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Automorphism group of lattice of fuzzy generalized topologies

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Abstract. In this paper we determine the automorphism group of lattice of all fuzzy generalized topologies $LFGT(X, L)$, when X is an arbitrary nonempty set and L is a finite chain.

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1. Introduction

Consider any lattice L. An $L-$fuzzy set is a mapping from X to L, where X is a nonempty ordinary set. The family of all $L-$fuzzy subsets on X is denoted by L^X, which consists of all mappings from X to L. In[5] it is proved that when X is infinite or X consists of atmost two elements, the lattice $LT(X, L)$ of all topologies on X is isomorphic to the symmetric group on X. From this it can be seen that if X is an infinite set and P is any topological property, then the set of all topologies in $LT(X, L)$ possessing the property P may be identified exclusively from the lattice structure of $LT(X, L)$ and hence the topological properties of elements of $LT(X, L)$ must be determined by the position of the topologies in $LT(X, L)$ [8]. Madhavan Namboothiri determined the automorphism group of lattice of fuzzy topologies when L is a finite chain and when L is a diamond type lattice[6]. We have already determined the automorphism group of lattice $LGT(X)$[7]. Here we consider the same problem in the lattice of fuzzy generalized topologies, $LFGT(X, L)$, on a set X and when L is a finite chain.

2. Preliminaries

Definition 2.1 ([2]). A completely distributive lattice is called an $F-$lattice, if L has an order reversing involution.

Definition 2.2 ([4]). Let X be a nonempty ordinary set, L an $F-$lattice. A family τ of $L-$fuzzy subsets on X is said to be $L-$fuzzy generalized topology on X, if $\underline{0} \in \tau$ and τ is closed under arbitrary union of $L-$fuzzy sets.

Consider the collection of all $L-$fuzzy generalized topologies $LFGT(X, L)$ on a set X. Note that it is a complete lattice under the order of set inclusion.

Definition 2.3 ([3]). The lattices $(L_0, \leq)$ and $(L_1, \leq^{'})$ are said to be isomorphic and the map $\phi : L_0 \to L_1$ is called an isomorphism, if
(i) ϕ is one-to-one and onto,
(ii) $a \leq b$ in L_0 if and only if $\phi(a) \leq^{'} \phi(b)$ in L_1.

Definition 2.4 ([3]). The lattices $(L_0, \wedge, \vee)$ and $(L_1, \wedge, \vee)$ are said to be isomorphic and the map $\phi : L_0 \to L_1$ is called an isomorphism, if
(i) ϕ is one-to-one and onto,
(ii) $\phi(a \vee b) = \phi(a) \vee \phi(b)$, $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$.

An isomorphism of a lattice with itself is called an automorphism.

It can be shown that the two isomorphism concepts in the preceding two definitions coincide[3].

Before proceeding to the main results, let us introduce some notations which will be using in the next section.

Notations. Throughout this paper X is any nonempty set and L is a finite chain unless otherwise stated. Also let us denote the set $\{0, l_1, l_2, \ldots, l_n, 1\}$ by L and let the order in L be $0 < l_1 < l_2 < \ldots < l_n < 1$. We define an involution $'$ in L as $0^{'} = 1, 1^{'} = 0$ and $l_i^{'} = l_{n-i+1}$ for every $i \in \{1, 2, \ldots, n\}$. Then L is an $F-$Lattice. Let us designate an atom of $LFGT(X, L)$ by $J_C = \{\underline{0}, C\}$, where $C \in L^X$ and $C \neq \underline{0}$.

For $l \in L, l \neq 0$ and $x \in X$,

$$x_l(t) = \begin{cases} l & when\ t = x \\ 0 & otherwise \end{cases}$$

and for $l \in L, l \neq 1$ and $x \in X$,

$$x^l(t) = \begin{cases} l & when\ t = x \\ 1 & otherwise. \end{cases}$$

For $i = 1, 2, \ldots, n$,

$$K_i = \{J_{x_{l_i}} : x \in X\},$$

$$M_i = \{J_{x^{l_i}} : x \in X\},$$

$$K_{n+1} = \{\{\underline{0}, x_1\} : x \in X\} \text{ and } M_{n+1} = \{\{\underline{0}, x^0\} : x \in X\}.$$

Remark 2.1. Note that an automorphism of $LFGT(X, L)$ map a $L-$fuzzy generalized topology containing n elements onto a $L-$fuzzy generalized topology containing same number of elements if n is finite. If $\{\mu_i\}_{i\in I} \subset LFGT(X,L)$ and A is an automorphism of $LFGT(X,L)$, then $A(\bigvee_{i\in I}\mu_i) = \bigvee_{i\in I} A(\mu_i)$, since L^X is a complete lattice.

