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Transitive and strongly transitive intuitionistic fuzzy matrices

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Abstract. In this paper, some general properties of transitive intuitionistic fuzzy matrices are studied. Here it is shown that, any intuitionistic fuzzy matrix can be represent as a sum of a nilpotent matrix and a symmetric matrix. The definition of strongly transitive intuitionistic fuzzy matrix is given. Finally, the canonical form of both the transitive and strongly transitive intuitonistic fuzzy matrices are given.

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1. Introduction

After the invention of fuzzy sets [28] and intuitionistic fuzzy sets [1], the theory of intuitionistic fuzzy matrix (IFM) is a very useful tool in the discussion of intuitionistic fuzzy relations. The intuitionistic fuzzy relations are represented in terms of IFM. Transitive relations are important in various applications and have many interesting properties. The transitive intuitionistic fuzzy matrices (TIFMs) correspond to the transitive intuitionistic fuzzy relations.

Properties of asymptotic forms of fuzzy matrices play an important role in the performance of fuzzy system modeling. Asymptotic forms of fuzzy matrices investigated in the literature include two types: the limiting behavior of consecutive powers of one fuzzy matrix [3] and the infinite products of a finite number of fuzzy matrices [4, 5, 6, 7]. Guu et al. [8, 10] proposed the notions of weak convergence and the strong convergence for the infinite products of a finite number of fuzzy matrices. Lur et al. [9] studied on nilpotent fuzzy matrices. Hashimoto [11, 12, 13] discussed different properties of transitive fuzzy matrices. Then many authors [14, 15, 24] studied on the properties of fuzzy matrices. First time Pal [18, 19] introduced intuitionistic fuzzy determinant and IFMs. Latter Pal et al. [2, 25, 26, 27] introduced the distance between IFMs. They also studied soft matrices [16] and incline IFMs [17]. Then Pradhan and Pal [20, 21] studied on the convergency of IFMs with respect to the different operations. They [22, 23] also studied on the intuitionistic fuzzy linear transformation and on the g-inverse of block IFM.

In this paper, some properties of TIFMs and strongly transitive intuitionistic fuzzy matrices (STIFMs) are examined. The canonical form of TIFMs and STIFMs are also presented.

2. Preliminaries

In this section, we define some operations for IFMs whose elements are intuitionistic fuzzy numbers (IFNs). For two intuitionistic fuzzy numbers x and y, we define $x \vee y$, $x \wedge y$ and $x \ominus y$ as follows:

$$x \vee y = \langle \max(x_\mu, y_\mu), \min(x_\nu, y_\nu)\rangle,$$
$$x \wedge y = \langle \min(x_\mu, y_\mu), \max(x_\nu, y_\nu)\rangle,$$
$$x \ominus y = \begin{cases} x & if \quad x > y \\ \langle 0,1\rangle & if \quad x \leq y \end{cases}.$$

Fuzzy set was not enough to study the hesitation about the membership degree of an element in a set. For dealing this situation Atanassov [1] introduced intuitionistic fuzzy sets (IFSs), which is defined below.

Definition 2.1 ([1])**.** An intuitionistic fuzzy set (IFS) A is defined as an object of the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\},$$

where the function $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degree of membership and degree of non-membership of an element $x \in X$ respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for every $x \in X$.

The value of $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the degree of non-determinacy (or hesitation) of the element $x \in X$ to the IFS A.

Let us define $\langle F\rangle = \{\langle x, y\rangle : x, y \in [0,1] \text{ and } 0 \leq x + y \leq 1\}$.

Definition 2.2. The subtraction '$\star$' is an arbitrary fixed binary operation on $\langle F\rangle$ satisfying $x \star y \leq x$ and $x \star \langle 0,1\rangle = x$ for all $x, y \in \langle F\rangle$.

In some particular situations, the operations '$\star$' and '$\ominus$' are same and that depend on the way we define the operation '$\star$'.

Let us defined the binary operation '$\star$' as,

$$x \star y = \langle \min(x_\mu, |x_\mu - y_\mu|), \max(x_\nu, |x_\nu - y_\nu|)\rangle.$$

Then,

(Case 1): If $\min(x_\mu, |x_\mu - y_\mu|) = x_\mu$ and $\max(x_\nu, |x_\nu - y_\nu|) = x_\nu$, then,

$$x \star y = \langle x_\mu, x_\nu\rangle = x.$$

(Case 2): If $\min(x_\mu, |x_\mu - y_\mu|) = x_\mu$ and $\max(x_\nu, |x_\nu - y_\nu|) = |x_\nu - y_\nu|$, then,

$$x \star y = \langle x_\mu, |x_\nu - y_\nu|\rangle \leq x.$$

(Case 3): If $\min(x_\mu, |x_\mu - y_\mu|) = |x_\mu - y_\mu|$ and $\max(x_\nu, |x_\nu - y_\nu|) = x_\nu$, then,

$$x \star y = \langle |x_\mu - y_\mu|, x_\nu \rangle \leq x.$$

(Case 4): If $\min(x_\mu, |x_\mu - y_\mu|) = |x_\mu - y_\mu|$ and $\max(x_\nu, |x_\nu - y_\nu|) = |x_\nu - y_\nu|$, then,

$$x \star y = \langle |x_\mu - y_\mu|, |x_\nu - y_\nu| \rangle < x.$$

Here, for any x, if $y = \langle y_\mu, y_\nu \rangle$ be such that, $y_\mu = 0$ and $x_\nu \geq |x_\nu - y_\nu|$, then the operation '$\star$' be same with '$\ominus$'.

Example 2.3. Let $x = \langle 0.2, 0.3 \rangle$ and $y = \langle 0.5, 0.4 \rangle$. Then, $\min(x_\mu, |x_\mu - y_\mu|) = 0.2$ and $\max(x_\nu, |x_\nu - y_\nu|) = 0.3$. Thus, $x \star y = \langle 0.2, 0.3 \rangle = x$.

Let $x = \langle 0.2, 0.1 \rangle$ and $y = \langle 0.5, 0.5 \rangle$. Then, $\min(x_\mu, |x_\mu - y_\mu|) = 0.2$ and $\max(x_\nu, |x_\nu - y_\nu|) = 0.4$. Thus, $x \star y = \langle 0.2, 0.4 \rangle \leq x$.

Let $x = \langle 0.3, 0.3 \rangle$ and $y = \langle 0.2, 0.5 \rangle$. Then, $\min(x_\mu, |x_\mu - y_\mu|) = 0.1$ and $\max(x_\nu, |x_\nu - y_\nu|) = 0.3$. Thus, $x \star y = \langle 0.1, 0.3 \rangle \leq x$.

