24.7, 21.5, 10.825 and 7.625, respectively. Here, maximum average width is 24.7 and minimum average width is 7.625 and hence membership function of the resulting IVFS is

$$\mu_{A+B}(x) = \begin{cases} \frac{x-12}{19}, x \in [12,31] \\ \frac{50-x}{19}, x \in [31,50] UMF \\ 0.7\frac{x-24}{6.375}, x \in [24, 30.375] \\ 0.7\frac{38-x}{6.375}, x \in [31.625, 38] LMF. \end{cases}$$

Similarly, membership functions for subtraction, multiplication and division can be obtained and which are respectively.

$$\mu_{A-B}(x)=\begin{cases}\frac{x}{19}, x\in[0,19]\\ \frac{38-x}{19}, x\in[19,38]UMF\\ 0.7\frac{x-12}{6.375}, x\in[12,18.375]\\ 0.7\frac{26-x}{6.575}, x\in[19.625,26]LMF,\end{cases}$$

$$\mu_{A\times B}(x)=\begin{cases}\frac{-35+\sqrt{25+60x}}{60}, x\in[20,150]UMF\\ \frac{155-\sqrt{25+60x}}{60}, x\in[150,400]\\ 0.7\frac{-46+\sqrt{324+22.4x}}{14}, x\in[80,146.25]LMF\\ 0.7\frac{76-\sqrt{400+22.4x}}{14}, x\in[153.75,240],\end{cases}$$

$$\mu_{A/B}(x)=\begin{cases}\frac{10x-10}{4x+15}, x\in[1,4.166]\\ \frac{40-2x}{4x+15}, x\in[4.166,20]UMF\\ 0.7\frac{8x-20}{2x+4.375}, x\in[2.5,4.0625]\\ 0.7\frac{30-x}{2x+4.375}, x\in[4.2708,7.5]LMF.\end{cases}$$

6. Conclusions

In general, real world problems are ill defined due to lack of data, imprecision, vagueness, partial information etc. More often, type-I fuzzy set is explored to deal with such real word problems. However, in some situations, generalized IVFNs may come into picture. In this paper, a new approach has been devised to combine generalized IVFNs based on average width of fuzzy set concept. In this study, it is found that the shape of resultant fuzzy number is generalized IVFN where shape of UMF is triangular type fuzzy set while LMF is trapezoidal type fuzzy number. This approach can also combine completely generalized IVFNs (i.e., IVFNs whose height of UMF and LMF are strictly less than 1). In this case, the resultant IVFN is completely generalized IVFNs in which shape of both the UMF and LMF are trapezoidal type fuzzy sets. Numerical illustrations have been exhibited to check the validity of the proposed approach.

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