3. Major section

Let us first prove some preliminary results which will be using in our main theorem.

Lemma 3.1. *Let X be a set with more than one point. If A is an automorphism of $LFGT(X,L)$, then $A(\{\underline{0},\underline{1}\}) = \{\underline{0},\underline{1}\}$.*

Proof. Let J_C be an atom of $LFGT(X,L)$ and $C \neq \underline{1}$.

Claim: There exists an $L-$fuzzy set $D \in L^X$ such that $J_C \vee J_D$ contain 4 elements.

Case (i): Suppose$C(y) = 0$ for some $y \in X$. Since $C \neq \underline{0}$, there exists an $x \in X$ such that $C(x) \neq 0$. Let $D = y_1$. Then $J_C \vee J_D = \{\underline{0},C\} \vee \{\underline{0},y_1\} = \{\underline{0},C,y_1,C\vee y_1\}$. Since $(C\vee y_1)(y) = 1,\ C\vee y_1 \neq C$. Since $(C\vee y_1)(x) \neq 0, C\vee y_1 \neq y_1$. Thus $J_C \vee J_D$ contain exactly 4 elements.

Case (ii): Suppose $C(y) \neq 0$ for every $y \in X$. Since $C \neq \underline{1}$, there exists an $x \in X$ such that $C(x) \neq 1$.

Now considering $D = x_1$ we can prove as above that $J_C \vee J_D$ contain 4 elements. Since $\underline{1}$ is comparable with every element of $LFGT(X,L)$, the join of $\{\underline{0},\underline{1}\}$ with any atom of $LFGT(X,L)$ contain exactly 3 elements. Thus if $A(\{\underline{0},\underline{1}\}) = \{\underline{0},C\}$ and $C \neq \underline{1}$, then by claim, there exists an $L-$fuzzy set $D \in L^X$ such that $\{\underline{0},C\} \vee \{\underline{0},D\}$ contain 4 elements. Let $A^{-1}(J_D) = J_H$. We have $|\{\underline{0},\underline{1}\} \vee \{\underline{0},H\}| = 3$. By Remark 2.1, $|A(\{\underline{0},\underline{1}\} \vee A(\{\underline{0},H\})| = |\{\underline{0},C\} \vee \{\underline{0},D\}| = 3$ which is a contradiction.Thus the proof is complete. □

Lemma 3.2. *Let X be a set with more than one point. Then every automorphism of $LFGT(X,L)$ maps strong fuzzy generalized topologies onto strong fuzzy generalized topologies of $LFGT(X,L)$.*

Proof. Let A be an automorphism of $LFGT(X,L)$ and μ be a strong fuzzy generalized topology on X. Then $\mu = \bigvee_{C\in\mu}\{\underline{0},C\}$ and $A(\mu) = A(\bigvee_{C\in\mu}\{\underline{0},C\}) = \bigvee_{C\in\mu} A(\{\underline{0},C\})$, by Remark 2.1. Since $\underline{1} \in \mu$ and $A(\{\underline{0},\underline{1}\}) = \{\underline{0},\underline{1}\}$ by Lemma 3.1, $\bigvee_{C\in\mu} A(\{\underline{0},C\})$ is a strong fuzzy generalized topology on X.

Similarly, the inverse image of a strong fuzzy generalized topology is a strong fuzzy generalized topology. □

Lemma 3.3. *Let X be a set with more than one point and let A be an automorphism of $LFGT(X,L)$. Then A maps M_n onto M_n.*

Proof. Consider the strong fuzzy generalized topologies of the form $\{\underline{0},x^{l_n},\underline{1}\}$ and let us denote this by $I_{x^{l_n}}$. Note that join of $I_{x^{l_n}}$ with any fuzzy generalized topology $I_C = \{\underline{0},C,\underline{1}\}$ contain exactly 4 elements. Now we claim that if $C \in L^X$ such that $I_C \notin \{I_{x^{l_n}}\}_{x\in X}$, then there exists an $L-$fuzzy set $D \in L^X$ such that $I_C \vee I_D$ contains 5 elements. Consider $I_C \notin \{I_{x^{l_n}}\}_{x\in X}$.

Case (i): Suppose for some $x \in X$, $C(x) = 0$. Since $C \neq \underline{0}$, there exists $y \in X$ such that $C(y) \neq 0$. Let us define $D \in L^X$ such that $D(x) = l_1$ and $D(y) = 0$. Since $(C \vee D)(x) = l_1, C \vee D \neq C$. Since $(C \vee D)(y) \neq 0$, we have $C \vee D \neq D$. Then $I_C \vee I_D = \{\underline{0}, C, D, C \vee D, \underline{1}\}$ contains 5 elements.