Let $x = \langle 0.3, 0.1 \rangle$ and $y = \langle 0.1, 0.5 \rangle$. Then, $\min(x_\mu, |x_\mu - y_\mu|) = 0.2$ and $\max(x_\nu, |x_\nu - y_\nu|) = 0.4$. Thus, $x \star y = \langle 0.2, 0.4 \rangle < x$.

Definition 2.4 ([2])**.** An intuitionistic fuzzy matrix (IFM) A of order $m \times n$ is defined as $A = [x_{ij}, \langle a_{ij\mu}, a_{ij\nu} \rangle]_{m \times n}$ where $a_{ij\mu}$, $a_{ij\nu}$ are called membership and non-membership values of x_{ij} in A, which maintains the condition $0 \leq a_{ij\mu} + a_{ij\nu} \leq 1$. For simplicity, we write $A = [a_{ij}]_{m \times n}$, where $a_{ij} = \langle a_{ij\mu}, a_{ij\nu} \rangle$.

In arithmetic operations, only the values of $a_{ij\mu}$ and $a_{ij\nu}$ are considered so from here we only consider the values of $a_{ij} = \langle a_{ij\mu}, a_{ij\nu} \rangle$. All elements of an IFM are the members of $\langle F \rangle$.

Definition 2.5 ([2])**.** An IFM is null if all elements of it are zero, that is, all elements are $\langle 0, 0 \rangle$ and it is denoted by $\mathbf{N}_{\langle 0,0 \rangle}$. An IFM is zero if all elements are $\langle 0, 1 \rangle$ and it is denoted by $\mathbf{O}$.

Definition 2.6 ([2])**.** An intuitionistic fuzzy identity matrix of order $n \times n$ is denoted by $\mathbf{I}_n$ and is defined by $\langle \delta_{ij\mu}, \delta_{ij\nu} \rangle$, where $\delta_{ij\mu} = \begin{cases} 1, \delta_{ij\nu} = 0 & if \quad i = j \\ 0, \delta_{ij\nu} = 1 & if \quad i \neq j \end{cases}$.
The universal IFM of order $m \times n$ is denoted by $\mathbf{J}$ and all the elements of it are $\langle 1, 0 \rangle$.

Let $M_{m \times n}$ be the set of all IFMs of order $m \times n$ and M_n that of order $n \times n$. Comparison between two IFMs is essential in our work, which is defined below.

Definition 2.7 ([2])**.** Let $A, B \in M_{m \times n}$ such that $A = (\langle a_{ij\mu}, a_{ij\nu} \rangle)$ and $B = (\langle b_{ij\mu}, b_{ij\nu} \rangle)$. Then we write $A \leq B$, if $a_{ij\mu} \leq b_{ij\mu}$ and $a_{ij\nu} \geq b_{ij\nu}$ for all i, j and we say that A is dominated by B or B dominates A. A and B are said to be comparable, if either $A \leq B$ or $B \leq A$.

Definition 2.8 ([2])**.** A square IFM is called intuitionistic fuzzy permutation matrix (IFPM), if every row and column contains exactly one element whose value is $\langle 1, 0 \rangle$ and all other entries are $\langle 0, 1 \rangle$.

Now we define the following matrix operations for the IFMs $A = [a_{ij}] = [\langle a_{ij\mu}, a_{ij\nu} \rangle]$ and $B = [b_{ij}] = [\langle b_{ij\mu}, b_{ij\nu} \rangle]$, where $A, B \in M_n$.

$A \vee B = [a_{ij} \vee b_{ij}] = [\langle \max(a_{ij\mu}, b_{ij\mu}), \min(a_{ij\nu}, a_{ij\nu})\rangle]$.
$A \wedge B = [a_{ij} \wedge b_{ij}] = [\langle \min(a_{ij\mu}, b_{ij\mu}), \max(a_{ij\nu}, a_{ij\nu})\rangle]$.
$A \star B = [\langle a_{ij} \star b_{ij}\rangle]$.
$A \ominus B = [\langle a_{ij} \ominus b_{ij}\rangle]$.
$A \times B = [\bigvee_{k=1}^{n} (a_{ik} \wedge b_{kj})]$.
$A^{k+1} = A^k \times A; (k = 1, 2, 3, ...)$.
$A' = [\langle a_{ji\mu}, a_{ji\nu}\rangle]$ (the transpose of A).
$\triangle A = A \ominus A'$.
$\nabla A = A \wedge A'$.

An IFM $A \in M_n$ is said to be transitive, if $A^2 \leq A$ and it is said to be idempotent, if $A^2 = A$. Accordingly, any idempotent IFM is transitive.

An IFM A is nilpotent, if $A^k = \mathbf{O}$ for some $k \in N$ (set of natural numbers).

An IFM is irreflexive, if $A \wedge \mathbf{I}_n = \mathbf{O}$, where $\mathbf{O}$ and $\mathbf{I}_n$ are the zero IFM and identity IFM respectively. As is well known, a nilpotent IFM must be irreflexive and an irreflexive transitive IFM is indeed nilpotent.

An IFM A is symmetric, if $A' = A$ and it said to be antisymmetric, if $A \wedge A' \leq \mathbf{I}_n$.

3. Some results on IFMs

In this section, we examine some basic properties of transitive and nilpotent IFMs.

Lemma 3.1. *Let B be a symmetric IFM. Then $\triangle(A \vee B) \leq \triangle A$ for any IFM A.*

Proof. Let

$$\begin{aligned} C &= [\langle c_{ij\mu}, c_{ij\nu}\rangle] \\ &= \triangle(A \vee B) \\ &= [(a_{ij} \vee b_{ij}) \ominus (a_{ji} \vee b_{ji})] \\ &= [(a_{ij} \vee b_{ij}) \ominus (a_{ji} \vee b_{ij})] \text{ (as } B \text{ is symmetric)} \\ &= [\langle \max(a_{ij\mu}, b_{ij\mu}), \min(a_{ij\nu}, b_{ij\nu})\rangle \ominus \langle \max(a_{ji\mu}, b_{ij\mu}), \min(a_{ji\nu}, b_{ij\nu})\rangle]. \end{aligned}$$

Suppose $\langle \max(a_{ij\mu}, b_{ij\mu}), \min(a_{ij\nu}, b_{ij\nu})\rangle > \langle \max(a_{ji\mu}, b_{ij\mu}), \min(a_{ji\nu}, b_{ij\nu})\rangle$. Then,

$$\begin{aligned} \langle c_{ij\mu}, c_{ij\nu}\rangle &= \langle \max(a_{ij\mu}, b_{ij\mu}), \min(a_{ij\nu}, b_{ij\nu})\rangle \\ &= \langle a_{ij\mu}, a_{ij\nu}\rangle \\ &= \langle a_{ij\mu}, a_{ij\nu}\rangle \ominus \langle a_{ji\mu}, a_{ji\nu}\rangle. \end{aligned}$$

