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Dual hesitant fuzzy subrings and ideals

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Abstract. In this paper, the notion of dual hesitant fuzzy subring and dual hesitant fuzzy ideal is introduced. For that purpose various new operations on dual hesitant fuzzy set such as score based union, score based intersection etc. are introduced. Moreover, some score based properties of dual hesitant fuzzy subring and ideal are obtained. Furthermore, an extension principle for dual hesitant fuzzy set is defined. Finally,the image and pre-image of dual hesitant fuzzy subring under a homomorphism are discussed.

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1. Introduction

Dual Hesitant Fuzzy set(DHFS), proposed by Bin Zhu et al. [17] in 2012, is an emerging research area due to its capability of representing opposite epistemic degrees together with hesitancy. DHFS, consists of the membership hesitancy function and the nonmembership hesitancy function, comprehends fuzzy sets, hesitant fuzzy sets, intuitionistic fuzzy sets and fuzzy multisets as special cases. Recently, research work on DHFS are at a rapid pace, and many fruitful results have been obtained in decision making.

In 2013, Wang et al.[14] defined the concept of correlation and correlation coefficient for DHFSs. They also proposed a clustering algorithm and applied it in a real example. In 2014, Zhu and Xu [18] introduced a principle to normalize a dual hesitant fuzzy element, which will be useful for handling problems involving decision-making factor. Pushpinder Singh [8] introduced the axiom definition of similarity measures between two DHFS and formulated a mathematical model of dual hesitant fuzzy assignment problem. In [6], a ranking of all alternatives in a group decision making can be determined through the weighted correlation coefficient between each alternative and the ideal alternative. Yu et al. [4] and Wang et al. [13] proposed Geometric Aggregation Operators on dual hesitant fuzzy environment and applied it to a real decision-making problem. In 2015, Z.Su et al. [11] proposes a no.of distance and similarity measures for DHFSs. They have shown that distance and similarity measures for DHFS are a better tool for pattern recognition by considering a building material recognition problem and a disease diagnostic problem. Based on a correctional score function and dice similarity function of dual hesitant fuzzy set, Ren et al. [9] solved a multi-attribute decision-making problem in a dual hesitant fuzzy environment in which the attributes are at different priority levels.

An extensive study of algebraic structures of the fuzzy set and the intuitionistic fuzzy set have been found in the literature. By introducing fuzzy subgroup, Rosenfield [10] starts this journey. Liu et al. [7] proposed fuzzy subrings and ideals. Later, Hur et al.[5] and Banerjee et al. [1] applied intuitionistic fuzzy set to ring structures. As a generalization of fuzzy set, it seems to be meaningful to apply dual hesitant fuzzy set to ring structures. This paper is an attempt to define the notion of dual hesitant fuzzy subring and dual hesitant fuzzy ideal.

The remainder of the paper is organized as follows: The second section begins with a discussion of some preliminaries of the hesitant fuzzy set and then moves onto the definition of the dual hesitant fuzzy set. Some score based operations of the dual hesitant fuzzy set such as score based dual hesitant fuzzy subset, score based dual hesitant fuzzy equal, score based dual hesitant fuzzy union and score based dual hesitant fuzzy intersection are introduced in section 3. The fourth section presents dual hesitant fuzzy subring and discusses score based intersection of two dual hesitant fuzzy subrings. Moreover, with the help of a counter example shows that the score based union of two hesitant fuzzy subrings need not be a hesitant fuzzy subring. In section 5, the concept of dual hesitant fuzzy ideal is introduced and discussed some algebraic properties. Finally, the homomorphic image and pre-image of dual hesitant fuzzy subring is examined.

2. Preliminaries

In this section, we review some notions of hesitant fuzzy set and dual hesitant fuzzy set.

Definition 2.1 ([12])**.** Let X be a reference set, a hesitant fuzzy set(HFS) E on X is defined in terms of a function h that when applied to X returns a subset of $[0, 1]$.

To be easily understood, Xu and Xia [15] expressed an HFS by the following mathematical form:

$$\mathrm{E} = \{< x, h(x) > / x \in \mathrm{X}\},$$

where $h(x)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees of the element $x \in X$ to the set E. For convenience, Xu and Xia[15] called $h(x)$ a hesitant fuzzy element(HFE).