Case (ii): Suppose $C(x) \neq 0$ for every $x \in X$. Note that $I_C \notin \{I_{x^{l_n}}\}_{x \in X}$. Then there exist elements $x, y \in X$ such that $C(x) = l_i$ where $i < n$ and $C(y) \neq 0$. Define $D \in L^X$ such that $D(x) = l_{i+1}$ and $D(y) = 0$. Then $C \vee D \neq C$ and $C \vee D \neq D$. Thus $I_C \vee I_D$ contains exactly 5 elements.

So the claim holds.

Now if $A(I_{x^{l_n}}) = I_C$ for some $I_C \notin \{I_{x^{l_n}}\}_{x \in X}$, then by above claim, there exists an $L-$fuzzy set $D \in L^X$ such that $I_C \vee I_D$ contains 5 elements. Since A is bijective, there exists an $L-$fuzzy set E such that $A(I_E) = I_D$. Thus $I_{x^{l_n}} \vee I_E$ contains 4 elements. But $A(I_{x^{l_n}} \vee I_E) = A(I_{x^{l_n}}) \vee A(I_E) = I_C \vee I_D$ contains 5 elements, which is not possible. Thus A map $\{I_{x^{l_n}}\}_{x \in X}$ onto itself. Now $I_{x^{l_n}} = \{\underline{0}, x^{l_n}, \underline{1}\} = \{\underline{0}, x^{l_n}\} \vee \{\underline{0}, \underline{1}\}$. Let $A(I_{x^{l_n}}) = I_{y^{l_n}}$ for some $y \in X$. Then

$$\begin{aligned} A(I_{x^{l_n}}) &= A(\{\underline{0}, x^{l_n}\} \vee \{\underline{0}, \underline{1}\}) = A(\{\underline{0}, x^{l_n}\}) \vee A(\{\underline{0}, \underline{1}\}) \\ &= I_{y^{l_n}} = \{\underline{0}, y^{l_n}\} \vee \{\underline{0}, \underline{1}\}. \end{aligned}$$

Thus $A(\{\underline{0}, x^{l_n}\}) = \{\underline{0}, y^{l_n}\}$, since $A(\{\underline{0}, \underline{1}\}) = \{\underline{0}, \underline{1}\}$ by Lemma 3.1. Since $x \in X$ is arbitrary, A map M_n onto itself. □

Lemma 3.4. *Let X be a set with more than one point. Then every automorphism of $LFGT(X, L)$ maps $\bigcup_{i=1}^{n+1} K_i$ onto itself.*

Proof. Let A be an automorphism of $LFGT(X, L)$ and $C \in L^X$. Then we can write C as $C = \bigvee\{x_l : x \in X \text{ and } l \in L \text{ such that } C(x) = l\}$, which implies

$$J_C \leq \bigvee\{J_{x_l} : x \in X \text{ and } l \in L \text{ such } \quad \text{that } C(x) = l\}.$$

Since A preserves order and arbitrary join,

$$\begin{aligned} A(J_C) &\leq A(\bigvee\{J_{x_l} : x \in X \text{ and } l \in L \text{ such that } C(x) = l\}) \\ &= \bigvee\{A(J_{x_l}) : x \in X \text{ and } l \in L \text{ such that } C(x) = l\}. \end{aligned}$$

Thus $A(J_C) \leq \bigvee\{A(J_{x_l}) : x \in X \text{ and } l \in L \text{ such that } C(x) = l\}$. An $L-$fuzzy generalized topology of the form J_{x_l} is less than or equal to the join of a collection of $L-$fuzzy generalized topologies if and only if J_{x_l} is already a member of that collection of $L-$fuzzy generalized topologies. This characterizes atoms of the form J_{x_l} for all $x \in X$ and $l \in L$. So A maps $\bigcup_{i=1}^{n+1} K_i = \{J_{x_l} : x \in X, l \in L\}$ onto itself. □

Lemma 3.5. *Let X be a set with more than one point and let A be an automorphism of the lattice $LFGT(X, L)$. If $C \in L^X$ and $A(J_C) = J_D$ for some $D \in L^X$, then for $x \in X$, $C(x) = 1$ if and only if there exists an element $y \in X$ such that $D(y) = 1$.*

Proof. Let $C(x) = 1$ for some $x \in X$. Then $J_C \vee J_{x^{l_n}}$ is a strong generalized topology. By Lemma 3.2, $A(J_C \vee J_{x^{l_n}}) = A(J_C) \vee A(J_{x^{l_n}})$ is a strong generalized topology. Since A map M_n onto itself, by Lemma 3.3, $A(J_{x^{l_n}}) = J_{y^{l_n}}$ for some $y \in X$. Let $A(J_C) = J_D$ for some $D \in L^X$. Then $J_D \vee J_{y^{l_n}}$ is a strong generalized topology implying $D(y) = 1$.

