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Bipolar fuzzy soft sets in $\Gamma-$semirings

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Abstract. In this paper, we have introduced the concept of bipolar fuzzy soft $\Gamma-$subsemiring and bipolar fuzzy soft $\Gamma-$ideal in a $\Gamma-$semiring. Also we study some of their algebraic properties. It is also proved that the collection of all bipolar fuzzy soft $\Gamma-$ideals over a $\Gamma-$semiring forms a complete distributive lattice with the special union and intersection.

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1. Introduction

The notion of a fuzzy subset of a set was given by Zadeh [30]. Since then it was made a number of generalizations of this concept. Lee [19] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is expanded from [0, 1] to [-1, 1]. If the membership degree is 0, then elements are irrelevant to the corresponding property. If the membership degree is (0, 1], then elements somewhat satisfy the property. If the membership degree is [-1, 0), then elements somewhat satisfy the implicit counter property.

In 1994, Zhang [31] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets [30]. In many domains, bipolar fuzzy set is important to be able to deal with bipolar information. Dubois et al. [10] proposed fuzzy and possibilistic formalisms for bipolar information. Jun et al. [17] introduced the notions of bipolar fuzzy CI-subalgebras, bipolar fuzzy ideals and (closed) bipolar fuzzy filters in CI-algebras, and investigated related properties of them. Ban et al. [8] introduced the notions of a bipolar Ω-fuzzy left (right) ideal, a bipolar Ω-fuzzy bi-ideal and a bipolar Ω-fuzzy (1; 2)-ideal in semigroups. Muhiuddin [22] introduced the notions of bipolar fuzzy KU-subalgebras and bipolar fuzzy KU-ideals in KU-algebras, and investigated related properties of them. Also he gave the relations between a bipolar fuzzy subalgebra and a bipolar fuzzy ideal.

The concepts of fuzzy bipolar soft sets and bipolar fuzzy soft sets have been introduced by Naz and Shabir [26]. They defined their special union and special intersection and also showed that the both notions are equivalent. Akram [2] introduced the notion of bipolar fuzzy soft Lie subalgebras and investigated some of their properties. Akram et al. [3] introduced the concept of bipolar fuzzy soft Γ-subsemigroup and bipolar fuzzy soft Γ-ideals in a Γ-semigroup. Saleem et al. [27] combined the concepts of a bipolar fuzzy set and a soft set. They also introduced the notion of bipolar fuzzy soft set and gave some properties of this concept. Aslam et al. [6] also worked on bipolar fuzzy soft sets and their special union and special intersection. Akram et al. [4] gave the concept of bipolar fuzzy soft $K-$algebras.

Notion of $\Gamma-$semiring, introduced by Rao [23], not only generalizes the notion of semiring and $\Gamma-$ring but also the notion of ternary semiring. The natural growth of $\Gamma-$semiring is influenced by two things. One is the generalization of results of $\Gamma-$rings and another is the generalization of results of semiring and ternary semirings. This notion provides an algebraic back ground to the non positive cones of the totally ordered rings.

The concept of soft set theory was introduced firstly by Molodtsov [24] as a new mathematical tool. After that algebraic properties of this concept were studied by some researchers. Initially, Aktaş and Çağman [5] defined the notion of soft groups and derived their basic properties using soft sets. Feng et al. [12] introduced the notions of soft semirings, soft ideals and idealistic soft semirings. Kazancı et al. [18] defined soft BCH-algebras. Acar et al. [1] defined soft rings, and introduced basic notions of soft rings. Çelik et al. [9] defined some new binary relation on soft sets, and also they investigated some new properties of soft rings. Yamak et al. [28] introduced the notion of soft hypergroupoids.

The algebraic properties of fuzzy soft sets were studied by some authors. Firstly, Maji et al. [20] defined fuzzy soft set, and established some results on them. Jinliang et al. [15] defined the operations on fuzzy soft groups. Aygünoğlu and Aygün [7] gave the concept of fuzzy soft group, and defined fuzzy soft function and fuzzy soft homomorphism. Majumdar and Samanta [21] defined generalised fuzzy soft sets, and studied some of their properties. Jun et al. [16] derived the notion of fuzzy soft BCK/BCI algebras. Jiang et al. [14] proposed the notion of the interval-valued intuitionistic fuzzy soft set theory. İnan and Öztürk [13] introduced the fuzzy soft ring, and $(\in, \in \vee q)$-fuzzy soft subring.

In this paper we introduce the notion of bipolar fuzzy soft $\Gamma-$subsemirings, bipolar fuzzy soft $\Gamma-$ideals over $\Gamma-$semirings and investigate some of their algebraic properties. Also we give some results about the lattice structures of bipolar fuzzy soft $\Gamma-$ideals.

2. Preliminaries

Definition 2.1 ([23])**.** Let $(M, +)$ and $(\Gamma, +)$ be two commutative semigroups. Then we call M a Γ-semiring, if there exist a mapping $M \times \Gamma \times M \to M$ written as (x, α, y) as $x\alpha y$ such that it satisfies the following axioms: for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,

(ii) $(x+y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha+\beta)y = x\alpha y + x\beta y$,
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Definition 2.2 ([23])**.** Let S be a Γ-semiring and A be a nonempty subset of S. A is called a Γ-subsemiring of S, if A is a subsemigroup of $(S,+)$ and $A\Gamma A \subset A$.

Definition 2.3 ([23])**.** Let S be a Γ-semiring. A subset A of S is called a left (right) ideal of S, if A is closed under addition and $S\Gamma A \subset A(A\Gamma S \subset A)$. A is called an ideal of S if it is both a left and right ideal.

Definition 2.4 ([25])**.** Let X, Y be two sets. A fuzzy subset μ of X (see [24]) is a map from X to $[0,1]$. If $\mathcal{F}(X)$ is the family of all fuzzy subsets of X, then, for all $\mu,\nu \in \mathcal{F}(X)$, $\omega \in \mathcal{F}(Y)$ and $x \in X, y \in Y$, the following operations were defined:

$$\begin{aligned}
(\mu \vee \nu)(x) &= \mu(x) \vee \nu(x),\\
(\mu \wedge \nu)(x) &= \mu(x) \wedge \nu(x),\\
(\mu \times \omega)(x,y) &= \mu(x) \wedge \omega(y)\\
f(\mu)(y) &= \begin{cases} \bigvee_{f(a)=y} \mu(a) & \text{if} \quad f^{-1}(y) \neq \varnothing,\\ 0 & \quad\text{otherwise,}\end{cases}\\
f^{-1}(\omega)(x) &= \omega(f(x)),
\end{aligned}$$

and we denote $\mu \le \nu$ if and only if $\mu(x) \le \nu(x)$, for every $x \in X$.

If X is a semigroup then it can be defined:

$$\begin{aligned}
(\mu+\nu)(x) &= \bigvee_{x=a+b} (\mu(a) \wedge \nu(b)),\\
(\mu \cdot \nu)(x) &= \bigvee_{x=\sum_{i=1}^{p} b_i . c_i} \{ \bigwedge_{1\le i\le p} (\mu(b_i) \wedge \nu(c_i)) \mid b_i, c_i \in X, \quad p \in \mathbb{N}\}.
\end{aligned}$$

If X is a group, then $-\mu(x) = \mu(-x)$.

For $T \subseteq X$, $\chi_T \in \mathcal{F}(X)$ is called characteristic function of T, where $\chi_T(x) = 1$, if $x \in T$ and $\chi_T(x) = 0$, otherwise.

Definition 2.5 ([25])**.** For any $\mu \in \mathcal{F}(X)$, the set $\{x \mid x \in X, \mu(x) \ge \alpha\}$ is called α-level subset of μ and denoted by μ_α. The mapping given by $Q : X \times X \to [0,1]$ is called a fuzzy relation over X.

Definition 2.6 ([11])**.** Let S be a Γ-semiring. A fuzzy subset μ of S is said to be a fuzzy Γ-subsemiring of S, if

(i) $\mu(x+y) \ge \min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \ge \min\{\mu(x), \mu(y)\}$ for all $x, y \in S$, $\alpha \in \Gamma$.

Definition 2.7 ([11])**.** A fuzzy subset μ of a Γ-semiring S is called a fuzzy left (right) ideal of S, if for all $x, y \in S, \alpha \in \Gamma$

(i) $\mu(x+y) \ge \min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \ge \mu(y)(\mu(x))$.

Definition 2.8 ([11])**.** A fuzzy subset μ of a Γ-semiring S is called a fuzzy ideal of S, if for all $x, y \in S, \alpha \in \Gamma$

(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \geq \max\{\mu(x), \mu(y)\}$.

Definition 2.9 ([31]). Let X be a nonempty set. A bipolar fuzzy set B in X is an object having the form $B = \{(x, \mu^+(x), \mu^-(x)) | x \in X\}$, where $\mu^+ : X \to [0,1]$ and $\mu^- : X \to [-1, 0]$.

For $x \in X$, $\mu^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B, $\mu^-(x)$ denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set B. For the bipolar fuzzy set $B = \{(x, \mu^+(x), \mu^-(x)) | x \in X\}$, we shall use the symbol $B = (\mu^+, \mu^-)$.

Definition 2.10 ([31]). Let $A = (\mu_A^+, \mu_A^-)$ and $B = (\mu_B^+, \mu_B^-)$ be two bipolar fuzzy sets in X. Then we define:
$(A \bigcap B)(x) = (\min(\mu_A^+(x), \mu_B^+(x)), \max(\mu_A^-(x), \mu_B^-(x))$,
$(A \bigcup B)(x) = (\max(\mu_A^+(x), \mu_B^+(x)), \min(\mu_A^-(x), \mu_B^-(x))$.