On the other hand, suppose

$$\langle \max(a_{ij\mu}, b_{ij\mu}), \min(a_{ij\nu}, b_{ij\nu})\rangle \leq \langle \max(a_{ji\mu}, b_{ij\mu}), \min(a_{ji\nu}, b_{ij\nu})\rangle.$$

Then,

$$\langle c_{ij\mu}, c_{ij\nu}\rangle = \langle 0, 1\rangle \leq \langle a_{ij\mu}, a_{ij\nu}\rangle \ominus \langle a_{ji\mu}, a_{ji\nu}\rangle.$$

Thus we have, $C \leq \triangle A$. So, $\triangle(A \vee B) \leq \triangle A$. □

Example 3.2. Let us consider the IFMs, $A = \begin{bmatrix} \langle 0.8, 0.1\rangle & \langle 0.6, 0.2\rangle \\ \langle 0.7, 0.1\rangle & \langle 0.5, 0.3\rangle \end{bmatrix}$ and $B = \begin{bmatrix} \langle 0.8, 0.1\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.7, 0.2\rangle \end{bmatrix}$. Here B is symmetric.

The transpose of A is $A' = \begin{bmatrix} \langle 0.8, 0.1\rangle & \langle 0.7, 0.1\rangle \\ \langle 0.6, 0.2\rangle & \langle 0.5, 0.3\rangle \end{bmatrix}$.

Then, $\triangle A = A \ominus A' = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.7, 0.1\rangle & \langle 0, 1\rangle \end{bmatrix}$.

Now, $A \vee B = \begin{bmatrix} \langle 0.8, 0.1\rangle & \langle 0.6, 0.2\rangle \\ \langle 0.7, 0.1\rangle & \langle 0.7, 0.2\rangle \end{bmatrix}$ and $(A \vee B)' = \begin{bmatrix} \langle 0.8, 0.1\rangle & \langle 0.7, 0.1\rangle \\ \langle 0.6, 0.2\rangle & \langle 0.7, 0.2\rangle \end{bmatrix}$.

Again, $\triangle(A \vee B) = (A \vee B) \ominus (A \vee B)' = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.7, 0.1\rangle & \langle 0, 1\rangle \end{bmatrix} = \triangle A$.

Lemma 3.3. *For any IFM A, $A = (\triangle A) \vee (\nabla A)$.*

Proof. Let $C = [\langle c_{ij\mu}, c_{ij\nu}\rangle] = (\triangle A) \vee (\nabla A)$, that is,

$$\langle c_{ij\mu}, c_{ij\nu}\rangle = (\langle a_{ij\mu}, a_{ij\nu}\rangle \ominus \langle a_{ji\mu}, a_{ji\nu}\rangle) \vee (\langle a_{ij\mu}, a_{ij\nu}\rangle \wedge \langle a_{ji\mu}, a_{ji\nu}\rangle).$$

If $\langle a_{ij\mu}, a_{ij\nu}\rangle > \langle a_{ji\mu}, a_{ji\nu}\rangle$, then,

$$\langle c_{ij\mu}, c_{ij\nu}\rangle = \langle a_{ij\mu}, a_{ij\nu}\rangle \vee \langle a_{ji\mu}, a_{ji\nu}\rangle = \langle a_{ij\mu}, a_{ij\nu}\rangle.$$

If $\langle a_{ij\mu}, a_{ij\nu}\rangle \leq \langle a_{ji\mu}, a_{ji\nu}\rangle$, then,

$$\langle c_{ij\mu}, c_{ij\nu}\rangle = \langle 0, 1\rangle \vee \langle a_{ij\mu}, a_{ij\nu}\rangle = \langle a_{ij\mu}, a_{ij\nu}\rangle.$$

Thus, $C = (\triangle A) \vee (\nabla A) = A$. □

Example 3.4. Let us consider the IFM $A = \begin{bmatrix} \langle 0.9, 0.1\rangle & \langle 0.6, 0.2\rangle \\ \langle 0.7, 0.2\rangle & \langle 0.8, 0.1\rangle \end{bmatrix}$.

Then $A' = \begin{bmatrix} \langle 0.9, 0.1\rangle & \langle 0.7, 0.2\rangle \\ \langle 0.6, 0.2\rangle & \langle 0.8, 0.1\rangle \end{bmatrix}$.

Now, $\triangle A = A \ominus A' = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.7, 0.2\rangle & \langle 0, 1\rangle \end{bmatrix}$

and

$$\nabla A = A \wedge A' = \begin{bmatrix} \langle 0.9, 0.1\rangle & \langle 0.6, 0.2\rangle \\ \langle 0.6, 0.2\rangle & \langle 0.8, 0.1\rangle \end{bmatrix}.$$

Thus, $\triangle A \vee \nabla A = \begin{bmatrix} \langle 0.9, 0.1\rangle & \langle 0.6, 0.2\rangle \\ \langle 0.7, 0.2\rangle & \langle 0.8, 0.1\rangle \end{bmatrix} = A$.

Remark 3.5. For any IFM A, it is easily seen that, $\triangle(\triangle A) = \triangle A$.

The above remark can be verified by the following example.

Let us consider the IFM $A = \begin{bmatrix} \langle 0.7, 0.2\rangle & \langle 0, 1\rangle & \langle 0.5, 0.4\rangle \\ \langle 0, 1\rangle & \langle 0.4, 0.5\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.8, 0.1\rangle & \langle 1, 0\rangle \end{bmatrix}$.

Then, $\triangle A = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0.8, 0.1\rangle & \langle 0, 1\rangle \end{bmatrix}$

and

$$(\triangle A)' = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.8, 0.1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \end{bmatrix}.$$

Thus, $\triangle(\triangle A) = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0.8, 0.1\rangle & \langle 0, 1\rangle \end{bmatrix} = \triangle A$.

Proposition 3.6. *If for any IFM A, there exists an IFM B ($A \leq B$) such that, $(A \star B')^2 \leq A \vee A' \vee \boldsymbol{I}_n$; $A \wedge B' \leq \boldsymbol{I}_n$ and $(A \star B')^3 \wedge \boldsymbol{I}_n = \boldsymbol{O}$ holds, then A is antisymmetric and transitive.*

Proof. The condition $A \leq B' \leq \mathbf{I}_n$ and $A \leq B$ imply that, $A \wedge A' \leq A \wedge B' \leq \mathbf{I}_n$. Then, from the definition of antisymmetric IFM, A is antisymmetric.