Similarly, If $A(J_C)(y) = 1$ for some $y \in X$, then $C(x) = 1$ for some $x \in X$. □

Lemma 3.6. *Let X be a set with more than one point. Then every automorphism of $LFGT(X, L)$ maps K_1 onto itself.*

Proof. Let A be an automorphism of $LFGT(X, L)$. By Lemma 3.4, A maps $\bigcup_{i=1}^{n+1} K_i$ onto itself. Suppose for some $x \in X$, $A(J_{x_{l_1}}) = J_{z_{l_i}}, i \geq 2$. Let $\mathfrak{C} = \{C \in L^X : C(x) \neq 0 \; for \; every \; x \in X\}$ and $\mathfrak{D} = \{D \in L^X : A(J_C) = J_D, C \in \mathfrak{C}\}$. Note that $J_C \vee J_{x_{l_1}}$ contains 3 elements for every $C \in \mathfrak{C}$. Then $A(J_C \vee J_{x_{l_1}}) = A(J_C) \vee A(J_{x_{l_1}}) = J_D \vee J_{z_{l_i}}$ contains 3 elements for every $D \in \mathfrak{D}$. Thus $\{\underline{0}, D\} \vee \{\underline{0}, z_{l_i}\} = \{\underline{0}, D, z_{l_i}\}$ and $D \vee z_{l_i} = D$ or z_{l_i}.

If $D \vee z_{l_i} = z_{l_i}$, then $D = J_{z_{l_j}}$ for some $j < i$. Thus there exists an element $C \in \mathfrak{C}$ such that $A(J_C) = J_{z_{l_j}}$ which is not possible, by Lemma 3.4. So $D \vee z_{l_i} = D$ which implies that $D(z) \geq l_i$ for every $D \in \mathfrak{D}$ and $i \geq 2$. Hence $z^{l_1} \notin \mathfrak{D}$. Therefore $A^{-1}(J_{z^{l_1}}) \notin \{J_C\}_{c \in \mathfrak{C}}$. Let $A^{-1}(J_{z^{l_1}}) = J_H$ for some $H \in L^X$. Then there exists $t \in X$ such that $H(t) = 0$, since $H \notin \mathfrak{C}$.

Define $f \in L^X$ such that

$$f(t) = \begin{cases} l_k & \text{whenever } H(t) = 0 \\ H(t) & \text{otherwise.} \end{cases}$$

Choose $k \in \{1, 2, \ldots, n\}$ such that $A(J_f) = J_D$ and $D \neq z^{l_j}$, where $j \geq i$ and $i \geq 2$. This is possible since A is a bijection and k has n choices and $z^{l_j}, j \geq i$ and $i \geq 2$, has $n-1$ choices. Then $f \in \mathfrak{C}$ and $A(J_f) = J_D$ for some $D \in \mathfrak{D}$. Now $|J_H \vee J_f| = 3$, since $H \leq f$, which implies $|A(J_H) \vee A(J_f)| = |J_{z^{l_1}} \vee J_D| = 3$. But $J_{z^{l_1}} \vee J_D = \{\underline{0}, z^{l_1}\} \vee \{\underline{0}, D\} = \{\underline{0}, z^{l_1}, D, z^{D(z)}\}$. Since $D(z) \geq l_i$ and $i \geq 2, z^{D(z)} \neq z^{l_1}$. Also we have chosen f such that $z^{D(z)} \neq D$, thus $|J_{z^{l_1}} \vee J_D| = 4$, which is a contradiction. So $A(J_{x_{l_1}})$ cannot be $J_{z_{l_i}}$ for any $i \geq 2$ and by Lemma 3.4 A map K_1 onto itself. □

Definition 3.1 ([1]). Let X be a nonempty set and L be an $F-$Lattice. If $p : X \to X$ is a bijection, then $H_p : L^X \to L^X$ defined by $H_p(C)(x) = C(p^{-1}(x))$ for all $C \in L^X$ and $x \in X$ is an automorphism of L^X.

