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Generalized hesitant fuzzy geometric aggregation operators and their applications in multicriteria decision making

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Abstract. In this paper, we defined some aggregation operators to aggregate generalized hesitant fuzzy elements and the relationship between our proposed operators and the existing ones are discussed in detail. Furthermore, the procedure of multicriteria decision making based on the proposed operators is given under generalized hesitant fuzzy environment. Finally, a practical example is provided to illustrate the developed method. The main advantage of our proposed method is that we can deal with the generalized hesitant fuzzy information which includes fuzzy information, intuitionistic fuzzy information, hesitant fuzzy information and hesitant intuitionistic fuzzy information.

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1. Introduction

Decision-making problems referring to evaluating, prioritizing or selecting over some available alternatives are very common in practice [14]. Since it was introduced by Zadeh [35], theories of fuzzy sets serve as an excellent resolution of decision-making under uncertainties. But the modeling tools of Zadeh's fuzzy sets (Z-FSs) are limited whereby two or more sources of vagueness appear simultaneously. Thus several generalizations and extensions of Z-FSs are developed, such as type-2 fuzzy sets [18, 5], type-n fuzzy sets [5], intuitionistic fuzzy sets (IFSs) [1], fuzzy multisets [17] and hesitant fuzzy sets (T-HFSs) [23, 24].

T-HFSs are quite suit for the situation where we have a set of possible values, rather than a margin of error (as in IFSs) or some possibility distribution on the possible values (as in type-2 fuzzy sets) [23, 24]. The motivation to propose the T-HFSs is that when people make a decision, they are usually hesitant and irresolute for one thing or another which makes it difficult to reach a final agreement. For example, three decision makers give the membership of x into A, and they want to assign 0.4, 0.5 and 0.7, which can be considered as a hesitant fuzzy element $\{0.4, 0.5, 0.7\}$ rather than the convex of 0.4 and 0.7, or the interval between 0.4 and 0.7.

There are some developments on T-HFSs. Torra and Narukawa [24] introduced the extension principle to apply it in decision-making. Xu and Xia [33] developed a series of aggregation operators for hesitant fuzzy information and applied to multi-criteria decision-making. Later, some induced aggregation operators in hesitant fuzzy setting are introduced by Xia et al. [28]. Based on Quasi arithmetic means, Xia et al. [29] discussed some ordered aggregation operators and induced ordered aggregation operators, as well as their application in group decision-making. Bedregal et al. [3] induced some aggregation functions for typical hesitant fuzzy elements. Later, Zhiming Zhang extended the hesitant fuzzy set to interval-valued intuitionistic fuzzy environments and proposed the concept of interval-valued intuitionistic hesitant fuzzy set and develop a series of aggregation operators for interval-valued intuitionistic hesitant fuzzy information in [36]. In [37], Zhou and Li developed several new hesitant fuzzy aggregation operators which are the extensions of the weighted geometric operator and the ordered weighted geometric (OWG) operator with hesitant fuzzy information. Zhu et al. proposed dual hesitant fuzzy sets (DHFSs) and investigated the basic operations and properties of DHFSs in [38]. And then, Wang et al. defined the correlation measures for dual hesitant fuzzy information and then discussed their properties in detail in [27]. Yu also introduced some hesitant fuzzy aggregation operators based on Einstein operations in [34].

Gang Qian et al. further generalized the concept of T-HFSs in practice needs and gave the definition of generalized hesitant fuzzy sets [20]. There are mainly three advantages of the extension. First, as the case in T-HFSs, it is very useful to consider all possible memberships with hesitancy rather than considering just an aggregation operator. Second, it can eliminate times of using aggregation operators during the group decision-making process, which can alleviate suffering from less robust decision led by times of aggregations. At last, individual expert can express his/her evaluations by either Z-FSs, IFSs, T-HFSs or the proposed fuzzy sets.

Information aggregation is an important research topic in many applications such as fuzzy logic systems and multiattribute decision-making [4, 12, 19, 21]. Research on aggregation operators with intuitionistic fuzzy information has received increasing attention as shown in the literatures[32, 6, 7, 8, 9, 10, 11].

Because of the uncertainty of the environment and the hesitancy of the decision makers, it is more appropriate to present the evaluated values by generalized hesitant fuzzy elements in some cases. And the generalized hesitant fuzzy information aggregation method is far from perfect. It should be noted that the basic operational laws of GHFEs for the above-mentioned aggregation operators are the algebraic operations. Although the algebraic product and algebraic sum are the basic algebraic operations, it is not the only operations. The Einstein product and Einstein sum are good alternative to the algebraic product and algebraic sum, respectively [25, 26].

The purpose of this paper is to research the generalized hesitant fuzzy information aggregation methods based on the Einstein operations. To do this, the remainder of this paper is organized as follows: In Section 2, we briefly review some basic concepts related to the IFSs and the existing aggregation operators for aggregating IFVs. We also introduce some Einstein operations of GHFEs and analyze some desirable properties of the proposed operations. In Section 3, we first develop a novel operator, e.g., the generalized hesitant fuzzy Einstein weighted geometric ($\text{GHFWG}^{\varepsilon}$) operator, to aggregate a collection of GHFEs. Then, we give some numerical examples to illustrate the developed operator. In Section 4, we give the definition of the operator, e.g., the generalized hesitant fuzzy Einstein ordered weighted geometric ($\text{GHFOWG}^{\varepsilon}$) operator, to aggregate a collection of GHFEs. In Section 5, we give the definition of generalized hesitant fuzzy Einstein hybrid geometric aggregation operator and discuss the relationship between the above two aggregation operators. In Section 6, we apply these operators to decision making with generalized hesitant fuzzy information. In Section 7, we have a conclusion.

2. Preliminaries

Definition 2.1. [1] Let X be a fixed set. Then an intuitionistic fuzzy set (IFS) A on X is represented in terms of two functions $\mu : X \to [0,1]$ and $\nu : X \to [0,1]$, with the condition $0 \leq \mu(x) + \nu(x) \leq 1$, for all $x \in X$.

Furthermore, $\pi(x) = 1-\mu(x)-\nu(x)$ is called a hesitancy degree or an intuitionistic index of x in A. In the special case $\pi(x) = 0$, that is, $\mu(x) + \nu(x) = 1$, the IFS A reduces to a FS.

Sometimes, it is difficult to determine the membership of an element into a fixed set which may be caused by a doubt among a set of different values. For the sake of a better description of this situation, Torra introduced the concept of T-HFS as a generalization of fuzzy sets. The membership degree of a T-HFS is presented by several possible values in [0, 1]. The definition is cited as follow.

Definition 2.2 ([23, 24])**.** Let X be a fixed set. Then a hesitant fuzzy set (T-HFS) A on X in terms of a function h is that when applied to X returns a subset of [0, 1], i.e., $A = \{\langle x, h_A(x)\rangle \mid x \in X\}$,
where $h_A(x)$ is a collection of some different values in [0, 1], representing the possible membership degrees of the element $x \in X$ to A. $h_A(x)$ is called a hesitant fuzzy element (HFE).

Later, Gang Qian et.al.[20] defined the generalized hesitant fuzzy set as follows.

Definition 2.3. Let X be a fixed set. Then a generalized hesitant fuzzy set (GHFS) G on X is described as:

$$G = \{\langle x, h(x)\rangle | x \in X\},$$

in which $h(x)$ is a set of some intuitionistic fuzzy sets, denoting the possible membership degrees and nonmembership degrees of the element $x \in X$ to the set G respectively with the conditions:

$$0 \leq \mu_i(x), \nu_i(x) \leq 1, 0 \leq \mu_i(x) + \nu_i(x) \leq 1,\ 1 \leq i \leq N_x = |h(x)|,$$

where $(\mu_i(x), \nu_i(x)) \in h(x)$, for all $x \in X$. And $|h(x)|$ denote the cardinality of the set $h(x)$. $h(x)$ is called a generalized hesitant fuzzy element (GHFE), for each $x \in X$.

For a give GHFE h, its upper and lower bound are denoted by:
- upper bound: $h^+ = max_{i=1,2,...N}\{1 - \nu_i(x)\}$.
- lower bound: $h^- = min_{i=1,2,...N}\{\mu_i(x)\}$.

Remark 2.4. Notice that the number of values in different GHFEs may be different. Suppose that $|h_M(x)|$ stands for the cardinality in $h_M(x)$. Hereafter, the following assumptions are made(see [13, 15, 16]):

(A1) All the elements in each $h_M(x)$ are arranged in increasing order with the membership function.

(A2) If, for some $x \in X$, $|h_M(x)| \neq |h_N(x)|$, then $l_x = max\{|h_M(x)|, |h_N(x)|\}$. To have a correct comparison, the two GHFEs $h_M(x)$ and $h_N(x)$ should have the same length l_x. If there are fewer elements in $h_M(x)$ than $h_N(x)$, an extension of $h_M(x)$ should be considered optimistically by repeating its maximum element until it has the same length with $h_N(x)$.

Remark 2.5. If we arrange the membership sequences in increasing order, then the corresponding non-membership sequence may not be in decreasing or increasing order.

Definition 2.6. Let M, N be two GHFSs on X. Then, M is a generalized hesitant fuzzy subset of N, denote by $M \sqsubseteq N$, if for each $x \in X$, $1 \leq \sigma(j) \leq l_x$, we have

$$\mu_M^{\sigma(j)}(x) \leq \mu_n^{\sigma(j)}(x) \text{ and } \nu_M^{\sigma(j)}(x) \geq \nu_n^{\sigma(j)}(x).$$

Example 2.7. Let $X = \{x_1, x_2, x_3\}$ be the discourse set, and

$M = \{\frac{x_1}{(0.4,0.5)}, \frac{x_2}{(0.4,0.5),(0.5,0.3)}, \frac{x_3}{(0.2,0.7),(0.3,0.6),(0.5,0.4),(0.6,0.3)}\}$,

$N = \{\frac{x_1}{(0.5,0.4),(0.7,0.2)}, \frac{x_2}{(0.6,0.3)}, \frac{x_3}{(0.5,0.3),(0.6,0.2)}\}$,

be two GHFSs on X. Then, in view of Remark 2.4, the GHF sets M and N can be respectively represented as

$M = \{\frac{x_1}{(0.4,0.5),(0.4,0.5)}, \frac{x_2}{(0.4,0.5),(0.5,0.3)}, \frac{x_3}{(0.2,0.7),(0.3,0.6),(0.5,0.4),(0.6,0.3)}\}$,

$N = \{\frac{x_1}{(0.5,0.4),(0.7,0.2)}, \frac{x_2}{(0.6,0.3),(0.6,0.3)}, \frac{x_3}{(0.5,0.3),(0.6,0.2),(0.6,0.2),(0.6,0.2)}\}$.

We can find that $\mu_M^{\sigma(j)}(x_i) \leq \mu_N^{\sigma(j)}(x_i)$ and $\nu_M^{\sigma(j)}(x_i) \geq \nu_N^{\sigma(j)}(x_i)$, for each $x_i \in X$ and each $1 \leq \sigma(j) \leq l_{x_i}$. Then, M is a generalized hesitant fuzzy subset of N and denote by $M \sqsubseteq N$.

Gang Qian et al.[20] also gave the score function and consistency function of GHFE and the comparison law for GHFEs as follows.

Definition 2.8 ([20])**.** Let $\alpha = (\mu_\alpha, \nu_\alpha)$ be an intuitionistic fuzzy set and h be a GHFE.

(i) The expect value of α is defined as:

$$E(\alpha) = \frac{1}{2}(\mu_\alpha + 1 - \nu_\alpha).$$

(ii) The score function of the GHFE h, denoted as $s(h)$, is defined as:

$$s(h) = \frac{1}{N}\sum_{i=1}^{N} E(\alpha_i), \text{ where } \alpha_i \in h, N = |h|.$$

(iii) The consistency function of the GHFE h, denoted as $c(h)$, is defined as:

$$c(h) = 1 - (h^+ - h^-).$$

Definition 2.9 ([20])**.** Let h_1 and h_2 be given two GHFEs.

(i) If $s(h_1) < s(h_2)$, then h_1 is small than h_2, denoted by $h_1 < h_2$.

(ii) Let $s(h_1) = s(h_2)$.

(a) If $c(h_1) < c(h_2)$, then h_1 is small than h_2, denoted by $h_1 < h_2$.

(b) If $c(h_1) = c(h_2)$, then h_1 and h_2 represent the same information, denoted by $h_1 = h_2$.

Gang Qian et al.[20] also defined some operations on the basis of GHFEs h, h_1 and h_2:

(i) $h^\lambda = \{\alpha^\lambda | \alpha \in h\} = \{(\mu_\alpha^\lambda, 1-(1-\nu_\alpha)^\lambda) | \alpha = (\mu_\alpha, \nu_\alpha) \in h\}$,

(ii) $\lambda h = \{\lambda\alpha | \alpha \in h\} = \{(1-(1-\mu_\alpha)^\lambda, \nu_\alpha^\lambda) | \alpha = (\mu_\alpha, \nu_\alpha) \in h\}$,

(iii) $h_1 \bigoplus h_2 = \{\alpha_1 \bigoplus \alpha_2 \mid \alpha_1 \in h_1, \alpha_2 \in h_2\}$
$= \{(\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2}, \nu_{\alpha_1}\nu_{\alpha_2}) \mid \alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1}) \in h_1,$
$\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2}) \in h_2\}$,

(iv) $h_1 \bigotimes h_2 = \{\alpha_1 \bigotimes \alpha_2 \mid \alpha_1 \in h_1, \alpha_2 \in h_2\}$
$= \{(\mu_{\alpha_1}\mu_{\alpha_2}, \nu_{\alpha_1} + \nu_{\alpha_2} - \nu_{\alpha_1}\nu_{\alpha_2}) \mid \alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1}) \in h_1,$
$\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2}) \in h_2\}$.

Recently, Wang and Liu introduced another operation for intuitionistic fuzzy values (IFVs) based on Einstein operations in [25, 26].

Definition 2.10. For three IFVs $\alpha, \alpha_1, \alpha_2$ and let $\lambda > 0$, some Einstein operations laws were given as follows:

(i) $\alpha^\lambda = (\frac{2\mu_\alpha^\lambda}{(2-\mu_\alpha)^\lambda + \mu_\alpha^\lambda}, \frac{(1+\nu_\alpha)^\lambda - (1-\nu_\alpha)^\lambda}{(1+\nu_\alpha)^\lambda + (1-\nu_\alpha)^\lambda})$,

(ii) $\lambda\alpha = (\frac{(1+\mu_\alpha)^\lambda - (1-\mu_\alpha)^\lambda}{(1+\mu_\alpha)^\lambda + (1-\mu_\alpha)^\lambda}, \frac{2\nu_\alpha^\lambda}{(2-\nu_\alpha)^\lambda + \nu_\alpha^\lambda})$,

(iii) $\alpha_1 \bigoplus_\varepsilon \alpha_2 = (\frac{\mu_{\alpha_1} + \mu_{\alpha_2}}{1 + \mu_{\alpha_1}\mu_{\alpha_2}}, \frac{\nu_{\alpha_1}\nu_{\alpha_2}}{1 + (1-\nu_{\alpha_1})(1-\nu_{\alpha_2})})$,

(iv) $\alpha_1 \bigotimes_\varepsilon \alpha_2 = (\frac{\mu_{\alpha_1}\mu_{\alpha_2}}{1 + (1-\mu_{\alpha_1})(1-\mu_{\alpha_2})}, \frac{\nu_{\alpha_1} + \nu_{\alpha_2}}{1 + \nu_{\alpha_1}\nu_{\alpha_2}})$.

Motivated by the idea of Xia, Xu [29], Yu [34] and [37] we defined some Einstein operations for generalized hesitant fuzzy information.