$A \subseteq B$ means that $\mu_A^+(x) \leq \mu_B^+(x)$ and $\mu_A^-(x) \geq \mu_B^-(x)$ for all $x \in X$.

Definition 2.11 ([24]). Let U be an initial universe and E be a set of parameters. Let $\mathcal{P}(U)$ denotes the power set of U and A be non-empty subset of E. If F is a mapping given by $F : A \to \mathcal{P}(U)$, then the pair (F, A) is called a soft set over U.

Definition 2.12 ([20]). Let U be an initial universe, E be the set of parameters, and $\mathcal{F}(U)$ be the collection of all fuzzy subsets of U. If F is a mapping given by $F : A \to \mathcal{F}(U)$, the pair (F, A) is called a fuzzy soft set over U.

Definition 2.13 ([29]). Let U be an initial universe and $A \subseteq E$ be a set of parameters. Let $BF(U)$ denotes the set of all bipolar fuzzy subset of U. If ϕ is a mapping given by $\phi : A \to BF(U)$, then the pair (ϕ, A) is called an bipolar fuzzy soft set over U. For any $a \in A$, $\phi_{(a)}$ is referred to as the set of a-approximate elements of (ϕ, A) and can be defined as

$$\phi_{(a)} = \{(\mu^+_{\phi_{(a)}}(x), \mu^-_{\phi_{(a)}}(x)) | x \in U, a \in A\},$$

where $\mu^+_{\phi_{(a)}}(x)$ denotes the degree of x keeping the parameter a, $\mu^-_{\phi_{(a)}}(x)$ denotes the degree of x keeping the non-parameter a.

Definition 2.14 ([29]). Let (ϕ, A) and (ψ, B) be two bipolar fuzzy soft sets over U. We say that (ϕ, A) is a bipolar fuzzy soft subset of (ψ, B) and write $(\phi, A) \widetilde{\precsim} (\psi, B)$, if $A \subseteq B$ and $\phi(a) \subseteq \psi(a)$ for all $a \in A$.

(ϕ, A) and (ψ, B) are said to be bipolar fuzzy soft equal and write $(\phi, A) = (\psi, B)$, if $(\phi, A) \widetilde{\precsim} (\psi, B)$ and $(\psi, B) \widetilde{\precsim} (\phi, A)$.

Example 2.15. Let $U = \{x_1, x_2, x_3\}$ be the universe set, $E = \{a, b, c, d\}$ be the set of parameters, $A = \{a, b\} \subseteq E$ and $B = \{a, b, c\} \subseteq E$. Define (ϕ, A) and (ψ, B) as

$$(\phi, A) = \{\phi(a), \phi(b)\} \text{ and } (\psi, B) = \{\psi(a), \psi(b), \psi(c)\},$$

where $\phi(a) = \{(x_1, 0.1, -0.5), (x_2, 0.2, -0.4), (x_3, 0.3, -0.3)\}$,
$\phi(b) = \{(x_1, 0.2, -0.5), (x_2, 0.3, -0.2), (x_3, 0.4, -0.4)\}$
and

$\psi(a) = \{(x_1, 0.2, -0.6), (x_2, 0.3, -0.5), (x_3, 0.4, -0.4)\}$,
$\psi(b) = \{(x_1, 0.3, -0.6), (x_2, 0.4, -0.4), (x_3, 0.5, -0.5)\}$,
$\psi(c) = \{(x_1, 0.2, -0.5), (x_2, 0.3, -0.2), (x_3, 0.6, -0.6)\}$.
Then (ϕ, A) and (ψ, B) are bipolar fuzzy soft sets over U. Also $A \subseteq B$ and $\phi(a) \subseteq \psi(a)$ for all $a \in A$. Hence $(\phi, A)\widetilde{\preceq}(\psi, B)$.

Definition 2.16 ([26], [29])**.** Let (ϕ, A) and (ψ, B) be two bipolar fuzzy soft sets over U. Then,

(i) The extended union of bipolar fuzzy soft sets (ϕ, A) and (ψ, B) is defined as the bipolar fuzzy soft set $(\phi\widetilde{\cup}\psi, C) = (\phi, A)\widetilde{\cup}(\psi, B)$ over U, where $C = A \cup B$ and

$$\phi\widetilde{\cup}\psi(c) = \begin{cases} \phi(c) & \text{if} \quad c \in A\backslash B \\ \psi(c) & \text{if} \quad c \in B\backslash A \\ \phi(c) \cup \psi(c) & \text{if} \quad c \in A \cap B \end{cases}$$

for all $c \in C$.

(ii) The $\wedge$-intersection of bipolar fuzzy soft sets (ϕ, A) and (ψ, B) is defined as the bipolar fuzzy soft set $(\phi\widetilde{\wedge}\psi, C) = (\phi, A)\widetilde{\wedge}(\psi, B)$ over U, where $C = A \times B$ and

$$\phi\widetilde{\wedge}\psi(a, b) = \phi(a) \cap \psi(b) \text{ for all } (a, b) \in A \times B.$$

(iii) The $\vee$-union of bipolar fuzzy soft sets (ϕ, A) and (ψ, B) is defined as the bipolar fuzzy soft set $(\phi\widetilde{\vee}\psi, C) = (\phi, A)\widetilde{\vee}(\psi, B)$ over U, where $C = A \times B$ and

$$\phi\widetilde{\vee}\psi(a, b) = \phi(a) \cup \psi(b) \text{ for all } (a, b) \in A \times B.$$

(iv) The restricted union of bipolar fuzzy soft sets (ϕ, A) and (ψ, B) is defined as the bipolar fuzzy soft set $(\phi\widetilde{\cup}_{\Re}\psi, C) = (\phi, A)\widetilde{\cup}_{\Re}(\psi, B)$ over U, where $C = A \cap B$ and

$$\phi\widetilde{\cup}_{\Re}\psi(c) = \phi(c) \cup \psi(c) \text{ for all } c \in C.$$

(v) The extended intersection of bipolar fuzzy soft sets (ϕ, A) and (ψ, B) is defined as the bipolar fuzzy soft set $(\phi\widetilde{\cap}_{\varepsilon}\psi, C) = (\phi, A)\widetilde{\cap}_{\varepsilon}(\psi, B)$ over U, where $C = A \cup B$ and

$$\phi\widetilde{\cap}_{\varepsilon}\psi(c) = \begin{cases} \phi(c) & \text{if} \quad c \in A\backslash B \\ \psi(c) & \text{if} \quad c \in B\backslash A \\ \phi(c) \cap \psi(c) & \text{if} \quad c \in A \cap B \end{cases}$$

for all $c \in C$.

(vi) The restricted intersection of bipolar fuzzy soft sets (ϕ, A) and (ψ, B) is defined as the bipolar fuzzy soft set $(\phi\widetilde{\cap}\psi, C) = (\phi, A)\widetilde{\cap}(\psi, B)$ over U, where $C = A \cap B$ and

$$\phi\widetilde{\cap}\psi(c) = \phi(c) \cap \psi(c) \text{ for all } c \in C.$$

3. Bipolar fuzzy soft Γ-subsemirings (ideals)

Throughout this paper, S will denotes Γ-semiring otherwise it will be specified.

Definition 3.1. A bipolar fuzzy soft set (ϕ, A) over a Γ-semiring S is called a bipolar fuzzy soft Γ-subsemiring over S, if

$\mu^+_{\phi(a)}(x + y) \geq \min\{\mu^+_{\phi(a)}(x), \mu^+_{\phi(a)}(y)\}$,
$\mu^-_{\phi(a)}(x + y) \leq \max\{\mu^-_{\phi(a)}(x), \mu^-_{\phi(a)}(y)\}$

and

$\mu^+_{\phi(a)}(x\alpha y) \geq \min\{\mu^+_{\phi(a)}(x), \mu^+_{\phi(a)}(y)\}$,
$\mu^-_{\phi(a)}(x\alpha y) \leq \max\{\mu^-_{\phi(a)}(x), \mu^-_{\phi(a)}(y)\}$,
for all $x, y \in S, \alpha \in \Gamma$ and $a \in A$.

Definition 3.2. A bipolar fuzzy soft set (ϕ, A) over a Γ-semiring S is called a bipolar fuzzy soft left (right) Γ-ideal over S, if
$\mu^+_{\phi(a)}(x+y) \geq \min\{\mu^+_{\phi(a)}(x), \mu^+_{\phi(a)}(y)\}$,
$\mu^-_{\phi(a)}(x+y) \leq \max\{\mu^-_{\phi(a)}(x), \mu^-_{\phi(a)}(y)\}$
and
$\mu^+_{\phi(a)}(x\alpha y) \geq \mu^+_{\phi(a)}(y)\ (\mu^+_{\phi(a)}(x))$,
$\mu^-_{\phi(a)}(x\alpha y) \leq \mu^-_{\phi(a)}(y)\ (\mu^-_{\phi(a)}(x))$,
for all $x, y \in S, \alpha \in \Gamma$ and $a \in A$.

A bipolar fuzzy soft set (ϕ, A) over a Γ-semiring S is called a bipolar fuzzy soft Γ-ideal over S, if it is a left and right Γ-ideal over S.

Remark 3.3. Every bipolar fuzzy soft left (right) Γ-ideal over a Γ-semiring S is a bipolar fuzzy soft Γ-subsemiring over S but the converse is not true.