Theorem 3.1. *Let X be a nonempty set and L be an $F-$Lattice. If μ is an $L-$fuzzy generalized topology on X, then the collection $H_p^*(\mu) = \{H_p(C) : C \in \mu\}$ is also an $L-$fuzzy generalized topology and H_p^* is an automorphism of $LFGT(X, L)$ where H_p is as in the Definition 3.1.*

Proof. Let μ be an $L-$fuzzy generalized topology on X. Then $\underline{0} \in H_p^*(\mu)$, because $H_p(\underline{0})(x) = \underline{0}(p^{-1}(x)) = \underline{0}$ for every $x \in X$. Let $\{C_i\}_{i \in I}$ be a collection of $L-$fuzzy sets in $H_p^*(\mu)$. Then for $i \in I$,

$$\begin{aligned} C_i &= H_p(K_i) \qquad \text{for some } K_i \in \mu \\ (\bigvee_{i \in I} C_i)(x) &= (\bigvee H_p(K_i))(x) \\ &= (\bigvee K_i)(p^{-1})(x) \\ &= H_p(\bigvee K_i)(x) \end{aligned}$$

Thus $H_p^*(\mu)$ is an $L-$fuzzy generalized topology on X and H_p^* map fuzzy generalized topologies onto fuzzy generalized topologies. Also note that H_p^* is bijective. For

$\mu, \tau \in LFGT(X)$; we have $\mu \leq \tau$ if and only if $H_p^*(\mu) \leq H_p^*(\tau)$ by definition itself. So H_p^* is an automorphism of $LFGT(X)$. □

Finally we are in a position to prove our main results. First we consider here the case when X is a singleton set.

Theorem 3.2. *Let X be a singleton set. Then the group of all automorphisms of the lattice $LFGT(X, L)$ is isomorphic to $S(L \setminus \{0\})$, the group of all permutations on $L \setminus \{0\}$.*

Proof. Let $X = \{x\}$ and L be as defined in the notation. Then the atoms of $LFGT(X, L)$ are $\{K_i\}_{i=1,2,...,n,n+1}$ where $K_i = \{\underline{0}, x_{l_i}\}$ for $i = 1, 2, \ldots, n, n+1$ where $l_{n+1} = 1$. In fact these are the only elements of $LFGT(X, L)$ other than $\underline{0}$ since $X = \{x\}$. Let p be a permutation on $\{1, 2, \ldots, n+1\}$.

Define a function A_p on L^X, $A_p : L^X \to L^X$, for $i = 1, 2, \ldots, n, n+1$

$$A_p(x_{l_i}) = x_{l_j} \text{ if and only if}, p(i) = j$$

and $A_p(\underline{0}) = \underline{0}$. For an L-fuzzy generalized topology $\mu \in LFGT(X, L)$, we define $A_p^*(\mu) = \{A_p(x_{l_i}) : x_{l_i} \in \mu\} \cup \{\underline{0}\}$. Then A_p^* is a bijection on $LFGT(X, L)$. Now for $\mu, \tau \in LFGT(X, L)$,

$$\mu \leq \tau \Leftrightarrow \mu \subseteq \tau \Leftrightarrow A_p^*(\mu) \subseteq A_p^*(\tau).$$

Hence A_p^* is an automorphism on $LFGT(X, L)$.

Conversely if M is an automorphism on $LFGT(X, L)$, M must map atoms onto atoms of $LFGT(X, L)$. Then it will induce a bijection on $\{x_{l_i} : i = 1, 2, \ldots, n+1\}$ and hence on $\{1, 2, \ldots, n+1\}$. Thus it defines a bijection between the group of all automorphisms of $LFGT(X, L)$ and the group of all permutations on $\{1, 2, \ldots, n+1\}$. Also if p and k are two permutations on $\{1, 2, \ldots, n+1\}$, then $A_{p\circ k}^* = A_p^* \circ A_k^*$. This defines an isomorphism between the group of all automorphisms of $LFGT(X, L)$ and the group of all permutations on $L \setminus \{0\}$. □

Theorem 3.3. *Let X be a set with more than one point. Then the group of all automorphisms of $LFGT(X, L)$ is precisely the collection $\{H_p^* : p$ is a bijection on $X\}$ where H_p^* is as in the Theorem 3.1.*

Proof. We have already proved in Theorem 3.1 that H_p^* is an automorphism on $LFGT(X, L)$. Now let A be an automorphism on $LFGT(X, L)$. We need to prove that $A = H_p^*$ for some bijection p on X. By Lemma 3.6, A maps K_1 onto itself. Let $x \in X$, consider $J_{x_{l_1}}$ and let $A(J_{x_{l_1}}) = J_{y_{l_1}}$ for some $y \in X$. This y is unique. Define $p : X \to X$ as $p(x) = y$ if and only if $A(J_{x_{l_1}}) = J_{y_{l_1}}$. For $t \in X$,

$$\begin{aligned} H_p(x_{l_1})(t) &= x_{l_1}(p^{-1}(t)) \\ &= \begin{cases} l_1 & if\ p^{-1}(t) = x \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} l_1 & if\ t = y \\ 0 & \text{otherwise} \end{cases} \\ &= y_{l_1}(t). \end{aligned}$$

Then $H_p(\underline{0}) = \underline{0}$. Thus $H_p^*(J_{x_{l_1}}) = \{H_p(\underline{0}), H_p(x_{l_1})\} = \{\underline{0}, y_{l_1}\} = J_{y_{l_1}}$. Since $x \in X$ is arbitrary, $A = H_p^*$ on K_1.