Let $C = [\langle c_{ij\mu}, c_{ij\nu}\rangle] = A \star B'$. Then,

$$\langle c_{ij\mu}, c_{ij\nu}\rangle = \langle a_{ij\mu}, a_{ij\nu}\rangle \star \langle b_{ji\mu}, b_{ji\nu}\rangle \leq \langle a_{ij\mu}, a_{ij\nu}\rangle.$$

If $\langle a_{ij\mu}, a_{ij\nu}\rangle = \langle 0, 1\rangle$, then $\langle c_{ij\mu}, c_{ij\nu}\rangle = \langle a_{ij\mu}, a_{ij\nu}\rangle$.

If $\langle a_{ij\mu}, a_{ij\nu}\rangle \neq \langle 0, 1\rangle$, then $\langle b_{ij\mu}, b_{ij\nu}\rangle = \langle 0, 1\rangle$.

Thus, $\langle c_{ij\mu}, c_{ij\nu}\rangle = \langle a_{ij\mu}, a_{ij\nu}\rangle \star \langle 0, 1\rangle = \langle a_{ij\mu}, a_{ij\nu}\rangle$.

So, $\langle c_{ij\mu}, c_{ij\nu}\rangle = \langle a_{ij\mu}, a_{ij\nu}\rangle$, for all $i \neq j$.

Now to prove $A^2 \leq A$, in view of

$$\langle a_{ii\mu}, a_{ii\nu}\rangle \wedge \langle a_{ij\mu}, a_{ij\nu}\rangle \leq \langle a_{ij\mu}, a_{ij\nu}\rangle$$

and

$$\langle a_{ij\mu}, a_{ij\nu}\rangle \wedge \langle a_{jj\mu}, a_{jj\nu}\rangle \leq \langle a_{ij\mu}, a_{ij\nu}\rangle,$$

it is sufficient to check $\langle a_{ik\mu}, a_{ik\nu}\rangle \wedge \langle a_{kj\mu}, a_{kj\nu}\rangle \leq \langle a_{ij\mu}, a_{ij\nu}\rangle$ for all $i \neq k \neq j$.

Here $\langle a_{ik\mu}, a_{ik\nu}\rangle = \langle c_{ik\mu}, c_{ik\nu}\rangle$ and $\langle a_{kj\mu}, a_{kj\nu}\rangle = \langle c_{kj\mu}, c_{kj\nu}\rangle$ and thus we check,

$$\langle c_{ik\mu}, c_{ik\nu}\rangle \wedge \langle c_{kj\mu}, c_{kj\nu}\rangle \leq \langle a_{ij\mu}, a_{ij\nu}\rangle.$$

Here, $\langle a_{ik\mu}, a_{ik\nu}\rangle = \langle c_{ik\mu}, c_{ik\nu}\rangle$ and $\langle a_{kj\mu}, a_{kj\nu}\rangle = \langle c_{kj\mu}, c_{kj\nu}\rangle$ and thus we check,

$$\langle c_{ik\mu}, c_{ik\nu}\rangle \wedge \langle c_{kj\mu}, c_{kj\nu}\rangle \leq \langle a_{ij\mu}, a_{ij\nu}\rangle.$$

(Case 1): If $\langle a_{ij\mu}, a_{ij\nu}\rangle \neq \langle 0, 1\rangle$, then $\langle a_{ji\mu}, a_{ji\nu}\rangle = \langle 0, 1\rangle$. From $C^2 \leq A \vee A' \vee \mathbf{I}_n$, we get,

$$\langle c_{ik\mu}, c_{ik\nu}\rangle \wedge \langle c_{kj\mu}, c_{kj\nu}\rangle \leq \langle a_{ij\mu}, a_{ij\nu}\rangle \vee \langle a_{ji\mu}, a_{ji\nu}\rangle \leq \langle a_{ij\mu}, a_{ij\nu}\rangle.$$

(Case 2): Let $\langle a_{ij\mu}, a_{ij\nu}\rangle = \langle 0, 1\rangle$.

If $\langle a_{ij\mu}, a_{ij\nu}\rangle = \langle 0, 1\rangle$, then from $C^2 \leq A \vee A' \vee \mathbf{I}_n$, we get,

$$\langle c_{ik\mu}, c_{ik\nu}\rangle \wedge \langle c_{kj\mu}, c_{kj\nu}\rangle = \langle 0, 1\rangle.$$

If $\langle a_{ji\mu}, a_{ji\nu}\rangle \neq \langle 0, 1\rangle$, then from $C^3 \wedge \mathbf{I}_n = \mathbf{O}$, we get,

$$\langle c_{ik\mu}, c_{ik\nu}\rangle \wedge \langle c_{kj\mu}, c_{kj\nu}\rangle \wedge \langle c_{ji\mu}, c_{ji\nu}\rangle = \langle 0, 1\rangle.$$

That imply, $\langle c_{ik\mu}, c_{ik\nu}\rangle \wedge \langle c_{kj\mu}, c_{kj\nu}\rangle = \langle 0, 1\rangle$, for $i \neq j$.

Again since A is antisymmetric, we have, $a_{ik} \wedge a_{ki} = \langle 0, 1\rangle \leq a_{ii}$, for all $k \neq i$ and $a_{ii} \wedge a_{ii} = a_{ii}$, which concludes the proof. □

Lemma 3.7. *If $A \wedge B' \leq \boldsymbol{I}_n$, then $(A \star B') \vee (A \wedge \boldsymbol{I}_n) = A$ for any IFMs A and B.*

Proof. Let $C = A \star B'$, i.e., $[\langle c_{ij\mu}, c_{ij\nu}\rangle] = [\langle a_{ij\mu}, a_{ij\nu}\rangle \star \langle b_{ji\mu}, b_{ji\nu}\rangle] \leq [\langle a_{ij\mu}, a_{ij\nu}\rangle]$. Again, $A \wedge B' \leq \mathbf{I}_n$, i.e., $[\langle a_{ij\mu}, a_{ij\nu}\rangle] \wedge [\langle b_{ji\mu}, b_{ji\nu}\rangle] \leq \mathbf{I}_n$, which imply, $A \star B' = A$. Then, $(A \star B') \vee (A \wedge \mathbf{I}_n) = A \vee (A \wedge \mathbf{I}_n) = A$. □

Example 3.8. Let us consider the IFMs $A = \begin{bmatrix} \langle 0.3, 0.5\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle \end{bmatrix}$ and $B = \begin{bmatrix} \langle 0,1\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.4, 0.5\rangle & \langle 0,1\rangle \end{bmatrix}$. Also consider the binary operation '$\star$' as

$$x \star y = \langle \min(x_\mu, |x_\mu - y_\mu|), \max(x_\nu, |x_\nu - y_\nu|)\rangle.$$

