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$T-$fuzzy ideals in ordered $\Gamma-$semirings

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Abstract. In this paper,we introduce the notion of $T-$fuzzy ideal, $T-$fuzzy left (right) $k-$ideal, $T-$fuzzy interior ideal, $T-$fuzzy quasi ideal, $T-$fuzzy bi-ideal in an ordered $\Gamma-$semiring. We characterize the regular ordered $\Gamma-$semiring in terms of $T-$fuzzy left (right) ideal, $T-$fuzzy quasi ideal, $T-$fuzzy bi-ideal, $T-$fuzzy interior ideal and study their properties and relations between them. We establish that $T-$fuzzy ideal, $T-$fuzzy bi-ideal, $T-$fuzzy quasi ideal and $T-$fuzzy interior ideal are equivalent in a regular ordered $\Gamma-$semiring and characterize the simple ordered $\Gamma-$semiring in terms of $T-$fuzzy interior ideal.

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Keywords: $\Gamma-$semiring, Ordered $\Gamma-$semiring, $T-$fuzzy ideal, $T-$fuzzy left (right) $k-$ideal, $T-$fuzzy interior ideal, $T-$fuzzy quasi ideal, $T-$fuzzy bi-ideal.

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1. Introduction

A semiring is an algebraic structure with two associative binary operations where one distributes over the other, first introduced by Vandiver [30] in 1934 but semirings have appeared in earlier studies on the theory of ideals of rings. A universal algebra $(S,+,\cdot)$ is called a semiring if and only if $(S,+),(S,\cdot)$ are semigroups which are connected by distributive laws, *i.e.*, $a(b+c)=ab+ac,\ \ (a+b)c=ac+bc$, for all $a,b,c\in S$.A natural example of semiring is the set of all natural numbers under usual addition and multiplication of numbers. In particular, if I is the unit interval on the real line then (I,max,min) is a semiring in which 0 is the additive identity and 1 is the multiplicative identity. Though semiring is a generalization of a ring, ideals of semiring do not coincide with ring ideals. For example ideals of a semiring need not be the kernel of some semiring homomorphism. To solve this problem, Herniksen [7] defined $k-$ideals in semirings to obtain analogous of ring results for semiring. In structure, semirings lie between semigroups and rings. The results which hold in rings but not in semigroups hold in semirings, since semiring is a generalization of ring. The study of rings shows that multiplicative structure of ring is an independent of additive structure whereas in semiring multiplicative structure is not independent of additive structure. The additive and the multiplicative structure of a semiring play an important role in determining the structure of a semiring. The theory of rings and theory of semigroups have considerable impact on the development of theory of semirings. Semirings play an important role in studying matrices and determinants. Semirings are useful in the areas of theoretical computer science as well as in graph theory, optimization theory, in particular for studying automata,coding theory and formal languages. Semiring theory has many applications in other branches.

As a generalization of ring, the notion of a $\Gamma-$ring was introduced by Nobusawa [23] in 1964. In 1981, Sen [26] introduced the notion of a $\Gamma-$semigroup as a generalization of semigroup. The notion of a ternary algebraic system was introduced by Lehmer [15] in 1932. In 1995, Murali Krishna Rao [18, 19, 21] introduced the notion of a $\Gamma-$semiring as a generalization of $\Gamma-$ring, ring, ternary semiring and semiring. Steinfeld [28] first introduced the notion of quasi ideals for semigroups and then for rings. Good and Hughes [6] first introduced the notion of bi-ideal in semigroups. Iseki [8] introduced the concept of quasi ideal for a semiring. Quasi ideals in $\Gamma-$semirings was studied by Jagtap and Pawar [9]. The concept of bi-ideal for semigroup and then for ring was given by Lajos and Szasz [14]. Quasi ideals are generalization of right ideals and left ideals whereas bi-ideals are generalization of quasi ideals.

The fuzzy set theory was developed by Zadeh [31] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. The fuzzification of algebraic structure was introduced by Rosenfeld [24] with the notion of fuzzy subgroups in 1971. K.L.N.Swamy and U.M.Swamy [29] studied fuzzy prime ideals in rings in 1988. In 1982, Liu [16] defined and studied fuzzy subrings as well as fuzzy ideals in rings. Applying the concept of fuzzy sets to the theory of $\Gamma-$ring, Jun and Lee [11] introduced the notion of fuzzy ideals in $\Gamma-$ring and studied the properties of fuzzy ideals of $\Gamma-$ring. Murali Krishna Rao et.al [22] studied $L-$fuzzy ideals in $\Gamma-$semirings. Ghosh et.al [10] studied Fuzzy soft rings and fuzzy soft ideals. Kehayopulu et.al [13] studied fuzzy ideals in ordered semigroups. Mandal [17] studied fuzzy ideals and fuzzy interior ideals in an ordered semiring. Dutta et.al [4] studied fuzzy ideals of $\Gamma-$semirings. Murali Krishna Rao [20] studied fuzzy soft $\Gamma-$semiring and fuzzy soft $k-$ideal over $\Gamma-$semiring. Kuroki [12] studied fuzzy interior ideals in semigroups. Ahsan et. al [2] characterized semigroups by their fuzzy bi-ideals.