Claim: If $A(J_{x_{l_1}}) = J_{y_{l_1}}$, then $A(J_{x_1}) = J_{y_1}$.

Suppose $A(J_{x_1}) = J_C = \{\underline{0}, C\}$, for some $L-$fuzzy set $C \in L^X$. Then

$$|\{\underline{0}, C\} \vee \{\underline{0}, z_{l_1}\}| = 3, \text{ for every } z \in X \text{ such that } C(z) \neq 0.$$

Thus $|J_C \vee J_{z_{l_1}}| = 3$, for every $z \in X$ such that $C(z) \neq 0$.
So $|A^{-1}(J_C) \vee A^{-1}(J_{z_{l_1}})| = 3$, for every $z \in X$ such that $C(z) \neq 0$.
Hence $|J_{x_1} \vee A^{-1}(J_{z_{l_1}})| = 3$, for every $z \in X$ such that $C(z) \neq 0$.

But $A^{-1}(J_{z_{l_1}}) \in K_1$. Then $A^{-1}(J_{z_{l_1}}) = J_{x_{l_1}}$ and thus $z = y$. So $C(z) \neq 0$ implying $z = y$. By Lemma 3.5, there exists an element $t \in X$ such that $C(t) = 1$. Hence $C = y_1$.

Claim: If $A(J_{x_{l_1}}) = J_{y_{l_1}}$ and $A(J_{x_1}) = J_{y_1}$, then

(1) $A(J_{x_{l_i}}) = J_{y_{l_i}}$, where $i \in \{2, 3, \ldots, n\}$,
(2) $A(J_{x^0}) = J_{y^0}$.

Proof of claim (1): By Lemma 3.4, A maps $\bigcup_{i=1}^{n+1} K_i$ onto itself. Suppose $A(J_{x_{l_i}}) = J_{z_{l_j}}$ for some $z \in X$ and $j \in \{2, 3, \ldots, n\}$. We know that $|J_{x_{l_i}} \vee J_{x_{l_1}}| = 3$. Then $|A(J_{x_{l_i}} \vee J_{x_{l_1}})| = |A(J_{x_{l_i}}) \vee A(J_{x_{l_1}})| = |J_{z_{l_j}} \vee J_{y_{l_1}}| = 3$. This happens only if $z = y$. Thus $A(J_{x_{l_i}}) = J_{y_{l_j}}$ for some $j = 2, 3, \ldots n$.

Now let $A(J_{x^{l_i}}) = \{\underline{0}, C\}$ for some $C \in L^X$. Then we have $J_{x^{l_i}} \vee J_{x_1} = \{\underline{0}, x^{l_i}, x_1, \underline{1}\}$ is a strong fuzzy generalized topology. Thus $A(J_{x^{l_i}} \vee J_{x_1}) = A(J_{x^{l_i}}) \vee A(J_{x_1}) = J_C \vee J_{y_1}$ is a strong fuzzy generalized topology. So $C \vee y_1 = \underline{1}$. Hence $C(t) = 1$ for every $t \neq y$.

Let $C = y^{l_k}$ for some $k \in \{1, 2, \ldots, n\}$. Then we have $A(J_{x_{l_i}}) = J_{y_{l_j}}$ and $A(J_{x^{l_i}}) = J_{y^{l_k}}$. Consider $J_{x_{l_i}} \vee J_{x^{l_i}} = \{\underline{0}, x_{l_i}, x^{l_i}\}$. Then

$$A(J_{x_{l_i}} \vee J_{x^{l_i}}) = A(J_{x_{l_i}}) \vee A(J_{x^{l_i}}) = J_{y_{l_j}} \vee J_{y^{l_k}} = \{\underline{0}, y_{l_j}, y^{l_k}\},$$

since $|J_{x_{l_i}} \vee J_{x^{l_i}}| = |A(J_{x_{l_i}} \vee J_{x^{l_i}})|$. But $\{\underline{0}, y_{l_j}, y^{l_k}\}$ is an $L-$fuzzy generalized topology. Thus $j \leq k$, otherwise $J_{y_{l_j}} \vee J_{y^{l_k}}$ contain 4 elements. Also we have $J_{x_{l_i}} \vee J_{x^{l_{i+1}}}$ contain 3 elements. So