Definition 2.2 ([16])**.** For an HFE h, status of h is defined as $S(h) = \dfrac{\sum_{\gamma \in h} \gamma}{\ell(h)}$, where $\ell(h)$ is the number of elements in h.

Definition 2.3 ([3]). Let h_1 and h_2 be two hesitant fuzzy sets on X.

(i) h_1 is hesitantly equal to h_2 (denoted by $h_1 \approx h_2$), if $S(h_1(x)) = S(h_2(x)), \forall x \in X$.

(ii) h_1 is a hesitant subset of h_2 (denoted by $h_1 \preceq h_2$), if $S(h_1(x)) \leq S(h_2(x)), \forall x \in X$.

Definition 2.4 ([2]). Let h_1 and h_2 be two hesitant fuzzy sets on X.Then a score based union of h_1 and h_2 (denoted by $h_1 \widetilde{\curlyvee} h_2$) is defined as

$$h_1 \widetilde{\curlyvee}\, h_2(x) = \begin{cases} h_1(x) & \text{if } \ h_1(x) \succ h_2(x) \\ h_2(x) & \text{if } \ h_2(x) \succ h_1(x) \\ h_1(x) \cup h_2(x) & \text{if } \ h_1(x) \approx h_2(x) \end{cases} \quad , \forall x \in X$$

and a score based intersection of h_1 and h_2(denoted by $h_1 \widetilde{\curlywedge} h_2$) is defined as

$$h_1 \widetilde{\curlywedge}\, h_2(x) = \begin{cases} h_1(x) & \text{if } \ h_1(x) \prec h_2(x) \\ h_2(x) & \text{if } \ h_2(x) \prec h_1(x) \\ h_1(x) \cup h_2(x) & \text{if } \ h_1(x) \approx h_2(x) \end{cases} \quad , \forall x \in X.$$

Definition 2.5 ([17]). Let X be a fixed set, then a dual hesitant fuzzy set (DHFS) d on X is described as:

$$d = \{< x, h(x), g(x) >, x \in X\},$$

in which $h(x)$ and $g(x)$ are two sets of some values in $[0, 1]$ denoting the possible membership degrees and nonmembership degrees of the element $x \in X$ to the set d, respectively with the conditions:

$$0 \leq \gamma, \eta \leq 1, \quad 0 \leq \gamma^+ + \eta^+ \leq 1,$$

where $\gamma \in h(x), \eta \in g(x), \gamma^+ \in h^+(x) = \cup_{\gamma \in h(x)} max\{\gamma\}$ and $\eta^+ \in g^+(x) = \cup_{\eta \in g(x)} max\{\eta\}$, for all $x \in X$. We write $d = (h, g)$ for short.

Denote by DHFS(U), the set of all Dual Hesitant fuzzy sets over U.

Remark 2.6 ([17]). For convenience, the pair $d(x) = (h(x), g(x))$ is called a dual hesitant fuzzy element(DHFE).

3. Score based operations - DHFS

In this section, some definitions for dual hesitant fuzzy set based on the score are introduced.

Let $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$ be two dual hesitant fuzzy sets on X.

Definition 3.1. We say that d_1 is a dual hesitant fuzzy subset of d_2 (denoted by $d_1 \lesssim d_2$), if $S(h_1(x)) \leq S(h_2(x)), \forall x \in X$ (denoted by $h_1 \preceq h_2$) and $S(g_1(x)) \geq S(g_2(x)), \forall x \in X$ (denoted by $g_1 \succeq g_2$).

Definition 3.2. We say that d_1 is dual hesitantly equal to d_2 (denoted by $d_1 \sim d_2$), if $S(h_1(x)) = S(h_2(x)), \forall x \in X$ (denoted by $h_1 \approx h_2$) and $S(g_1(x)) = S(g_2(x)), \forall x \in X$ (denoted by $g_1 \approx g_2$).