Definition 2.11. Let h, h_1 and h_2 be three GHFEs and $\lambda > 0$. Then the Einstein operations are defined as follows:

(i) $h^\lambda = \{\alpha^\lambda | \alpha \in h\} = \{(\frac{2\mu_\alpha^\lambda}{(2-\mu_\alpha)^\lambda + \mu_\alpha^\lambda}, \frac{(1+\nu_\alpha)^\lambda - (1-\nu_\alpha)^\lambda}{(1+\nu_\alpha)^\lambda + (1-\nu_\alpha)^\lambda}) | \alpha = (\mu_\alpha, \nu_\alpha) \in h\}$,

(ii) $\lambda h = \{\lambda\alpha | \alpha \in h\} = \{(\frac{(1+\mu_\alpha)^\lambda - (1-\mu_\alpha)^\lambda}{(1+\mu_\alpha)^\lambda + (1-\mu_\alpha)^\lambda}, \frac{2\nu_\alpha^\lambda}{(2-\nu_\alpha)^\lambda + \nu_\alpha^\lambda}) | \alpha = (\mu_\alpha, \nu_\alpha) \in h\}$,

(iii) $h_1 \bigoplus h_2 = \{\alpha_1 \bigoplus \alpha_2 \mid \alpha_1 \in h_1, \alpha_2 \in h_2\} = \{(\frac{\mu_{\alpha_1} + \mu_{\alpha_2}}{1 + \mu_{\alpha_1}\mu_{\alpha_2}}, \frac{\nu_{\alpha_1}\nu_{\alpha_2}}{1 + (1-\nu_{\alpha_1})(1-\nu_{\alpha_2})}) \mid$
$\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1}) \in h_1, \alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2}) \in h_2\}$,

(iv) $h_1 \bigotimes h_2 = \{\alpha_1 \bigotimes \alpha_2 \mid \alpha_1 \in h_1, \alpha_2 \in h_2\} = \{(\frac{\mu_{\alpha_1}\mu_{\alpha_2}}{1 + (1-\mu_{\alpha_1})(1-\mu_{\alpha_2})}, \frac{\nu_{\alpha_1} + \nu_{\alpha_2}}{1 + \nu_{\alpha_1}\nu_{\alpha_2}}) \mid$
$\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1}) \in h_1, \alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2}) \in h_2\}$.

It's not hard to find that all operation results are GHFE, too. By the definition, we could get the following properties.

Lemma 2.12. *Let h, h_1 and h_2 be three GHFEs and $\lambda > 0$. Then*

(1) $h_1 \bigotimes h_2 = h_2 \bigotimes h_1$,
(2) $(h_1 \bigotimes h_2) \bigotimes h_3 = h_1 \bigotimes (h_2 \bigotimes h_3)$,
(3) $(h_1 \bigotimes h_2)^\lambda = h_1^\lambda \bigotimes h_2^\lambda$.

3. Generalized hesitant fuzzy Einstein weighted geometric aggregation operators

Motivated by the idea of IFWG operator proposed by Xu and Yager (i.e., Equation 8 described in Section 3.2 of [31]), we defined some Einstein operations for generalized hesitant fuzzy information.

Definition 3.1. Let $h_j (j = 1, 2, \cdots, n)$ be a collection of GHFEs. Then a generalized hesitant fuzzy Einstein weighted geometric (GHFWG$^\varepsilon$) operator of dimension n is a mapping GHFWG$^\varepsilon$: $GHFE^n \to GHFE$, and

$$\mathrm{GHFWG}^\varepsilon_\omega(h_1, h_2, ..., h_n) = \bigotimes_{j=1}^n h_j^{\omega_j} = h_1^{\omega_1} \bigotimes h_2^{\omega_2} \bigotimes ... \bigotimes h_n^{\omega_n},$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma_{j=1}^n \omega_j$ =1.

Especially, if $\omega = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$, then the GHFWG$^\varepsilon$ operator is reduced to the generalized hesitant fuzzy Einstein geometric averaging (GHFGA$^\varepsilon$) operator of dimension n, which is defined as follows:

$$\mathrm{GHFGA}^\varepsilon_\omega(h_1, h_2, ..., h_n) = \bigotimes_{j=1}^n h_j^{\frac{1}{n}} = h_1^{\frac{1}{n}} \bigotimes h_2^{\frac{1}{n}} \bigotimes ... \bigotimes h_n^{\frac{1}{n}}.$$

Clearly, the basic steps of the GHFWG$^\varepsilon$ operator are that based on Einstein scalar multiplication it first weights all the given GHFEs by normalized weight vector, and then aggregates these weighted GHFEs by Einstein product operation.

Theorem 3.2. *Let $h_j (j = 1, 2...n)$ be a collection of GHFEs, and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma_{j=1}^n \omega_j$ =1. Then, their aggregated value by using the GHFWG$^\varepsilon$ operator is also a GHFE, and*

$$GHFWG^\varepsilon_\omega(h_1, h_2, ..., h_n)$$
$$= \{(\frac{2\prod_{j=1}^n \mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^n (2-\mu_{\alpha_j})^{\omega_j} + \prod_{j=1}^n \mu_{\alpha_j}^{\omega_j}}, \frac{\prod_{j=1}^n (1+\nu_{\alpha_j})^{\omega_j} - \prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^n (1+\nu_{\alpha_j})^{\omega_j} + \prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}})$$
$$|\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1...n\}.$$

Proof. In the following, we prove it by using mathematical induction on n.

For $n = 2$: Clearly

$$h_1^{\omega_1} = \{(\frac{2\mu_{\alpha_1}^{\omega_1}}{(2-\mu_{\alpha_1})^{\omega_1} + \mu_{\alpha_1}^{\omega_1}}, \frac{(1+\nu_{\alpha_1})^{\omega_1} - (1-\nu_{\alpha_1})^{\omega_1}}{(1+\nu_{\alpha_1})^{\omega_1} + (1-\nu_{\alpha_1})^{\omega_1}})|\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1}) \in h_1\}$$

and

$$h_2^{\omega_2} = \{(\frac{2\mu_{\alpha_2}^{\omega_2}}{(2-\mu_{\alpha_2})^{\omega_2} + \mu_{\alpha_2}^{\omega_2}}, \frac{(1+\nu_{\alpha_2})^{\omega_2} - (1-\nu_{\alpha_2})^{\omega_2}}{(1+\nu_{\alpha_2})^{\omega_2} + (1-\nu_{\alpha_2})^{\omega_2}})|\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2}) \in h_2\}.$$

Then, by Definition 2.11 (iv), we have

$$h_1^{\omega_1} \bigotimes h_2^{\omega_2}$$
$$= \{(\frac{\frac{2\mu_{\alpha_1}^{\omega_1}}{(2-\mu_{\alpha_1})^{\omega_1} + \mu_{\alpha_1}^{\omega_1}} \frac{2\mu_{\alpha_2}^{\omega_2}}{(2-\mu_{\alpha_2})^{\omega_2} + \mu_{\alpha_2}^{\omega_2}}}{1 + (1 - \frac{2\mu_{\alpha_1}^{\omega_1}}{(2-\mu_{\alpha_1})^{\omega_1} + \mu_{\alpha_1}^{\omega_1}})(1 - \frac{2\mu_{\alpha_2}^{\omega_2}}{(2-\mu_{\alpha_2})^{\omega_2} + \mu_{\alpha_2}^{\omega_2}})},$$

$$\frac{\frac{(1+\nu_{\alpha_1})^{\omega_1}-(1-\nu_{\alpha_1})^{\omega_1}}{(1+\nu_{\alpha_1})^{\omega_1}+(1-\nu_{\alpha_1})^{\omega_1}}+\frac{(1+\nu_{\alpha_2})^{\omega_2}-(1-\nu_{\alpha_2})^{\omega_2}}{(1+\nu_{\alpha_2})^{\omega_2}+(1-\nu_{\alpha_2})^{\omega_2}}}{1+\frac{(1+\nu_{\alpha_1})^{\omega_1}-(1-\nu_{\alpha_1})^{\omega_1}}{(1+\nu_{\alpha_1})^{\omega_1}+(1-\nu_{\alpha_1})^{\omega_1}}\frac{(1+\nu_{\alpha_2})^{\omega_2}-(1-\nu_{\alpha_2})^{\omega_2}}{(1+\nu_{\alpha_2})^{\omega_2}+(1-\nu_{\alpha_2})^{\omega_2}}}) \mid \alpha_1=(\mu_{\alpha_1},\nu_{\alpha_1})\in h_1, \alpha_2=(\mu_{\alpha_2},\nu_{\alpha_2})\in h_2\}$$

$$=\{(\frac{2\prod_{j=1}^{2}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{2}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{2}\mu_{\alpha_j}^{\omega_j}},\frac{\prod_{j=1}^{2}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{2}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{2}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{2}(1-\nu_{\alpha_j})^{\omega_j}})| \alpha_j=(\mu_{\alpha_j},\nu_{\alpha_j})\in h_j, j=1,2\}.$$

If it holds for $n=k$, that is,

$$\bigotimes_{j=1}^{k}h_j^{\omega_j}=\{(\frac{2\prod_{j=1}^{k}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{k}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{k}\mu_{\alpha_j}^{\omega_j}}, \frac{\prod_{j=1}^{k}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{k}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{k}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{k}(1-\nu_{\alpha_j})^{\omega_j}})|\alpha_j=(\mu_{\alpha_j},\nu_{\alpha_j})\in h_j, j=1...k\}.$$

then, when $n=k+1$, by the Einstein operations of GHFEs, we have

$$\bigotimes_{j=1}^{k+1}h_j^{\omega_j}=\bigotimes_{j=1}^{k}h_j^{\omega_j}\bigotimes(h_{k+1}^{\omega_{k+1}})$$
$$=\{(\frac{2\prod_{j=1}^{k}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{k}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{k}\mu_{\alpha_j}^{\omega_j}}, \frac{\prod_{j=1}^{k}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{k}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{k}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{k}(1-\nu_{\alpha_j})^{\omega_j}})|\alpha_j=(\mu_{\alpha_j},\nu_{\alpha_j})\in h_j, j=1...k\}$$

and

$$\bigotimes\{(\frac{2\mu_{\alpha_{k+1}}^{\omega_{k+1}}}{(2-\mu_{\alpha_{k+1}})^{\omega_{k+1}}+\mu_{\alpha_{k+1}}^{\omega_{k+1}}}, \frac{(1+\nu_{\alpha_{k+1}})^{\omega_{k+1}}-(1-\nu_{\alpha_{k+1}})^{\omega_{k+1}}}{(1+\nu_{\alpha_{k+1}})^{\omega_{k+1}}+(1-\nu_{\alpha_{k+1}})^{\omega_{k+1}}})|\alpha_{k+1}=(\mu_{\alpha_{k+1}},\nu_{\alpha_{k+1}})\in h_{k+1}\}$$
$$=\{(\frac{2\prod_{j=1}^{k+1}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{k+1}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{k+1}\mu_{\alpha_j}^{\omega_j}}, \frac{\prod_{j=1}^{k+1}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{k+1}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{k+1}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{k+1}(1-\nu_{\alpha_j})^{\omega_j}})|\alpha_j=(\mu_{\alpha_j},\nu_{\alpha_j})\in h_j, j=1...k+1\}.$$

That is, it holds for $n=k+1$. Thus, it holds for all n. So

$$\text{GHFWG}_{\omega}^{\varepsilon}(h_1,h_2,...,h_n)$$
$$=\{(\frac{2\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}, \frac{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}) |\alpha_j=(\mu_{\alpha_j},\nu_{\alpha_j})\in h_j, j=1...n\}.$$ □

Note. Especially, if $\nu_{\alpha_j}=1-\mu_{\alpha_j}$, for all $j=1,2,...,n$, e.g., if $h_j(j=1,2...n)$ is a collection of HFEs, then Theorem 3.2 is reduced to the following form:

$$\text{HFEWG}_{\omega}^{\varepsilon}(h_1,h_2,...,h_n)=\{\frac{2\prod_{j=1}^{n}\mu_j^{\omega_j}}{\prod_{j=1}^{n}(2-\mu_j)^{\omega_j}+\prod_{j=1}^{n}\mu_j^{\omega_j}}|\mu_j\in h_j, j=1...n\}.$$

which is the Definition 11 in [37] and Theorem 3 in [34].

Now we shall analyze the relationship between the GHFWG operator and IHFWG operator proposed by Zhang (i.e., Equation 68 in [36]). Firstly we recall the following lemma.

Lemma 3.3. [30, 22] *Let* $x_j>0$, $\omega_j>0$, $(j=1,2,...,n)$ *and* $\Sigma_{j=1}^{n}\omega_j=1$. *Then,*

$\prod_{j=1}^{n}x_j^{\omega_j}\leq\sum_{j=1}^{n}\omega_jx_j$, *with equality if and only if* $x_1=x_2=...=x_n$.

Theorem 3.4. *Let* $h_j(j=1,2...n)$ *be a collection of GHFEs, and* $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ *be the weight vector of* $h_j(j=1,2,...,n)$ *with* $\omega_j\in[0,1]$ *and* $\Sigma_{j=1}^{n}\omega_j=1$. *Then,*

$$IHFWG_{\omega}(h_1,h_2,...,h_n)\leq GHFWG_{\omega}^{\varepsilon}(h_1,h_2,...,h_n).$$

Proof. Since

$$\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}\leq\sum_{j=1}^{n}\omega_j(2-\mu_{\alpha_j})+\sum_{j=1}^{n}\omega_j\mu_{\alpha_j}=2,$$

by lemma 3.3,

$\frac{2\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}\geq\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}$ with equality if and only if $\mu_{\alpha_j}(j=1,2,...,n)$ are equal.

Also since

$$\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}\leq\sum_{j=1}^{n}\omega_j(1+\nu_{\alpha_j})+\sum_{j=1}^{n}\omega_j(1-\nu_{\alpha_j})=2,$$

$$\begin{aligned}\frac{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}&=1-\frac{2\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}\\&\leq 1-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}\end{aligned}$$

with equality if and only if $\nu_{\alpha_j}(j=1,2,...,n)$ are equal.

Note that $h_j=\{\alpha_j=(\mu_{\alpha_j},\nu_{\alpha_j})\}$ and by Definition 2.9, we complete the proof of Theorem 3.4. □

Theorem 3.4 tell us that the $GHFWG^{\varepsilon}$ operator shows the decision maker's more optimistic attitude than the $IHFWG$ operator proposed by Zhang [36] in aggregation process. To illustrate that, we give the following example.