Example 3.4. Let $S = \{x_1, x_2, x_3\}$ and $\Gamma = \{\alpha\}$, then S is a Γ-semiring with the following tables,

$+$	x_1	x_2	x_3
x_1	x_1	x_2	x_3
x_2	x_2	x_2	x_3
x_3	x_3	x_3	x_2

and

α	x_1	x_2	x_3
x_1	x_1	x_3	x_3
x_2	x_3	x_2	x_3
x_3	x_3	x_3	x_3

Let $E = \{a, b, c\}$ and $B = \{a, b\}$. Then (ψ, B) is bipolar fuzzy soft set defined as

$$(\psi, B) = \{\psi(a), \psi(b)\},$$

where $\psi(a) = \{(x_1, 0.2, -0.2), (x_2, 0.6, -0.4), (x_3, 0.3, -0.3)\}$
and
$\psi(c) = \{(x_1, 0.4, -0.3), (x_2, 0.7, -0.8), (x_3, 0.5, -0.4)\}$.
Then (ψ, B) is a bipolar fuzzy soft Γ-subsemiring but it is neither a bipolar fuzzy soft left Γ-ideal nor a bipolar fuzzy soft right Γ-ideal over S. Thus we can see below:

$$\mu^+_{\psi_{(a)}}(x_1\alpha x_2) = \mu^+_{\psi_{(a)}}(x_3) = 0.3 \ngeq 0.6 = \mu^+_{\psi_{(a)}}(x_2)$$

and

$$\mu^+_{\psi_{(a)}}(x_2\alpha x_1) = \mu^+_{\psi_{(a)}}(x_3) = 0.3 \ngeq 0.6 = \mu^+_{\psi_{(a)}}(x_2).$$

Let $C = \{b, c\} \subseteq E$ and $(\varphi, C) = \{\varphi(b), \varphi(c)\}$, where

$$\varphi(b) = \{(x_1, 0.5, -0.8), (x_2, 0.8, -0.5), (x_3, 0.1, -0.4)\}$$

and

$$\varphi(c) = \{(x_1, 0.7, -0.5), (x_2, 0.6, -0.7), (x_3, 0.5, -0.2)\}.$$

Then (φ, C) is a bipolar fuzzy soft set over S but it is not a Γ-subsemiring over S as $\mu^+_{\varphi_{(b)}}(x_1\alpha x_2) = \mu^+_{\varphi_{(b)}}(x_3) = 0.1 \ngeq 0.5 = \min\{\mu^+_{\varphi_{(b)}}(x_1), \mu^+_{\varphi_{(b)}}(x_2)\}$.

Theorem 3.5. *If (ϕ, A) and (ψ, B) are two bipolar fuzzy soft Γ-subsemirings (left Γ-ideals, right Γ-ideals) over a Γ-semiring S, then so are: $(\phi, A)\widetilde{\wedge}(\psi, B)$, $(\phi, A)\widetilde{\vee}(\psi, B)$, $(\phi, A)\widetilde{\cap}(\psi, B)$, $(\phi, A)\widetilde{\cap}_\varepsilon(\psi, B)$, $(\phi, A)\widetilde{\cup}(\psi, B)$ and $(\phi, A)\widetilde{\cup}_\Re(\psi, B)$.*

Proof. Let (ϕ, A) and (ψ, B) be two bipolar fuzzy soft Γ-subsemirings over a Γ-semiring S. Then, by Definition 2.16 (2), $(\phi, A)\widetilde{\wedge}(\psi, B) = (\phi\widetilde{\wedge}\psi, C)$, where $C = A \times B$ and $\phi\widetilde{\wedge}\psi(a, b) = \phi(a) \cap \psi(b)$, for all $(a, b) \in C$. Since (ϕ, A) and (ψ, B) are bipolar fuzzy soft Γ-subsemirings over S, by Definitions 2.10 and 3.1, we get, for all $(a, b) \in C$, $x, y \in S$ and $\alpha \in \Gamma$,

$$\begin{aligned}
\mu^+_{\phi\widetilde{\wedge}\psi(a,b)}(x+y) &= \mu^+_{\phi(a)\cap\psi(b)}(x+y) \\
&= \min\{\mu^+_{\phi(a)}(x+y), \mu^+_{\psi(b)}(x+y)\} \\
&\geq \min\{\min\{\mu^+_{\phi(a)}(x), \mu^+_{\phi(a)}(y)\}, \min\{\mu^+_{\psi(b)}(x), \mu^+_{\psi(b)}(y)\}\} \\
&= \min\{\min\{\mu^+_{\phi(a)}(x), \mu^+_{\psi(b)}(x)\}, \min\{\mu^+_{\phi(a)}(y), \mu^+_{\psi(b)}(y)\}\} \\
&= \min\{\mu^+_{\phi\widetilde{\wedge}\psi(a,b)}(x), \mu^+_{\phi\widetilde{\wedge}\psi(a,b)}(y)\},
\end{aligned}$$

$$\begin{aligned}
\mu^-_{\phi\widetilde{\wedge}\psi(a,b)}(x+y) &= \mu^-_{\phi(a)\cap\psi(b)}(x+y) \\
&= \max\{\mu^-_{\phi(a)}(x+y), \mu^-_{\psi(b)}(x+y)\} \\
&\leq \max\{\max\{\mu^-_{\phi(a)}(x), \mu^-_{\phi(a)}(y)\}, \max\{\mu^-_{\psi(b)}(x), \mu^-_{\psi(b)}(y)\}\} \\
&= \max\{\max\{\mu^-_{\phi(a)}(x), \mu^-_{\psi(b)}(x)\}, \max\{\mu^-_{\phi(a)}(y), \mu^-_{\psi(b)}(y)\}\} \\
&= \max\{\mu^-_{\phi\widetilde{\wedge}\psi(a,b)}(x), \mu^-_{\phi\widetilde{\wedge}\psi(a,b)}(y)\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mu^+_{\phi\widetilde{\wedge}\psi(a,b)}(x\alpha y) &= \mu^+_{\phi(a)\cap\psi(b)}(x\alpha y) \\
&= \min\{\mu^+_{\phi(a)}(x\alpha y), \mu^+_{\psi(b)}(x\alpha y)\} \\
&\geq \min\{\min\{\mu^+_{\phi(a)}(x), \mu^+_{\phi(a)}(y)\}, \min\{\mu^+_{\psi(b)}(x), \mu^+_{\psi(b)}(y)\}\} \\
&= \min\{\min\{\mu^+_{\phi(a)}(x), \mu^+_{\psi(b)}(x)\}, \min\{\mu^+_{\phi(a)}(y), \mu^+_{\psi(b)}(y)\}\} \\
&= \min\{\mu^+_{\phi\widetilde{\wedge}\psi(a,b)}(x), \mu^+_{\phi\widetilde{\wedge}\psi(a,b)}(y)\},
\end{aligned}$$

$$\begin{aligned}
\mu^-_{\phi\widetilde{\wedge}\psi(a,b)}(x\alpha y) &= \mu^-_{\phi(a)\cap\psi(b)}(x\alpha y) \\
&= \max\{\mu^-_{\phi(a)}(x\alpha y), \mu^-_{\psi(b)}(x\alpha y)\} \\
&\leq \max\{\max\{\mu^-_{\phi(a)}(x), \mu^-_{\phi(a)}(y)\}, \max\{\mu^-_{\psi(b)}(x), \mu^-_{\psi(b)}(y)\}\} \\
&= \max\{\max\{\mu^-_{\phi(a)}(x), \mu^-_{\psi(b)}(x)\}, \max\{\mu^-_{\phi(a)}(y), \mu^-_{\psi(b)}(y)\}\} \\
&= \max\{\mu^-_{\phi\widetilde{\wedge}\psi(a,b)}(x), \mu^-_{\phi\widetilde{\wedge}\psi(a,b)}(y)\}.
\end{aligned}$$

Thus $(\phi, A)\widetilde{\wedge}(\psi, B)$ is a bipolar fuzzy soft $\Gamma-$subsemiring over S. The other two cases for $(\phi, A)\widetilde{\wedge}(\psi, B)$ can be proved in similar way.

The statements for $(\phi, A)\widetilde{\cap}(\psi, B)$ can be proved as for $(\phi, A)\widetilde{\wedge}(\psi, B)$.

Let (ϕ, A) and (ψ, B) be two bipolar fuzzy soft Γ-subsemirings over a Γ-semiring S. Then, by Definition 2.16 (3), $(\phi, A)\widetilde{\vee}(\psi, B) = (\phi\widetilde{\vee}\psi, C)$, where $C = A \times B$ and $\phi\widetilde{\vee}\psi(a, b) = \phi(a) \cup \psi(b)$, for all $(a, b) \in C$. Since (ϕ, A) and (ψ, B) are bipolar fuzzy soft Γ-subsemirings over S, then, by Definitions 2.10 and 3.1, we get, for all