Triangular norm was introduced by Schweizer and Skla [25]. In fuzzy set theory, triangular norm ($t-$norm) is extensively used to model the logical connective conjunction (AND), which has applications in several fields of Mathematics and Artificial intelligence. Dudek and Jun [5] studied fuzzy subgroups over a $t-$norm. Srinivas and Nagaiah [27] studied $T-$Fuzzy ideals of Gamma near-rings. Akram [3] studied $T-$fuzzy ideals in near rings. Throughout this paper, χ_M is the characteristic function. In this paper we introduce the notion of $T-$fuzzy ideal, $T-$fuzzy left (right) $k-$ideal, $T-$fuzzy interior ideal, $T-$fuzzy quasi ideal, $T-$fuzzy bi-ideal in an ordered $\Gamma-$semiring. We characterize the regular ordered $\Gamma-$semiring in terms of $T-$fuzzy quasi ideal, $T-$fuzzy bi-ideal and $T-$fuzzy interior ideal and we study their properties and relation between them. We prove that $T-$fuzzy ideal, $T-$fuzzy bi-ideal, $T-$fuzzy quasi ideal and $T-$fuzzy interior ideal are equivalent in a regular ordered $\Gamma-$semiring.

2. Preliminaries

In this section we recall some of the fundamental concepts and definitions which are necessary for this paper.

Definition 2.1 ([1])**.** A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) is called a semiring, if

(i) addition is a commutative operation,

(ii)m ultiplication distributes over addition both from the left and from the right,

(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

Definition 2.2 ([26])**.** Let M and Γ be two non-empty sets. Then M is called a $\Gamma-$semigroup, if it satisfies

(i) $x\alpha y \in M$,

(ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 2.3 ([21])**.** Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. If there exists a mapping $M \times \Gamma \times M \to M$ (images to be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) satisfying the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,

(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$,

(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$,

then M is called a $\Gamma-$semiring.

A $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0,$ for all $x \in M$ and $\alpha \in \Gamma$. Every semiring M is a $\Gamma-$semiring with $\Gamma = M$ and ternary operation is defined as the usual semiring multiplication.

Example 2.4 ([21])**.** Let M be the additive semigroup of all $m \times n$ matrices over the set of non negative rational numbers and Γ be the additive semigroup of all $n \times m$ matrices over the set of non negative integers and ternary operation is defined as $M \times \Gamma \times M \to M$ by $(x, \alpha, y) \to x\alpha y$ using usual matrix multiplication for all $x, y \in M$ and $\alpha \in \Gamma$. Then M is a $\Gamma-$semiring.

Definition 2.5. A non-empty subset A of a $\Gamma-$semiring M is called:

(i) A $\Gamma-$subsemiring of M, if $(A, +)$ is a subsemigroup of $(M, +)$ and $A\Gamma A \subseteq A$.

(ii) A quasi ideal of M, if A is a $\Gamma-$subsemiring of M and $A\Gamma M \cap M\Gamma A \subseteq A$.

(iii) A bi-ideal of M, if A is a $\Gamma-$subsemiring of M and $A\Gamma M\Gamma A \subseteq A$.

(iv) An interior ideal of M, if A is a $\Gamma-$subsemiring of M and $M\Gamma A\Gamma M \subseteq A$.

(v) A left (right) ideal of M, if A is a $\Gamma-$subsemiring of M and $M\Gamma A \subseteq A$ $(A\Gamma M \subseteq A)$.

(vi) An ideal if A is a $\Gamma-$subsemiring of $M, A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.

(vii) A $k-$ideal if A is a $\Gamma-$subsemiring of $M, A\Gamma M \subseteq A, M\Gamma A \subseteq A$ and $x \in M,\ x + y \in A$, for $y \in A$ implies $x \in A$.

Let M be a non-empty set. A mapping $\mu : M \to [0,1]$ is called a fuzzy subset of M. If μ is a fuzzy subset of M and $t \in [0,1]$ then the set $\mu_t = \{x \in M \mid \mu(x) \geq t\}$ is called a level subset of M with respect to the fuzzy subset μ. A fuzzy subset $\mu : M \to [0,1]$ is a non-empty fuzzy subset if μ is not a constant function. For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$ for all $x \in M$. Let A be a non-empty subset of M. The characteristic function of A is defined by

$$\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Definition 2.6 ([19])**.** Let M be a $\Gamma-$semiring . A fuzzy subset μ of M is said to be fuzzy $\Gamma-$subsemiring of M, if it satisfying the following conditions:
(i) $\mu(x+y) \geq min\ \{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \geq min\ \{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.7 ([19])**.** A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy left (right) ideal of M, if it satisfying the following conditions:
(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \geq \mu(y)\ (\mu(x))$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.8 ([19])**.** A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy ideal of M, if it satisfying the following conditions:
(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \geq max\ \{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.9 ([17])**.** Let μ and γ be two fuzzy subsets of an ordered semiring M and $z, y \in M$. We define
(i) $\mu \circ \gamma(x) = \begin{cases} \sup\limits_{x \leq yz} & \{min(\mu(y), \gamma(z))\}, \\ 0, & \text{otherwise.} \end{cases}$
(ii) $\mu + \gamma(x) = \begin{cases} \sup\limits_{x \leq y+z} & \{\ min(\mu(y), \gamma(z))\}, \\ 0, & \text{otherwise.} \end{cases}$
(iii) $\mu \cap \gamma(x) = \min\{\mu(x), \gamma(x)\},$ for all $x \in M$.