$$A(J_{x_{l_i}} \vee J_{x^{l_{i+1}}}) = A(J_{x_{l_i}}) \vee A(J_{x^{l_{i+1}}}) = J_{y_{l_j}} \vee J_{y^{l_{k_1}}}$$

also contain 3 elements, where $A(J_{x^{l_{i+1}}}) = J_{y^{l_{k_1}}}$ for some $k_1 \in \{1, 2, \ldots, n\}$. Hence k_1 must be greater than or equal to j. This is true for $J_{x^{l_{i+2}}}, J_{x^{l_{i+3}}}, \ldots, J_{x^{l_n}}$. Therefore for example,

we have $A(J_{x_{l_1}}) = J_{y_{l_1}}$, let $A(J_{x_{l_2}}) = J_{y_{l_j}}$, where $j \geq 2$.

$$A(J_{x^{l_2}}) = J_{y^{l_{k_2}}},\ k_2 \geq j$$
$$A(J_{x^{l_3}}) = J_{y^{l_{k_3}}},\ k_3 \geq j$$
$$A(J_{x^{l_4}}) = J_{y^{l_{k_4}}},\ k_4 \geq j$$
$$\vdots$$
$$A(J_{x^{l_n}}) = J_{y^{l_{k_n}}},\ k_n \geq j.$$

Since A is a bijection j must be equal to 2. Then $A(J_{x_{l_2}}) = J_{y_{l_2}}$.

Similarly, $A(J_{x_{l_i}}) = J_{y_{l_i}}$ for every $i \in \{1, 2, \ldots, n\}$.

Proof of claim (2): Suppose $A(J_{x^0}) = \{\underline{0}, C\}$ for some $L-$fuzzy set $C \in L^X$. We know that $J_{x^0} \vee J_{x_1} = \{\underline{0}, x^0, x_1, \underline{1}\}$ is a strong generalized topology. Then by Lemma 3.2, $A(J_{x^0}) \vee A(J_{x_1})$ is a strong fuzzy generalized topology on X. Thus $\{\underline{0}, C\} \vee A(J_{y_1})$ is a strong fuzzy generalized topology implying $C(t) = 1$ for every $t \neq y$. Now if $C(y) \neq 0$, then $J_C \vee J_{y_{l_1}}$ contain 3 elements and thus $J_{x^0} \vee J_{x_{l_1}}$ contain 3 elements, which is a contradiction. So $C(y) = 0$, proving $A(J_{x^0}) = J_{y^0}$.

Claim: If $A(J_{x_{l_i}}) = J_{y_{l_i}}$ for every $i \in \{1, 2, \ldots, n\}$, then $A(J_{x^{l_i}}) = J_{y^{l_i}}$ for every $i \in \{1, 2, \ldots, n\}$.

Suppose $A(J_{x^{l_i}}) = \{\underline{0}, C\}$ for some $L-$fuzzy subset $C \in L^X$. Then $J_{x^{l_i}} \vee J_{x_1}$ is a strong generalized topology and thus $J_C \vee J_{y_1}$ is a strong generalized topology, which implies that $C(t) = 1$ for every $t \neq y$. Let $C(y) = l_j$ for some $j = 1, 2, \ldots, n$. Then $C = y^{l_j}$. Also $|J_{x_{l_i}} \vee J_{x^{l_i}}| = 3$. Thus $|J_{y_{l_i}} \vee J_C| = |J_{y_{l_i}} \vee J_{y^{l_j}}| = 3$ implying $j \geq i$. So if $A(J_{x^{l_i}}) = J_{y^{l_j}}$, then $j \geq i$. But A map M_n onto itself. Thus $A(J_{x^{l_{n-1}}}) = J_{y^{l_{n-1}}}, A(J_{x^{l_{n-2}}}) = J_{y^{l_{n-2}}}$ and so on. Hence $A(J_{x^{l_i}}) = J_{y^{l_i}}$ for every $i \in \{1, 2, \ldots, n\}$.