Definition 3.3. The score based union of d_1 and d_2, denoted by $d_1 \widetilde{\vee} d_2$, is defined to be the dual hesitant fuzzy set

$$d_1 \widetilde{\vee} d_2 = (h_1 \ \widetilde{\curlyvee}\ h_2, g_1 \ \widetilde{\curlywedge}\ g_2),$$

where

$$h_1 \widetilde{\curlyvee}\ h_2(x) = \begin{cases} h_1(x) & \text{if } \ h_1(x) \succ h_2(x) \\ h_2(x) & \text{if } \ h_2(x) \succ h_1(x) \\ h_1(x) \cup h_2(x) & \text{if } \ h_1(x) \approx h_2(x) \end{cases} \quad , \forall x \in X$$

$$g_1 \widetilde{\curlywedge} g_2(x) = \begin{cases} g_1(x) & \text{if } \ g_1(x) \prec g_2(x) \\ g_2(x) & \text{if } \ g_2(x) \prec g_1(x) \\ g_1(x) \cup g_2(x) & \text{if } \ g_1(x) \approx g_2(x) \end{cases} \quad , \forall x \in X.$$

Definition 3.4. The score based intersection of d_1 and d_2, denoted by $d_1 \widetilde{\wedge}\ d_2$, is defined to be the dual hesitant fuzzy set

$$d_1 \widetilde{\wedge}\ d_2 = (h_1 \widetilde{\curlywedge}\ h_2, g_1 \widetilde{\curlyvee} g_2),$$

where

$$h_1 \widetilde{\curlywedge}\ h_2(x) = \begin{cases} h_1(x) & \text{if } \ h_1(x) \prec h_2(x) \\ h_2(x) & \text{if } \ h_2(x) \prec h_1(x) \\ h_1(x) \cup h_2(x) & \text{if } \ h_1(x) \approx h_2(x) \end{cases} \quad , \forall x \in X$$

$$g_1 \widetilde{\curlyvee} g_2(x) = \begin{cases} g_1(x) & \text{if } \ g_1(x) \succ g_2(x) \\ g_2(x) & \text{if } \ g_2(x) \succ g_1(x) \\ g_1(x) \cup g_2(x) & \text{if } \ g_1(x) \approx g_2(x) \end{cases} \quad , \forall x \in X.$$

Remark 3.5. Let $d = (h, g)$ be a dual hesitant fuzzy set on X and $x \in X$. If $d(x) = \big(h(x), g(x)\big)$, where $h(x) = \{0\}$ and $g(x) = \{1\}$, then we write $d(x) = \varnothing$ for short.This notation will be used only in section6.

4. Dual hesitant fuzzy subring

In this section, dual hesitant fuzzy subring is defined and discussed its algebraic properties.

Definition 4.1. Let R be a ring. A dual hesitant fuzzy set d on R is called a Dual hesitant fuzzy subring of R if it satisfies the following conditions: $\forall x, y \in R$,

(i) $d(x + y) \gtrsim d(x) \widetilde{\wedge}\ d(y)$,
(ii) $d(-x) \quad \gtrsim d(x)$,
(iii) $d(xy) \quad \gtrsim d(x) \widetilde{\wedge}\ d(y)$.

Denote by DHFR(R), the set of all Dual Hesitant fuzzy subrings of R.

Lemma 4.2. *Let d be a dual hesitant fuzzy set on R. Then $d \in DHFR(R)$ iff*
(1) $d(x - y) \gtrsim d(x) \widetilde{\wedge}\ d(y)$,
(2) $d(xy) \quad \gtrsim d(x) \widetilde{\wedge} d(y)$.

Proof. Proof is straightforward. □

Example 4.3. Let $R = \mathbb{P}(X)$, Power set of X, where $X = \{a, b\}$.
Then R together with the operations

$$X + Y = (X \cup Y) - (X \cap Y)$$
$$X.Y = X \cap Y \qquad\qquad \forall X, Y \in R$$

is a ring with additive identity ϕ, and additive inverse of X is X itself for every X$\in R$. Here $R = \{\phi, \{a\}, \{b\}, \{a, b\}\}$.