Definition 2.10 ([17])**.** A semiring M is called an ordered semiring, if it admits a compatible relation $\leq$, i.e.,$\leq$ is a partial ordering on M satisfying the following conditions: For all $a, b, c, d \in M$, if $a \leq b$ and $c \leq d$, then
(i) $a + c \leq b + d$, (ii) $ac \leq bd$, (iii) $ca \leq db$.

Definition 2.11. A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm ($t-$norm), if it satisfying the following conditions: For all $x, y, z, w \in [0,1]$,
(i) $T(x,1) = x = T(1,x)$,
(ii) $T(x,y) = T(y,x)$,
(iii)$T\big(x, T(y,z)\big) = T\big(T(x,y), z\big)$,
(iv) $T(x,y) \leq T(z,w)$, if $x \leq z$ and $y \leq w$.

It is obvious that $t-$norm T satisfies the following properties: For all $x, y, z, w \in [0,1]$,
(1) $T(x,y) \leq min\{x,y\}$,
(2) $T(x,0) = 0$,

(3) $T(0,0) = 0, T(1,1) = 1$,
(4) $T\big(T(x,y), T(z,w)\big) = T\big(T(x,z), T(y,w)\big)$.

Note: In general $T(x,y) \leq \min\{x,y\}$. The $t-$norm minimum is defined by $T(a,b) = \min\{a,b\}$. Some standard $t-$norms are

$$T_p(a,b) = ab,\ T_m(a,b) = \max\{a+b-1, 0\} \text{ and } T_w(a,b) = \begin{cases} a, & \text{if } b = 1, \\ b, & \text{if } a = 1, \\ 0, & \text{otherwise .} \end{cases}$$

Definition 2.12 ([25])**.** Let T be a $t-$norm and μ be a fuzzy subset of M. Then μ has an imaginable property, if $Im(\mu) \subseteq \Delta_T = \{a \in [0,1] \mid T(a,a) = a\}$.

Definition 2.13 ([25])**.** A triangular norm (t-norm) is said to be combined translation, if

(i) $T(a+c, b+c) = T(a,b) + c$,
(ii) $T(ad, bd) = d\ T(a,b)$, where $c \in [0,\ 1 - \sup\limits_{a,b\in[0,1]} \{T(a,b)\}], d \in [0,1]$.

3. $T-$fuzzy ideals

In this section, we introduce the notion of $T-$fuzzy ideal, $T-$fuzzy left (right) $k-$ideal and normal $T-$fuzzy left $k-$ideal in an ordered $\Gamma-$semiring and study their properties. We characterize a regular ordered $\Gamma-$semiring in terms of $T-$fuzzy left (right) ideals.

Definition 3.1. A $\Gamma-$semiring M is called an ordered $\Gamma-$semiring, if it admits a compatible relation $\leq$, i.e., $\leq$ is a partial ordering on M satisfying the following conditions: For all $a, b, c, d \in M, \alpha \in \Gamma$, if $a \leq b$ and $c \leq d$, then

(i) $a + c \leq b + d$, (ii) $a\alpha c \leq b\alpha d$, (iii) $c\alpha a \leq d\alpha b$.

Example 3.2. Let $M = [0,1], \Gamma = N,\ +$ and ternary operation be defined as $x+y = \max\{x,y\}, x\gamma y = \min\{x,\gamma,y\}$ for all $x, y \in M, \gamma \in \Gamma$. Then M is an ordered $\Gamma-$semiring with respect to usual ordering.

Definition 3.3. Let M be an ordered $\Gamma-$semiring and A be a non-empty subset of M. A is called a $\Gamma-$subsemiring of an ordered $\Gamma-$semiring M, if A is a sub-semigroup of $(M,+)$ and $A\Gamma A \subseteq A$.

Definition 3.4. Let M be an ordered $\Gamma-$semiring and A be a non-empty subset of M.

(i) A is called a left (right) ideal of an ordered $\Gamma-$semiring M, if A is closed under addition and $M\Gamma A \subseteq A$ ($A\Gamma M \subseteq A$) and for any $a \in M, b \in A, a \leq b$, then $a \in A$.

(ii) A is called an ideal of M, if it is both left ideal and right ideal.

Definition 3.5. Let M be an ordered $\Gamma-$semiring. A fuzzy subset μ of M is called a fuzzy $\Gamma-$subsemiring of M, if it satisfies the following conditions: For all $x, y \in M, \alpha \in \Gamma$,

(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \geq min\{\mu(x), \mu(y)\}$,
(iii) if $x \leq y$, then $\mu(x) \geq \mu(y)$.

Definition 3.6. Let μ be a non-empty fuzzy subset of an ordered $\Gamma-$semiring M. Then μ is called a fuzzy left (right) ideal of M, if it satisfies the following conditions: For all $x, y \in M, \alpha \in \Gamma$,
(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y) \geq \mu(y)\big(\mu(x)\big)$, (iii) if $x \leq y$, then $\mu(x) \geq \mu(y)$.

By fuzzy ideal we mean, it is both a fuzzy left ideal as well as a fuzzy right ideal of an ordered $\Gamma-$semiring M.