Example 3.5. Let $h_1=\{(0.1,0.4),(0.3,0.5),(0.4,0.2)\}$ and $h_2=\{(0.5,0.2),(0.6,0.1)\}$ be two $GHFEs$, $\omega=(0.4,0.6)^T$ be the weight vector of them, then by Theorem 3.2, we have

$$\begin{aligned}&GHFWG_{\omega}^{\varepsilon}(h_1,h_2)\\&=\{(\frac{2\prod_{j=1}^{2}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{2}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{2}\mu_{\alpha_j}^{\omega_j}},\frac{\prod_{j=1}^{2}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{2}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{2}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{2}(1-\nu_{\alpha_j})^{\omega_j}})\\&\qquad|\alpha_j=(\mu_{\alpha_j},\nu_{\alpha_j})\in h_j,j=1,2\}\\&=\{(\frac{2\times0.1^{0.4}\times0.5^{0.6}}{(2-0.1)^{0.4}\times(2-0.5)^{0.6}+0.1^{0.4}\times0.5^{0.6}},\frac{(1+0.4)^{0.4}\times(1+0.2)^{0.6}-(1-0.4)^{0.4}\times(1-0.2)^{0.6}}{(1+0.4)^{0.4}\times(1+0.2)^{0.6}+(1-0.4)^{0.4}\times(1-0.2)^{0.6}}),\\&\quad(\frac{2\times0.1^{0.4}\times0.6^{0.6}}{(2-0.1)^{0.4}\times(2-0.6)^{0.6}+0.1^{0.4}\times0.6^{0.6}},\frac{(1+0.4)^{0.4}\times(1+0.1)^{0.6}-(1-0.4)^{0.4}\times(1-0.1)^{0.6}}{(1+0.4)^{0.4}\times(1+0.1)^{0.6}+(1-0.4)^{0.4}\times(1-0.1)^{0.6}}),\\&\quad(\frac{2\times0.3^{0.4}\times0.5^{0.6}}{(2-0.3)^{0.4}\times(2-0.5)^{0.6}+0.3^{0.4}\times0.5^{0.6}},\frac{(1+0.5)^{0.4}\times(1+0.2)^{0.6}-(1-0.5)^{0.4}\times(1-0.2)^{0.6}}{(1+0.5)^{0.4}\times(1+0.2)^{0.6}+(1-0.5)^{0.4}\times(1-0.2)^{0.6}}),\\&\quad(\frac{2\times0.3^{0.4}\times0.6^{0.6}}{(2-0.3)^{0.4}\times(2-0.6)^{0.6}+0.3^{0.4}\times0.6^{0.6}},\frac{(1+0.5)^{0.4}\times(1+0.1)^{0.6}-(1-0.5)^{0.4}\times(1-0.1)^{0.6}}{(1+0.5)^{0.4}\times(1+0.1)^{0.6}+(1-0.5)^{0.4}\times(1-0.1)^{0.6}}),\\&\quad(\frac{2\times0.4^{0.4}\times0.5^{0.6}}{(2-0.4)^{0.4}\times(2-0.5)^{0.6}+0.4^{0.4}\times0.5^{0.6}},\frac{(1+0.2)^{0.4}\times(1+0.2)^{0.6}-(1-0.2)^{0.4}\times(1-0.2)^{0.6}}{(1+0.2)^{0.4}\times(1+0.2)^{0.6}+(1-0.2)^{0.4}\times(1-0.2)^{0.6}}),\\&\quad(\frac{2\times0.4^{0.4}\times0.6^{0.6}}{(2-0.4)^{0.4}\times(2-0.6)^{0.6}+0.4^{0.4}\times0.6^{0.6}},\frac{(1+0.2)^{0.4}\times(1+0.1)^{0.6}-(1-0.2)^{0.4}\times(1-0.1)^{0.6}}{(1+0.2)^{0.4}\times(1+0.1)^{0.6}+(1-0.2)^{0.4}\times(1-0.1)^{0.6}})\}\\&=\{(0.2748,0.2831),(0.3126,0.2257),(0.4108,0.3287),(0.4622,0.2728),\\&\quad(0.4581,0.2),(0.5135,0.1404)\}.\end{aligned}$$

And by Equation 68 in [36], we have

$$\begin{aligned}&IHFWG_{\omega}(h_1,h_2)\\&=\{(\prod_{j=1}^{2}\mu_{\alpha_j}^{\omega_j},1-\prod_{j=1}^{2}(1-\nu_{\alpha_j})^{\omega_j})|\alpha_j=(\mu_{\alpha_j},\nu_{\alpha_j})\in h_j,j=1,2\}\\&=\{(0.1^{0.4}\times0.5^{0.6},1-(1-0.4)^{0.4}\times(1-0.2)^{0.6}),(0.1^{0.4}\times0.6^{0.6},\\&\qquad1-(1-0.4)^{0.4}\times(1-0.1)^{0.6}),\\&\quad(0.3^{0.4}\times0.5^{0.6},1-(1-0.5)^{0.4}\times(1-0.2)^{0.6}),(0.3^{0.4}\times0.6^{0.6},\\&\qquad1-(1-0.5)^{0.4}\times(1-0.1)^{0.6}),\\&\quad(0.4^{0.4}\times0.5^{0.6},1-(1-0.2)^{0.4}\times(1-0.2)^{0.6}),(0.4^{0.4}\times0.6^{0.6},\\&\qquad1-(1-0.2)^{0.4}\times(1-0.1)^{0.6})\}\\&=\{(0.2627,0.2870),(0.2930,0.2347),(0.4076,0.3371),(0.4547,0.2886),\end{aligned}$$

$(0.4573, 0.2), (0.5102, 0.1414)\}.$

By simple computing,

$$s(GHFWG^{\varepsilon}_{\omega}(h_1, h_2))$$
$$= \frac{1}{6}\cdot\frac{1}{2}[(6 + 0.2748 + 0.3126 + 0.4108 + 0.4622 + 0.4581 + 0.5135) - (0.2831 + 0.2257 + 0.3287 + 0.2728 + 0.2 + 0.1404)]$$
$$= \frac{1}{12}(8.45948 - 1.4507) = 0.584065$$

and

$$s(IHFWG_{\omega}(h_1, h_2))$$
$$= \frac{1}{6}\cdot\frac{1}{2}[(6 + 0.2627 + 0.2930 + 0.4076 + 0.4547 + 0.4573 + 0.5102) - (0.2870 + 0.2347 + 0.3371 + 0.2886 + 0.2 + 0.1414)]$$
$$= \frac{1}{12}(8.3855 - 1.4888) = 0.574725.$$

Thus we have

$$IHFWG_{\omega}(h_1, h_2) \le GHFWG^{\varepsilon}_{\omega}(h_1, h_2).$$

Theorem 3.6. *Let $h_j (j = 1, 2...n)$ be a collection of GHFEs, and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma^n_{j=1}\omega_j = 1$. Then,*

$$GHFWG^{\varepsilon}_{\omega}(h^{\lambda}_1, h^{\lambda}_2, ..., h^{\lambda}_n) = (GHFWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n))^{\lambda}, \text{ where } \lambda > 0.$$

Proof. Since for all $j = 1, 2, ..., n$,

$$h^{\lambda}_j = \{(\frac{2\mu^{\lambda}_{\alpha_j}}{(2-\mu_{\alpha_j})^{\lambda}+\mu^{\lambda}_{\alpha_j}}, \frac{(1+\nu_{\alpha_j})^{\lambda}-(1-\nu_{\alpha_j})^{\lambda}}{(1+\nu_{\alpha_j})^{\lambda}+(1-\nu_{\alpha_j})^{\lambda}})|\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j\},$$

by Theorem 3.2, we have

$$GHFWG^{\varepsilon}_{\omega}(h^{\lambda}_1, h^{\lambda}_2, ..., h^{\lambda}_n)$$
$$= \{(\frac{2\prod^n_{j=1}(\frac{2\mu^{\lambda}_{\alpha_j}}{(2-\mu_{\alpha_j})^{\lambda}+\mu^{\lambda}_{\alpha_j}})^{\omega_j}}{\prod^n_{j=1}(2-\frac{2\mu^{\lambda}_{\alpha_j}}{(2-\mu_{\alpha_j})^{\lambda}+\mu^{\lambda}_{\alpha_j}})^{\omega_j}+\prod^n_{j=1}(\frac{2\mu^{\lambda}_{\alpha_j}}{(2-\mu_{\alpha_j})^{\lambda}+\mu^{\lambda}_{\alpha_j}})^{\omega_j}},$$
$$\frac{\prod^n_{j=1}(1+\frac{(1+\nu_{\alpha_j})^{\lambda}-(1-\nu_{\alpha_j})^{\lambda}}{(1+\nu_{\alpha_j})^{\lambda}+(1-\nu_{\alpha_j})^{\lambda}})^{\omega_j}-\prod^n_{j=1}(1-\frac{(1+\nu_{\alpha_j})^{\lambda}-(1-\nu_{\alpha_j})^{\lambda}}{(1+\nu_{\alpha_j})^{\lambda}+(1-\nu_{\alpha_j})^{\lambda}})^{\omega_j}}{\prod^n_{j=1}(1+\frac{(1+\nu_{\alpha_j})^{\lambda}-(1-\nu_{\alpha_j})^{\lambda}}{(1+\nu_{\alpha_j})^{\lambda}+(1-\nu_{\alpha_j})^{\lambda}})^{\omega_j}+\prod^n_{j=1}(1-\frac{(1+\nu_{\alpha_j})^{\lambda}-(1-\nu_{\alpha_j})^{\lambda}}{(1+\nu_{\alpha_j})^{\lambda}+(1-\nu_{\alpha_j})^{\lambda}})^{\omega_j}})|$$
$$\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1...n\}$$
$$= \{(\frac{2\prod^n_{j=1}\mu^{\lambda\omega_j}_{\alpha_j}}{\prod^n_{j=1}(2-\mu_{\alpha_j})^{\lambda\omega_j}+\prod^n_{j=1}\mu^{\lambda\omega_j}_{\alpha_j}}, \frac{\prod^n_{j=1}(1+\nu_{\alpha_j})^{\lambda\omega_j}-\prod^n_{j=1}(1-\nu_{\alpha_j})^{\lambda\omega_j}}{\prod^n_{j=1}(1+\nu_{\alpha_j})^{\lambda\omega_j}+\prod^n_{j=1}(1-\nu_{\alpha_j})^{\lambda\omega_j}})|$$
$$\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1...n\}.$$

Since for all $j = 1, 2, ..., n$,

$$h^{\lambda}_j = \{(\frac{2\mu^{\lambda}_{\alpha_j}}{(2-\mu_{\alpha_j})^{\lambda}+\mu^{\lambda}_{\alpha_j}}, \frac{(1+\nu_{\alpha_j})^{\lambda}-(1-\nu_{\alpha_j})^{\lambda}}{(1+\nu_{\alpha_j})^{\lambda}+(1-\nu_{\alpha_j})^{\lambda}})|\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j\},$$

$$GHFWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n)$$
$$= \{(\frac{2\prod^n_{j=1}\mu^{\omega_j}_{\alpha_j}}{\prod^n_{j=1}(2-\mu_{\alpha_j})^{\omega_j}+\prod^n_{j=1}\mu^{\omega_j}_{\alpha_j}}, \frac{\prod^n_{j=1}(1+\nu_{\alpha_j})^{\omega_j}-\prod^n_{j=1}(1-\nu_{\alpha_j})^{\omega_j}}{\prod^n_{j=1}(1+\nu_{\alpha_j})^{\omega_j}+\prod^n_{j=1}(1-\nu_{\alpha_j})^{\omega_j}})|$$
$$\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1...n\}.$$

Then, by Definition 2.11(i), we have

$$(GHFWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n))^{\lambda}$$

$$= \{(\frac{2(\frac{2\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}})^{\lambda}}{(2-\frac{2\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}})^{\lambda}+(\frac{2\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}})^{\lambda}},$$

$$\frac{(1+\frac{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}})^{\lambda}-(1-\frac{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}})^{\lambda}}{(1+\frac{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}})^{\lambda}+(1-\frac{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}})^{\lambda}})|$$

$$\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j\}.$$

$$= \{(\frac{2\prod_{j=1}^{n}\mu_{\alpha_j}^{\lambda\omega_j}}{\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\lambda\omega_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{\lambda\omega_j}}, \frac{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\lambda\omega_j}-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\lambda\omega_j}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\lambda\omega_j}+\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\lambda\omega_j}})|$$

$$\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1...n\}.$$

So, we complete the proof. □

Corollary 3.7. *Let $h_j (j = 1, 2...n)$ be a collection of HFEs, and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma_{j=1}^{n}\omega_j$ =1. Then*

$$HFWG_{\omega}^{\varepsilon}(h_1^{\lambda}, h_2^{\lambda}, ..., h_n^{\lambda}) = (HFWG_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n))^{\lambda}, \text{ where } \lambda > 0.$$

Proof. By Theorem 3.6 and Note, it can be easily proved. □

Theorem 3.8. *Let $h_j (j = 1, 2...n)$ be a collection of GHFEs, and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma_{j=1}^{n}\omega_j$ =1. Then,*

$$GHFWG_{\omega}^{\varepsilon}(h_1 \bigotimes h, h_2 \bigotimes h, ..., h_n \bigotimes h) = GHFWG_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n) \bigotimes h.$$

Proof. Since

$$h_1 \bigotimes h_2 = \{\alpha_1 \bigotimes \alpha_2 \mid \alpha_1 \in h_1, \alpha_2 \in h_2\}$$

$$= \{(\frac{\mu_{\alpha_1}\mu_{\alpha_2}}{1+(1-\mu_{\alpha_1})(1-\mu_{\alpha_2})}, \frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{1+\nu_{\alpha_1}\nu_{\alpha_2}}) \mid \alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1}) \in h_1, \alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2}) \in h_2\},$$

by Theorem 3.2,

$$GHFWG_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n)$$

$$= \{(\frac{2\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}, \frac{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}})$$

$$|\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1...n\}.$$

Then

$$GHFWG_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n) \bigotimes h$$

$$= \{(\frac{\frac{2\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}\mu}{1+(1-\frac{2\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}})(1-\mu)}, \frac{\frac{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}+\nu}{1+\frac{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}\nu}) \mid$$

$$\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1...n, \alpha = (\mu, \nu) \in h\}$$

$$= \{(\frac{2\mu\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}{(2-\mu)\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\omega_j}+\mu\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}, \frac{(1+\nu)\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}-(1-\nu)\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}{(1+\nu)\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}+(1-\nu)\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}})$$

$$|\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1...n, \alpha = (\mu, \nu) \in h\}.$$

Since

$$h_j \bigotimes h$$

$$= \{(\frac{\mu_{\alpha_j}\mu}{1+(1-\mu_{\alpha_j})(1-\mu)}, \frac{\nu_{\alpha_j}+\nu}{1+\nu_{\alpha_j}\nu}) \mid \alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1...n, \alpha = (\mu, \nu) \in h\},$$

by Theorem 3.2

$$GHFWG_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n)$$

$$= \{(\frac{2\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}, \frac{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}})| \alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1...n\}.$$

Thus

$$GHFWG_{\omega}^{\varepsilon}(h_1 \bigotimes h, h_2 \bigotimes h, ..., h_n \bigotimes h)$$

$$= \{(\frac{2\prod_{j=1}^{n}(\frac{\mu_{\alpha_j}\mu}{1+(1-\mu_{\alpha_j})(1-\mu)})^{\omega_j}}{\prod_{j=1}^{n}(2-\frac{\mu_{\alpha_j}\mu}{1+(1-\mu_{\alpha_j})(1-\mu)})^{\omega_j}+\prod_{j=1}^{n}(\frac{\mu_{\alpha_j}\mu}{1+(1-\mu_{\alpha_j})(1-\mu)})^{\omega_j}},$$

$$\frac{\prod_{j=1}^{n}(1+\frac{\nu_{\alpha_j}+\nu}{1+\nu_{\alpha_j}\nu})^{\omega_j}-\prod_{j=1}^{n}(1-\frac{\nu_{\alpha_j}+\nu}{1+\nu_{\alpha_j}\nu})^{\omega_j}}{\prod_{j=1}^{n}(1+\frac{\nu_{\alpha_j}+\nu}{1+\nu_{\alpha_j}\nu})^{\omega_j}+\prod_{j=1}^{n}(1-\frac{\nu_{\alpha_j}+\nu}{1+\nu_{\alpha_j}\nu})^{\omega_j}})|$$

$$\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1...n, \alpha = (\mu, \nu) \in h\}$$

$$= \{(\frac{2\prod_{j=1}^{n}\mu^{\omega_j}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{n}(2-\mu)^{\omega_j}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}\mu^{\omega_j}\mu_{\alpha_j}^{\omega_j}},$$

$$\frac{\prod_{j=1}^{n}(1+\nu)^{\omega_j}(1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}(1-\nu)^{\omega_j}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{n}(1+\nu)^{\omega_j}(1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}(1-\nu)^{\omega_j}(1-\nu_{\alpha_j})^{\omega_j}})$$

$$|\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1...n, \alpha = (\mu, \nu) \in h\}$$

$$= \{(\frac{2\mu^{\sum_{j=1}^{n}\omega_j}\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}{(2-\mu)^{\sum_{j=1}^{n}\omega_j}\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\omega_j}+\mu^{\sum_{j=1}^{n}\omega_j}\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}},$$

$$\frac{(1+\nu)^{\sum_{j=1}^{n}\omega_j}\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}-(1-\nu)^{\sum_{j=1}^{n}\omega_j}\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}{(1+\nu)^{\sum_{j=1}^{n}\omega_j}\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}+(1-\nu)^{\sum_{j=1}^{n}\omega_j}\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}})$$

$$|\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1...n, \alpha = (\mu, \nu) \in h\}.$$

$$= \{(\frac{2\mu\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}{(2-\mu)\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\omega_j}+\mu\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}, \frac{(1+\nu)\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}-(1-\nu)\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}{(1+\nu)\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j}+(1-\nu)\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}})$$

$$|\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1...n, \alpha = (\mu, \nu) \in h\}.$$

So we complete the proof. □

Corollary 3.9 ([37]). *Let $h_j (j = 1, 2...n)$ be a collection of HFEs, and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma_{j=1}^{n}\omega_j = 1$. Then,*

$$HFWG_{\omega}^{\varepsilon}(h_1 \bigotimes h, h_2 \bigotimes h, ..., h_n \bigotimes h) = HFWG_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n) \bigotimes h.$$

Proof. By Theorem 3.8 and Note, it can be easily proved. □

Theorem 3.10. *Let $h_j (j = 1, 2...n)$ be a collection of GHFEs, and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma_{j=1}^{n}\omega_j = 1$. Then,*

$$GHFWG_{\omega}^{\varepsilon}(h_1^{\lambda} \bigotimes h^{\lambda}, h_2^{\lambda} \bigotimes h^{\lambda}, ..., h_n^{\lambda} \bigotimes h^{\lambda}) = (GHFWG_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n) \bigotimes h)^{\lambda},$$

where $\lambda > 0$.