Let $d = (h, g) \in DHFS(R)$ defined by

$$h(x) = \begin{cases} \{.7, .75\} & , if \quad x = \phi \\ \{.1, .2, .3\} & , if \quad x = \{a\} \ or \ \{a, b\} \\ \{.5, .55, .6\} & , if \quad x = \{b\} \end{cases}$$

$$g(x) = \begin{cases} \{.15, .2\} & , if \quad x = \phi \\ \{.6, .7\} & , if \quad x = \{a\} \ or \ \{a, b\} \\ \{.35, .4\} & , if \quad x = \{b\}. \end{cases}$$

We can easily verify that $d \in DHFR(R)$.

Proposition 4.4. *Let $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$ be two dual hesitant fuzzy subrings of a ring R, then $d_1 \mathbin{\widetilde{\curlywedge}} d_2$ is a dual hesitant Fuzzy subring of R.*

Proof. For any x,y∈R, it is enough to prove the following:

(1) $h_1 \mathbin{\widetilde{\curlywedge}} h_2\ (x - y) \succeq [h_1 \mathbin{\widetilde{\curlywedge}} h_2\ (x)] \mathbin{\widetilde{\curlywedge}} [h_1 \mathbin{\widetilde{\curlywedge}} h_2\ (y)]$,

(2) $h_1 \mathbin{\widetilde{\curlywedge}} h_2\ (xy) \succeq [h_1 \mathbin{\widetilde{\curlywedge}} h_2\ (x)] \mathbin{\widetilde{\curlywedge}} [h_1 \mathbin{\widetilde{\curlywedge}} h_2\ (y)]$,

(3) $g_1 \mathbin{\widetilde{\curlyvee}} g_2\ (x - y) \preceq [g_1 \mathbin{\widetilde{\curlyvee}} g_2\ (x)] \mathbin{\widetilde{\curlyvee}} [g_1 \mathbin{\widetilde{\curlyvee}} g_2\ (y)]$,

(4) $g_1 \mathbin{\widetilde{\curlyvee}} g_2\ (xy) \preceq [g_1 \mathbin{\widetilde{\curlyvee}} g_2\ (x)] \mathbin{\widetilde{\curlyvee}} [g_1 \mathbin{\widetilde{\curlyvee}} g_2\ (y)]$.

We shall only prove (1).
We need to consider the following three cases.

Case(i): Suppose $h_1(x - y) \prec h_2(x - y)$. Then

$$\begin{aligned} h_1 \mathbin{\widetilde{\curlywedge}} h_2\ (x - y) &= h_1\ (x - y) \\ &\succeq h_1\ (x) \mathbin{\widetilde{\curlywedge}} h_1\ (y) \\ &\succeq \left[h_1 \mathbin{\widetilde{\curlywedge}} h_2\ (x)\right] \mathbin{\widetilde{\curlywedge}} \left[h_1 \mathbin{\widetilde{\curlywedge}} h_2\ (y)\right]. \end{aligned}$$

Case(ii): Suppose $h_1(x - y) \succ h_2(x - y)$. Then

$$\begin{aligned} h_1 \mathbin{\widetilde{\curlywedge}} h_2\ (x - y) &= h_2\ (x - y) \\ &\succeq h_2\ (x) \mathbin{\widetilde{\curlywedge}} h_2\ (y) \\ &\succeq \left[h_1 \mathbin{\widetilde{\curlywedge}} h_2\ (x)\right] \mathbin{\widetilde{\curlywedge}} \left[h_1 \mathbin{\widetilde{\curlywedge}} h_2\ (y)\right]. \end{aligned}$$

Case(iii): Suppose $h_1(x-y) \approx h_2(x-y)$. Then

$$\begin{aligned} h_1 \mathbin{\widetilde{\curlywedge}} h_2\ (x-y) &= h_1\ (x-y)\ \cup\ h_2\ (x-y) \\ &\approx h_1(x-y) \\ &\succeq\ \big[h_1\ (x) \mathbin{\widetilde{\curlywedge}} h_1\ (y)\big] \\ &\succeq\ \big[h_1 \mathbin{\widetilde{\curlywedge}} h_2\ (x)\big] \mathbin{\widetilde{\curlywedge}} \big[h_1 \mathbin{\widetilde{\curlywedge}} h_2\ (y)\big]. \end{aligned}$$

In a similar manner, we prove (2),(3) and (4) and this completes the proof. □

Remark 4.5. The score based union of two dual hesitant fuzzy subrings does not need to be a dual hesitant fuzzy subring. This is illustrated in the following example.