Definition 3.7. Let M be an ordered Γ- semiring and $x \in M$. We define

$$M_x = \{(y, z) \in M \times M / x \leq y\alpha z, \alpha \in \Gamma\}.$$

For any fuzzy subsets μ and γ of M, $\mu \circ \gamma : M \to [0, 1]$ is defined by

$$\mu \circ \gamma(x) = \begin{cases} \sup\limits_{(y,z) \in M_x} \{T(\mu(y), \gamma(z))\}, \text{if } M_x \neq \phi. \\ 0, \text{ otherwise.} \end{cases}$$

Let μ and γ be two fuzzy subsets of an ordered $\Gamma-$semiring M. We define

$$\mu \wedge \gamma(x) = T\big(\mu(x), \gamma(x)\big),\ \mu \cap \gamma(x) = \min\{\mu(x), \gamma(x)\},\ \text{for all } x \in M.$$

Definition 3.8. A fuzzy subset μ of an ordered $\Gamma-$semiring M is called a $T-$fuzzy left (right) ideal, if it satisfies the following conditions: For all $x, y \in M, \alpha \in \Gamma$,
(i) $\mu(x + y) \geq T(\mu(x), \mu(y))$,
(ii) $\mu(x\alpha y) \geq \mu(y)\big(\mu(x)\big)$,
(iii) if $x \leq y$, then $\mu(x) \geq \mu(y)$.

Definition 3.9. If μ is a $T-$fuzzy right ideal and a $T-$fuzzy left ideal of an ordered $\Gamma-$semiring M, then μ is called a $T-$fuzzy ideal of M.

Example 3.10. Let M be an abelian semigroup of all non negative integers and Γ be an abelian semigroup of all natural numbers with usual addition, define the ternary operation $M \times \Gamma \times M \to M$ by $(a, \alpha, b) \to a\alpha b$ using usual multiplication. Then M is ordered $\Gamma-$semiring with respect to usual ordering $\leq$.
Define $\mu : M \to [0, 1]$ by

$$\mu(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0.2, & \text{if } x \text{ is even}, \\ 0.1, & \text{if } x \text{ is odd}. \end{cases}$$

Then μ is a $T-$fuzzy ideal of M with respect to $t-$norm T_m.

Definition 3.11. A fuzzy subset μ of an ordered $\Gamma-$semiring M is called a $T-$left (right) fuzzy $k-$ideal of M, if it satisfying the following conditions: For all $x, y \in M, \alpha \in \Gamma$,
(i) $\mu(x + y) \geq T(\mu(x), \mu(y))$,
(ii) $\mu(x\alpha y) \geq \mu(y)\big(\mu(x)\big)$,
(iii) $\mu(x) \geq T\big(\mu(x + y), \mu(y)\big)$,
(iv) if $x \leq y$, then $\mu(x) \geq \mu(y)$.

Definition 3.12. If μ is a $T-$fuzzy left $k-$ideal and a $T-$fuzzy right $k-$ideal of an ordered $\Gamma-$semiring M, then μ is called a $T-$fuzzy $k-$ideal of M.

Definition 3.13. An ordered $\Gamma-$semiring M is called a simple, if its only ideals are M and (0).

Let μ be a fuzzy subset of an ordered $\Gamma-$semiring M and $a \in M$. The set $\{b \in M \mid \mu(b) \geq \mu(a)\}$ is denoted by I_a.

Theorem 3.1. *If μ is a $T-$fuzzy right ideal of an ordered $\Gamma-$semiring M, $T(\mu(x), \mu(x)) \geq \mu(x)$ for all $x \in M$ and $a \in M$, then I_a is a right ideal of M.*

Proof. Let μ be a $T-$fuzzy right ideal of an ordered $\Gamma-$semiring M and $T(\mu(x), \mu(x)) \geq \mu(x)$, for all $x \in M$. Then $I_a \neq \phi$, since $a \in I_a$.

Let $b, c \in I_a$ and $\alpha \in \Gamma$. Then $\mu(b) \geq \mu(a)$ and $\mu(c) \geq \mu(a)$. Thus

$$\mu(b + c) \geq T(\mu(b), \mu(c)) \geq T(\mu(a), \mu(a)) \geq \mu(a).$$

So $b + c \in I_a$. On one hand, $\mu(b\alpha x) \geq \mu(b) \geq \mu(a)$, for all $x \in M$. Hence $b\alpha x \in I_a$.

Let $b \in I_a$ and $c \leq b$. Then $\mu(b) \geq \mu(a)$ and $\mu(c) \geq \mu(b)$. Thus $\mu(c) \geq \mu(b) \geq \mu(a)$. So $c \in I_a$. Therefore I_a is a right ideal of M. □

Corollary 3.2. *If μ is a $T-$fuzzy left ideal of an ordered $\Gamma-$semiring M, $T(\mu(x), \mu(x)) \geq \mu(x)$, for all $x \in M$ and $a \in M$ then I_a is a left ideal of M.*

Corollary 3.3. *Let μ be a $T-$fuzzy ideal of an ordered $\Gamma-$semiring M, $T(\mu(x), \mu(x)) \geq \mu(x)$, for all $x \in M$ and $a \in M$. Then I_a is an ideal of M.*

Theorem 3.4. *Every fuzzy ideal of an ordered $\Gamma-$semiring M is a $T-$fuzzy ideal of M.*

Proof. Suppose μ is a fuzzy ideal of an ordered $\Gamma-$semiring M and $x, y \in M, \alpha \in \Gamma$. Then

$$\mu(x + y) \geq \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$

and

$$\mu(x\alpha y) \geq \max\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y)).$$

Thus μ is a $T-$fuzzy ideal of M. □

The following theorem is a straightforward verification.