Proof. By Theorem 3.6, Theorem 3.8 and Prop 2.12 (3), we have

$$(GHFWG_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n) \bigotimes h)^{\lambda}$$
$$= (GHFWG_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n))^{\lambda} \bigotimes h^{\lambda}$$
$$= GHFWG_{\omega}^{\varepsilon}(h_1^{\lambda}, h_2^{\lambda}, ..., h_n^{\lambda}) \bigotimes h^{\lambda}$$
$$= GHFWG_{\omega}^{\varepsilon}(h_1^{\lambda} \bigotimes h^{\lambda}, h_2^{\lambda} \bigotimes h^{\lambda}, ..., h_n^{\lambda} \bigotimes h^{\lambda}).$$

□

Corollary 3.11 ([37]). *Let $h_j (j = 1, 2...n)$ be a collection of HFEs, and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma_{j=1}^{n}\omega_j = 1$. Then,*

$$HFWG^{\varepsilon}_{\omega}(h_1^{\lambda}\bigotimes h^{\lambda}, h_2^{\lambda}\bigotimes h^{\lambda}, ..., h_n^{\lambda}\bigotimes h^{\lambda}) = (HFWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n)\bigotimes h)^{\lambda},$$

where $\lambda > 0$.

Proof. By Theorem 3.10 and Note, it can be easily proved. □

Theorem 3.12. *Let* $h_j(j = 1, 2...n)$ *be a collection of GHFEs, and* $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ *be the weight vector of* $h_j(j = 1, 2, ..., n)$ *with* $\omega_j \in [0, 1]$ *and* $\Sigma_{j=1}^n \omega_j = 1$. *Then,*

$$GHFWG^{\varepsilon}_{\omega}(h_1\bigotimes h_1^{'}, h_2\bigotimes h_2^{'}, ..., h_n\bigotimes h_n^{'}) = GHFWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n)\bigotimes GHFWG^{\varepsilon}_{\omega}(h_1^{'}, h_2^{'}, ..., h_n^{'}).$$

Proof. Since for all $j = 1...n$,

$$h_j\bigotimes h_j^{'} = \{(\frac{\mu_{\alpha_j}\mu^{'}_{\alpha_j^{'}}}{1+(1-\mu_{\alpha_j})(1-\mu^{'}_{\alpha_j^{'}})}, \frac{\nu_{\alpha_j}+\nu^{'}_{\alpha_j^{'}}}{1+\nu_{\alpha_j}\nu^{'}_{\alpha_j^{'}}}) \mid \alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, \alpha_j^{'} = (\mu^{'}_{\alpha_j^{'}}, \nu^{'}_{\alpha_j^{'}}) \in h_j^{'}\},$$

by Theorem 3.2,

$$GHFWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n) = \{(\frac{2\prod_{j=1}^n \mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^n (2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^n \mu_{\alpha_j}^{\omega_j}}, \frac{\prod_{j=1}^n (1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^n (1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}}) |\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1...n\}.$$

Then, we have

$$GHFWG^{\varepsilon}_{\omega}(h_1\bigotimes h_1^{'}, h_2\bigotimes h_2^{'}, ..., h_n\bigotimes h_n^{'})$$

$$= \{(\frac{2\prod_{j=1}^n (\frac{\mu_{\alpha_j}\mu^{'}_{\alpha_j^{'}}}{1+(1-\mu_{\alpha_j})(1-\mu^{'}_{\alpha_j^{'}})})^{\omega_j}}{\prod_{j=1}^n (2-(\frac{\mu_{\alpha_j}\mu^{'}_{\alpha_j^{'}}}{1+(1-\mu_{\alpha_j})(1-\mu^{'}_{\alpha_j^{'}})}))^{\omega_j}+\prod_{j=1}^n (\frac{\mu_{\alpha_j}\mu^{'}_{\alpha_j^{'}}}{1+(1-\mu_{\alpha_j})(1-\mu^{'}_{\alpha_j^{'}})})^{\omega_j}},$$

$$\frac{\prod_{j=1}^n (1+\frac{\nu_{\alpha_j}+\nu^{'}_{\alpha_j^{'}}}{1+\nu_{\alpha_j}\nu^{'}_{\alpha_j^{'}}})^{\omega_j}-\prod_{j=1}^n (1-\frac{\nu_{\alpha_j}+\nu^{'}_{\alpha_j^{'}}}{1+\nu_{\alpha_j}\nu^{'}_{\alpha_j^{'}}})^{\omega_j}}{\prod_{j=1}^n (1+\frac{\nu_{\alpha_j}+\nu^{'}_{\alpha_j^{'}}}{1+\nu_{\alpha_j}\nu^{'}_{\alpha_j^{'}}})^{\omega_j}+\prod_{j=1}^n (1-\frac{\nu_{\alpha_j}+\nu^{'}_{\alpha_j^{'}}}{1+\nu_{\alpha_j}\nu^{'}_{\alpha_j^{'}}})^{\omega_j}})|\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j,$$

$$\alpha_j^{'} = (\mu^{'}_{\alpha_j^{'}}, \nu^{'}_{\alpha_j^{'}}) \in h_j^{'}, j = 1...n\}$$

$$= \{(\frac{2\prod_{j=1}^n (\mu_{\alpha_j}\mu^{'}_{\alpha_j^{'}})^{\omega_j}}{\prod_{j=1}^n (2-\mu_{\alpha_j})^{\omega_j}(2-\mu^{'}_{\alpha_j^{'}})^{\omega_j}+\prod_{j=1}^n (\mu_{\alpha_j}\mu^{'}_{\alpha_j^{'}})^{\omega_j}},$$

$$\frac{\prod_{j=1}^n (1+\nu_{\alpha_j})^{\omega_j}(1+\nu^{'}_{\alpha_j^{'}})^{\omega_j}-\prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}(1-\nu^{'}_{\alpha_j^{'}})^{\omega_j}}{\prod_{j=1}^n (1+\nu_{\alpha_j})^{\omega_j}(1+\nu^{'}_{\alpha_j^{'}})^{\omega_j}+\prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}(1-\nu^{'}_{\alpha_j^{'}})^{\omega_j}})$$

$$|\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, \alpha_j^{'} = (\mu^{'}_{\alpha_j^{'}}, \nu^{'}_{\alpha_j^{'}}) \in h_j^{'}, j = 1...n\}.$$

Since

$$GHFWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n)\bigotimes GHFWG^{\varepsilon}_{\omega}(h_1^{'}, h_2^{'}, ..., h_n^{'})$$

$$= \{(\frac{\frac{2\prod_{j=1}^n \mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^n (2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^n \mu_{\alpha_j}^{\omega_j}} \frac{2\prod_{j=1}^n \mu'^{\omega_j}_{\alpha'_j}}{\prod_{j=1}^n (2-\mu'_{\alpha'_j})^{\omega_j}+\prod_{j=1}^n \mu'^{\omega_j}_{\alpha'_j}}}{1+(1-\frac{2\prod_{j=1}^n \mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^n (2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^n \mu_{\alpha_j}^{\omega_j}})(1-\frac{2\prod_{j=1}^n \mu'^{\omega_j}_{\alpha'_j}}{\prod_{j=1}^n (2-\mu'_{\alpha'_j})^{\omega_j}+\prod_{j=1}^n \mu'^{\omega_j}_{\alpha'_j}})},$$

$$\frac{\frac{\prod_{j=1}^n (1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^n (1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}}+\frac{\prod_{j=1}^n (1+\nu'_{\alpha'_j})^{\omega_j}-\prod_{j=1}^n (1-\nu'_{\alpha'_j})^{\omega_j}}{\prod_{j=1}^n (1+\nu'_{\alpha'_j})^{\omega_j}+\prod_{j=1}^n (1-\nu'_{\alpha'_j})^{\omega_j}}}{1+\frac{\prod_{j=1}^n (1+\nu_{\alpha_j})^{\omega_j}-\prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^n (1+\nu_{\alpha_j})^{\omega_j}+\prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}}\frac{\prod_{j=1}^n (1+\nu'_{\alpha'_j})^{\omega_j}-\prod_{j=1}^n (1-\nu'_{\alpha'_j})^{\omega_j}}{\prod_{j=1}^n (1+\nu'_{\alpha'_j})^{\omega_j}+\prod_{j=1}^n (1-\nu'_{\alpha'_j})^{\omega_j}}})$$

$$|\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, \alpha'_j = (\mu'_{\alpha'_j}, \nu'_{\alpha'_j}) \in h'_j, j = 1...n\}$$

$$= \{(\frac{2\prod_{j=1}^n (\mu_{\alpha_j}\mu'_{\alpha'_j})^{\omega_j}}{\prod_{j=1}^n (2-\mu_{\alpha_j})^{\omega_j}(2-\mu'_{\alpha'_j})^{\omega_j}+\prod_{j=1}^n (\mu_{\alpha_j}\mu'_{\alpha'_j})^{\omega_j}},$$

$$\frac{\prod_{j=1}^n (1+\nu_{\alpha_j})^{\omega_j}(1+\nu'_{\alpha'_j})^{\omega_j}-\prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}(1-\nu'_{\alpha'_j})^{\omega_j}}{\prod_{j=1}^n (1+\nu_{\alpha_j})^{\omega_j}(1+\nu'_{\alpha'_j})^{\omega_j}+\prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}(1-\nu'_{\alpha'_j})^{\omega_j}})$$

$$|\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, \alpha'_j = (\mu'_{\alpha'_j}, \nu'_{\alpha'_j}) \in h'_j, j = 1...n\}.$$

We complete the proof. □

Corollary 3.13 ([37])**.** *Let $h_j (j = 1, 2...n)$ be a collection of HFEs, and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma_{j=1}^n \omega_j = 1$. Then,*

$$HFWG_\omega^\varepsilon(h_1 \bigotimes h'_1, h_2 \bigotimes h'_2, ..., h_n \bigotimes h'_n)$$
$$= HFWG_\omega^\varepsilon(h_1, h_2, ..., h_n) \bigotimes HFWG_\omega^\varepsilon(h'_1, h'_2, ..., h'_n).$$

Proof. By Theorem 3.12 and Note, it can be easily proved. □

Based on Theorem 3.2, we have the following properties of the $GHFWG^\varepsilon$ operator as follows.

Theorem 3.14. *Let $h_j (j = 1, 2...n)$ be a collection of GHFEs, and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma_{j=1}^n \omega_j = 1$. Then,*

(1) $h^-_{min} \sqsubseteq GHFWG_\omega^\varepsilon(h_1, h_2, ..., h_n) \sqsubseteq h^+_{max}$,
where $h^-_{min} = \{\alpha_{min} = (\mu_{\alpha_{min}}, \nu_{\alpha_{min}}) = min_j\{\alpha^-_j | \alpha^-_j = min\{\alpha_j \in h_j\}\}\}$,
and

$$h^+_{max} = \{\alpha_{max} = (\mu_{\alpha_{max}}, \nu_{\alpha_{max}}) = max_j\{\alpha^+_j | \alpha^+_j = max\{\alpha_j \in h_j\}\}\}.$$

(2) Let $h^*_j (j = 1, 2...n)$ be another collection of GHFEs,
where $h^*_j = \{\alpha^*_j = (\mu^*_{\alpha_j}, \nu^*_{\alpha_j})\}$, with $h_j \sqsubseteq h^*_j$ for all j, then

$$GHFWG_\omega^\varepsilon(h_1, h_2, ..., h_n) \leq GHFWG_\omega^\varepsilon(h^*_1, h^*_2, ..., h^*_n).$$

Proof. (1) For any $\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j$, $j = 1, 2, ..., n$, we have $\alpha_{min} \leq \alpha_j \leq \alpha_{max}$.

(i) Let $f(x) = \frac{2-x}{x}, x \in (0, 1]$. Then $f'(x) = \frac{-2}{x^2} < 0$ and $f(x)$ is a decreasing function on $(0, 1]$. Since $\alpha_{min} \leq \alpha_j \leq \alpha_{max}$ for any j,

$$\mu_{\alpha_{min}} \leq \mu_{\alpha_j} \leq \mu_{\alpha_{max}}$$

and

$$f(\mu_{\alpha_{max}}) \leq f(\mu_{\alpha_j}) \leq f(\mu_{\alpha_{min}}),\ j = 1, 2, ..., n,$$

i.e.,

$$\tfrac{2-\mu_{\alpha_{max}}}{\mu_{\alpha_{max}}} \leq \tfrac{2-\mu_{\alpha_j}}{\mu_{\alpha_j}} \leq \tfrac{2-\mu_{\alpha_{min}}}{\mu_{\alpha_{min}}}, j = 1, 2, ..., n.$$

Let $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma_{j=1}^n \omega_j = 1$. Then, we have

$$(\tfrac{2-\mu_{\alpha_{max}}}{\mu_{\alpha_{max}}})^{\omega_j} \leq (\tfrac{2-\mu_{\alpha_j}}{\mu_{\alpha_j}})^{\omega_j} \leq (\tfrac{2-\mu_{\alpha_{min}}}{\mu_{\alpha_{min}}})^{\omega_j}, j = 1, 2, ..., n.$$

Thus

$$\prod_{j=1}^n (\tfrac{2-\mu_{\alpha_{max}}}{\mu_{\alpha_{max}}})^{\omega_j} \leq \prod_{j=1}^n (\tfrac{2-\mu_{\alpha_j}}{\mu_{\alpha_j}})^{\omega_j} \leq \prod_{j=1}^n (\tfrac{2-\mu_{\alpha_{min}}}{\mu_{\alpha_{min}}})^{\omega_j}, j = 1, 2, ..., n$$

$$\Leftrightarrow (\tfrac{2-\mu_{\alpha_{max}}}{\mu_{\alpha_{max}}})^{\sum_{j=1}^n \omega_j} \leq \prod_{j=1}^n (\tfrac{2-\mu_{\alpha_j}}{\mu_{\alpha_j}})^{\omega_j} \leq (\tfrac{2-\mu_{\alpha_{min}}}{\mu_{\alpha_{min}}})^{\sum_{j=1}^n \omega_j}$$

$$\Leftrightarrow \tfrac{2-\mu_{\alpha_{max}}}{\mu_{\alpha_{max}}} \leq \prod_{j=1}^n (\tfrac{2-\mu_{\alpha_j}}{\mu_{\alpha_j}})^{\omega_j} \leq \tfrac{2-\mu_{\alpha_{min}}}{\mu_{\alpha_{min}}} \text{ [Since } \Sigma_{j=1}^n \omega_j = 1]$$

$$\Leftrightarrow \tfrac{2}{\mu_{\alpha_{max}}} \leq \prod_{j=1}^n (\tfrac{2-\mu_{\alpha_j}}{\mu_{\alpha_j}})^{\omega_j} + 1 \leq \tfrac{2}{\mu_{\alpha_{min}}}$$

$$\Leftrightarrow \tfrac{\mu_{\alpha_{min}}}{2} \leq \tfrac{1}{\prod_{j=1}^n (\frac{2-\mu_{\alpha_j}}{\mu_{\alpha_j}})^{\omega_j} + 1} \leq \tfrac{\mu_{\alpha_{max}}}{2}$$

$$\Leftrightarrow \mu_{\alpha_{min}} \leq \tfrac{2}{\prod_{j=1}^n (\frac{2-\mu_{\alpha_j}}{\mu_{\alpha_j}})^{\omega_j} + 1} \leq \mu_{\alpha_{max}}.$$

So $\mu_{\alpha_{min}} \leq \frac{2 \prod_{j=1}^n \mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^n (2-\mu_{\alpha_j})^{\omega_j} + \prod_{j=1}^n \mu_{\alpha_j}^{\omega_j}} \leq \mu_{\alpha_{max}}$.