$(a,b) \in C$, $x, y \in S$ and $\alpha \in \Gamma$,

$$\begin{aligned}
\mu^{+}_{\phi\widetilde{\vee}\psi(a,b)}(x+y) &= \mu^{+}_{\phi(a)\cup\psi(b)}(x+y) \\
&= \max\{\mu^{+}_{\phi(a)}(x+y), \mu^{+}_{\psi(b)}(x+y)\} \\
&\geq \max\{\min\{\mu^{+}_{\phi(a)}(x), \mu^{+}_{\phi(a)}(y)\}, \min\{\mu^{+}_{\psi(b)}(x), \mu^{+}_{\psi(b)}(y)\}\} \\
&= \min\{\max\{\mu^{+}_{\phi(a)}(x), \mu^{+}_{\psi(b)}(x)\}, \max\{\mu^{+}_{\phi(a)}(y), \mu^{+}_{\psi(b)}(y)\}\} \\
&= \min\{\mu^{+}_{\phi\widetilde{\vee}\psi(a,b)}(x), \mu^{+}_{\phi\widetilde{\vee}\psi(a,b)}(y)\},
\end{aligned}$$

$$\begin{aligned}
\mu^{-}_{\phi\widetilde{\vee}\psi(a,b)}(x+y) &= \mu^{-}_{\phi(a)\cup\psi(b)}(x+y) \\
&= \min\{\mu^{-}_{\phi(a)}(x+y), \mu^{-}_{\psi(b)}(x+y)\} \\
&= \min\{\max\{\mu^{-}_{\phi(a)}(x), \mu^{-}_{\phi(a)}(y)\}, \max\{\mu^{-}_{\psi(b)}(x), \mu^{-}_{\psi(b)}(y)\}\} \\
&\leq \max\{\max\{\mu^{-}_{\phi(a)}(x), \mu^{-}_{\psi(b)}(x)\}, \max\{\mu^{-}_{\phi(a)}(y), \mu^{-}_{\psi(b)}(y)\}\} \\
&= \max\{\mu^{-}_{\phi\widetilde{\vee}\psi(a,b)}(x), \mu^{-}_{\phi\widetilde{\vee}\psi(a,b)}(y)\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mu^{+}_{\phi\widetilde{\vee}\psi(a,b)}(x\alpha y) &= \mu^{+}_{\phi(a)\cup\psi(b)}(x\alpha y) \\
&= \max\{\mu^{+}_{\phi(a)}(x\alpha y), \mu^{+}_{\psi(b)}(x\alpha y)\} \\
&\geq \max\{\min\{\mu^{+}_{\phi(a)}(x), \mu^{+}_{\phi(a)}(y)\}, \min\{\mu^{+}_{\psi(b)}(x), \mu^{+}_{\psi(b)}(y)\}\} \\
&= \min\{\max\{\mu^{+}_{\phi(a)}(x), \mu^{+}_{\psi(b)}(x)\}, \max\{\mu^{+}_{\phi(a)}(y), \mu^{+}_{\psi(b)}(y)\}\} \\
&= \min\{\mu^{+}_{\phi\widetilde{\vee}\psi(a,b)}(x), \mu^{+}_{\phi\widetilde{\vee}\psi(a,b)}(y)\},
\end{aligned}$$

$$\begin{aligned}
\mu^{-}_{\phi\widetilde{\vee}\psi(a,b)}(x\alpha y) &= \mu^{-}_{\phi(a)\cup\psi(b)}(x\alpha y) \\
&= \min\{\mu^{-}_{\phi(a)}(x\alpha y), \mu^{-}_{\psi(b)}(x\alpha y)\} \\
&= \min\{\max\{\mu^{-}_{\phi(a)}(x), \mu^{-}_{\phi(a)}(y)\}, \max\{\mu^{-}_{\psi(b)}(x), \mu^{-}_{\psi(b)}(y)\}\} \\
&\leq \max\{\max\{\mu^{-}_{\phi(a)}(x), \mu^{-}_{\psi(b)}(x)\}, \max\{\mu^{-}_{\phi(a)}(y), \mu^{-}_{\psi(b)}(y)\}\} \\
&= \max\{\mu^{-}_{\phi\widetilde{\vee}\psi(a,b)}(x), \mu^{-}_{\phi\widetilde{\vee}\psi(a,b)}(y)\},
\end{aligned}$$

Thus $(\phi, A)\widetilde{\vee}(\psi, B)$ is a bipolar fuzzy soft $\Gamma-$subsemiring over S. The other two cases for $(\phi, A)\widetilde{\vee}(\psi, B)$ can be proved in similar way.

For $(\phi, A)\widetilde{\cup}_{\Re}(\psi, B)$, the proofs are similar.

Let (ϕ, A) and (ψ, B) be two bipolar fuzzy soft Γ-subsemirings over a Γ-semiring S. Then, by Definition 2.16 (1), $(\phi, A)\widetilde{\cup}(\psi, B) = (\phi\widetilde{\cup}\psi, C)$, where $C = A \cup B$ and

$$\phi\widetilde{\cup}\psi(c) = \begin{cases} \phi(c) & \text{if } \ c \in A\backslash B \\ \psi(c) & \text{if } \ c \in B\backslash A \\ \phi(c)\cup\psi(c) & \text{if } \ c \in A\cap B \end{cases}$$

for all $c \in C$.

Since (ϕ, A) and (ψ, B) are bipolar fuzzy soft Γ-subsemirings over S, then, by Definitions 2.10 and 3.1, for all $c \in C$, $x, y \in S$ and $\alpha \in \Gamma$, we have the following cases:

Case (i): If $c \in A \backslash B$, then $\phi \widetilde{\cup} \psi(c) = \phi(c)$. Thus we have

$$\mu^{+}_{\phi\widetilde{\cup}\psi(c)}(x+y) = \mu^{+}_{\phi(c)}(x+y) \geq \min\{\mu^{+}_{\phi(c)}(x), \mu^{+}_{\phi(c)}(y)\} = \min\{\mu^{+}_{\phi\widetilde{\cup}\psi(c)}(x), \mu^{+}_{\phi\widetilde{\cup}\psi(c)}(y)\},$$

$$\mu^{-}_{\phi\widetilde{\cup}\psi(c)}(x+y) = \mu^{-}_{\phi(c)}(x+y) \leq \max\{\mu^{-}_{\phi(c)}(x), \mu^{-}_{\phi(c)}(y)\} = \max\{\mu^{-}_{\phi\widetilde{\cup}\psi(c)}(x), \mu^{-}_{\phi\widetilde{\cup}\psi(c)}(y)\}.$$

Similarly,

$$\mu^{+}_{\phi\widetilde{\cup}\psi(c)}(x\alpha y) \geq \min\{\mu^{+}_{\phi\widetilde{\cup}\psi(c)}(x), \mu^{+}_{\phi\widetilde{\cup}\psi(c)}(y)\},$$

$$\mu^{-}_{\phi\widetilde{\cup}\psi(c)}(x\alpha y) \leq \max\{\mu^{-}_{\phi\widetilde{\cup}\psi(c)}(x), \mu^{-}_{\phi\widetilde{\cup}\psi(c)}(y)\}.$$

Case (ii) If $c \in B \backslash A$, then $\phi \widetilde{\cup} \psi(c) = \psi(c)$. Thus we have

$$\mu^{+}_{\phi\widetilde{\cup}\psi(c)}(x+y) = \mu^{+}_{\psi(c)}(x+y) \geq \min\{\mu^{+}_{\psi(c)}(x), \mu^{+}_{\psi(c)}(y)\} = \min\{\mu^{+}_{\phi\widetilde{\cup}\psi(c)}(x), \mu^{+}_{\phi\widetilde{\cup}\psi(c)}(y)\},$$

$$\mu^{-}_{\phi\widetilde{\cup}\psi(c)}(x+y) = \mu^{-}_{\psi(c)}(x+y) \leq \max\{\mu^{-}_{\psi(c)}(x), \mu^{-}_{\psi(c)}(y)\} = \max\{\mu^{-}_{\phi\widetilde{\cup}\psi(c)}(x), \mu^{-}_{\phi\widetilde{\cup}\psi(c)}(y)\}.$$

Similarly,

$$\mu^{+}_{\phi\widetilde{\cup}\psi(c)}(x\alpha y) \geq \min\{\mu^{+}_{\phi\widetilde{\cup}\psi(c)}(x), \mu^{+}_{\phi\widetilde{\cup}\psi(c)}(y)\},$$

$$\mu^{-}_{\phi\widetilde{\cup}\psi(c)}(x\alpha y) \leq \max\{\mu^{-}_{\phi\widetilde{\cup}\psi(c)}(x), \mu^{-}_{\phi\widetilde{\cup}\psi(c)}(y)\}.$$

Case (iii) If $c \in A \cap B$, then $\phi \widetilde{\cup} \psi(c) = \phi(c) \cup \psi(c)$. Thus we have

$$\begin{aligned}
\mu^{+}_{\phi\widetilde{\cup}\psi(c)}(x+y) &= \mu^{+}_{\phi(c)\cup\psi(c)}(x+y) \\
&= \max\{\mu^{+}_{\phi(c)}(x+y), \mu^{+}_{\psi(c)}(x+y)\} \\
&\geq \max\{\min\{\mu^{+}_{\phi(c)}(x), \mu^{+}_{\phi(c)}(y)\}, \min\{\mu^{+}_{\psi(c)}(x), \mu^{+}_{\psi(c)}(y)\}\} \\
&= \min\{\max\{\mu^{+}_{\phi(c)}(x), \mu^{+}_{\psi(c)}(x)\}, \max\{\mu^{+}_{\phi(c)}(y), \mu^{+}_{\psi(c)}(y)\}\} \\
&= \min\{\mu^{+}_{\phi\widetilde{\cup}\psi(c)}(x), \mu^{+}_{\phi\widetilde{\cup}\psi(c)}(y)\},
\end{aligned}$$

$$\begin{aligned}
\mu^{-}_{\phi\widetilde{\cup}\psi(c)}(x+y) &= \mu^{-}_{\phi(c)\cup\psi(c)}(x+y) \\
&= \min\{\mu^{-}_{\phi(c)}(x+y), \mu^{-}_{\psi(c)}(x+y)\} \\
&= \min\{\max\{\mu^{-}_{\phi(c)}(x), \mu^{-}_{\phi(c)}(y)\}, \max\{\mu^{-}_{\psi(c)}(x), \mu^{-}_{\psi(c)}(y)\}\} \\
&\leq \max\{\max\{\mu^{-}_{\phi(c)}(x), \mu^{-}_{\psi(c)}(x)\}, \max\{\mu^{-}_{\phi(c)}(y), \mu^{-}_{\psi(c)}(y)\}\} \\
&= \max\{\mu^{-}_{\phi\widetilde{\cup}\psi(c)}(x), \mu^{-}_{\phi\widetilde{\cup}\psi(c)}(y)\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mu^{+}_{\phi\widetilde{\cup}\psi(c)}(x\alpha y) &= \mu^{+}_{\phi(c)\cup\psi(c)}(x\alpha y) \\
&= \max\{\mu^{+}_{\phi(c)}(x\alpha y), \mu^{+}_{\psi(c)}(x\alpha y)\} \\
&\geq \max\{\min\{\mu^{+}_{\phi(c)}(x), \mu^{+}_{\phi(c)}(y)\}, \min\{\mu^{+}_{\psi(c)}(x), \mu^{+}_{\psi(c)}(y)\}\} \\
&= \min\{\max\{\mu^{+}_{\phi(c)}(x), \mu^{+}_{\psi(c)}(x)\}, \max\{\mu^{+}_{\phi(c)}(y), \mu^{+}_{\psi(c)}(y)\}\} \\
&= \min\{\mu^{+}_{\phi\widetilde{\cup}\psi(c)}(x), \mu^{+}_{\phi\widetilde{\cup}\psi(c)}(y)\},
\end{aligned}$$