Then, $A \star B' = \begin{bmatrix} \langle 0.3, 0.5\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle \end{bmatrix} = A$. Thus,

$$\begin{aligned} &(A \star B') \vee (A \wedge \mathbf{I}_n) \\ &= A \vee (A \wedge \mathbf{I}_n) \\ &= \begin{bmatrix} \langle 0.3, 0.5\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle \end{bmatrix} \vee \left(\begin{bmatrix} \langle 0.3, 0.5\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle \end{bmatrix} \wedge \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 1,0\rangle \end{bmatrix} \right) \\ &= A. \end{aligned}$$

Proposition 3.9. *Let A and B be two IFMs such that, $A \leq B$; $(A \star B')^2 \leq A \vee A' \vee \boldsymbol{I}_n$; $A \wedge B' \leq \boldsymbol{I}_n$ and $(A \star B')^3 = \boldsymbol{O}$, then $A^2 \leq (A \star B') \vee (A \wedge \boldsymbol{I}_n)$.*

Proof. From Proposition 3.6, A is a transitive IFM, that is, $A^2 \leq A$. Again, from Lemma 3.7, $A = (A \star B') \vee (A \wedge \mathbf{I}_n)$. Then, $A^2 \leq A$, that is, $A^2 \leq (A \star B') \vee (A \wedge \mathbf{I}_n)$. □

Lemma 3.10. *If the IFM A be nilpotent, then $\triangle A = A$ and $\nabla A = \boldsymbol{O}$.*

Proof. Since the IFM A is nilpotent, i.e., $A^n = \mathbf{O}$ for some $n \in N$, A must be an irreflexive IFM, i.e., $A \wedge \mathbf{I}_n = \mathbf{O}$. Then, A^2 must be irreflexive IFM [12]. Thus,

$$\langle a_{ij\mu}, a_{ij\nu}\rangle \wedge \langle a_{ji\mu}, a_{ji\nu}\rangle = \langle 0, 1\rangle. \tag{3.10.1}$$

Again, $\triangle A = [\langle a_{ij\mu}, a_{ij\nu}\rangle] \ominus [\langle a_{ji\mu}, a_{ji\nu}\rangle]$.
Now, if $\langle a_{ij\mu}, a_{ij\nu}\rangle \neq \langle 0, 1\rangle$, then $\langle a_{ji\mu}, a_{ji\nu}\rangle = \langle 0, 1\rangle$ and thus $\triangle A = [\langle a_{ij\mu}, a_{ij\nu}\rangle]$.
If $\langle a_{ij\mu}, a_{ij\nu}\rangle = \langle 0, 1\rangle$, then from the definition of $x \ominus y$,

$$\langle a_{ij\mu}, a_{ij\nu}\rangle \ominus \langle a_{ji\mu}, a_{ji\nu}\rangle = \langle 0, 1\rangle.$$

So, $\triangle A = [\langle a_{ij\mu}, a_{ij\nu}\rangle]$. Hence, $\triangle A = A$.
Again, $\nabla A = [\langle a_{ij\mu}, a_{ij\nu}\rangle] \wedge [\langle a_{ji\mu}, a_{ji\nu}\rangle] = \mathbf{O}$, by (3.10.1). □

Lemma 3.11. *Let A be a nilpotent IFM and B be a symmetric IFM. If $C = A \vee B$, then $\triangle C \leq A$ and $\nabla C = B$.*

Proof. Here, $\triangle C = \triangle(A \vee B) \leq \triangle A = A$ [By Lemma 3.1 and Lemma 3.10]. Then,

$$\begin{aligned} \nabla C &= \nabla(A \vee B) \\ &= (A \vee B) \wedge (A' \vee B) (\text{as } B = B') \\ &= (A \wedge A') \vee (A \wedge B) \vee (B \wedge A') \vee B \\ &= \mathbf{O} \vee (A \wedge B) \vee (B \wedge A') \vee B \\ &= A \wedge (B \vee B) \wedge A' \vee B \\ &= (A \wedge B \wedge A') \vee B \\ &= (A \wedge A' \wedge B) \vee B \\ &= \mathbf{O} \vee B \\ &= B. \end{aligned}$$
□

Proposition 3.12. *If an IFM A be transitive and irreflexive, then it is nilpotent.*

Proof. Since the IFM A is irreflexive, $A \wedge \mathbf{I}_n = \mathbf{O}$, i.e., $a_{ii} = \langle 0, 1\rangle$, for all $i \in N$.

Again, by the hypothesis, A is transitive, i.e., $A^2 \leq A$. Then it follows that, $A^k \geq A^{k+1}$ (for some $k \in N$). Thus, $A^n \geq A^{n+1}$ and $a_{ii} \geq a_{ii}^s$ (for some $s \in N$). Since $a_{ii} = \langle 0, 1\rangle$, we can get $a_{ii} = a_{ii}.\ldots.a_{ii} \leq a_{ii}^s$ (for some $s \in N$). So, $a_{ii} = a_{ii}^s = \langle 0, 1\rangle$, for all $i \in N$. Hence, $A^k = (a_{ij}^k) = \langle 0, 1\rangle$, for all $i \in N$ and $k \in N$. Therefore, A is nilpotent. □

4. Canonical form of transitive IFM

In this section, we will discuss the canonical form of an transitive IFM. Let A be an $(n \times n)$ transitive IFM. If there exists an $(n \times n)$ IFPM P such that, $F = PAP^T = (f_{ij})$ satisfying $f_{ij} \geq f_{ji}$, for all $i > j$, then F is called a canonical form of A.