(ii) Let $g(y) = \frac{1-y}{1+y}, y \in [0, 1]$. Then $g'(y) = \frac{-2}{(1+y)^2} < 0$ and $g(y)$ is a decreasing function on $(0, 1]$. Since $\alpha_{min} \leq \alpha_j \leq \alpha_{max}$ for any j,

$$\nu_{\alpha_{max}} \leq \nu_{\alpha_j} \leq \nu_{\alpha_{min}},$$

and

$$g(\nu_{\alpha_{min}}) \leq g(\nu_{\alpha_j}) \leq g(\nu_{\alpha_{max}}), j = 1, 2, ..., n,$$

i.e.,

$$\tfrac{1-\nu_{\alpha_{min}}}{1+\nu_{\alpha_{min}}} \leq \tfrac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}} \leq \tfrac{1-\nu_{\alpha_{max}}}{1+\nu_{\alpha_{max}}}, j = 1, 2, ..., n.$$

Let $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma_{j=1}^n \omega_j = 1$. Then, we have

$$(\tfrac{1-\nu_{\alpha_{min}}}{1+\nu_{\alpha_{min}}})^{\omega_j} \leq (\tfrac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}})^{\omega_j} \leq (\tfrac{1-\nu_{\alpha_{max}}}{1+\nu_{\alpha_{max}}})^{\omega_j}, j = 1, 2, ..., n.$$

Thus,

$$\prod_{j=1}^n (\tfrac{1-\nu_{\alpha_{min}}}{1+\nu_{\alpha_{min}}})^{\omega_j} \leq \prod_{j=1}^n (\tfrac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}})^{\omega_j} \leq \prod_{j=1}^n (\tfrac{1-\nu_{\alpha_{max}}}{1+\nu_{\alpha_{max}}})^{\omega_j}, j = 1, 2, ..., n$$

$$\Leftrightarrow (\tfrac{1-\nu_{\alpha_{min}}}{1+\nu_{\alpha_{min}}})^{\sum_{j=1}^n \omega_j} \leq \prod_{j=1}^n (\tfrac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}})^{\omega_j} \leq (\tfrac{1-\nu_{\alpha_{max}}}{1+\nu_{\alpha_{max}}})^{\sum_{j=1}^n \omega_j}$$

$$\Leftrightarrow \tfrac{1-\nu_{\alpha_{min}}}{1+\nu_{\alpha_{min}}} \leq \prod_{j=1}^n (\tfrac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}})^{\omega_j} \leq \tfrac{1-\nu_{\alpha_{max}}}{1+\nu_{\alpha_{max}}} \text{ [Since } \Sigma_{j=1}^n \omega_j = 1]$$

$\Leftrightarrow \frac{2}{1+\nu_{\alpha_{min}}} \leq 1 + \prod_{j=1}^{n}(\frac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}})^{\omega_j} \leq \frac{2}{1+\nu_{\alpha_{max}}}$

$\Leftrightarrow \frac{1+\nu_{\alpha_{max}}}{2} \leq \frac{1}{1+\prod_{j=1}^{n}(\frac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}})^{\omega_j}} \leq \frac{1+\nu_{\alpha_{min}}}{2}$

$\Leftrightarrow 1+\nu_{\alpha_{max}} \leq \frac{2}{1+\prod_{j=1}^{n}(\frac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}})^{\omega_j}} \leq 1+\nu_{\alpha_{min}}$

$\Leftrightarrow \nu_{\alpha_{max}} \leq \frac{2}{1+\prod_{j=1}^{n}(\frac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}})^{\omega_j}} - 1 \leq \nu_{\alpha_{min}}.$

So $\nu_{\alpha_{max}} \leq \frac{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j} - \prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^{n}(1+\nu_{\alpha_j})^{\omega_j} + \prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}} \leq \nu_{\alpha_{min}}$.

Let $GHFWG_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n) = h = \{\alpha = (\mu_\alpha, \nu_\alpha)\}$. Then (i) and (ii) are transformed into the following forms: for any $\alpha \in h$.

$$\mu_{\alpha_{min}} \leq \mu_\alpha \leq \mu_{\alpha_{max}}$$

and

$$\nu_{\alpha_{max}} \leq \nu_\alpha \leq \nu_{\alpha_{min}}.$$

Thus, by Definition 2.6, we complete the proof of

$$h_{min}^{-} \sqsubseteq GHFWG_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n) \sqsubseteq h_{max}^{+}.$$

This property establishes that the GHFWG$^{\varepsilon}$ operator is a function that yields a GHFE between the GHFE h_{min}^{-} and h_{max}^{+}.

(2) Let $\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j$. Because $h_j \sqsubseteq h_j^*$ for all j, let $\alpha_j^* = (\mu_{\alpha_j^*}, \nu_{\alpha_j^*}) \in h_j^*$ be the corresponding element such that $\alpha_j \leq \alpha_j^*$, i.e., $\mu_{\alpha_j} \leq \mu_{\alpha_j^*}$ and $\nu_{\alpha_j} \geq \nu_{\alpha_j^*}$.

(i) Let $f(x) = \frac{2-x}{x}, x \in (0,1]$. Then $f^{'}(x) = \frac{-2}{x^2} < 0$ and $f(x)$ is a decreasing function on $(0,1]$. If $\mu_{\alpha_j} \leq \mu_{\alpha_j^*}$ for all j, then $f(\mu_{\alpha_j^*}) \leq f(\mu_{\alpha_j})$ for all j, i.e.,

$$\frac{2-\mu_{\alpha_j^*}}{\mu_{\alpha_j^*}} \leq \frac{2-\mu_{\alpha_j}}{\mu_{\alpha_j}},\ j = 1, 2, ..., n.$$

Let $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0,1]$ and $\Sigma_{j=1}^{n}\omega_j$ =1. Then, we have

$$(\frac{2-\mu_{\alpha_j^*}}{\mu_{\alpha_j^*}})^{\omega_j} \leq (\frac{2-\mu_{\alpha_j}}{\mu_{\alpha_j}})^{\omega_j}, j = 1, 2, ..., n.$$

Thus

$$\prod_{j=1}^{n}(\frac{2-\mu_{\alpha_j^*}}{\mu_{\alpha_j^*}})^{\omega_j} \leq \prod_{j=1}^{n}(\frac{2-\mu_{\alpha_j}}{\mu_{\alpha_j}})^{\omega_j}$$

$$\Leftrightarrow \prod_{j=1}^{n}(\frac{2-\mu_{\alpha_j^*}}{\mu_{\alpha_j^*}})^{\omega_j} + 1 \leq \prod_{j=1}^{n}(\frac{2-\mu_{\alpha_j}}{\mu_{\alpha_j}})^{\omega_j} + 1$$

$$\Leftrightarrow \frac{1}{\prod_{j=1}^{n}(\frac{2-\mu_{\alpha_j}}{\mu_{\alpha_j}})^{\omega_j}+1} \leq \frac{1}{\prod_{j=1}^{n}(\frac{2-\mu_{\alpha_j^*}}{\mu_{\alpha_j^*}})^{\omega_j}+1}$$

$$\Leftrightarrow \frac{2}{\prod_{j=1}^{n}(\frac{2-\mu_{\alpha_j}}{\mu_{\alpha_j}})^{\omega_j}+1} \leq \frac{2}{\prod_{j=1}^{n}(\frac{2-\mu_{\alpha_j^*}}{\mu_{\alpha_j^*}})^{\omega_j}+1}.$$

So $\frac{2\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}}{\prod_{j=1}^{n}(2-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j}} \leq \frac{2\prod_{j=1}^{n}[\mu_{\alpha_j^*}]^{\omega_j}}{\prod_{j=1}^{n}(2-\mu_{\alpha_j^*})^{\omega_j}+\prod_{j=1}^{n}[\mu_{\alpha_j^*}]^{\omega_j}}$.

(ii) Let $g(y) = \frac{1-y}{1+y}, y \in [0,1]$. Then $g^{'}(y) = \frac{-2}{(1+y)^2} < 0$ and $g(y)$ is a decreasing function on $(0,1]$. If $\nu_{\alpha_j} \geq \nu_{\alpha_j^*}$ for all j, then $g(\nu_{\alpha_j}) \leq g(\nu_{\alpha_j^*})$ for all j, i.e.,

$$\frac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}} \leq \frac{1-\nu_{\alpha_j^*}}{1+\nu_{\alpha_j^*}}, j = 1, 2, ..., n.$$

Let $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0,1]$ and $\Sigma_{j=1}^n \omega_j$ =1. Then, we have

$$(\frac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}})^{\omega_j} \leq (\frac{1-\nu_{\alpha_j^*}}{1+\nu_{\alpha_j^*}})^{\omega_j}, j = 1, 2, ..., n.$$

Thus

$$\begin{aligned}
&\prod_{j=1}^n (\frac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}})^{\omega_j} \leq \prod_{j=1}^n (\frac{1-\nu_{\alpha_j^*}}{1+\nu_{\alpha_j^*}})^{\omega_j} \\
\Leftrightarrow\ & 1 + \prod_{j=1}^n (\frac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}})^{\omega_j} \leq 1 + \prod_{j=1}^n (\frac{1-\nu_{\alpha_j^*}}{1+\nu_{\alpha_j^*}})^{\omega_j} \\
\Leftrightarrow\ & \frac{1}{1+\prod_{j=1}^n (\frac{1-\nu_{\alpha_j^*}}{1+\nu_{\alpha_j^*}})^{\omega_j}} \leq \frac{1}{1+\prod_{j=1}^n (\frac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}})^{\omega_j}} \\
\Leftrightarrow\ & \frac{2}{1+\prod_{j=1}^n (\frac{1-\nu_{\alpha_j^*}}{1+\nu_{\alpha_j^*}})^{\omega_j}} \leq \frac{2}{1+\prod_{j=1}^n (\frac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}})^{\omega_j}} \\
\Leftrightarrow\ & \frac{2}{1+\prod_{j=1}^n (\frac{1-\nu_{\alpha_j^*}}{1+\nu_{\alpha_j^*}})^{\omega_j}} - 1 \leq \frac{2}{1+\prod_{j=1}^n (\frac{1-\nu_{\alpha_j}}{1+\nu_{\alpha_j}})^{\omega_j}} - 1.
\end{aligned}$$

So

$$\frac{\prod_{j=1}^n (1+\nu_{\alpha_j^*})^{\omega_j} - \prod_{j=1}^n (1-\nu_{\alpha_j^*})^{\omega_j}}{\prod_{j=1}^n (1+\nu_{\alpha_j^*})^{\omega_j} + \prod_{j=1}^n (1-\nu_{\alpha_j^*})^{\omega_j}} \leq \frac{\prod_{j=1}^n (1+\nu_{\alpha_j})^{\omega_j} - \prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}}{\prod_{j=1}^n (1+\nu_{\alpha_j})^{\omega_j} + \prod_{j=1}^n (1-\nu_{\alpha_j})^{\omega_j}}.$$

Let

$$GHFWG_\omega^\varepsilon(h_1, h_2, ..., h_n) = h = \{\alpha = (\mu_\alpha, \nu_\alpha)\}$$

and

$$GHFWG_\omega^\varepsilon(h_1^*, h_2^*, ..., h_n^*) = h^* = \{\alpha^* = (\mu_{\alpha^*}, \nu_{\alpha^*})\}.$$

Then (i) and (ii) are transformed into the following forms:

$$\mu_\alpha \leq \mu_{\alpha^*}.$$

and

$$\nu_{\alpha^*} \leq \nu_\alpha, \text{ respectivelly.}$$

Thus, by Definition 2.9, we complete the proof of

$$GHFWG_\omega^\varepsilon(h_1, h_2, ..., h_n) \leq GHFWG_\omega^\varepsilon(h_1^*, h_2^*, ..., h_n^*).$$

□

Theorem 3.15. *If all $h_j (j = 1, 2, ..., n)$ are equal, i.e., $h_j = h$, for all $j = 1, 2, ..., n$, and the number of values in h_j is only one, that is, $h_j = h = \{\alpha\}$ for all $j = 1, 2, ..., n$, then $GHFWG_\omega^\varepsilon(h_1, h_2, ..., h_n) = h$.*

Proof. By the definition of $GHFWG^\varepsilon$ and Theorem 3.2. □

Remark 3.16. Note that the $GHFWG^{\varepsilon}$ operator is not idempotent in general. We give the following example to illustrate this case.

Example 3.17. Let $h_1 = h_2 = h_3 = h = \{(0.6, 0.3), (0.4, 0.5)\}$ be a collection of GHFEs, where $\omega = (0.4, 0.25, 0.35)^T$ is the weight vector of $h_j (j = 1, 2, 3)$. Then

$$GHFWG^{\varepsilon}_{\omega}(h_1, h_2, h_3)$$
$$= \{(0.6, 0.3), (0.5239, 0.3743), (0.5450, 0.3535), (0.4735, 0.4247), (0.5135, 0.3846), (0.4449, 0.4537), (0.4638, 0.4345), (0.4, 0.5)\}.$$

Thus, by Definitions 2.8 and 2.9, s($GHFWG^{\varepsilon}_{\omega}(h_1, h_2, h_3)$)=0.546206 and $s(h) = 0.55$. So, $GHFWG^{\varepsilon}_{\omega}(h_1, h_2, h_3) < h$.

Remark 3.18. Let $h^*_j (j = 1, 2...n)$ be another collection of GHFEs, where $h^*_j = \{\alpha^*_j = (\mu^*_{\alpha_j}, \nu^*_{\alpha_j})\}$, with $h_j < h^*_j$ for all j. Then

$$GHFWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n) < GHFWG^{\varepsilon}_{\omega}(h^*_1, h^*_2, ..., h^*_n)$$

does not hold necessarily in general. To illustrate that, an example is given as follows.

Example 3.19. Let $h_1 = \{(0.45, 0.1), (0.6, 0.1)\}$, $h_2 = \{(0.6, 0.1), (0.7, 0.1)\}$, $h_3 = \{(0.5, 0.1), (0.6, 0.1)\}$, $h^*_1 = \{(0.2, 0.1), (0.9, 0.1)\}$, $h^*_2 = \{(0.45, 0.1), (0.9, 0.1)\}$, $h^*_3 = \{(0.35, 0.1), (0.8, 0.1)\}$ and $\omega = (0.5, 0.3, 0.2)^T$.
Then,

$$GHFWA^{\varepsilon}_{\omega}(h_1, h_2, h_3)$$
$$= \{(0.5024, 0.1), (0.5215, 0.1), (0.5286, 0.1), (0.5483, 0.1), (0.5791, 0.1), (0.6000, 0.1), (0.6077, 0.1), (0.6291, 0.1)\}$$

and

$$GHFWA^{\varepsilon}_{\omega}(h^*_1, h^*_2, h^*_3)$$
$$= \{(0.2887, 0.1), (0.3500, 0.1), (0.3742, 0.1), (0.4489, 0.1), (0.6280, 0.1), (0.7306, 0.1), (0.7689, 0.1), (0.8798, 0.1)\}.$$

Thus, by Definition 2.9, we have

$$s(GHFWA^{\varepsilon}_{\omega}(h_1, h_2, h_3)) = 0.7323 \text{ and } s(GHFWA^{\varepsilon}_{\omega}(h^*_1, h^*_2, h^*_3)) = 0.7293.$$

It follows that $GHFWA^{\varepsilon}_{\omega}(h_1, h_2, h_3) > GHFWA^{\varepsilon}_{\omega}(h^*_1, h^*_2, h^*_3)$.
Clearly, $h_j < h^*_j$ for each $j = 1, 2, 3$, but $GHFWA^{\varepsilon}_{\omega}(h_1, h_2, h_3) > GHFWA^{\varepsilon}_{\omega}(h^*_1, h^*_2, h^*_3)$.

By definition 2.9 and Theorem 3.14, it's not hard to prove the following properties.

Theorem 3.20. *Let $h_j (j = 1, 2...n)$ be a collection of GHFEs and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma^n_{j=1}\omega_j$ =1. Then,*

$$h^-_{min} \leq GHFWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n) \leq h^+_{max}.$$

Proof. By Theorem 3.14 and Definition 2.9, it can be easily proved. □

Corollary 3.21 ([37])**.** *. Let $h_j (j = 1, 2...n)$ be a collection of HFEs, and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma^n_{j=1}\omega_j$ =1. Then,*

$$h^-_{min} \leq HFEWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n) \leq h^+_{max}.$$

Proof. By Theorem 3.20 and Note, it can be easily proved. □

4. Generalized hesitant fuzzy Einstein ordered weighted geometric aggregation operators

Motivated by the idea of IFOWG operator proposed by Xu and Yager (i.e., Equation 22 described in Section 3.3 of [31]), we defined some Einstein operations for generalized hesitant fuzzy information.