Theorem 3.5. *If I is an ideal of an ordered $\Gamma-$semiring M then χ_I is a $T-$fuzzy ideal of M.*

Theorem 3.6. *A fuzzy subset μ of an ordered $\Gamma-$semiring M is a $T-$fuzzy left ideal of M if and only if a fuzzy subset μ satisfying the following conditions:*

(1) $\chi_M \circ \mu \subseteq \mu$.

(2) $\mu(x + y) \geq T(\mu(x), \mu(y))$, *for all* $x, y \in M$.

Proof. Suppose μ is a $T-$fuzzy left ideal of an ordered $\Gamma-$semiring M and $x \in M$.

Case(i): If $\chi_M \circ \mu(x) = 0$, then $\chi_M \circ \mu(x) = 0 \leq \mu(x)$. Thus $\chi_M \circ \mu \subseteq \mu$.

Case(ii): If $\chi_M \circ \mu(x) \neq 0$ then there exist $y, z \in M$ such that $x \leq y\alpha z, \alpha \in \Gamma$. Thus

$$\begin{aligned}\chi_M \circ \mu(x) &= \sup_{(y,z)\in M_x} \{T(\chi_M(y), \mu(z))\} \\ &= \sup_{(y,z)\in M_x} \{\mu(z)\} \\ &\leq \mu(y\alpha z), \text{since } \mu \text{ is a } T-\text{fuzzy left ideal} \\ &\leq \mu(x).\end{aligned}$$

So $\chi_M \circ \mu \subseteq \mu$.

Conversely, suppose that the given conditions hold and $x, y \in M, \alpha \in \Gamma$. Since $\chi_M \circ \mu \subseteq \mu$,

$$\begin{aligned}\mu(x\alpha y) &\geq \chi_M \circ \mu(x\alpha y) \\ &= \sup_{(a,b)\in M_{x\alpha y}} \{T(\chi_M(a), \mu(b))\} \\ &\geq T(\chi_M(x), \mu(y)) \\ &= \mu(y).\end{aligned}$$

Let $x, y \in M$ and $x \leq y, \alpha \in \Gamma$. Then

$$\mu(x) \geq \mu(x\alpha y) = \sup_{(y,y)\in M_{x\alpha y}} \{T(\chi_M(y), \mu(y))\} = \mu(y).$$

Thus $\mu(x) \geq \mu(y)$. So μ is a $T-$fuzzy left ideal of M. □

Corollary 3.7. *In an ordered $\Gamma-$semiring M, the following are equivalent:*
(1) *μ is a $T-$fuzzy right ideal of M.*
(2) *$\mu \circ \chi_M \subseteq \mu$, $\mu(x + y) \geq T(\mu(x), \mu(y))$, for all $x, y \in M$.*

Theorem 3.8. *M is a regular ordered $\Gamma-$semiring if and only if $A\Gamma B = A \cap B$ for any right ideal A and left ideal B of M.*

Proof. Let A, B be a right ideal and a left ideal of a regular ordered $\Gamma-$semiring M respectively. Obviously $A\Gamma B \subseteq A \cap B$. Let $x \in A \cap B$. Since M is a regular, there exist $\alpha, \beta \in \Gamma$ and $y \in M$ such that $x \leq x\alpha y\beta x$. Since $x\alpha y \in A$ and $x \in B$, $x\alpha y\beta x \in A\Gamma B$. Thus $x \in A\Gamma B$. So $A\Gamma B = A \cap B$.

Conversely, suppose that $A\Gamma B = A\cap B$ for any right ideal A and left ideal B of M. Let $x \in M$ and I be the right ideal generated by x and J be the left ideal generated by x. Then we have $x \in I \cap J = I\Gamma J$. Thus $x = x\alpha y = z\beta x, \alpha, \beta \in \Gamma, y, z \in M$. So $x = x\alpha y\gamma z\beta x$, for some $\gamma \in \Gamma$. Hence M is a regular ordered $\Gamma-$semiring. □

Theorem 3.9. *An ordered $\Gamma-$semiring M is a regular if and only if $\lambda \circ \mu = \lambda \wedge \mu$ for any $T-$fuzzy right ideal λ and $T-$fuzzy left ideal μ of M.*

Proof. Let λ, μ be $T-$fuzzy right and $T-$fuzzy left ideals of an ordered $\Gamma-$semiring M respectively and $x \in M$. Then $\lambda \circ \mu(x) = \sup\limits_{(y,z)\in M_x} \{T(\lambda(y), \mu(z))\}$. Since $x \leq y\alpha z$, $\mu(x) \geq \mu(y\alpha z) \geq \mu(z)$. Then $\lambda(x) \geq \lambda(y\alpha z) \geq \lambda(y)$. Thus $T(\lambda(y), \mu(z)) \leq T(\lambda(x), \mu(x))$. So $\lambda \circ \mu(x) = \sup\limits_{(y,z)\in M_x} \{T(\lambda(y), \mu(z))\} \leq T(\lambda(x), \mu(x)) = \lambda \wedge \mu(x)$. Hence $\lambda \circ \mu \subseteq \lambda \wedge \mu$.