$$\begin{aligned}
\mu^-_{\phi\widetilde{\cup}\psi(c)}(x\alpha y) &= \mu^-_{\phi(c)\cup\psi(c)}(x\alpha y)\\
&= \min\{\mu^-_{\phi(c)}(x\alpha y), \mu^-_{\psi(c)}(x\alpha y)\}\\
&= \min\{\max\{\mu^-_{\phi(c)}(x), \mu^-_{\phi(c)}(y)\}, \max\{\mu^-_{\psi(c)}(x), \mu^-_{\psi(c)}(y)\}\}\\
&\leq \max\{\max\{\mu^-_{\phi(c)}(x), \mu^-_{\psi(c)}(x)\}, \max\{\mu^-_{\phi(c)}(y), \mu^-_{\psi(c)}(y)\}\}\\
&= \max\{\mu^-_{\phi\widetilde{\cup}\psi(c)}(x), \mu^-_{\phi\widetilde{\cup}\psi(c)}(y)\}.
\end{aligned}$$

So $(\phi, A)\widetilde{\cup}(\psi, B)$ is a bipolar fuzzy soft $\Gamma-$subsemiring over S. The other two cases for $(\phi, A)\widetilde{\cup}(\psi, B)$ can be proved in similar way.

For $(\phi, A)\widetilde{\cap}_\varepsilon(\psi, B)$ the proofs are similar. □

Example 3.6. Let $S = \{x_1, x_2, x_3\}$ and $\Gamma = \{\alpha\}$ be the $\Gamma-$semiring along with the table given in Example 3.4. Let $E = \{a, b, c, d\}$ and $A = \{a, b\} \subseteq E$. Then (ϕ, A) is a bipolar fuzzy soft set defined as, $(\phi, A) = \{\phi(a), \phi(b)\}$, where

$$\phi(a) = \{(x_1, 0.1, -0.2), (x_2, 0.5, -0.5), (x_3, 0.3, -0.4)\},$$
$$\phi(b) = \{(x_1, 0.3, -0.3), (x_2, 0.8, -0.6), (x_3, 0.5, -0.5)\}.$$

Let $B = \{a, c\} \subseteq E$. Then (ψ, B) is a bipolar fuzzy soft set defined as, $(\psi, B) = \{\psi(a), \psi(c)\}$, where

$$\psi(a) = \{(x_1, 0.2, -0.2), (x_2, 0.6, -0.4), (x_3, 0.4, -0.3)\},$$
$$\psi(c) = \{(x_1, 0.4, -0.3), (x_2, 0.7, -0.8), (x_3, 0.5, -0.4)\}.$$

Clearly, it is seen that (ϕ, A) and (ψ, B) are bipolar fuzzy soft $\Gamma-$subsemiring of S.

Now let $(\phi, A)\widetilde{\wedge}(\psi, B) = (\phi\widetilde{\wedge}\psi, C)$, where

$C = A \times B = \{a, b\} \times \{a, c\} = \{(a, a), (a, c), (b, a), (b, c)\}$

and

$\phi\widetilde{\wedge}\psi(a, b) = \phi(a) \cap \psi(b)$, for all $(a, b) \in C$.

Then $(\phi\widetilde{\wedge}\psi, C) = \{\phi\widetilde{\wedge}\psi(a, a), \phi\widetilde{\wedge}\psi(a, c), \phi\widetilde{\wedge}\psi(b, a), \phi\widetilde{\wedge}\psi(b, c)\}$, where

$$\phi\widetilde{\wedge}\psi(a, a) = \{(x_1, 0.1, -0.2), (x_2, 0.5, -0.4), (x_3, 0.3, -0.3)\},$$
$$\phi\widetilde{\wedge}\psi(a, c) = \{(x_1, 0.1, -0.2), (x_2, 0.5, -0.5), (x_3, 0.3, -0.4)\},$$
$$\phi\widetilde{\wedge}\psi(b, a) = \{(x_1, 0.2, -0.2), (x_2, 0.6, -0.4), (x_3, 0.4, -0.3)\},$$
$$\phi\widetilde{\wedge}\psi(b, c) = \{(x_1, 0.3, -0.3), (x_2, 0.7, -0.6), (x_3, 0.5, -0.4)\}.$$

Obviously, for all $x, y \in S, \alpha \in \Gamma$ and $(a, b) \in C$,

$$\mu^+_{\phi\widetilde{\wedge}\psi(a,b)}(x + y) \geq \min\{\mu^+_{\phi\widetilde{\wedge}\psi(a,b)}(x), \mu^+_{\phi\widetilde{\wedge}\psi(a,b)}(y)\},$$
$$\mu^-_{\phi\widetilde{\wedge}\psi(a,b)}(x + y) \leq \max\{\mu^-_{\phi\widetilde{\wedge}\psi(a,b)}(x), \mu^-_{\phi\widetilde{\wedge}\psi(a,b)}(y)\},$$
$$\mu^+_{\phi\widetilde{\wedge}\psi(a,b)}(x\alpha y) \geq \min\{\mu^+_{\phi\widetilde{\wedge}\psi(a,b)}(x), \mu^+_{\phi\widetilde{\wedge}\psi(a,b)}(y)\},$$
$$\mu^-_{\phi\widetilde{\wedge}\psi(a,b)}(x\alpha y) \leq \max\{\mu^-_{\phi\widetilde{\wedge}\psi(a,b)}(x), \mu^-_{\phi\widetilde{\wedge}\psi(a,b)}(y)\}.$$

Thus $(\phi\widetilde{\wedge}\psi, C) = (\phi, A)\widetilde{\wedge}(\psi, B)$ is a bipolar fuzzy soft $\Gamma-$subsemiring over S.

Now let $(\phi, A)\widetilde{\vee}(\psi, B) = (\phi\widetilde{\vee}\psi, C)$, where

$C = A \times B = \{a, b\} \times \{a, c\} = \{(a, a), (a, c), (b, a), (b, c)\}$ and

$\phi\widetilde{\vee}\psi(a,b) = \phi(a) \cup \psi(b)$, for all $(a,b) \in C$.
Then $(\phi\widetilde{\vee}\psi, C) = \{\phi\widetilde{\vee}\psi(a,a), \phi\widetilde{\vee}\psi(a,c), \phi\widetilde{\vee}\psi(b,a), \phi\widetilde{\vee}\psi(b,c)\}$, where

$$\phi\widetilde{\vee}\psi(a,a) = \{(x_1, 0.2, -0.2), (x_2, 0.6, -0.5), (x_3, 0.4, -0.4)\},$$
$$\phi\widetilde{\vee}\psi(a,c) = \{(x_1, 0.4, -0.3), (x_2, 0.7, -0.8), (x_3, 0.5, -0.4)\},$$
$$\phi\widetilde{\vee}\psi(b,a) = \{(x_1, 0.3, -0.3), (x_2, 0.8, -0.6), (x_3, 0.5, -0.5)\},$$
$$\phi\widetilde{\vee}\psi(b,c) = \{(x_1, 0.4, -0.3), (x_2, 0.8, -0.8), (x_3, 0.5, -0.5)\}.$$

Obviously, for all $x, y \in S, \alpha \in \Gamma$ and $(a,b) \in C$,

$$\mu^+_{\phi\widetilde{\vee}\psi(a,b)}(x+y) \geq \min\{\mu^+_{\phi\widetilde{\vee}\psi(a,b)}(x), \mu^+_{\phi\widetilde{\vee}\psi(a,b)}(y)\},$$
$$\mu^-_{\phi\widetilde{\vee}\psi(a,b)}(x+y) \leq \max\{\mu^-_{\phi\widetilde{\vee}\psi(a,b)}(x), \mu^-_{\phi\widetilde{\vee}\psi(a,b)}(y)\},$$
$$\mu^+_{\phi\widetilde{\vee}\psi(a,b)}(x\alpha y) \geq \min\{\mu^+_{\phi\widetilde{\vee}\psi(a,b)}(x), \mu^+_{\phi\widetilde{\vee}\psi(a,b)}(y)\},$$
$$\mu^-_{\phi\widetilde{\vee}\psi(a,b)}(x\alpha y) \leq \max\{\mu^-_{\phi\widetilde{\vee}\psi(a,b)}(x), \mu^-_{\phi\widetilde{\vee}\psi(a,b)}(y)\}.$$

So $(\phi\widetilde{\vee}\psi, C) = (\phi, A)\widetilde{\vee}(\psi, B)$ is a bipolar fuzzy soft $\Gamma-$subsemiring over S.

Definition 3.7. A bipolar fuzzy soft $\Gamma-$semiring (ϕ, A) of S is called a bipolar fuzzy soft interior $\Gamma-$ideal over S, if
(i) $\mu^+_{\phi(a)}(x\alpha z\beta y) \geq \mu^+_{\phi(a)}(z)$,
(ii) $\mu^-_{\phi(a)}(x\alpha z\beta y) \leq \mu^-_{\phi(a)}(z)$, for all $x, y, z \in S, \alpha, \beta \in \Gamma$ and $a \in A$.