We define the operation '$\times'$ for two intuitionistic fuzzy number x and y as,

$$x \times y = \langle min(x_\mu, y_\mu), max(x_\nu, y_\nu)\rangle.$$

Theorem 4.1. *Let A be a nilpotent IFM and B be a symmetric IFM. For an IFM C given by $C = A \vee B$ there exists an IFPM P such that $D = [\langle d_{ij\mu}, d_{ij\nu}\rangle] = P \times C \times P'$ satisfies $\langle d_{ij\mu}, d_{ij\nu}\rangle \geq \langle d_{ji\mu}, d_{ji\nu}\rangle$ for all $i > j$.*

Proof. Here,

$$\begin{aligned} D &= P \times C \times P' \\ &= P \times (A \vee B) \times P' \\ &= (P \times A \times P') \vee (P \times B \times P') \end{aligned}$$

Since A is nilpotent IFM, $(P \times A \times P')$ becomes strictly lower triangular for some IFPM P [12]. Again, as B is symmetric IFM , $(P \times B \times P')$ is symmetric IFM [12]. Then, the IFM D satisfies $\langle d_{ij\mu}, d_{ij\nu}\rangle \geq \langle d_{ji\mu}, d_{ji\nu}\rangle$ for all $i > j$ by choosing such an IFPM P. □

Lemma 4.2. *If A be a transitive IFM, then $\triangle A$ is a nilpotent IFM and ∇A is a symmetric IFM.*

Proof. Since A is transitive , $A^2 \leq A$, i.e., $\max_k(a_{ik} \wedge a_{kj}) \leq a_{ij}$. From definition, $\triangle A = [\triangle a_{ij}]$, where $\triangle a_{ij} = a_{ij} \ominus a_{ji} \leq a_{ij}$. Now, $(\triangle A)^2 = [(\triangle a_{ij})^2]$, where

$$\begin{aligned} (\triangle a_{ij})^2 &= \triangle a_{ij} \times \triangle a_{ij} \\ &= (a_{ij} \ominus a_{ji}) \times (a_{ij} \ominus a_{ji}) \\ &= (a_{ij} \ominus a_{ji}) \\ &= (\triangle a_{ij}) \text{ for } i \neq j \end{aligned}$$

Then, $(\triangle A)^2 = \triangle A$, i.e., $\triangle A$ is an idempotent IFM. Since idempotent IFM is transitive, $\triangle A$ is transitive.

Again, for $i = j$, $\triangle A = [\triangle a_{ii}]$, where $\triangle a_{ii} = \langle a_{ii} \ominus a_{ii}\rangle = \langle 0, 1\rangle$. Thus the diagonal elements of $\triangle A$ are $\langle 0, 1\rangle$. So, $\triangle A \wedge \mathbf{I} = \mathbf{O}$, i.e., $\triangle A$ is irreflexive. Hence, $\triangle A$ is transitive and irreflexive and thus $\triangle A$ is nilpotent IFM (by Proposition 3.12).

From the definition, $\nabla A = A \wedge A'$. Then, $(\nabla A)' = (A \wedge A')' = A' \wedge A = \nabla A$. Thus, ∇A is a symmetric IFM. □

Example 4.3. Let us consider the IFM $A = \begin{bmatrix} \langle 0.2, 0.4\rangle & \langle 0, 1\rangle & \langle 0.2, 0.4\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.2, 0.3\rangle & \langle 0.5, 0.2\rangle \\ \langle 0.4, 0.4\rangle & \langle 0, 1\rangle & \langle 0.3, 0.4\rangle \end{bmatrix}$.

Then, $A^2 = \begin{bmatrix} \langle 0.2, 0.4\rangle & \langle 0, 1\rangle & \langle 0.2, 0.4\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.2, 0.3\rangle & \langle 0.3, 0.3\rangle \\ \langle 0.3, 0.4\rangle & \langle 0, 1\rangle & \langle 0.3, 0.4\rangle \end{bmatrix}$ and $A^2 \leq A$ holds.

Thus the IFM A is transitive.

Now, $\triangle A = A \ominus A' = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.6, 0.3\rangle & \langle 0, 1\rangle & \langle 0.5, 0.2\rangle \\ \langle 0.4, 0.4\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \end{bmatrix}$,

$$(\triangle A)^2 = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.4, 0.4\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \end{bmatrix} \leq (\triangle A),$$

$$(\triangle A)^3 = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \end{bmatrix} = \mathbf{O}.$$

So, $(\triangle A)$ is nilpotent of index 3.

Again, $\nabla A = A \wedge A' = \begin{bmatrix} \langle 0.2, 0.4\rangle & \langle 0, 1\rangle & \langle 0.2, 0.4\rangle \\ \langle 0, 1\rangle & \langle 0.2, 0.3\rangle & \langle 0, 1\rangle \\ \langle 0.2, 0.4\rangle & \langle 0, 1\rangle & \langle 0.3, 0.4\rangle \end{bmatrix}$ is a symmetric IFM.

Theorem 4.4. *For a transitive IFM A, there exists an IFPM P, such that,* $D = [\langle d_{ij\mu}, d_{ij\nu}\rangle] = P \times A \times P'$ *satisfies,* $d_{ij} \geq d_{ji}$ *for all* $i > j$.

Proof. Since A is a transitive IFM, A can be expressed as, $A = (\triangle A) \vee (\nabla A)$ (By Lemma 3.3). Again, $\triangle A$ is nilpotent and ∇A is a symmetric IFM (by Lemma 4.2). Hence by Theorem 4.1,

$$\begin{aligned} D = [\langle d_{ij\mu} d_{ij\nu}\rangle] &= (P \times \triangle A \times P') \vee (P \times \nabla A \times P') \\ &= P \times (\triangle A \vee \nabla A) \times P' \\ &= P \times A \times P' \end{aligned}$$

satisfies, $d_{ij} \geq d_{ji}$ for all $i > j$. □

Example 4.5. Let us consider the IFM $A = \begin{bmatrix} \langle 0.6, 0.2\rangle & \langle 0, 1\rangle & \langle 0.5, 0.2\rangle \\ \langle 0.5, 0.3\rangle & \langle 0.4, 0.4\rangle & \langle 0.5, 0.3\rangle \\ \langle 0.7, 0.1\rangle & \langle 0, 1\rangle & \langle 0.6, 0.2\rangle \end{bmatrix}$.

Then $A^2 = \begin{bmatrix} \langle 0.6, 0.2\rangle & \langle 0, 1\rangle & \langle 0.5, 0.2\rangle \\ \langle 0.5, 0.3\rangle & \langle 0.4, 0.4\rangle & \langle 0.5, 0.3\rangle \\ \langle 0.6, 0.2\rangle & \langle 0, 1\rangle & \langle 0.6, 0.2\rangle \end{bmatrix}$ and the transpose

$$A' = \begin{bmatrix} \langle 0.6, 0.2\rangle & \langle 0.5, 0.3\rangle & \langle 0.7, 0.1\rangle \\ \langle 0, 1\rangle & \langle 0.4, 0.4\rangle & \langle 0, 1\rangle \\ \langle 0.5, 0.2\rangle & \langle 0.5, 0.3\rangle & \langle 0.6, 0.2\rangle \end{bmatrix}.$$

As $A^2 \leq A$, A is transitive.

Now, $\triangle A = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.5, 0.3\rangle & \langle 0, 1\rangle & \langle 0.5, 0.3\rangle \\ \langle 0.7, 0.1\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \end{bmatrix}$

and

$$(\triangle A)^3 = \begin{bmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{bmatrix} = \mathbf{O}.$$

Then, $(\triangle A)$ is nilpotent of index 3.