Now

$$\begin{aligned} H_p^*(J_{x_{l_i}}) &= \{H_p(\underline{0}), H_p(x_{l_i})\} \\ &= \{\underline{0}, y_{l_i}\} \\ &= J_{y_{l_i}} \\ &= A(J_{x_{l_i}}). \end{aligned}$$

Then $A = H_p^*$ on K_i, where $i \in \{1, 2, \ldots, n\}$,

$$\begin{aligned} H_p^*(J_{x_1}) &= \{H_p(\underline{0}), H_p(x_1)\} \\ &= \{\underline{0}, y_1\} \\ &= J_{y_1} \\ &= A(J_{x_1}). \end{aligned}$$

Thus $A = H_p^*$ on K_{n+1},

$$\begin{aligned} H_p^*(J_{x^0}) &= \{H_p(\underline{0}), H_p(x^0)\} \\ &= \{\underline{0}, y^0\} \\ &= J_{y^0} \\ &= A(J_{x^0}). \end{aligned}$$

So $A = H_p^*$ on M_{n+1},

$$\begin{aligned} H_p^*(J_{x^{l_i}}) &= \{H_p(\underline{0}), H_p(x^{l_i})\} \\ &= \{\underline{0}, y^{l_i}\} \\ &= J_{y^{l_i}} \\ &= A(J_{x^{l_i}}). \end{aligned}$$

Hence $A = H_p^*$ on M_i, where $i \in \{1, 2, \ldots, n\}$. Therefore $A = H_p^*$ on $\{K_i \cup M_i\}_{i=1,2,\ldots,n+1}$.

Now let $C \notin \{K_i \cup M_i\}_{i=1,2,\ldots,n+1}$ be an $L-$fuzzy set. Suppose $A(\{\underline{0}, C\}) = \{\underline{0}, D\}$ for some $L-$fuzzy set $D \notin \{K_i \cup M_i\}_{i=1,2,\ldots,n+1}$ and $H_p^*(\{\underline{0}, C\}) = \{\underline{0}, H_p(C)\}$. To prove that $H_p(C) = D$, it is enough to prove the following results:

(a) $C(p^{-1}(y)) = 0$ if and only if $D(y) = 0$.
(b) $C(p^{-1}(y)) = l_i$ if and only if $D(y) = l_i$ where $i = 1, 2, \ldots, n$.
(c) $C(p^{-1}(y)) = 1$ if and only if $D(y) = 1$.

Proof of (a): We have $H_p(C) = \{C(p^{-1})(t) : t \in X\}$. Then $C(p^{-1}(y)) = 0 \Leftrightarrow C(x) = 0 \Leftrightarrow C \leq x^0 \Leftrightarrow |\{\underline{0}, C\} \vee \{\underline{0}, x^0\}| = 3 \Leftrightarrow |A(\{\underline{0}, C\}) \vee A(\{\underline{0}, x^0\})| = 3 \Leftrightarrow |\{\underline{0}, D\} \vee \{\underline{0}, y^0\}| = 3 \Leftrightarrow D \leq y^0$, for otherwise, if $D > y^0$, then $D = \underline{1}$, which is not possible, since by Lemma 3.1 $A(\underline{0}, \underline{1}) = \{\underline{0}, \underline{1}\}$. Thus $D(y) = 0$.

Proof of (b): Assume $C(p^{-1}(y)) = l_i$. But $p^{-1}(y) = x$, implying $C(x) = l_i$. Then $|\{\underline{0}, C\} \vee \{\underline{0}, x^{l_i}\}| = 3$. By Remark 2.1, $|\{\underline{0}, D\} \vee \{\underline{0}, y^{l_i}\}| = 3$ which implies $D(y) \leq l_i$, since $D \notin M_i$ for every i.

Also if $C(x) = l_i$, then $|\{\underline{0}, C\} \vee \{\underline{0}, x_{l_i}\}| = 3$. By Remark 2.1, $|\{\underline{0}, D\} \vee \{\underline{0}, y_{l_i}\}| = 3$ implying $D(y) \geq l_i$, since $D \notin K_i$ for every $i \in \{1, 2, \ldots, n\}$.
Thus we get $D(y) = l_i$. So if $C(p^{-1}(y)) = l_i$, then $D(y) = l_i$.

Similarly, it is also easy to show that, if $D(y) = l_i$, then $C(p^{-1}(y)) = l_i$ for every $i = 1, 2, \ldots, n$.

Proof of (c): Consider $C(p^{-1}(y)) = 1 \Leftrightarrow C(x) = 1 \Leftrightarrow |\{\underline{0}, C\} \vee \{\underline{0}, x_1\}| = 3 \Leftrightarrow |\{\underline{0}, D\} \vee \{\underline{0}, y_1\}| = 3 \Leftrightarrow D \geq y_1$(since $D \notin K_{n+1}$)$\Leftrightarrow D(y) = 1$. Since x and y are arbitrary $A = H_p^*$ on all atoms in $LFGT(X, L)$. Also $LFGT(X, L)$ is atomistic. then $A = H_p^*$ on $LFGT(X, L)$. Thus the proof is complete.. □

4. Conclusions

We determined the automorphism group of lattice of all fuzzy generalized topologies $LFGT(X, L)$, when X is an arbitrary nonempty set and L is a finite chain.

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