Remark 3.8. Every bipolar fuzzy soft $\Gamma-$ideal over a $\Gamma-$semiring S is a bipolar fuzzy interior $\Gamma-$ideal over S but the converse is not true.

Example 3.9. Let $S = \{x_1, x_2, x_3, x_4\}$ and $\Gamma = \{\alpha, \beta\}$, then S is a Γ-semiring with the following tables:

$+$	x_1	x_2	x_3	x_4
x_1	x_1	x_2	x_3	x_4
x_2	x_2	x_2	x_3	x_4
x_3	x_3	x_3	x_3	x_4
x_4	x_4	x_4	x_4	x_3

and

α	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_3	x_1	x_2
x_3	x_1	x_3	x_1	x_4
x_4	x_1	x_1	x_1	x_3

and

β	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_2	x_2	x_2	x_2
x_3	x_1	x_1	x_1	x_1
x_4	x_1	x_1	x_1	x_4

Let $E = \{a, b, c, d\}$, $A = \{a, b\}$. Then (ϕ, A) is bipolar fuzzy soft set defined as $(\phi, A) = \{\phi(a), \phi(b)\}$, where

$$\phi(a) = \{(x_1, 0.9, -0.7), (x_2, 0.7, -0.5), (x_3, 0.6, -0.3), (x_4, 0.1, -0.1)\}$$

and

$$\phi(b) = \{(x_1, 0.6, -0.8), (x_2, 0.4, -0.6), (x_3, 0.3, -0.4), (x_3, 0.1, -0.2)\}.$$

Then (ϕ, A) is a bipolar fuzzy soft interior $\Gamma-$ideal over S but it is not a $\Gamma-$ideal over S as we can see below:

$$\mu^+_{\phi(a)}(x_3\alpha x_4\beta x_4) = \mu^+_{\phi(a)}(x_4\beta x_4) = \mu^+_{\phi(a)}(x_4) = 0.1 \ngeq 0.6 = \mu^+_{\phi(a)}(x_3).$$

Thus (ϕ, A) is not a bipolar fuzzy soft right $\Gamma-$ideal over S and hence not a bipolar fuzzy soft $\Gamma-$ideal over S.

Definition 3.10. A bipolar fuzzy soft $\Gamma-$subsemiring (ϕ, A) of S is called a *bipolar fuzzy soft bi-*$\Gamma-$*ideal* over S, if

(i) $\mu^{+}_{\phi(a)}(x\alpha z\beta y) \geq \min\{\mu^{+}_{\phi(a)}(x), \mu^{+}_{\phi(a)}(y)\}$,

(ii) $\mu^{-}_{\phi(a)}(x\alpha z\beta y)) \leq \max\{\mu^{-}_{\phi(a)}(x), \mu^{-}_{\phi(a)}(y)\}$,

for all $x, y, z \in S, \alpha, \beta \in \Gamma$ and $a \in A$.

Example 3.11. Let $S = \{x_1, x_2, x_3, x_4\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$, then S is a Γ-semiring with the following tables:

$+$	x_1	x_2	x_3	x_4
x_1	x_1	x_2	x_3	x_4
x_2	x_2	x_2	x_3	x_4
x_3	x_3	x_3	x_3	x_4
x_4	x_4	x_4	x_4	x_3

and

α	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_2	x_2	x_2	x_2
x_3	x_1	x_1	x_1	x_3
x_4	x_1	x_1	x_1	x_4

β	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_2	x_1	x_1
x_3	x_1	x_1	x_3	x_1
x_4	x_1	x_1	x_1	x_4

and

γ	x_1	x_2	x_3	x_4
x_1	x_1	x_1	x_1	x_1
x_2	x_1	x_3	x_1	x_2
x_3	x_1	x_3	x_1	x_4
x_4	x_1	x_1	x_1	x_3

Let $E = \{a, b, c, d\}$ and $A = \{a, b\}$. Then (ϕ, A) is bipolar fuzzy soft set defined as $(\phi, A) = \{\phi(a), \phi(b)\}$, where

$$\phi(a) = \{(x_1, 0.8, -0.9), (x_2, 0.6, -0.7), (x_3, 0.4, -0.6), (x_4, 0.2, -0.3)\}$$

and

$$\phi(b) = \{(x_1, 0.7, -0.6), (x_2, 0.5, -0.4), (x_3, 0.3, -0.2), (x_3, 0.1, -0.1)\}.$$

Thus (ϕ, A) is a bipolar fuzzy soft bi-$\Gamma-$ideal over S.

Theorem 3.12. *If (ϕ, A) and (ψ, B) are two bipolar fuzzy soft bi-Γ-ideals (interior Γ-ideals) over a Γ-semiring S, then so are: $(\phi, A)\widetilde{\wedge}(\psi, B)$, $(\phi, A)\widetilde{\vee}(\psi, B)$, $(\phi, A)\widetilde{\cap}(\psi, B)$, $(\phi, A)\widetilde{\cap}_{\varepsilon}(\psi, B)$, $(\phi, A)\widetilde{\cup}(\psi, B)$ and $(\phi, A)\widetilde{\cup}_{\Re}(\psi, B)$.*

Proof. Let (ϕ, A) and (ψ, B) be two bipolar fuzzy soft bi-Γ-ideals over a Γ-semiring S. Then they are also bipolar fuzzy soft Γ-subsemirings. Thus, by Theorem 3.5, $(\phi, A)\widetilde{\wedge}(\psi, B)$ is also a bipolar fuzzy soft Γ-subsemiring. So, by Definition 2.16 (2), $(\phi, A)\widetilde{\wedge}(\psi, B) = (\phi\widetilde{\wedge}\psi, C)$, where $C = A \times B$ and $\phi\widetilde{\wedge}\psi(a, b) = \phi(a) \cap \psi(b)$, for all $(a, b) \in C$. Since (ϕ, A) and (ψ, B) are bipolar fuzzy soft bi-Γ-ideals over S, by Definitions 2.10, 3.8 and 3.11, we get, for all $(a, b) \in C$, $x, y \in S$ and $\alpha \in \Gamma$,

$$\begin{aligned}
\mu^{+}_{\phi\widetilde{\wedge}\psi(a,b)}(x\alpha z\beta y) &= \mu^{+}_{\phi(a)\cap\psi(b)}(x\alpha z\beta y) \\
&= \min\{\mu^{+}_{\phi(a)}(x\alpha z\beta y), \mu^{+}_{\psi(b)}(x\alpha z\beta y)\} \\
&\geq \min\{\min\{\mu^{+}_{\phi(a)}(x), \mu^{+}_{\phi(a)}(y)\}, \min\{\mu^{+}_{\psi(b)}(x), \mu^{+}_{\psi(b)}(y)\}\} \\
&= \min\{\min\{\mu^{+}_{\phi(a)}(x), \mu^{+}_{\psi(b)}(x)\}, \min\{\mu^{+}_{\phi(a)}(y), \mu^{+}_{\psi(b)}(y)\}\} \\
&= \min\{\mu^{+}_{\phi\widetilde{\wedge}\psi(a,b)}(x), \mu^{+}_{\phi\widetilde{\wedge}\psi(a,b)}(y)\},
\end{aligned}$$

$$\begin{aligned}
\mu^-_{\phi\widetilde{\wedge}\psi(a,b)}(x\alpha z\beta y) &= \mu^-_{\phi(a)\cap\psi(b)}(x\alpha z\beta y) \\
&= \max\{\mu^-_{\phi(a)}(x\alpha z\beta y), \mu^-_{\psi(b)}(x\alpha z\beta y)\} \\
&\leq \max\{\max\{\mu^-_{\phi(a)}(x), \mu^-_{\phi(a)}(y)\}, \max\{\mu^-_{\psi(b)}(x), \mu^-_{\psi(b)}(y)\}\} \\
&= \max\{\max\{\mu^-_{\phi(a)}(x), \mu^-_{\psi(b)}(x)\}, \max\{\mu^-_{\phi(a)}(y), \mu^-_{\psi(b)}(y)\}\} \\
&= \max\{\mu^-_{\phi\widetilde{\wedge}\psi(a,b)}(x), \mu^-_{\phi\widetilde{\wedge}\psi(a,b)}(y)\}.
\end{aligned}$$

Hence $(\phi, A)\widetilde{\wedge}(\psi, B) = (\varphi, C)$ is a bipolar fuzzy soft bi-$\Gamma-$ideal over S. The other case for $(\phi, A)\widetilde{\wedge}(\psi, B)$ can be proved in similar way.

The statements for $(\phi, A)\widetilde{\cap}(\psi, B)$ can be proved as for $(\phi, A)\widetilde{\wedge}(\psi, B)$.