Definition 4.1. Let $h_j(j=1,2...n)$ be a collection of $GHFEs$. Then a generalized hesitant fuzzy Einstein ordered weighted geometric ($GHFOWG^{\varepsilon}$) operator of dimension n is a mapping $GHFOWG^{\varepsilon}$: $GHFE^n \to GHFE$, where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector of $h_{\sigma(j)}(j=1,2,...,n)$ with $\omega_j \in [0,1]$ and $\Sigma_{j=1}^n \omega_j = 1$.

$$GHFOWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n) = \bigotimes_{j=1}^n h_{\sigma(j)}^{\omega_j} = h_{\sigma(1)}^{\omega_1} \bigotimes h_{\sigma(2)}^{\omega_2} \bigotimes ... \bigotimes h_{\sigma(n)}^{\omega_n},$$

where $(\sigma(1), \sigma(2), ..., \sigma(n))$ is a permutation of $(1,2,...,n)$ such that $h_{\sigma(j)} \leq h_{\sigma(j-1)}$ for all j and $h_{\sigma(j)}$ is the $j-$th largest of $h_j(j=1,2...n)$.

Especially, if $\omega = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$, then the $GHFOWG^{\varepsilon}$ operator is reduced to a generalized hesitant fuzzy Einstein ordered geometric averaging ($GHFOGA\varepsilon$) operator of dimension n, which is defined as follows:

$$GHFOGA^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n) = \bigotimes_{j=1}^n h_{\sigma(j)}^{\frac{1}{n}} = h_{\sigma(1)}^{\frac{1}{n}} \bigotimes h_{\sigma(2)}^{\frac{1}{n}} \bigotimes ... \bigotimes h_{\sigma(n)}^{\frac{1}{n}}.$$

The fundamental aspect of the $GHFOWG^{\varepsilon}$ operator is its reordering step. More specifically, the $GHFOWG^{\varepsilon}$ operator first ranks all the given $GHFEs$ in descending order, and then additively aggregates these $GHFEs$ together with the weights of their ordered positions, where the corresponding operations are Einstein operations. Moreover, we have the following result.

Theorem 4.2. *Let $h_j(j=1,2...n)$ be a collection of GHFEs, and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_{\sigma(j)}(j=1,2,...,n)$ with $\omega_j \in [0,1]$ and $\Sigma_{j=1}^n \omega_j = 1$. Then, their aggregated value by using the $GHFOWG^{\varepsilon}$ operator is also a GHFE, and*

$$GHFOWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n)$$
$$= \{(\frac{2\prod_{j=1}^n \mu_{\alpha_{\sigma(j)}}^{\omega_j}}{\prod_{j=1}^n (2-\mu_{\alpha_{\sigma(j)}})^{\omega_j} + \prod_{j=1}^n \mu_{\alpha_{\sigma(j)}}^{\omega_j}}, \frac{\prod_{j=1}^n (1+\nu_{\alpha_{\sigma(j)}})^{\omega_j} - \prod_{j=1}^n (1-\nu_{\alpha_{\sigma(j)}})^{\omega_j}}{\prod_{j=1}^n (1+\nu_{\alpha_{\sigma(j)}})^{\omega_j} + \prod_{j=1}^n (1-\nu_{\alpha_{\sigma(j)}})^{\omega_j}}) |\alpha_{\sigma(j)} = (\mu_{\alpha_{\sigma(j)}}, \nu_{\alpha_{\sigma(j)}}) \in h_{\sigma(j)}, j = 1...n\},$$

where $(\sigma(1), \sigma(2), ..., \sigma(n))$ is a permutation of $(1,2,...,n)$ such that $h_{\sigma(j)} \leq h_{\sigma(j-1)}$ for all j.

Proof. Similar to the Theorem 3.2. □

Remark 4.3. Especially, if $\nu_{\alpha_j} = 1 - \mu_{\alpha_j}$, for all $j=1,2,...,n$, e.g., if $h_j(j=1,2...n)$ is a collection of $HFEs$, then Theorem 4.2 is reduced to the following form:

$$HFEOWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n) = \{\frac{2\prod_{j=1}^n \mu_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^n (2-\mu_{\sigma(j)})^{\omega_j} + \prod_{j=1}^n \mu_{\sigma(j)}^{\omega_j}} |\mu_{\sigma(j)} \in h_{\sigma(j)}, j = 1...n\},$$

which is Definition 24 in [37] and Theorem 5 in [34].

Similar to the proof of Theorem 3.7, Theorem 3.9, Theorem 3.11 and Theorem 3.13, we can get the following properties of $GHFOWG^{\varepsilon}_{\omega}$.

Theorem 4.4. *Let* $h_j(j=1,2...n)$ *be a collection of* $GHFEs$, *and* $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ *be the weight vector of* $h_{\sigma(j)}(j=1,2,...,n)$ *with* $\omega_j\in[0,1]$ *and* $\Sigma_{j=1}^n\omega_j=1$, $\lambda>0$. *Then*

(1) $GHFOWG_\omega^\varepsilon(h_1^\lambda,h_2^\lambda,...,h_n^\lambda)=(GHFOWG_\omega^\varepsilon(h_1,h_2,...,h_n))^\lambda$,

(2) $GHFOWG_\omega^\varepsilon(h_1\bigotimes h,h_2\bigotimes h,...,h_n\bigotimes h)=GHFOWG_\omega^\varepsilon(h_1,h_2,...,h_n)\bigotimes h$,

(3) $GHFOWG_\omega^\varepsilon(h_1^\lambda\bigotimes h^\lambda,h_2^\lambda\bigotimes h^\lambda,...,h_n^\lambda\bigotimes h^\lambda)$
$=(GHFOWG_\omega^\varepsilon(h_1,h_2,...,h_n)\bigotimes h)^\lambda$,

(4) $GHFOWG_\omega^\varepsilon(h_1\bigotimes h_1^{'},h_2\bigotimes h_2^{'},...,h_n\bigotimes h_n^{'})$
$=GHFOWG_\omega^\varepsilon(h_1,h_2,...,h_n)\bigotimes GHFOWG_\omega^\varepsilon(h_1^{'},h_2^{'},...,h_n^{'})$.

Theorem 4.5. *Let* $h_j(j=1,2...n)$ *be a collection of GHFEs and* $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ *be the weight vector of* $h_{\sigma(j)}(j=1,2,...,n)$ *with* $\omega_j\in[0,1]$ *and* $\Sigma_{j=1}^n\omega_j=1$. *Then,*

(1) $h_{min}^-\sqsubseteq GHFOWG_\omega^\varepsilon(h_1,h_2,...,h_n)\sqsubseteq h_{max}^+$,
where $h_{min}^-=\{\alpha_{min}=(\mu_{\alpha_{min}},\nu_{\alpha_{min}})=min_j\{\alpha_j^-|\alpha_j^-=min\{\alpha_j\in h_j\}\}\}$
and

$$h_{max}^+=\{\alpha_{max}=(\mu_{\alpha_{max}},\nu_{\alpha_{max}})=max_j\{\alpha_j^+|\alpha_j^+=max\{\alpha_j\in h_j\}\}\}.$$

(2) *Let* $h_j^*(j=1,2...n)$ *be another collection of GHFEs, where* $h_j^*=\{\alpha_j^*=(\mu_{\alpha_j}^*,\nu_{\alpha_j}^*)\}$, *with* $h_j\sqsubseteq h_j^*$ *for all* j. *then*

$$GHFOWG_\omega^\varepsilon(h_1,h_2,...,h_n)\le GHFOWG_\omega^\varepsilon(h_1^*,h_2^*,...,h_n^*).$$

To analyze the relationship between the $GHFOWG$ operator and $IHFOWG$ operator proposed by Zhang (i.e., Equation 81 in [36], we have the following Theorem.

Theorem 4.6. *Let* $h_j(j=1,2...n)$ *be a collection of GHFEs and* $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ *be the weight vector of* $h_{\sigma(j)}(j=1,2,...,n)$ *with* $\omega_j\in[0,1]$ *and* $\Sigma_{j=1}^n\omega_j=1$. *Then,*

$$IHFOWG_\omega(h_1,h_2,...,h_n)\le GHFOWG_\omega^\varepsilon(h_1,h_2,...,h_n).$$

Proof. Similar to the Theorem 3.4. □

Theorem 4.6 tell us that the $GHFOWG^\varepsilon$ operator shows the decision maker's more optimistic attitude than the $IHFOWG$ operator proposed by Zhang [36] in aggregation process. To illustrate that, we give the following example.

Example 4.7. Let $h_1=\{(0.1,0.4),(0.3,0.5),(0.4,0.2)\}$ and $h_2=\{(0.5,0.2),(0.6,0.1)\}$ be two $GHFEs$, $\omega=(0.4,0.6)^T$ be the weight vector of them. Then We can get $s(h_1)=0.45$ and $s(h_2)=0.7$. Thus, by Definition 2.9, $h_2>h_1$. So, $h_{\sigma(1)}=h_2$ and $h_{\sigma(2)}=h_1$. Hence, by Definition 4.1 and Theorem 4.2, we have

$$GHFOWG_\omega^\varepsilon(h_{\sigma(1)},h_{\sigma(2)})$$
$$=\{(\frac{2\prod_{j=1}^2\mu_{\alpha_{\sigma(j)}}^{\omega_j}}{\prod_{j=1}^2(2-\mu_{\alpha_{\sigma(j)}})^{\omega_j}+\prod_{j=1}^2\mu_{\alpha_{\sigma(j)}}^{\omega_j}},\frac{\prod_{j=1}^2(1+\nu_{\alpha_{\sigma(j)}})^{\omega_j}-\prod_{j=1}^2(1-\nu_{\alpha_{\sigma(j)}})^{\omega_j}}{\prod_{j=1}^2(1+\nu_{\alpha_{\sigma(j)}})^{\omega_j}+\prod_{j=1}^2(1-\nu_{\alpha_{\sigma(j)}})^{\omega_j}})$$
$$|\alpha_{\sigma(j)}=(\mu_{\alpha_{\sigma(j)}},\nu_{\alpha_{\sigma(j)}})\in h_{\sigma(j)},j=1...n\}$$
$$=\{(\frac{2\times0.5^{0.4}\times0.1^{0.6}}{(2-0.5)^{0.4}\times(2-0.1)^{0.6}+0.5^{0.4}\times0.1^{0.6}},\frac{(1+0.2)^{0.4}\times(1+0.4)^{0.6}-(1-0.2)^{0.4}\times(1-0.4)^{0.6}}{(1+0.2)^{0.4}\times(1+0.4)^{0.6}+(1-0.2)^{0.4}\times(1-0.4)^{0.6}}),$$
$$(\frac{2\times0.5^{0.4}\times0.3^{0.6}}{(2-0.5)^{0.4}\times(2-0.3)^{0.6}+0.5^{0.4}\times0.3^{0.6}},\frac{(1+0.2)^{0.4}\times(1+0.5)^{0.6}-(1-0.2)^{0.4}\times(1-0.5)^{0.6}}{(1+0.2)^{0.4}\times(1+0.5)^{0.6}+(1-0.2)^{0.4}\times(1-0.5)^{0.6}}),$$
$$(\frac{2\times0.5^{0.4}\times0.4^{0.6}}{(2-0.5)^{0.4}\times(2-0.4)^{0.6}+0.5^{0.4}\times0.4^{0.6}},\frac{(1+0.2)^{0.4}\times(1+0.2)^{0.6}-(1-0.2)^{0.4}\times(1-0.2)^{0.6}}{(1+0.2)^{0.4}\times(1+0.2)^{0.6}+(1-0.2)^{0.4}\times(1-0.2)^{0.6}}),$$
$$(\frac{2\times0.6^{0.4}\times0.1^{0.6}}{(2-0.6)^{0.4}\times(2-0.1)^{0.6}+0.6^{0.4}\times0.1^{0.6}},\frac{(1+0.1)^{0.4}\times(1+0.4)^{0.6}-(1-0.1)^{0.4}\times(1-0.4)^{0.6}}{(1+0.1)^{0.4}\times(1+0.4)^{0.6}+(1-0.1)^{0.4}\times(1-0.4)^{0.6}}),$$

$$(\frac{2\times0.6^{0.4}\times0.3^{0.6}}{(2-0.6)^{0.4}\times(2-0.3)^{0.6}+0.6^{0.4}\times0.3^{0.6}}, \frac{(1+0.1)^{0.4}\times(1+0.5)^{0.6}-(1-0.1)^{0.4}\times(1-0.5)^{0.6}}{(1+0.1)^{0.4}\times(1+0.5)^{0.6}+(1-0.1)^{0.4}\times(1-0.5)^{0.6}}),$$
$$(\frac{2\times0.6^{0.4}\times0.4^{0.6}}{(2-0.6)^{0.4}\times(2-0.4)^{0.6}+0.6^{0.4}\times0.4^{0.6}}, \frac{(1+0.1)^{0.4}\times(1+0.2)^{0.6}-(1-0.1)^{0.4}\times(1-0.2)^{0.6}}{(1+0.1)^{0.4}\times(1+0.2)^{0.6}+(1-0.1)^{0.4}\times(1-0.2)^{0.6}})\}$$
$$= \{(0.1984, 0.3233), (0.3708, 0.3890), (0.4381, 0.2), (0.2171, 0.2861), (0.4021, 0.3537), (0.4735, 0.1604)\}.$$

Then, by Equation 81 in [36], we have

$$IHFOWG_{\omega}(h_{\sigma(1)}, h_{\sigma(2)})$$
$$= \{(\prod_{j=1}^{2} \mu_{\alpha_{\sigma(j)}}^{\omega_j}, 1-\prod_{j=1}^{2}(1-\nu_{\alpha_{\sigma(j)}})^{\omega_j})|\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j}) \in h_j, j = 1, 2\}$$
$$= \{(0.5^{0.4} \times 0.1^{0.6}, 1-(1-0.2)^{0.4} \times (1-0.4)^{0.6}),$$
$$(0.5^{0.4} \times 0.3^{0.6}, 1-(1-0.4)^{0.4} \times (1-0.1)^{0.6}),$$
$$(0.50.4 \times 0.4^{0.6}, 1-(1-0.2)^{0.4} \times (1-0.2)^{0.6}),$$
$$(0.6^{0.4} \times 0.1^{0.6}, 1-(1-0.1)^{0.4} \times (1-0.4)^{0.6}),$$
$$(0.60.4 \times 0.3^{0.6}, 1-(1-0.1)^{0.4} \times (1-0.5)^{0.6}),$$
$$(0.6^{0.4} \times 0.4^{0.6}, 1-(1-0.1)^{0.4} \times (1-0.2)^{0.6})\}$$
$$= \{(0.1904, 0.3268), (0.3680, 0.3966), (0.4373, 0.2), (0.2048, 0.2944), (0.3959, 0.3675), (0.4704, 0.1614)\}.$$

Thus, by simple computing,

$$s(GHFOWG_{\omega}^{\varepsilon}(h_1, h_2))$$
$$= \frac{1}{6} \cdot \frac{1}{2}[(6 + 0.1984 + 0.3708 + 0.4381 + 0.2171 + 0.4021 + 0.4735) -(0.3233 + 0.389 + 0.2 + 0.2861 + 0.3537 + 0.1604)]$$
$$= \frac{1}{12}(8.1 - 1.7125) = 0.5323$$

and

$$s(IHFOWG_{\omega}(h_1, h_2))$$
$$= \frac{1}{6} \cdot \frac{1}{2}[(6 + 0.1904 + 0.368 + 0.4373 + 0.2048 + 0.3959 + 0.4704) \; 0.3268 + 0.3966 + 0.2 + 0.2944 + 0.3675 + 0.1614)]$$
$$= \frac{1}{12}(8.0668 - 1.7467) = 0.526675.$$

So, we have

$$IHFOWG_{\omega}(h_1, h_2) \leq GHFOWG_{\omega}^{\varepsilon}(h_1, h_2).$$

Theorem 4.8. *Let $h_j (j = 1, 2...n)$ be a collection of GHFEs and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_{\sigma(j)} (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma_{j=1}^{n}\omega_j$ =1. Then,*

$$h_{min}^{-} \leq GHFOWG_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n) \leq h_{max}^{+}.$$

Proof. By the Theorem 4.5 and Definition 2.9, it can be easily proved. □

Besides the above properties similar to $GHFWG_{\omega}^{\varepsilon}$ above, $GHFOWG_{\omega}^{\varepsilon}$ has the following properties.