Example 4.6. Let $(\mathbb{Z}, +, .)$ be the ring of integers. Consider $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2) \in DHFS(\mathbb{Z})$ defined by

$$h_1(x) = \begin{cases} \{.1, .2, .4\} & \text{if} \quad \text{x is odd} \\ \{.6, .7\} & \text{if} \quad \text{x is even} \end{cases}$$

$$g_1(x) = \begin{cases} \{.4, .5, .6\} & \text{if} \quad \text{x is odd} \\ \{.1, .2\} & \text{if} \quad \text{x is even} \end{cases}$$

$$h_2(x) = \begin{cases} \{.6, .8\} & \text{if} \quad x = 7n, n \in \mathbb{Z} \\ \{.1, .2\} & \text{otherwise} \end{cases}$$

$$g_2(x) = \begin{cases} \{.1, .2\} & \text{if} \quad x = 7n, n \in \mathbb{Z} \\ \{.5, .7\} & \text{otherwise.} \end{cases}$$

We can easily verify that $d_1, d_2 \in DHFR(\mathbb{Z})$.

Take $x = 7$ and $y = 2$. Then we have

$$h_1 \mathbin{\widetilde{\curlyvee}} h_2\ (7-2) = h_1 \mathbin{\widetilde{\curlyvee}} h_2\ (5) = \{.1, .2, .4\}.$$

But

$$[h_1 \mathbin{\widetilde{\curlyvee}} h_2\ (7)] \mathbin{\widetilde{\curlywedge}} [h_1 \mathbin{\widetilde{\curlyvee}} h_2\ (2)] = \{.6, .7\}.$$

Thus

$$h_1 \mathbin{\widetilde{\curlyvee}} h_2\ (7-2)\ \prec\ [h_1 \mathbin{\widetilde{\curlyvee}} h_2\ (7)] \mathbin{\widetilde{\curlywedge}} [h_1 \mathbin{\widetilde{\curlyvee}} h_2\ (2)].$$

So, the first condition of dual hesitant fuzzy subring is violated. Hence, $d_1 \mathbin{\widetilde{\vee}} d_2 \notin DHFR(\mathbb{Z})$.

5. Dual hesitant fuzzy ideal

In this section, the notion of Dual Hesitant Fuzzy Ideal is introduced and discussed its algebraic properties.

Definition 5.1. Let R be a ring and let $d = (h, g) \in DHFR(R)$. Then d is called

(i) a dual hesitant fuzzy left ideal if $h(ax) \succeq h(x)$ and $g(ax) \preceq g(x)$, for every a,x $\in R$,

(ii) a dual hesitant fuzzy right ideal if $h(xa) \succeq h(x)$ and $g(xa) \preceq g(x)$, for every a,x $\in R$,

(iii) a dual hesitant fuzzy ideal if it is both dual hesitant fuzzy left ideal and dual hesitant fuzzy right ideal.

Denote by DHFI(R), the set of all Dual Hesitant Fuzzy Ideals of R.

Lemma 5.2. *Let d be a dual hesitant fuzzy subring of R. Then $d \in DHFI(R)$ iff $d(xy) \gtrsim d(x) \,\widetilde{\vee}\, d(y)$.*

Proof. Proof is left to the reader. □

Proposition 5.3. *Let $d_1, d_2 \in DHFI(R)$. Then $d_1 \,\widetilde{\wedge}\, d_2 \in DHFI(R)$.*

Proof. Clearly, by Proposition 4.4, $d_1 \,\widetilde{\wedge}\, d_2 \in DHFR(R)$.
It is enough to prove the following:

(1) $h_1 \,\widetilde{\curlywedge}\, h_2\ (xy) \succeq [h_1 \,\widetilde{\curlywedge}\, h_2\ (x)] \,\widetilde{\curlyvee}\, [h_1 \,\widetilde{\curlywedge}\, h_2\ (y)]$.

(2) $g_1 \,\widetilde{\curlyvee}\, g_2\ (xy) \preceq [g_1 \,\widetilde{\curlyvee}\, g_2\ (x)] \,\widetilde{\curlywedge}\, [g_1 \,\widetilde{\curlyvee}\, g_2\ (y)]$.