Let (ϕ, A) and (ψ, B) be two bipolar fuzzy soft bi-Γ-ideals over a Γ-semiring S. Then, by Definition 2.16 (3), $(\phi, A)\widetilde{\vee}(\psi, B) = (\phi\widetilde{\vee}\psi, C)$, where $C = A \times B$ and $\phi\widetilde{\vee}\psi(a, b) = \phi(a) \cup \psi(b)$, for all $(a, b) \in C$. Since (ϕ, A) and (ψ, B) are bipolar fuzzy soft bi-Γ-ideals over S, then, by Definitions 2.10, 3.8 and 3.11, we get, for all $(a, b) \in C$, $x, y \in S$ and $\alpha \in \Gamma$,

$$\begin{aligned}
\mu^+_{\phi\widetilde{\vee}\psi(a,b)}(x\alpha z\beta y) &= \mu^+_{\phi(a)\cup\psi(b)}(x\alpha z\beta y) \\
&= \max\{\mu^+_{\phi(a)}(x\alpha z\beta y), \mu^+_{\psi(b)}(x\alpha z\beta y)\} \\
&\geq \max\{\min\{\mu^+_{\phi(a)}(x), \mu^+_{\phi(a)}(y)\}, \min\{\mu^+_{\psi(b)}(x), \mu^+_{\psi(b)}(y)\}\} \\
&= \min\{\max\{\mu^+_{\phi(a)}(x), \mu^+_{\psi(b)}(x)\}, \max\{\mu^+_{\phi(a)}(y), \mu^+_{\psi(b)}(y)\}\} \\
&= \min\{\mu^+_{\phi\widetilde{\vee}\psi(a,b)}(x), \mu^+_{\phi\widetilde{\vee}\psi(a,b)}(y)\},
\end{aligned}$$

$$\begin{aligned}
\mu^-_{\phi\widetilde{\vee}\psi(a,b)}(x\alpha z\beta y) &= \mu^-_{\phi(a)\cup\psi(b)}(x\alpha z\beta y) \\
&= \min\{\mu^-_{\phi(a)}(x\alpha z\beta y), \mu^-_{\psi(b)}(x\alpha z\beta y)\} \\
&= \min\{\max\{\mu^-_{\phi(a)}(x), \mu^-_{\phi(a)}(y)\}, \max\{\mu^-_{\psi(b)}(x), \mu^-_{\psi(b)}(y)\}\} \\
&\leq \max\{\max\{\mu^-_{\phi(a)}(x), \mu^-_{\psi(b)}(x)\}, \max\{\mu^-_{\phi(a)}(y), \mu^-_{\psi(b)}(y)\}\} \\
&= \max\{\mu^-_{\phi\widetilde{\vee}\psi(a,b)}(x), \mu^-_{\phi\widetilde{\vee}\psi(a,b)}(y)\}.
\end{aligned}$$

So $(\phi, A)\widetilde{\vee}(\psi, B)$ is a bipolar fuzzy soft bi-Γ-ideal over S. The other case for $(\phi, A)\widetilde{\vee}(\psi, B)$ can be proved in similar way.

The statements for $(\phi, A)\widetilde{\cup}_{\Re}(\psi, B)$ can be proved as for $(\phi, A)\widetilde{\vee}(\psi, B)$.

The proofs of $(\phi, A)\widetilde{\cup}(\psi, B)$ and $(\phi, A)\widetilde{\cap}_{\varepsilon}(\psi, B)$ are shown similarly. □

Corollary 3.13. *The restricted union $\widetilde{\cup}_{\Re}$ and restricted intersection $\widetilde{\cap}$ of two bipolar fuzzy soft Γ-semirings (left Γ-ideals, right Γ-ideals, bi-Γ-ideals, interior Γ-ideals) over a Γ-semiring S is also a bipolar fuzzy Γ-semiring (left Γ-ideal, right Γ-ideal, bi-Γ-ideal, interior Γ-ideal) over S.*

Corollary 3.14. *The extended union $\widetilde{\cup}$ and extended intersection $\widetilde{\cap}_{\varepsilon}$ of two bipolar fuzzy soft Γ-semirings (left Γ-ideals, right Γ-ideals, bi-Γ-ideals, interior Γ-ideals) over a Γ-semiring S is also a bipolar fuzzy Γ-semiring (left Γ-ideal, right Γ-ideal, bi-Γ-ideal, interior Γ-ideal) over S.*

Theorem 3.15. *Let I be a parameter set and $\sum_{\boldsymbol{I}}(S)$ be the set of all bipolar fuzzy soft left $\Gamma-$ ideals over S. Then $(\sum_{\boldsymbol{I}}(S), \widetilde{\cup}, \widetilde{\cap})$ forms a complete distributive lattice along with the relation $\widetilde{\preceq}$.*

Proof. Consider two bipolar fuzzy soft left $\Gamma-$ ideals (ϕ, A) and (ψ, B) over S with $A \subseteq \mathbf{I}$ and $B \subseteq \mathbf{I}$. If $(\phi, A), (\psi, B) \in \sum_{\mathbf{I}}(S)$, then $(\phi, A)\widetilde{\cap}(\psi, B)$, $(\phi, A)\widetilde{\cup}(\psi, B) \in \sum_{\mathbf{I}}(S)$, by Theorem 3.9 and 3.10.
Obviously, $(\phi, A)\widetilde{\cap}(\psi, B)$ is the greatest lower bound of $\{(\phi, A), (\psi, B)\}$ and $(\phi, A)\widetilde{\cup}(\psi, B)$ is the least upper bound of $\{(\phi, A), (\psi, B)\}$.
Thus every sub collection of $\sum_{\mathbf{I}}(S)$ has a least upper bound as well as a greatest lower bound. So $\sum_{\mathbf{I}}(S)$ is a complete lattice.

Now let, $(\phi, A), (\psi, B), (\varphi, C) \in \sum_{\mathbf{I}}(S)$. Then

$$(\phi, A)\widetilde{\cap}((\psi, B)\widetilde{\cup}(\varphi, C)) = (f, A \cap (B \cup C)).$$

Also $((\phi, A)\widetilde{\cap}(\psi, B))\widetilde{\cup}((\phi, A)\widetilde{\cap}(\varphi, C)) = (g, (A \cap B) \cup (A \cap C)) = (g, A \cap (B \cup C))$. Clearly, $f(x) = g(x)$ for $x \in A \cap (B \cup C)$. Hence,

$$(\phi, A)\widetilde{\cap}((\psi, B)\widetilde{\cup}(\varphi, C)) = ((\phi, A)\widetilde{\cap}(\psi, B))\widetilde{\cup}((\phi, A)\widetilde{\cap}(\varphi, C)).$$

This implies that $\sum_{\mathbf{I}}(S)$ forms a complete distributive lattice over S.

Similarly, it can be proved the above result for right $\Gamma-$ideals. □

Theorem 3.16. *Let I be a parameter set and $\sum_{\boldsymbol{I}}(S)$ be the set of all bipolar fuzzy soft left $\Gamma-$ ideals over S. Then $(\sum_{\boldsymbol{I}}(S), \widetilde{\cup}_{\Re}, \widetilde{\cap}_{\varepsilon})$ forms a complete distributive lattice along with the relation $\widetilde{\preceq}$.*

Proof. Similar as above Theorem. □

Let I be a parameter set and $D \subseteq I$. If $\sum_D(S)$ be the collection of all bipolar fuzzy soft left (right) $\Gamma-$ideals S with parameter set D. Then for $D_1, D_2 \subseteq D$. We have the following results.

Lemma 3.17. *If $(\phi, D_1), (\psi, D_2) \in \sum_D(S)$, then $(\phi, D_1)\widetilde{\cap}(\psi, D_2) \in \sum_D(S)$ and $(\phi, D_1)\widetilde{\cup}_{\Re}(\psi, D_2) \in \sum_D(S)$.*

Proof. Straightforward. □

Lemma 3.18. *If $(\phi, D_1), (\psi, D_2) \in \sum_D(S)$, then $(\phi, D_1)\widetilde{\cap}_{\varepsilon}(\psi, D_2) \in \sum_D(S)$ and $(\phi, D_1)\widetilde{\cup}(\psi, D_2) \in \sum_D(S)$.*

Proof. Straightforward. □

Remark 3.19. The collection $(\sum_{\mathbf{D}}(S), \widetilde{\cup}, \widetilde{\cap})$ forms a sublattice of the collection $(\sum_{\mathbf{I}}(S), \widetilde{\cup}, \widetilde{\cap})$.

Remark 3.20. The collection $(\sum_{\mathbf{D}}(S), \widetilde{\cup}_{\Re}, \widetilde{\cap}_{\varepsilon})$ forms a sublattice of the collection $(\sum_{\mathbf{I}}(S), \widetilde{\cup}_{\Re}, \widetilde{\cap}_{\varepsilon})$.

Definition 3.21. Let (ϕ, A) and (ψ, B) be two bipolar fuzzy soft sets over a Γ-semiring S. The product (ϕ, A) and (ψ, B) is defined as $(\phi, A)\circ_{\Gamma}(\psi, B) = (\phi \circ_{\Gamma} \psi, C)$, where $C = A \cup B$ and

$$\mu^+_{\phi\circ_\Gamma\psi_{(c)}}(x) = \begin{cases} \mu^+_{\phi_{(c)}}(x) & \text{if} \quad c \in A\backslash B \\ \mu^+_{\psi_{(c)}}(x) & \text{if} \quad c \in B\backslash A \\ \sup_{x=y\alpha z}\{\min\{\mu^+_{\phi_{(c)}}(y), \mu^+_{\psi_{(c)}}(z)\}\} & \text{if} \quad c \in A \cap B \end{cases}$$

and

$$\mu^-_{\phi\circ_\Gamma\psi_{(c)}}(x) = \begin{cases} \mu^-_{\phi_{(c)}}(x) & \text{if} \quad c \in A\backslash B \\ \mu^-_{\psi_{(c)}}(x) & \text{if} \quad c \in B\backslash A \\ \inf_{x=y\alpha z}\{\max\{\mu^-_{\phi_{(c)}}(y), \mu^-_{\psi_{(c)}}(z)\}\} & \text{if} \quad c \in A \cap B \end{cases}$$

for all $c \in C$, $x, y, z \in S, \alpha \in \Gamma$.