Let $x \in M$. Since M is a regular ordered $\Gamma-$semiring , there exist $\alpha, \beta \in M, a \in M$ such that $x \leq x\alpha a\beta x$. Suppose $y, z \in M, \gamma \in \Gamma$ and $x \leq y\gamma z$. Then $\lambda(x) \geq \lambda(y\gamma z) \geq \lambda(y)$. Thus $\mu(x) \geq \mu(y\gamma z) \geq \mu(z)$.

On the other hand, $T(\lambda(y), \mu(z)) \leq T(\lambda(x\alpha a), \mu(x))$. Then

$$\lambda \circ \mu(x) = \sup_{(y,z)\in M_x} \{T(\lambda(y), \mu(z))\} \geq T(\lambda(x\alpha a), \mu(x)).$$

Thus

$$\lambda \circ \mu(x) \geq T(\lambda(x\alpha a), \mu(x)) \geq T(\lambda(x), \mu(x)) = \lambda \wedge \mu(x).$$

So $\lambda \circ \mu \supseteq \lambda \wedge \mu$. Hence $\lambda \circ \mu = \lambda \wedge \mu$.

Conversely, suppose that $\lambda \circ \mu = \lambda \wedge \mu$ for $T-$fuzzy right ideal λ and $T-$fuzzy left ideal μ of an ordered $\Gamma-$semiring M. Let A, B be a right ideal and a left ideal of M respectively. Then χ_A, χ_B are $T-$fuzzy right ideal and $T-$fuzzy left ideal of ordered $\Gamma-$semiring M respectively. Thus $\chi_A \circ \chi_B = \chi_A \wedge \chi_B$. Obviously, $A\Gamma B \subseteq A \cap B$.

Suppose $x \in A \cap B$. Then $\chi_A(x) = \chi_B(x) = 1$. Thus

$$\chi_A \wedge \chi_B(x) = T(\chi_A(x), \chi_B(x)) = T(1,1) = 1,$$

i.e., $\chi_A \circ \chi_B(x) = 1$. So there exist $a \in A, b \in B$ such that $x \leq a\alpha b, \quad a\alpha b \in A\Gamma B$. Hence $x \in A\Gamma B$ and thus $A\Gamma B = A \cap B$. Therefore by Theorem 3.8, M is a regular ordered $\Gamma-$semiring. □

Corollary 3.10. *An ordered $\Gamma-$semiring M is a regular if and only if $\lambda \circ \mu = \lambda \wedge \mu$ for any fuzzy ideals λ and μ of M.*

Theorem 3.11. *If μ and λ are $T-$fuzzy left $k-$ideals of an ordered $\Gamma-$semiring M, then $\mu \wedge \lambda$ is a $T-$fuzzy left $k-$ideal of M.*

Proof. Let μ and λ be $T-$fuzzy left $k-$ideals of an ordered $\Gamma-$semiring M and $x, y \in M, \alpha \in \Gamma$. Then

$$\begin{aligned} \mu \wedge \lambda(x+y) &= T(\mu(x+y), \lambda(x+y)) \\ &\geq T(T(\mu(x), \mu(y)), T(\lambda(x), \lambda(y))) \\ &= T(T(\mu(x), \lambda(x)), T(\mu(y), \lambda(y))) \\ &= T(\mu \wedge \lambda(x), \mu \wedge \lambda(y)). \end{aligned}$$

Since μ and λ are $T-$fuzzy left $k-$ideals , we have

$\mu(x\alpha y) \geq \mu(y)$ and $\lambda(x\alpha y) \geq \lambda(y)$,

$$\begin{aligned} \mu \wedge \lambda(x\alpha y) &= T(\mu(x\alpha y), \lambda(x\alpha y)) \\ &\geq T(\mu(y), \lambda(y)) \\ &= \mu \wedge \lambda(y). \end{aligned}$$

Suppose $x, y \in M$ and $x \leq y$. Then $\mu(x) \geq \mu(y)$ and $\lambda(x) \geq \lambda(y)$. Thus

$$\mu \wedge \lambda(x) = T(\mu(x), \lambda(x)) \geq T(\mu(y), \lambda(y)) = \mu \wedge \lambda(y).$$

So $\mu \wedge \lambda$ is a $T-$fuzzy left ideal of an ordered $\Gamma-$semiring M. Since μ and λ are $T-$fuzzy left $k-$ideals, we have

$\mu(x) \geq T(\mu(x+y), \mu(y))$ and $\lambda(x) \geq T(\lambda(x+y), \lambda(y))$, for all $x, y \in M$.