Theorem 4.9. *Let $h_j (j = 1, 2...n)$ be a collection of GHFEs and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_{\sigma(j)} (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma_{j=1}^{n}\omega_j$ =1. Then,*

$$GHFOWG_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n) = GHFOWG_{\omega}^{\varepsilon}(h_1', h_2', ..., h_n'),$$

where $h_j' (j = 1, 2, ..., n)$ is any permutation of $h_j (j = 1, 2, ..., n)$.

Proof. Clearly,

$$GHFOWG_{\omega}^{\varepsilon}(h_1, h_2, ..., h_n) = \bigotimes_{j=1}^{n} h_{\sigma(j)}^{\omega_j}$$

and

$$GHFOWG^{\varepsilon}_{\omega}(h_1^{'}, h_2^{'}, ..., h_n^{'}) = \bigotimes_{j=1}^{n} h_{\sigma(j)}^{'\omega_j}.$$

Since $h_1^{'}, h_2^{'}, ..., h_n^{'}$ is any permutation of $h_1, h_2, ..., h_n$, $h_{\sigma(j)}^{\omega_j} = h_{\sigma(j)}^{'\omega_j} (j = 1, 2, ..., n)$. Thus we have

$$GHFOWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n) = GHFOWG^{\varepsilon}_{\omega}(h_1^{'}, h_2^{'}, ..., h_n^{'}).$$

□

Theorem 4.10. *Let $h_j (j = 1, 2...n)$ be a collection of GHFEs and $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the weight vector of $h_{\sigma(j)} (j = 1, 2, ..., n)$ with $\omega_j \in [0, 1]$ and $\Sigma_{j=1}^{n} \omega_j$ =1.*

(1) *If $\omega = (1, 0, ..., 0)^T$, then $GHFOWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n) = h_{\sigma_{(1)}}$.*

(2) *If $\omega = (0, 0, ..., 1)^T$, then $GHFOWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n) = h_{\sigma_{(n)}}$.*

(3) *If $\omega_j = 1$ and $\omega_i = 0 (i \neq j)$, then $GHFOWG^{\varepsilon}_{\omega}(h_1, h_2, ..., h_n) = h_{\sigma_{(j)}}$, where $h_{\sigma_{(j)}}$ is the $j-$th largest of $h_i (i = 1, ..., n)$.*

5. Generalized hesitant fuzzy Einstein hybrid geometric aggregation operators

From Definitions 3.1 and 4.1, we know that the $GHFWG^{\varepsilon}$ operator weights only the generalized hesitant fuzzy argument itself, but ignores the importance of the ordered position of the argument, while the $GHFOWG^{\varepsilon}$ operator weights only the ordered positions of each given argument, but ignores the importance of the argument. In this section, we develop an $GHFEHG^{\varepsilon}$ operator, which weights both the given generalized hesitant fuzzy argument and its ordered position.

Definition 5.1. Let $h_j (j = 1, 2...n)$ be a collection of $GHFEs$ and $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ be the weight vector of $h_j (j = 1, 2, ..., n)$ with $\varpi_j \in [0, 1]$ and $\Sigma_{j=1}^{n} \varpi_j$ =1, n be the balancing coefficient. Then we define the following aggregation operator based on the mapping $GHFE^n \to GHFE$ with an associated vector $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ such that $\omega_j \in [0, 1]$ and $\Sigma_{j=1}^{n} \omega_j$ =1.
Thus, the generalized hesitant fuzzy Einstein hybrid geometric ($GHFEHG^{\varepsilon}$) operator of dimension n is a mapping $GHFEHG^{\varepsilon}$: $GHFE^n \to GHFE$

$$\text{GHFEHG}^{\varepsilon}_{\varpi,\omega}(h_1, h_2, ..., h_n) = \bigotimes_{j=1}^{n} \dot{h}_{\sigma(j)}^{\omega_j} = \dot{h}_{\sigma(1)}^{\omega_1} \bigotimes \dot{h}_{\sigma(2)}^{\omega_2} \bigotimes ... \bigotimes \dot{h}_{\sigma(n)}^{\omega_n},$$

where $\dot{h}_{\sigma(j)}$ is the $j-$th largest of $\dot{h}_k = h_k^{n\varpi_k} (k = 1, 2, ..., n)$.

Theorem 5.2. *Let $h_j (j = 1, 2...n)$ be a collection of GHFEs. Then, their aggregated value by using the $GHFEHG^{\varepsilon}$ operator is also a GHFE and*

$$GHFEHG^{\varepsilon}_{\varpi,\omega}(h_1, h_2, ..., h_n)$$
$$= \{ (\frac{2 \prod_{j=1}^{n} \mu_{\dot{\alpha}_{\sigma(j)}}^{\omega_j}}{\prod_{j=1}^{n} (2-\mu_{\dot{\alpha}_{\sigma(j)}})^{\omega_j} + \prod_{j=1}^{n} \mu_{\dot{\alpha}_{\sigma(j)}}^{\omega_j}}, \frac{\prod_{j=1}^{n} (1+\nu_{\dot{\alpha}_{\sigma(j)}})^{\omega_j} - \prod_{j=1}^{n} (1-\nu_{\dot{\alpha}_{\sigma(j)}})^{\omega_j}}{\prod_{j=1}^{n} (1+\nu_{\dot{\alpha}_{\sigma(j)}})^{\omega_j} + \prod_{j=1}^{n} (1-\nu_{\dot{\alpha}_{\sigma(j)}})^{\omega_j}})$$
$$| \dot{\alpha}_{\sigma(j)} = (\mu_{\dot{\alpha}_{\sigma(j)}}, \nu_{\dot{\alpha}_{\sigma(j)}}) \in \dot{h}_{\sigma(j)}, j = 1...n \}.$$

Proof. Similar to the Theorem 3.2 and 4.2. □

Example 5.3. Let $h_1 = \{(0.1, 0.4), (0.3, 0.5), (0.5, 0.2)\}$ and $h_2 = \{(0.5, 0.2), (0.6, 0.1)\}$ be two $GHFEs$, and suppose that the weight vector of them is $\varpi = (0.37, 0.63)^T$ and the aggregation associated vector is $\omega = (0.7, 0.3)^T$. On one hand,

$$\begin{aligned}\dot{h}_1 &= h_1^{n\varpi_1}\\ &= \{(\tfrac{2\times 0.1^{2\times 0.37}}{(2-0.1)^{2\times 0.37}+0.1^{2\times 0.37}}, \tfrac{(1+0.4)^{2\times 0.37}-(1-0.4)^{2\times 0.37}}{(1+0.4)^{2\times 0.37}+(1-0.4)^{2\times 0.37}}),\\ &\qquad (\tfrac{2\times 0.3^{2\times 0.37}}{(2-0.3)^{2\times 0.37}+0.3^{2\times 0.37}}, \tfrac{(1+0.5)^{2\times 0.37}-(1-0.5)^{2\times 0.37}}{(1+0.5)^{2\times 0.37}+(1-0.5)^{2\times 0.37}}),\\ &\qquad (\tfrac{2\times 0.5^{2\times 0.37}}{(2-0.5)^{2\times 0.37}+0.5^{2\times 0.37}}, \tfrac{(1+0.2)^{2\times 0.37}-(1-0.2)^{2\times 0.37}}{(1+0.2)^{2\times 0.37}+(1-0.2)^{2\times 0.37}})\}\\ &= \{(0.2033, 0.3036), (0.4339, 0.3855), (0.6145, 0.1489)\}\end{aligned}$$

and

$$\begin{aligned}\dot{h}_2 &= h_2^{n\varpi_2}\\ &= \{(\tfrac{2\times 0.5^{2\times 0.63}}{(2-0.5)^{2\times 0.63}+0.5^{2\times 0.63}}, \tfrac{(1+0.2)^{2\times 0.63}-(1-0.2)^{2\times 0.63}}{(1+0.2)^{2\times 0.63}+(1-0.2)^{2\times 0.63}}),\\ &\qquad (\tfrac{2\times 0.6^{2\times 0.63}}{(2-0.6)^{2\times 0.63}+0.6^{2\times 0.63}}, \tfrac{(1+0.1)^{2\times 0.63}-(1-0.1)^{2\times 0.63}}{(1+0.1)^{2\times 0.63}+(1-0.1)^{2\times 0.63}})\}\\ &= \{(0.4007, 0.25), (0.5117, 0.1258)\}.\end{aligned}$$

Then $s(\dot{h}_1) = 0.56895$, $s(\dot{h}_2) = 0.63415$. Since $s(\dot{h}_2) > s(\dot{h}_1)$,

$$\dot{h}_{\sigma(1)} = \dot{h}_2 = \{(0.4007, 0.25), (0.5117, 0.1258)\}$$

and

$$\dot{h}_{\sigma(2)} = \dot{h}_1 = \{(0.2033, 0.3036), (0.4339, 0.3855), (0.6145, 0.1489)\}.$$

Based on the definition of the $GHFEHG^{\varepsilon}$ and Theorem 5.2, we have

$$\begin{aligned}&GHFEHG^{\varepsilon}_{\varpi,\omega}(h_1, h_2)\\ &= \bigotimes_{j=1}^{2} \dot{h}_{\sigma(j)}^{\omega_j} = \dot{h}_{\sigma(1)}^{\omega_1} \bigotimes \dot{h}_{\sigma(2)}^{\omega_2}\\ &= \{(\tfrac{2\prod_{j=1}^{n}\mu_{\dot{\alpha}_{\sigma(j)}}^{\omega_j}}{\prod_{j=1}^{n}(2-\mu_{\dot{\alpha}_{\sigma(j)}})^{\omega_j}+\prod_{j=1}^{n}\mu_{\dot{\alpha}_{\sigma(j)}}^{\omega_j}}, \tfrac{\prod_{j=1}^{n}(1+\nu_{\dot{\alpha}_{\sigma(j)}})^{\omega_j}-\prod_{j=1}^{n}(1-\nu_{\dot{\alpha}_{\sigma(j)}})^{\omega_j}}{\prod_{j=1}^{n}(1+\nu_{\dot{\alpha}_{\sigma(j)}})^{\omega_j}+\prod_{j=1}^{n}(1-\nu_{\dot{\alpha}_{\sigma(j)}})^{\omega_j}})\\ &\qquad\qquad |\dot{\alpha}_{\sigma(j)} = (\mu_{\dot{\alpha}_{\sigma(j)}}, \nu_{\dot{\alpha}_{\sigma(j)}}) \in \dot{h}_{\sigma(j)}, j = 1, 2\}\\ &= \{(\tfrac{2\times 0.4007^{0.7}\times 0.2033^{0.3}}{(2-0.4007)^{0.7}\times(2-0.2033)^{0.3}+0.4007^{0.7}\times 0.2033^{0.3}},\\ &\qquad \tfrac{(1+0.25)^{0.7}\times(1+0.3036)^{0.3}-(1-0.25)^{0.7}\times(1-0.3036)^{0.3}}{(1+0.25)^{0.7}\times(1+0.3036)^{0.3}+(1-0.25)^{0.7}\times(1-0.3036)^{0.3}}),\\ &\quad (\tfrac{2\times 0.4007^{0.7}\times 0.4339^{0.3}}{(2-0.4007)^{0.7}\times(2-0.4339)^{0.3}+0.4007^{0.7}\times 0.4339^{0.3}},\\ &\qquad \tfrac{(1+0.25)^{0.7}\times(1+0.3855)^{0.3}-(1-0.25)^{0.7}\times(1-0.3855)^{0.3}}{(1+0.25)^{0.7}\times(1+0.3855)^{0.3}+(1-0.25)^{0.7}\times(1-0.3855)^{0.3}}),\\ &\quad (\tfrac{2\times 0.4007^{0.7}\times 0.6145^{0.3}}{(2-0.4007)^{0.7}\times(2-0.6145)^{0.3}+0.4007^{0.7}\times 0.6145^{0.3}},\\ &\qquad \tfrac{(1+0.25)^{0.7}\times(1+0.1489)^{0.3}-(1-0.25)^{0.7}\times(1-0.1489)^{0.3}}{(1+0.25)^{0.7}\times(1+0.1489)^{0.3}+(1-0.25)^{0.7}\times(1-0.1489)^{0.3}}),\\ &\quad (\tfrac{2\times 0.5177^{0.7}\times 0.2033^{0.3}}{(2-0.5177)^{0.7}\times(2-0.2033)^{0.3}+0.5177^{0.7}\times 0.2033^{0.3}},\\ &\qquad \tfrac{(1+0.1258)^{0.7}\times(1+0.3036)^{0.3}-(1-0.1258)^{0.7}\times(1-0.3036)^{0.3}}{(1+0.1258)^{0.7}\times(1+0.3036)^{0.3}+(1-0.1258)^{0.7}\times(1-0.3036)^{0.3}}),\\ &\quad (\tfrac{2\times 0.5177^{0.7}\times 0.4339^{0.3}}{(2-0.5177)^{0.7}\times(2-0.4339)^{0.3}+0.5177^{0.7}\times 0.4339^{0.3}},\\ &\qquad \tfrac{(1+0.1258)^{0.7}\times(1+0.3855)^{0.3}-(1-0.1258)^{0.7}\times(1-0.3855)^{0.3}}{(1+0.1258)^{0.7}\times(1+0.3855)^{0.3}+(1-0.1258)^{0.7}\times(1-0.3855)^{0.3}}),\\ &\quad (\tfrac{2\times 0.5177^{0.7}\times 0.6145^{0.3}}{(2-0.5177)^{0.7}\times(2-0.6145)^{0.3}+0.5177^{0.7}\times 0.6145^{0.3}},\\ &\qquad \tfrac{(1+0.1258)^{0.7}\times(1+0.1489)^{0.3}-(1-0.1258)^{0.7}\times(1-0.1489)^{0.3}}{(1+0.1258)^{0.7}\times(1+0.1489)^{0.3}+(1-0.1258)^{0.7}\times(1-0.1489)^{0.3}})\}\\ &= \{(0.3297, 0.2663), (0.4105, 0.2920), (0.4584, 0.2201),\\ &\qquad (0.3953, 0.1806), (0.4874, 0.2074), (0.5413, 0.1327)\}.\end{aligned}$$

Theorem 5.4. *The GHFWG$^{\varepsilon}$ operator is a special case of the GHFEHG$^{\varepsilon}$ operator.*

Proof. Let the associated vector $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ of the GHFEHG$^{\varepsilon}$ operator be $\omega=(\frac{1}{n},\frac{1}{n},...,\frac{1}{n})^T$. Then,

$$\begin{aligned}&GHFEHG^{\varepsilon}_{\varpi,\omega}(h_1,h_2,...,h_n)\\ &=\bigotimes_{j=1}^{n}\dot{h}^{\omega_j}_{\sigma(j)}=\dot{h}^{\omega_1}_{\sigma(1)}\bigotimes\dot{h}^{\omega_2}_{\sigma(2)}\bigotimes...\bigotimes\dot{h}^{\omega_n}_{\sigma(n)}\\ &=\dot{h}^{\frac{1}{n}}_{\sigma(1)}\bigotimes\dot{h}^{\frac{1}{n}}_{\sigma(2)}\bigotimes...\bigotimes\dot{h}^{\frac{1}{n}}_{\sigma(n)}\\ &=(\dot{h}_{\sigma(1)}\bigotimes\dot{h}_{\sigma(2)}\bigotimes...\bigotimes\dot{h}_{\sigma(n)})^{\frac{1}{n}}\\ &=h_1^{\varpi_1}\bigotimes h_2^{\varpi_2}\bigotimes...\bigotimes h_k^{\varpi_k}\\ &=GHFWG^{\varepsilon}_{\varpi}(h_1,h_2,...,h_n).\end{aligned}$$

□

Theorem 5.5. *The GHFOWG$^{\varepsilon}$ operator is a special case of the GHFEHG$^{\varepsilon}$ operator.*

Proof. Let the associated vector $\varpi=(\varpi_1,\varpi_2,...,\varpi_n)^T$ of $h_j(j=1,2,...,n)$ be $\varpi=(\frac{1}{n},\frac{1}{n},...,\frac{1}{n})^T$. Then,

$$\begin{aligned}&GHFEHG^{\varepsilon}_{\varpi,\omega}(h_1,h_2,...,h_n)\\ &=\bigotimes_{j=1}^{n}\dot{h}^{\omega_j}_{\sigma(j)}=\dot{h}^{\omega_1}_{\sigma(1)}\bigotimes\dot{h}^{\omega_2}_{\sigma(2)}\bigotimes...\bigotimes\dot{h}^{\omega_n}_{\sigma(n)}\\ &=h^{\omega_1}_{\sigma(1)}\bigotimes h^{\omega_2}_{\sigma(2)}\bigotimes...\bigotimes h^{\omega_n}_{\sigma(n)}\\ &=GHFOWG^{\varepsilon}_{\omega}(h_1,h_2,...,h_n).\end{aligned}$$

□

6. Evaluation of the performance and development review based on generalized hesitant fuzzy information

Consider a group decision-making problem under uncertainty. Suppose that there are m potential alternatives $Y_1,Y_2,...,Y_m$, the evaluation process is responsibility of the p experts $e_1,e_2,...,e_p$ that evaluate the n aspects $C_1,C_2,...,C_n$. The evaluation information provided by the experts group is expressed by the generalized hesitant fuzzy information. Based on the information, we shall construct the generalized hesitant fuzzy group matrix $H=(h_{ij})_{m\times n}$.