We shall only prove (1).
We need to consider the following three cases.

Case(i): Suppose $h_1(xy) \prec h_2(xy)$. Then

$$\begin{aligned} h_1 \,\widetilde{\curlywedge}\, h_2\ (xy) &= h_1\ (xy) \\ &\succeq h_1\ (x) \,\widetilde{\curlyvee}\, h_1\ (y) \\ &\succeq \left[h_1 \,\widetilde{\curlywedge}\, h_2\ (x)\right] \widetilde{\curlyvee} \left[h_1 \,\widetilde{\curlywedge}\, h_2\ (y)\right]. \end{aligned}$$

Case(ii): Suppose$h_1(xy) \succ h_2(xy)$. Then

$$\begin{aligned} h_1 \,\widetilde{\curlywedge}\, h_2\ (xy) &= h_2\ (xy) \\ &\succeq h_2\ (x) \,\widetilde{\curlyvee}\, h_2\ (y) \\ &\succeq \left[h_1 \,\widetilde{\curlywedge}\, h_2\ (x)\right] \widetilde{\curlyvee} \left[h_1 \,\widetilde{\curlywedge}\, h_2\ (y)\right]. \end{aligned}$$

Case(iii): Suppose $h_1(xy) \approx h_2(xy)$. Then

$$\begin{aligned} h_1 \,\widetilde{\curlywedge}\, h_2\ (xy) &= h_1\ (xy) \cup h_2\ (xy) \\ &\approx h_1(xy) \\ &\succeq \left[h_1\ (x) \,\widetilde{\curlyvee}\, h_1\ (y)\right] \\ &\succeq \left[h_1 \,\widetilde{\curlywedge}\, h_2\ (x)\right] \widetilde{\curlyvee} \left[h_1 \,\widetilde{\curlywedge}\, h_2\ (y)\right]. \end{aligned}$$

In a similar manner, we prove (2) and this completes the proof. □

Remark 5.4. The score based union of two dual hesitant fuzzy ideal does not need to be a dual hesitant fuzzy ideal. This is illustrated in the following example.

Example 5.5. Consider the ring $(\mathbb{Z}_{10}, \underset{10}{+}, \underset{10}{\times})$. Let $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2) \in DHFS(\mathbb{Z}_{10})$ defined by

$$h_1(x) = \begin{cases} \{.6, .7\} & \text{if} \quad x \in \{0, 2, 4, 6, 8\} \\ \{.1, .2\} & \text{otherwise} \end{cases}$$

$$g_1(x) = \begin{cases} \{.2\} & \text{if} \quad x \in \{0, 2, 4, 6, 8\} \\ \{.5, .6\} & \text{otherwise} \end{cases}$$

$$h_2(x) = \begin{cases} \{.4, .5, .6\} & \text{if} \quad x \in \{0, 5\} \\ \{.2, .3\} & \text{otherwise} \end{cases}$$

$$g_2(x) = \begin{cases} \{.4\} & \text{if} \quad x \in \{0, 5\} \\ \{.5, .6\} & \text{otherwise.} \end{cases}$$

We can easily verify that $d_1, d_2 \in DHFI(\mathbb{Z}_{10})$.

Take $x = 5$ and $y = 4$. Then we have

$$h_1 \,\widetilde{\Upsilon}\, h_2 \,(5-4) = h_1 \,\widetilde{\Upsilon}\, h_2 \,(1) = \{.2, .3\}.$$

But

$$[h_1 \,\widetilde{\Upsilon}\, h_2 \,(5)] \,\widetilde{\Upsilon}\, [h_1 \,\widetilde{\Upsilon}\, h_2 \,(4)] = \{.4, .5, .6\}.$$

Clearly, $h_1 \,\widetilde{\Upsilon}\, h_2 \,(5-4) \prec [h_1 \,\widetilde{\Upsilon}\, h_2 \,(5)] \,\widetilde{\Upsilon}\, [h_1 \,\widetilde{\Upsilon}\, h_2 \,(4)]$. Thus, the first condition of dual hesitant fuzzy ideal is violated. So, $d_1 \,\widetilde{\vee}\, d_2 \notin DHFI(\mathbb{Z}_{10})$

6. Homomorphism of dual hesitant fuzzy subring and ideal

In this section, the image and pre-image of a dual hesitant fuzzy set under a mapping is defined . After that, the image and pre-image of a dual hesitant fuzzy subring under a homomorphism are discussed. Moreover, the image and pre-image of a dual hesitant fuzzy ideal under a homomorphism are examined.