Hence

$$\begin{aligned} \mu \wedge \lambda(x) &= T(\mu(x), \lambda(x)) \\ &\geq T(T(\mu(x+y), \mu(y)), T(\lambda(x+y), \lambda(y))) \\ &= T(T(\mu(x+y), \lambda(x+y)), T(\mu(y), \lambda(y))) \\ &= T(\mu \wedge \lambda(x+y), \mu \wedge \lambda(y)) \text{ for all } x, y \in M. \end{aligned}$$

Therefore $\mu \wedge \lambda$ is a $T-$fuzzy left $k-$ideal of M. □

Theorem 3.12. *Let μ be an imaginable fuzzy subset of an ordered $\Gamma-$semiring M. If μ is a $T-$fuzzy left $k-$ideal of M, then each level subset $\mu_a (\neq \varnothing), a \in [0,1]$ of μ is a left $k-$ideal of M.*

Proof. Suppose μ is an imaginable $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring M. If $x \in M, b, c \in \mu_a, \alpha \in \Gamma$, then $\mu(b) \geq a$ and $\mu(c) \geq a$.

Thus $\mu(b+c) \geq T(\mu(b), \mu(c)) \geq T(a,a) = a$. So $b+c \in \mu_a$.

Now $\mu(x\alpha c) \geq \mu(c) \geq a$. Then $x\alpha c \in \mu_a$. If $y \in \mu_a, x \in M$ and $x \leq y$, then $\mu(y) \geq a$ and $\mu(x) \geq \mu(y)$. Thus $\mu(x) \geq \mu(y) \geq a$. So $x \in \mu_a$.

Let $x+y \in \mu_a$ and $y \in \mu_a$. Then $\mu(x+y) \geq a$ and $\mu(y) \geq a$. Thus

$$\mu(x) \geq T(\mu(x+y), \mu(y)) \geq T(a,a) = a.$$

So $x \in \mu_a$. Hence μ_a is a left $k-$ideal of M. □

Definition 3.14 ([22]). Let μ be a fuzzy subset of X and $a \in [0, 1 - sup\{\mu(x) \mid x \in X\}], b \in [0,1]$. The mappings

$$\mu_a^T : X \to [0,1], \quad \mu_b^M : X \to [0,1] \quad \text{and} \quad \mu_{b,a}^{MT} : X \to [0,1]$$

are called fuzzy translation, fuzzy multiplication and fuzzy magnified translation of μ respectively, if

$$\mu_a^T(x) = \mu(x) + a, \mu_b^M = b\mu(x), \mu_{b,a}^{MT} = b\mu(x) + a, \quad \text{for all } x \in X.$$

Theorem 3.13. *A fuzzy subset μ is a $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring M if and only if μ_a^T, the fuzzy translation of μ is a $T-$fuzzy left $k-$ideal of M, provided $t-$norm T is a combined translation.*

Proof. Suppose μ is a $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring M with $t-$norm is a combined translation and $x, y \in M, \alpha \in \Gamma$. Then

$$\begin{aligned}
\mu_a^T(x+y) &= \mu(x+y) + a \\
&\geq T(\mu(x), \mu(y)) + a \\
&= T(\mu(x)+a, \mu(y)+a) \\
&= T(\mu_a^T(x), \mu_a^T(y)), \\
\mu_a^T(x\alpha y) &= \mu(x\alpha y) + a \\
&\geq \mu(y) + a \\
&= \mu_a^T(y), \\
\mu_a^T(x) &= \mu(x) + a \\
&\geq T(\mu(x+y), \mu(y)) + a \\
&= T(\mu(x+y)+a, \mu(y)+a) \\
&= T(\mu_a^T(x+y), \mu_a^T(y)).
\end{aligned}$$

Suppose $x, y \in M$ and $x \leq y$. Then $\mu(x) \geq \mu(y)$ and thus $\mu(x) + a \geq \mu(y) + a$. So $\mu_a^T(x) \geq \mu_a^T(y)$. Hence μ_a^T is a $T-$fuzzy left $k-$ideal.

Conversely suppose that μ_a^T is a $T-$fuzzy left $k-$ideal. Then obviously μ is a $T-$fuzzy left ideal. Let $\mu(y) = t_1$ and $\mu(x+y) = t_2$ and $t = min\{t_1, t_2\} \geq T(t_1, t_2)$. Then $y \in \mu_t, x+y \in \mu_t$. Since μ_t is a $k-$ideal, $x \in \mu_t$ which implies that

$$\mu(x) \geq t = \min\{t_1, t_2\} \geq T(t_1, t_2) = T(\mu(y), \mu(x+y)).$$

Thus μ is a $T-$fuzzy left $k-$ideal of M. □

Theorem 3.14. *Let M be an ordered $\Gamma-$semiring. Then μ is a $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring M if and only if μ_b^M, the fuzzy multiplication of μ is a $T-$fuzzy left $k-$ideal of M, provided $t-$norm T is a combined translation and $b \in [0,1]$.*

Proof. Suppose μ is a $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring M with $t-$norm is a combined translation and $x, y \in M, \alpha \in \Gamma$.

$$\begin{aligned} \mu_b^M(x+y) &= b\mu(x+y) \\ &\geq bT(\mu(x),\mu(y)) \\ &= T(b\mu(x), b\mu(y)) \\ &= T(\mu_b^M(x), \mu_b^M(y)), \\ \mu_b^M(x\alpha y) &= b\mu(x\alpha y) \\ &\geq b\mu(y) \\ &= \mu_b^M(y), \\ \mu_b^M(x) &= b\mu(x) \\ &\geq bT(\mu(x+y),\mu(y)) \\ &= T(b\mu(x+y), b\mu(y)) \\ &= T(\mu_b^M(x+y), \mu_b^M(y)). \end{aligned}$$

If $x \leq y$, then $\mu(x) \geq \mu(y)$ and thus $b\mu(x) \geq b\mu(y)$. So $\mu_b^M(x) \geq \mu_b^M(y)$. Hence μ_b^M is a $T-$fuzzy left $k-$ideal of ordered $\Gamma-$semiring M.