Step 1. All the decision makers provide their evaluations about the alternative Y_i under the attribute C_j, denoted by the $GHFEh_{ij}(i=1,2,...,m;j=1,2,...,n)$, and construct the group decision matrix $H=(h_{ij})_{m\times n}$.

Step 2. Aggregate the generalized hesitant fuzzy values h_{ij} for each alternative Y_i by the $GHFWG^{\varepsilon}$ (or $GHFOWG^{\varepsilon}$, $GHFEHG^{\varepsilon}$) operator.

For example, let

$$\begin{aligned}h_i=&GHFWG^{\varepsilon}_{\omega}(h_{i1},h_{i2},...,h_{in})\\ =&\{(\frac{2\prod_{j=1}^{n}\mu_{\alpha_{ij}}^{\omega_j}}{\prod_{j=1}^{n}(2-\mu_{\alpha_{ij}})^{\omega_j}+\prod_{j=1}^{n}\mu_{\alpha_{ij}}^{\omega_j}},\frac{\prod_{j=1}^{n}(1+\nu_{\alpha_{ij}})^{\omega_j}-\prod_{j=1}^{n}(1-\nu_{\alpha_{ij}})^{\omega_j}}{\prod_{j=1}^{n}(1+\nu_{\alpha_{ij}})^{\omega_j}+\prod_{j=1}^{n}(1-\nu_{\alpha_{ij}})^{\omega_j}})\\ &|\alpha_{ij}=(\mu_{\alpha_{ij}},\nu_{\alpha_{ij}})\in h_{ij},j=1...n\},\ (i=1,2,...,m).\end{aligned}$$

Step 3. Rank the $GHFEsh_i(i=1,2,...,m)$.

In what follows, we give an example adapted from [34] to illustrate the above algorithm for decision-making.

Example 6.1. The performance and development review (PADR) is an opportunity to recognize and acknowledge employee's individual successes over the past year and to review and discuss opportunities to increase the performance. The PADR is intended to facilitate an open dialogue between employees and managers about individual performance-what is working and what is not. It also brings into focus key areas of development that will encourage people to do their best work. Suppose that there are four alternatives $Y_i (i = 1, 2, 3, 4)$ to be considered and three attributes to be considered: c_1: business appraisal; c_2: professional competence assessment; c_3: an attitude appraisal. (more details about them can be found in [34]). Several decision makers are invited to evaluate the performance of the four alternatives.

Suppose the weight vector of the three criteria is $\omega = (\omega_1, \omega_2, \omega_3)^T =$ $(0.27, 0.46, 0.27)^T$. The evaluation team will review employees' work attitude and a range of responsibilities through their daily work.

Step 1. The members of the evaluation team provide their evaluations about the four candidates under business appraisal, work abilities, attitude appraisal, respectively. And we denote every evaluation by the GHFE $h_{ij} (i = 1, 2, 3, 4; j = 1, 2, 3)$, and construct the following generalized hesitant fuzzy group decision $H = (h_{ij})_{m\times n}$ (see Table I).

Table I. Generalized hesitant fuzzy decision matrix

	c_1	c_2	c_3
Y_1	$\{(0.6, 0.3), (0.8, 0.1), (0.9, 0.1)\}$	$\{(0.2, 0.3), (0.6, 0.2), (0.8, 0.1)\}$	$\{(0.7, 0.1), (0.9, 0.1)\}$
Y_2	$\{(0.4, 0.5), (0.7, 0.2), (0.9, 0.05)\}$	$\{(0.4, 0.3), (0.5, 0.2), (0.7, 0.3)\}$	$\{(0.6, 0.2)\}$
Y_3	$\{(0.5, 0.4), (0.6, 0.3), (0.7, 0.2)\}$	$\{(0.5, 0.2), (0.8, 0.1)\}$	$\{(0.5, 0.3), (0.7, 0.1), (0.8, 0.2)\}$
Y_4	$\{(0.7, 0.1), (0.8, 0.2)\}$	$\{(0.2, 0.5), (0.4, 0.3)\}$	$\{(0.6, 0.3), (0.7, 0.2)\}$

Step 2. Utilize the $GHFWG^{\varepsilon}$ operator to aggregate all the preference values $h_{ij} (i = 1, 2, 3, 4)$ in the ith line of H and get the overall preference values. Then, we can get

$$\begin{aligned} h_1 = \{&(0.3935, 0.2478), (0.4305, 0.2478), (0.6262, 0.2011),\\ &(0.6758, 0.2011), (0.7167, 0.1555), (0.7695, 0.1555),\\ &(0.4326, 0.1941), (0.4721, 0.1941), (0.6786, 0.1464),\\ &(0.7301, 0.1464), (0.7725, 0.1), (0.8267, 0.1),\\ &(0.4516, 0.1941), (0.4924, 0.1941), (0.7036, 0.1464),\\ &(0.7560, 0.1464), (0.7989, 0.1), (0.8536, 0.1)\},\end{aligned}$$

$$\begin{aligned} h_2 = \{&(0.4486, 0.3323), (0.4963, 0.2879), (0.5831, 0.3323),\\ &(0.5248, 0.2467), (0.5776, 0.2), (0.6722, 0.2467),\\ &(0.5697, 0.2076), (0.6251, 0.1601), (0.7235, 0.2076)\},\end{aligned}$$

$$\begin{aligned} h_3 = \{&(0.5, 0.2833), (0.5501, 0.2305), (0.5734, 0.2565),\\ &(0.6287, 0.2394), (0.6859, 0.1855), (0.7121, 0.2120),\\ &(0.5259, 0.2547), (0.5776, 0.2011), (0.6016, 0.2275),\\ &(0.6584, 0.2101), (0.7167, 0.1555), (0.7435, 0.1824),\\ &(0.5501, 0.2547), (0.6032, 0.2011), (0.6278, 0.2275),\\ &(0.6859, 0.2101), (0.7453, 0.1555), (0.7725, 0.1824)\}\end{aligned}$$

and

$$\begin{aligned} h_4 = \{&(0.3935, 0.3482), (0.4134, 0.3226), (0.5248, 0.2478),\\ &(0.5490, 0.2205), (0.4121, 0.3722), (0.4326, 0.3471),\\ &(0.5474, 0.2735), (0.5722, 0.2467)\}.\end{aligned}$$

Step 3. Calculate the scores of $h_i (i = 1, 2, 3, 4)$, respectively.

$$s(h_1) = 0.7392,\ s(h_2) = 0.6667,\ s(h_3) = 0.7108,\ s(h_4) = 0.5917.$$

Since $s(h_1) > s(h_3) > s(h_2) > s(h_4)$, we have

$$Y_1 \succ Y_3 \succ Y_2 \succ Y_4.$$

The best option is candidate Y_1.

In the above decision-making problem, if we choose the HFEWG operator (introduced by Yu in [34],which can only deal with the hesitant fuzzy information). The scores of $h_i (i = 1, 2, 3, 4)$ are

$$s(h_1) = 0.6434,\ s(h_2) = 0.5801,\ s(h_3) = 0.6366,\ s(h_4) = 0.4806.$$

Then. we have

$$Y_1 \succ Y_3 \succ Y_2 \succ Y_4.$$

The best option is Y_1.

If we choose the HFEWA operator (introduced by Yu in [34],which can only deal with the hesitant fuzzy information). The scores of $h_i (i = 1, 2, 3, 4)$ are

$$s(h_1) = 0.6706,\ s(h_2) = 0.5652,\ s(h_3) = 0.6166,\ s(h_4) = 0.4950.$$

Then, we have

$$Y_1 \succ Y_3 \succ Y_2 \succ Y_4.$$

The best option is also candidate Y_1.
Thus the result using our method in this example is the same to Yu's.

7. Conclusions

Under the framework of GHFE, a new multicriteria decision-making method is proposed in this paper. Some generalized hesitant fuzzy operational rules have been developed based on the Einstein operations. To aggregate the generalized hesitant fuzzy information, a series of aggregation operators have been proposed under various situations. Moreover, on the basis of the proposed aggregation operators, we presented a new multicriteria decision-making method. Finally, we applied our decision-making method successfully to solve the evaluation of PADR in a company. Future research is intended to find more suitable application area for our proposed generalized hesitant fuzzy operators, for example, in the area of image fusion as [2].

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References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1) (1986) 87–96.
[2] P. Balasubramaniam and V. P. Ananthi, Image fusion using intuitionistic fuzzy sets, Information Fusion 20 (1) (2014) 21–30.
[3] Bedregal, Benjamin, Reiser, Renata, Bustince, Humberto, Lopez-Molina, Carlos, Torra and Vicen, Aggregation functions for typical hesitant fuzzy elements and the action of automorphisms, Inform. Sci. 255 (2014) 82–99.
[4] I. Deli and N. Cagman, Application of soft sets in decision making based on game theory, Ann. Fuzzy Math. Inform. 11 (3) (2016) 425–438.
[5] Didier Dubois and H. Prade, Fuzzy Sets and Systems: Theory and Applications, Kluwer Academic, New York 1980.

[6] Y. D. He, H. Y. Chen, L. Zhou, J. Liu and Z. Tao, Generalized Interval-Valued Atanassov's intuitionistic fuzzy power geometric operators and their application to group decision making, nt. J. Fuzzy Syst. 15 (4) (2013) 401–411.
[7] Y. D. He, Z. He, G. D. Wang and H. Y. Chen, Hesitant Fuzzy Power Bonferroni Means and their Application to Multiple Attribute Decision Making, IEEE Transactions on Fuzzy Systems 23 (2015) 1655–1668.
[8] Y. D. He, Z. He and H. Y. Chen, Intuitionistic Fuzzy Interaction Bonferroni Means and Its Application to Multiple Attribute Decision Making, IEEE Transactions on Cybernetics 45 (2015) 116–128.
[9] Y. D. He, H. Y. Chen, Z. He and L. G. Zhou, Multi-attribute Decision Making Based on Neutral Averaging Operators for Intuitionistic Fuzzy Information, Applied Soft Computing 27 (2015) 64–76.
[10] Y. D. He, Z. He, Y. J. Deng and P. P. Zhou, IFPBMs and Their Application to Multiple Attribute Group Decision Making, Int. J. Oper. Res. 67 (1) (2016) 127–147.
[11] Y. D. He and Z. He, Extensions of Atanassoví≫s intuitionistic fuzzy interaction Bonferroni means and Their Application to Multiple attribute decision making, IEEE Transactions on Fuzzy Systems 24 (3) (2016) 558–573.
[12] F. Karaaslan, Intuitionistic fuzzy parameterized intuitionistic fuzzy soft sets with applications in decision making, Ann. Fuzzy Math. Inform. 11 (4) (2016) 607–619.
[13] H. W. Liu and G. J. Wang, Multi-criteria decision-making methods based on intuitionistic fuzzy sets, European J. Oper. Res. 197 (2007) 220–233.
[14] J. Ma, J. Lu and G. Zhang, Decider: a fuzzy multi-criteria group decision support system, Knowledge-Based Systems 23 (1) (2010) 23–31.
[15] J. M. Merigo and A. M. Gil-Lafuente, The induced generalized OWA operator, Inform. Sci. 179 (2009) 729–741.
[16] J. M. Merigo and M. Casanovas, Induced aggregation operators in decision-making with the Dempster Shafer belief structure, International Journal of Intelligent Systems24 (2009) 934–954.
[17] S. Miyamoto, Fuzzy multisets and their generalizations, Multiset Processing 2235 (2001) 225–235.
[18] M. Mizumoto and K. Tanaka, Some properties of fuzzy sets of type-2, Information and Control 31 (1976) 312–340.
[19] X. D. Peng and Y. Yang, Approaches to interval-valued intuitionistic hesitant fuzzy soft sets based decision making, Ann. Fuzzy Math. Inform. 10 (4) (2015) 657–680.
[20] Gang Qian, Hai Wanga and Xiangqian Feng, Generalized hesitant fuzzy sets and their application in decision support system, Information and Control 37 (2013) 357–365.
[21] P. Rajarajeswari and P. Dhanalakshmi, An application of interval valued intuitionistic fuzzy soft matrix theory in medical diagnosis, Ann. Fuzzy Math. Inform. 9 (3) (2015) 463–472.
[22] V. Torra and Y. Narukawa., Modeling decisions: Information fusion and aggregation operators, Berlin, Germany: Springer 2007.
[23] V. Torra and Y. Narukawa, On hesitant fuzzy sets and decision, 2009 IEEE International Conference on Fuzzy Systems (2009) 1378–1382.
[24] V. Torra, Hesitant fuzzy sets, International Journal of Intelligent Systems 25 (6) (2010) 529–539.
[25] W. Wang and X. Liu, Intuitionistic fuzzy geometric aggregation operators based on einstein operations, International Journal of Intelligent Systems 26 (11) (2011) 1049–1075.
[26] W. Wang and X. Liu, Intuitionistic fuzzy information aggregation using einstein operations, IEEE Transactions on Fuzzy Systems20 (5) (2012) 923–938.
[27] Lei Wang, Mingfang Ni and Lei Zhu, Correlation Measures of Dual Hesitant Fuzzy Sets, J. Appl. Math. 2013 (2013) Article ID 593739, 12 pages.
[28] M. M. Xia, Z. S. Xu and N. Chen, Induced aggregation under confidence levels, International Journal of Uncertainty Fuzziness and Knowledge-Based Systems 19 (2) (2011) 201–227.
[29] M. M. Xia, Z. S. Xu and N. Chen, Some hesitant fuzzy aggregation operators with their application in group decision-making, Group Decision and Negotiation 22 (2) (2013) 259–279.

[30] Z. Xu, On consistency of the weighted geometric mean complex judgment matrix in AHP, Eur. J. Oper. Res. 126 (2000) 683–687.
[31] Z. Xu and R. R. Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, International Journal of General Systems 35 (6) (2006) 417–433.
[32] Z. Xu, Intuitionistic fuzzy aggregation operators, IEEE Transactions on Fuzzy Systems 15 (6) (2007) 1179–1187.
[33] Z. S. Xu and M. M. Xia Hesitant fuzzy information aggregation in decision-making, Internat. J. Approx. Reason. 52 (3) (2011) 395–407.
[34] Dejian Yu, Some Hesitant Fuzzy Information Aggregation Operators Based on Einstein Operational Laws, International Journal of Intelligent Systems 29 (2014) 1–21.
[35] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[36] Zhiming Zhang, Interval-Valued Intuitionistic Hesitant Fuzzy Aggregation Operators and Their Application in Group Decision-Making, J. Appl. Math. (2013)Article ID 670285, 33 pages.
[37] Xiaoqiang Zhou, Qingguo Li, Multiple Attribute Decision Making Based on Hesitant Fuzzy Einstein Geometric Aggregation Operators, J. Appl. Math. 2014 (2014) Article ID 745617, 14 pages.
[38] Bin Zhu, Zeshui Xuand Meimei Xia, Dual Hesitant Fuzzy Sets, J. Appl. Math. 2012 (2012) Article ID 879629, 13 pages.

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