Definition 6.1. Let f be a function from X into Y and let $d_1 = (h_1, g_1) \in DHFS(X)$. Then the image of d_1 under f is the Dual Hesitant fuzzy set $f(d_1) = (f(h_1), f(g_1)) \in DHFS(Y)$ defined by, $\forall y \in Y$,

$$f(h_1)(y) = \begin{cases} \underset{\substack{f(x)=y \\ x \in X}}{\widetilde{\Upsilon}} h_1(x) & if \quad f^{-1}(y) \neq \phi \\ \{0\} & if \quad f^{-1}(y) = \phi \end{cases}$$

and

$$f(g_1)(y) = \begin{cases} \underset{\substack{f(x)=y \\ x\in X}}{\widetilde{\curlywedge}} g_1(x) & if \quad f^{-1}(y) \neq \phi \\ \\ \{1\} & if \quad f^{-1}(y) = \phi. \end{cases}$$

In other words,

$$f(d_1)(y) = \begin{cases} \underset{\substack{f(x)=y \\ x\in X}}{\widetilde{\curlyvee}} d_1(x) & if \quad f^{-1}(y) \neq \phi \\ \\ \varnothing & if \quad f^{-1}(y) = \phi. \end{cases}$$

Definition 6.2. Let f be a function from X into Y and let $d_2 = (h_2, g_2) \in DHFS(Y)$. Then the pre-image of d_2 under f is the dual hesitant fuzzy set $f^{-1}(d_2) = (f^{-1}(h_2), f^{-1}(g_2)) \in DHFS(X)$ defined by, $\forall x \in X$,

$$f^{-1}(h_2)(x) = h_2(f(x)),$$

$$f^{-1}(g_2)(x) = g_2(f(x)).$$

In other words,

$$f^{-1}(d_2)(x) = d_2(f(x)).$$

Theorem 6.3. *Let d be a dual hesitant fuzzy subring of a ring R and f is a homomorphism from R to a ring S. Then $f(d)$ is a dual hesitant fuzzy subring of S.*

Proof. Let $y_1, y_2 \in S$.

If $f^{-1}(y_1) = \phi$ or $f^{-1}(y_2) = \phi$, then we have $f(d)(y_1) \ \widetilde{\wedge} \ f(d)(y_2) = \varnothing$. Thus

$$f(d)(y_1 - y_2) \gtrsim f(d)(y_1) \ \widetilde{\wedge} \ f(d)(y_2)$$

and

$$f(d)(y_1.y_2) \gtrsim f(d)(y_1) \ \widetilde{\wedge} \ f(d)(y_2).$$

So $f(d)$ is a dual hesitant fuzzy subring of a ring S.

If $f^{-1}(y_1) \neq \phi$ and $f^{-1}(y_2) \neq \phi$, then $f^{-1}(y_1 - y_2) \neq \phi$ and $f^{-1}(y_1.y_2) \neq \phi$. Let us assume that there exist $x_1, x_2 \in R$ such that $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. Then

$$\begin{aligned} f(d)(y_1 - y_2) &= \underset{f(x)=y_1-y_2}{\widetilde{\vee}} d(x) \\ &\gtrsim d(x_1 - x_2) \\ &\gtrsim d(x_1) \ \widetilde{\wedge} \ d(x_2). \end{aligned}$$

Because this inequality holds true for each $x_1, x_2 \in R$ satisfying $f(x_1) = y_1$ and $f(x_2) = y_2$, we have

$$f(d)(y_1 - y_2) \gtrsim \left[\underset{f(x)=y_1}{\widetilde{\vee}} d(x) \right] \widetilde{\wedge} \left[\underset{f(x)=y_2}{\widetilde{\vee}} d(x) \right]$$

$$= f(d)(y_1) \ \widetilde{\wedge} \ f(d)(y_2).$$

Similarly, we can prove that

$$f(d)(y_1.y_2) \gtrsim f(d)(y_1) \ \widetilde{\wedge} \ f(d)(y_2).$$

Thus $f(d)$ is a dual hesitant fuzzy subring of a ring S. □

Theorem 6.4. *Let d be a dual hesitant fuzzy subring of a ring S and f is a homomorphism from a ring R to S. Then $f^{-1}(d)$ is a dual hesitant fuzzy subring of R.*