Also, $\nabla A = \begin{bmatrix} \langle 0.6,0.2\rangle & \langle 0,1\rangle & \langle 0.5,0.2\rangle \\ \langle 0,1\rangle & \langle 0.4,0.4\rangle & \langle 0,1\rangle \\ \langle 0.5,0.2\rangle & \langle 0,1\rangle & \langle 0.6,0.2\rangle \end{bmatrix}$ is a symmetric IFM.

Now, for the IFPM $P = \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,0\rangle \\ \langle 0,1\rangle & \langle 1,0\rangle & \langle 0,1\rangle \end{bmatrix}$,

$$P(\triangle A)P' = \begin{bmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0.7,0.1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0.5,0.3\rangle & \langle 0.5,0.3\rangle & \langle 0,1\rangle \end{bmatrix}$$

and

$$P(\nabla A)P' = \begin{bmatrix} \langle 0.6,0.2\rangle & \langle 0.5,0.2\rangle & \langle 0,1\rangle \\ \langle 0.5,0.2\rangle & \langle 0.6,0.2\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0.4,0.4\rangle \end{bmatrix}.$$

Thus, $D = P \times A \times P' = \begin{bmatrix} \langle 0.6,0.2\rangle & \langle 0.5,0.2\rangle & \langle 0,1\rangle \\ \langle 0.7,0.1\rangle & \langle 0.6,0.2\rangle & \langle 0,1\rangle \\ \langle 0.5,0.3\rangle & \langle 0.5,0.3\rangle & \langle 0.4,0.4\rangle \end{bmatrix}$, which satisfies the condition $d_{ij} \geq d_{ji}$ for all $i > j$.

5. Properties of strongly transitive IFMs

In this section, we discuss about some general properties of strongly transitive IFMs.

Definition 5.1. An IFM A is strongly transitive (s-transitive) if and only if for any indices $i, j, k \in \{1, 2, 3, ..., n\}$ with $i \neq j \neq k$, such that, $a_{ik} > a_{ki}$ and $a_{kj} > a_{jk}$ imply $a_{ij} > a_{ji}$.

If an IFM A be symmetric, the conditions $a_{ik} > a_{ki}$ and $a_{kj} > a_{jk}$ are false for any $i, j, k \in \{1, 2, 3, ..., n\}$ and hence A is also s-transitive. In particular, ∇A is s-transitive for any IFM A.

The above definition of s-transitive is not suitable for practical use. Now, we give a new characterization of s-transitivity of IFMs which seems to be more appropriate for application. In this purpose, we define the relation $\prec$ in the set of IFMs in the following way:
$B \prec A$ if and only if $\langle a_{ij\mu}, a_{ij\nu}\rangle = \langle 0,1\rangle$ imply $\langle b_{ij\mu}, b_{ij\nu}\rangle = \langle 0,1\rangle$ for all $i, j \in \{1, 2, 3, ..., n\}$.

Theorem 5.2. *An IFM A is s-transitive if and only if $(\triangle A)^2 \prec \triangle A$.*

Proof. Let A be a s-transitive IFM and $\triangle a_{kh} = \langle 0,1\rangle$ for some $k, h \in \{1, 2, 3, ..., n\}$, where $\triangle A = [\triangle a_{ij}]$. We have to prove that, $\max_i\{\triangle a_{ki} \wedge \triangle a_{ih}\} = \langle 0,1\rangle$.

Let us assume that, $\triangle a_{kj} \wedge \triangle a_{jh} > \langle 0,1\rangle$ for some $j \in \{1, 2, 3, ..., n\}$. Then, $\triangle a_{kj} = a_{kj} \ominus a_{jk} > \langle 0,1\rangle$ imply $a_{kj} > a_{jk}$. Similarly, $\triangle a_{jh} = a_{jh} \ominus a_{hj} > \langle 0,1\rangle$ imply $a_{jh} > a_{hj}$. By the definition of s-transitivity of A, we obtain $a_{kh} > a_{hk}$ and $\triangle a_{kh} > \langle 0,1\rangle$. This is a contradiction of our assumption. Thus, $\max_i\{\triangle a_{ki} \wedge \triangle a_{ih}\} = \langle 0,1\rangle$.

Conversely, let $(\triangle A)^2 \prec \triangle A$. We have to prove that A is s-transitive.

Let A is not s-transitive, then there exist integers $i, j, k \in \{1, 2, 3, ..., n\}$ such that, $a_{ik} > a_{ki}$, $a_{kj} > a_{jk}$ and $a_{ij} \leq a_{ji}$. Then, $\triangle a_{ik} > \langle 0, 1\rangle$, $\triangle a_{kj} > \langle 0, 1\rangle$ and $\max_h\{\triangle a_{ih} \wedge \triangle a_{hj}\} \geq \triangle a_{ik} \wedge \triangle a_{kj} > \langle 0, 1\rangle$. The element which lies in the entry (i, j) of the matrix $(\triangle A)^2$ is greater than $\langle 0, 1\rangle$ while $\triangle a_{ij} > \langle 0, 1\rangle$. This contradicts the definition of the relation $\prec$. Thus, the IFM A is s-transitive. □

Theorem 5.3. *If A is s-transitive IFM then,*
(1) *$\triangle A$ is s-transitive,*
(2) *$\triangle A$ is nilpotent.*

Proof. (1) By Theorem 5.2 and Remark 3.5, we have,

$$[\triangle(\triangle A)]^2 = (\triangle A)^2 \prec \triangle A = \triangle(\triangle A),$$

which means $\triangle A$ is s-transitive IFM.