Theorem 3.22. *Let (ϕ, A) and (ψ, B) be two bipolar fuzzy soft left (right) Γ-ideals over the Γ-semiring S. Then $(\phi, A) \circ_\Gamma (\psi, B)$ is also bipolar fuzzy soft left (right) Γ-ideal over S.*

Proof. For any $c \in C$, we have the following cases:

Case (i): If $c \in A\backslash B$, then for $x, y \in S$ and $\alpha \in \Gamma$,

$$\mu^+_{\phi\circ_\Gamma\psi_{(c)}}(x+y) = \mu^+_{\phi_{(c)}}(x+y) \geq \mu^+_{\phi_{(c)}}(y) = \mu^+_{\phi\circ_\Gamma\psi_{(c)}}(y),$$

$$\mu^-_{\phi\circ_\Gamma\psi_{(c)}}(x+y) = \mu^-_{\phi_{(c)}}(x+y) \leq \mu^-_{\phi_{(c)}}(y) = \mu^-_{\phi\circ_\Gamma\psi_{(c)}}(y),$$

$$\mu^+_{\phi\circ_\Gamma\psi_{(c)}}(x\alpha y) = \mu^+_{\phi_{(c)}}(x\alpha y) \geq \mu^+_{\phi_{(c)}}(y) = \mu^+_{\phi\circ_\Gamma\psi_{(c)}}(y),$$

$$\mu^-_{\phi\circ_\Gamma\psi_{(c)}}(x\alpha y) = \mu^-_{\phi_{(c)}}(x\alpha y) \leq \mu^-_{\phi_{(c)}}(y) = \mu^-_{\phi\circ_\Gamma\psi_{(c)}}(y).$$

Case (ii): If $c \in B\backslash A$, then same as Case (i).

Case (iii): If $c \in A \cap B$, then for $x, y \in S$ and $\alpha \in \Gamma$,

$$\begin{aligned} \mu^+_{\phi\circ_\Gamma\psi_{(c)}}(y) &= \sup_{y=a\beta b}\{\min\{\mu^+_{\phi_{(c)}}(a), \mu^+_{\psi_{(c)}}(b)\}\} \\ &\leq \sup_{x\alpha y=x\alpha a\beta b}\{\min\{\mu^+_{\phi_{(z)}}(x\alpha a), \mu^+_{\psi_{(c)}}(b)\}\} \\ &= \sup_{x\alpha y=a_1\beta b}\{\min\{\mu^+_{\phi_{(z)}}(a_1), \mu^+_{\psi_{(c)}}(b)\}\} \\ &= \mu^+_{\phi\circ_\Gamma\psi_{(c)}}(x\alpha y) \end{aligned}$$

and

$$\begin{aligned} \mu^-_{\phi\circ_\Gamma\psi_{(c)}}(y) &= \inf_{y=a\beta b}\{\max\{\mu^-_{\phi_{(c)}}(a), \mu^-_{\psi_{(c)}}(b)\}\} \\ &\geq \inf_{x\alpha y=x\alpha a\beta b}\{\max\{\mu^-_{\phi_{(z)}}(x\alpha a), \mu^-_{\psi_{(c)}}(b)\}\} \\ &= \inf_{x\alpha y=a_1\beta b}\{\max\{\mu^-_{\phi_{(z)}}(a_1), \mu^-_{\psi_{(c)}}(b)\}\} \\ &= \mu^-_{\phi\circ_\Gamma\psi_{(c)}}(x\alpha y). \end{aligned}$$

In all cases, we have

$$\mu^+_{\phi\circ_\Gamma\psi_{(c)}}(x+y) \geq \mu^+_{\phi\circ_\Gamma\psi_{(c)}}(y),$$

$$\mu^-_{\phi\circ_\Gamma\psi_{(c)}}(x+y) \leq \mu^-_{\phi\circ_\Gamma\psi_{(c)}}(y),$$

$$\mu^+_{\phi\circ_\Gamma\psi_{(c)}}(x\alpha y) \geq \mu^+_{\phi\circ_\Gamma\psi_{(c)}}(y)$$
$$\mu^-_{\phi\circ_\Gamma\psi_{(c)}}(x\alpha y) \leq \mu^-_{\phi\circ_\Gamma\psi_{(c)}}(y).$$

This implies that, $(\phi, A) \circ_\Gamma (\psi, B)$ is bipolar fuzzy soft left Γ-ideal over S. Similarly, we can prove above result for bipolar fuzzy soft right Γ-ideals. □

Theorem 3.23. *Consider a Γ-semiring S with identity e. Let $\Omega_T(S)$ denotes the set of all bipolar fuzzy soft left (right) Γ-ideals over S such that $(\phi, A) \in \Omega_T(S)$. If $\mu^+_{\phi(c)}(e) = 1$ and $\mu^-_{\phi(c)}(e) = -1$, then $(\Omega_T(S), \circ_\Gamma, \widetilde{\cap})$ forms a complete lattice under $\widetilde{\precsim}$.*

Proof. It is clear that for any (ϕ, A), $(\psi, B) \in \Omega_T(S)$,

$$\mu^+_{\phi_{(c)}}(e) = \mu^+_{\psi_{(c)}}(e) = 1 \text{ and } \mu^-_{\phi_{(c)}}(e) = \mu^-_{\psi_{(c)}}(e) = -1.$$

Then clearly, $(\phi, A)\widetilde{\cap}(\psi, B)$ and $(\phi, A) \circ_\Gamma (\psi, B)$ are bipolar fuzzy soft left Γ-ideals over S. Obviously,

$$\mu^+_{\phi\circ_\Gamma\psi_{(c)}}(e) = 1,\ \mu^-_{\phi\circ_\Gamma\psi_{(c)}}(e) = -1$$

and

$$\mu^+_{\phi\circ_\Gamma\psi_{(c)}}(e) = 1,\ \mu^-_{\phi\circ_\Gamma\psi_{(c)}}(e) = -1.$$

Also $(\phi, A)\widetilde{\cap}(\psi, B) \in \Omega_T(S)$ and $(\phi, A) \circ_\Gamma (\psi, B) \in \Omega_T(S)$. Clearly, $(\phi, A)\widetilde{\cap}(\psi, B)$ is the greatest lower bound of $\{(\phi, A), (\psi, B)\}$.

Now let $c \in A \cup B$ and $x \in S$, then we have following cases:

Case (i): If $c \in A\backslash B$, then

$$\mu^+_{\phi\circ_\Gamma\psi_{(c)}}(x) = \mu^+_{\phi_{(c)}}(x) \text{ and } \mu^-_{\phi\circ_\Gamma\psi_{(c)}}(x) = \mu^-_{\phi_{(c)}}(x).$$

Case (ii): If $c \in B\backslash A$, then

$$\mu^+_{\phi\circ_\Gamma\psi_{(c)}}(x) = \mu^+_{\psi_{(c)}}(x) \text{ and } \mu^-_{\phi\circ_\Gamma\psi_{(c)}}(x) = \mu^-_{\psi_{(c)}}(x).$$

Case (iii): If $c \in A \cap B$, then

$$\begin{aligned}\mu^+_{\phi\circ_\Gamma\psi_{(c)}}(x) &= \sup_{x=x\alpha e}\{\min\{\mu^+_{\phi_{(c)}}(x), \mu^+_{\psi_{(c)}}(e)\}\} \\ &\geq \min\{\mu^+_{\phi_{(c)}}(x), \mu^+_{\psi_{(c)}}(e)\}\} \\ &= \mu^+_{\phi_{(c)}}(x)\end{aligned}$$

and

$$\begin{aligned}\mu^-_{\phi\circ_\Gamma\psi_{(c)}}(x) &= \inf_{x=x\alpha e}\{\max\{\mu^-_{\phi_{(c)}}(x), \mu^-_{\psi_{(c)}}(e)\}\} \\ &\leq \max\{\mu^-_{\phi_{(c)}}(x), \mu^-_{\psi_{(c)}}(e)\}\} \\ &= \mu^-_{\phi_{(c)}}(x).\end{aligned}$$

From the above, we can write,

$$\mu^+_{\phi\circ_\Gamma\psi_{(c)}}(x) \geq \mu^+_{\phi_{(c)}}(x)$$
$$\mu^-_{\phi\circ_\Gamma\psi_{(c)}}(x) \leq \mu^-_{\phi_{(c)}}(x)$$

for all $x \in S$.
This shows that $(\phi, A)\widetilde{\preceq}(\phi, A) \circ_\Gamma (\psi, B)$.

Similarly, $(\psi, B)\widetilde{\preceq}(\phi, A) \circ_\Gamma (\psi, B)$. This implies that $(\phi, A) \circ_\Gamma (\psi, B)$ is an upper bound of $\{(\phi, A), (\psi, B)\}$.

To show that $(\phi, A) \circ_\Gamma (\psi, B)$ is least in the sense of upper bound, let $(\varphi, C) \in \Omega_T(S)$ be another upper bound of $\{(\phi, A), (\psi, B)\}$. Then $(\phi, A)\widetilde{\preceq}(\varphi, C)$ and $(\psi, B)\widetilde{\preceq}(\varphi, C)$ implies that $(\phi, A) \circ_\Gamma (\psi, B)\widetilde{\preceq}(\varphi, C) \circ_\Gamma (\varphi, C)\widetilde{\preceq}(\varphi, C)$. This situation shows that $(\phi, A) \circ_\Gamma (\psi, B)$ is the least upper bound of $\{(\phi, A), (\psi, B)\}$. Consequently, we conclude that $(\Omega_T(S), \circ_\Gamma, \cap)$ is a complete lattice. □

4. Conclusions

In soft computing and uncertain modeling, soft sets can be combined with other mathematical tools. Bipolar fuzzy sets and soft sets are two different mathematical tools for representing vagueness and uncertainty. In this study, we have introduced the concept of bipolar fuzzy soft $\Gamma-$subsemiring and bipolar fuzzy soft $\Gamma-$ideal in a $\Gamma-$semiring. Also we study some of their algebraic properties. It is also proved that the collection of all bipolar fuzzy soft $\Gamma-$ideals over a $\Gamma-$semiring forms a complete distributive lattice with the special union and intersection.

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