Proof. We have $f^{-1}(d)(x) = d(f(x)) \quad \forall x \in R$. Then for every $x_1, x_2 \in R$,

$$\begin{aligned} f^{-1}(d)\ (x_1 - x_2) &= d(f(x_1 - x_2)) \\ &= d\,[f(x_1) - f(x_2)] \\ &\gtrsim d(f(x_1)) \,\widetilde{\wedge}\, d(f(x_2)) \\ &= f^{-1}(d)(x_1) \,\widetilde{\wedge}\, f^{-1}(d)(x_2). \end{aligned}$$

Similarly, we can prove that

$$f^{-1}(d)\ (x_1.x_2) \gtrsim f^{-1}(d)(x_1) \,\widetilde{\wedge}\, f^{-1}(d)(x_2).$$

Thus $f^{-1}(d)$ is a dual hesitant fuzzy subring of a ring R. □

Theorem 6.5. *Let d be a dual hesitant fuzzy ideal of a ring R and f is a surjective homomorphism from R to a ring S. Then $f(d)$ is a dual hesitant fuzzy ideal of S.*

Proof. Let $y_1, y_2 \in S$. Let us assume that there exist x_1,$x_2 \in R$ such that $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. Then

$$\begin{aligned} f(d)(y_1.y_2) &= \widetilde{\bigvee_{f(x)=y_1.y_2}} \quad d(x) \\ &\gtrsim d(x_1.x_2) \\ &\gtrsim d(x_1) \quad \widetilde{\vee} \quad d(x_2). \end{aligned}$$

Because this inequality holds true for each $x_1, x_2 \in R$ satisfying $f(x_1) = y_1$ and $f(x_2) = y_2$, we have

$$\begin{aligned} f(d)(y_1.y_2) &\gtrsim \left[\widetilde{\bigvee_{f(x)=y_1}} \quad d(x)\right] \widetilde{\vee} \left[\widetilde{\bigvee_{f(x)=y_2}} \quad d(x)\right] \\ &= f(d)(y_1) \ \widetilde{\vee} \ f(d)(y_2). \end{aligned}$$

Similarly, we can prove that

$$f(d)(y_1 - y_2) \gtrsim f(d)(y_1) \ \widetilde{\wedge} \ f(d)(y_2).$$

Thus, $f(d)$ is a dual hesitant fuzzy ideal of S. □

Theorem 6.6. *Let d be a dual hesitant fuzzy ideal of a ring S and f is a homomorphism from a ring R to S. Then $f^{-1}(d)$ is a dual hesitant fuzzy ideal of R.*

Proof. We have $f^{-1}(d)(x) = d(f(x)), \quad \forall x \in R$. Then for every $x_1, x_2 \in R$,

$$\begin{aligned} f^{-1}(d)\ (x_1.x_2) &= d(f(x_1.x_2)) \\ &= d\,[f(x_1).f(x_2)] \\ &\gtrsim d(f(x_1))\ \widetilde{\vee}\ d(f(x_2)) \\ &= f^{-1}(d)(x_1)\ \widetilde{\vee}\ f^{-1}(d)(x_2). \end{aligned}$$

Similarly, we can prove that

$$f^{-1}(d)\ (x_1 - x_2)\ \gtrsim\ f^{-1}(d)(x_1)\ \widetilde{\wedge}\ f^{-1}(d)(x_2).$$

Thus, $f^{-1}(d)$ is a dual hesitant fuzzy ideal of R. □

7. Conclusions

The study of dual hesitant fuzzy subring and ideal will set a theoretical base for further development of the algebraic structures of dual hesitant fuzzy set. Dual hesitant fuzzy subrings can be further developed in the case of dual hesitant fuzzy soft set to widen its scope.

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