(2) Let $(\triangle A)^n = [\langle \triangle a^n_{ij\mu}, \triangle a^n_{ij\nu}\rangle]$ and assume that there exist indices $i, j \in \{1, 2, 3, ..., n\}$ such that $\langle \triangle a^n_{ij\mu}, \triangle a^n_{ij\nu}\rangle > \langle 0, 1\rangle$. Then,

$$\begin{aligned}&\langle \triangle a^n_{ij\mu}, \triangle a^n_{ij\nu}\rangle \\ &= \langle \triangle a_{i_0 i_1\mu}, \triangle a_{i_0 i_1\nu}\rangle \wedge \langle \triangle a_{i_1 i_2\mu}, \triangle a_{i_1 i_2\nu}\rangle \wedge \ldots \wedge \langle \triangle a_{i_{n-1} i_n\mu}, \triangle a_{i_{n-1} i_n\nu}\rangle \\ &> \langle 0, 1\rangle,\end{aligned}$$

for some integers $i_0, i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n\}$ such that $i_0 = i$ and $i_n = j$. Thus

$$i_a = i_b \text{ for some } a,\ b\ (a < b)$$

and

$$\begin{aligned}&\langle \triangle a_{i_a i_{a+1}\mu}, \triangle a_{i_a i_{a+1}\nu}\rangle > \langle 0, 1\rangle \\ &\Rightarrow \langle \triangle a_{i_{a+1} i_a\mu}, \triangle a_{i_{a+1} i_a\nu}\rangle, \langle \triangle a_{i_{a+2} i_{a+1}\mu}, \triangle a_{i_{a+2} i_{a+1}\nu}\rangle > \langle 0, 1\rangle \\ &\Rightarrow \langle \triangle a_{i_{a+1} i_{a+2}\mu}, \triangle a_{i_{a+1} i_{a+2}\nu}\rangle, \ldots, \langle \triangle a_{i_{b-1} i_b\mu}, \triangle a_{i_{b-1} i_b\nu}\rangle > \langle 0, 1\rangle \\ &\Rightarrow \langle \triangle a_{i_b i_{b-1}\mu}, \triangle a_{i_b i_{b-1}\nu}\rangle > \langle 0, 1\rangle.\end{aligned}$$

Using the s-transitivity of the IFM $\triangle A$, we obtain $\triangle a_{i_a i_a} = \triangle a_{i_a i_b} > \triangle a_{i_b i_a} = \triangle a_{i_a i_a}$, which is not possible. So our assumption is wrong and hence $\triangle A$ is nilpotent IFM. □

Like canonical form of transitive IFMs, a similar result is also holds for s-transitive IFMs. The related theorem is stated below.

Theorem 5.4. *If A is s-transitive IFM then there exists a IFPM P, such that, $C = [\langle c_{ij\mu}, c_{ij\nu}\rangle] = P \times A \times P'$ satisfies $c_{ij} \geq c_{ji}$ for all $i > j$.*

Proof. Using Lemma 3.3, we can write $A = \triangle A \vee \nabla A$, where ∇A is symmetric IFM of course. From Theorem 5.3, $\triangle A$ is nilpotent. Then, for an IFPM P, $(P \times \triangle A \times P')$ is strictly lower triangular and $(P \times \nabla A \times P')$ is symmetric. Thus by Theorem 4.1, there exists an IFPM P such that,

$$C = (P \times \triangle A \times P') \vee (P \times \nabla A \times P') = P \times (\triangle A \vee \nabla A) \times P' = P \times A \times P',$$

which satisfies $c_{ij} \geq c_{ji}$ for all $i > j$. □

Example 5.5. Let us consider an IFM $A = \begin{bmatrix} \langle 1,0\rangle & \langle 0.6,0.2\rangle & \langle 0.8,0.1\rangle \\ \langle 0.4,0.5\rangle & \langle 0,1\rangle & \langle 0.5,0.3\rangle \\ \langle 0.1,0.5\rangle & \langle 0.9,0.1\rangle & \langle 0,1\rangle \end{bmatrix}$.

Then the transpose of A is $A' = \begin{bmatrix} \langle 1,0\rangle & \langle 0.4,0.5\rangle & \langle 0.1,0.5\rangle \\ \langle 0.6,0.2\rangle & \langle 0,1\rangle & \langle 0.9,0.1\rangle \\ \langle 0.8,0.1\rangle & \langle 0.5,0.3\rangle & \langle 0,1\rangle \end{bmatrix}$.

Also, $\triangle A = \begin{bmatrix} \langle 0,1\rangle & \langle 0.6,0.2\rangle & \langle 0.8,0.1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0.9,0.1\rangle & \langle 0,1\rangle \end{bmatrix}$

and

$$(\triangle A)^2 = \begin{bmatrix} \langle 0,1\rangle & \langle 0.8,0.1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{bmatrix}.$$

Here it is observed that $(\triangle A)^2 \prec \triangle A$. Thus by the Theorem 5.2, the IFM A is s-transitive.

Again, $(\triangle A)^3 = \begin{bmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{bmatrix} = \mathbf{O}$, i.e., $(\triangle A)$ is nilpotent.

Also, $\nabla A = \begin{bmatrix} \langle 1,0\rangle & \langle 0.4,0.5\rangle & \langle 0.1,0.5\rangle \\ \langle 0.4,0.5\rangle & \langle 0,1\rangle & \langle 0.5,0.3\rangle \\ \langle 0.1,0.5\rangle & \langle 0.5,0.3\rangle & \langle 0,1\rangle \end{bmatrix}$ is a symmetric IFM.

Now, for the IFPM $P = \begin{bmatrix} \langle 0,1\rangle & \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 0,1\rangle & \langle 0,1\rangle & \langle 1,0\rangle \\ \langle 1,0\rangle & \langle 0,1\rangle & \langle 0,1\rangle \end{bmatrix}$,

$$P(\triangle A)P' = \begin{bmatrix} \langle 0,1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0.9,0.1\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ \langle 0.6,0.2\rangle & \langle 0.8,0.1\rangle & \langle 0,1\rangle \end{bmatrix}$$

and

$$P(\nabla A)P' = \begin{bmatrix} \langle 0,1\rangle & \langle 0.5,0.3\rangle & \langle 0.4,0.5\rangle \\ \langle 0.5,0.3\rangle & \langle 0,1\rangle & \langle 0.1,0.5\rangle \\ \langle 0.4,0.5\rangle & \langle 0.1,0.5\rangle & \langle 1,0\rangle \end{bmatrix}.$$

So, $C = P \times A \times P' = \begin{bmatrix} \langle 0,1\rangle & \langle 0.5,0.3\rangle & \langle 0.4,0.5\rangle \\ \langle 0.9,0.1\rangle & \langle 0,1\rangle & \langle 0.1,0.5\rangle \\ \langle 0.6,0.2\rangle & \langle 0.8,0.1\rangle & \langle 1,0\rangle \end{bmatrix}$ which satisfy the required conditions.

6. Conclusions

Transitive intuitionistic fuzzy relation which correspond to the transitive intuitionistic fuzzy matrices are important both in theory of intuitionistic fuzzy relations and as in their applications in various areas of the research. Here, we examine some basic properties of transitive intuitionistic fuzzy matrix. Then we decompose a transitive and strongly transitive IFMs into the sum of a nilpotent IFM and a symmetric IFM. Some interesting results are also presented regarding nilpotent, transitive and s-transitive IFMs. Now, the question arises, is it possible to represent any IFM to its canonical form? We shall try to discuss about this matter in our next paper.

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