The converse is obvious. □

Theorem 3.15. *Let M be an ordered $\Gamma-$semiring. Then μ is a $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring M if and only if $\mu_{b,a}^{MT}$, fuzzy magnified translation of μ is a $T-$fuzzy left $k-$ideal of M, provided $t-$norm T is a combined translation.*

Proof. Suppose T is a combined translation. Then

μ is a $T-$fuzzy left $k-$ideal of M

$\Leftrightarrow \mu_b^M$ is a $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring M, by Theorem 3.14 .

$\Leftrightarrow \mu_{b,a}^{MT}$ is a $T-$fuzzy left $k-$ideal of M, by Theorem 3.13 .

Thus the Theorem holds. □

Theorem 3.16. *If μ is an imaginable $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring M then μ is a fuzzy left $k-$ideal of M.*

Proof. Suppose μ is an imaginable $T-$fuzzy left $k-$ideal of M and $x, y \in M, \alpha \in \Gamma$. Since μ is an imaginable, we have

$$\begin{aligned} \min\{\mu(x),\mu(y)\} &= T(\min\{\mu(x),\mu(y)\}, \min\{\mu(x),\mu(y)\}) \\ &\leq T(\mu(x),\mu(y)) \\ &\leq \min\{\mu(x),\mu(y)\}. \end{aligned}$$

Then $T(\mu(x),\mu(y)) = \min\{\mu(x),\mu(y)\}$. Thus

$\mu(x+y) \geq T(\mu(x),\mu(y)) = \min\{\mu(x),\mu(y)\}$ and $\mu(x\gamma y) \geq \mu(y)$.

If $x \leq y$, then $\mu(x) \geq \mu(y)$. So

$$\mu(x) \geq T(\mu(x+y),\mu(y)) = \min\{\mu(x+y),\mu(y)\}.$$

Hence μ is a fuzzy $k-$ideal of M. □

Definition 3.15. A fuzzy subset μ of an ordered $\Gamma-$semiring M is said to be normal fuzzy subset, if $\mu(0) = 1$.

Theorem 3.17. *Let μ be a $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring M and $\mu^+(x) = \mu(x) + 1 - \mu(0)$ for all $x \in M$. Then μ^+ is a normal $T-$fuzzy left $k-$ideal of ordered $\Gamma-$semiring M containing μ, provided $t-$norm T is a combined translation.*

Proof. Let μ be a $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring M.

$$\text{Define } \ \mu^+(x) = \mu(x) + 1 - \mu(0), \quad \text{for all } \ x \in M.$$

Put $1 - \mu(0) = a$. Then $\mu^+(x) = \mu(x) + a$ and thus $\mu^+(x) = \mu_a^T(x)$. By Theorem 3.13, μ^+ is a $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring M. By definition of μ^+, $\mu \subseteq \mu^+$ and $\mu^+(0) = 1$. Thus the Theorem holds. □

The following theorem can be verified easily.

Theorem 3.18. *Let M, S be ordered $\Gamma-$semirings. If we define*
(i) $(x, y) + (z, w) = (x + z, y + w)$,
(ii) $(x, y)\alpha(z, w) = (x\alpha z, y\alpha w)$, *for all* $(x, y), (z, w) \in M \times S, \alpha \in \Gamma$,
then $M \times S$ is ordered $\Gamma-$semiring.

Definition 3.16. Let μ and γ be fuzzy subsets of X. The cartesian product of μ and γ is defined by $\mu \times \gamma(x, y) = T(\mu(x), \gamma(y))$, for all $(x, y) \in X \times X$.

Definition 3.17. Let μ and γ be $T-$fuzzy $k-$ideals of an ordered $\Gamma-$semiring M. Then $\mu \times \gamma$ is said to be $T-$fuzzy left (right) $k-$ideal of $M \times M$, if
(i) $\mu \times \gamma((x, z) + (y, w)) \geq T(\mu(x + y),\ \gamma(z + w))$,
(ii) $\mu \times \gamma((x, z)\alpha(y, w)) \geq T\big(\mu \times \gamma(y, w),\ \mu \times \gamma(x, z)\big)$,
(iii) if $(x, z) \leq (y, w)$, then $\mu \times \gamma(x, z) \geq \mu \times \gamma(y, w)$,
(iv) $\mu\times\gamma(x, z) \geq T(\mu\times\gamma(x+y, z+y), \mu\times\gamma(y, y))$, for all $(x, z), (y, w) \in M\times M, \alpha \in \Gamma$.

Theorem 3.19. *Let μ be an imaginable fuzzy subset of an ordered $\Gamma-$semiring M. If μ is a $T-$fuzzy left $k-$ideal, then $\mu \times \mu$ is an imaginable $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring $M \times M$.*

Proof. Suppose μ is an imaginable $T-$fuzzy left $k-$left ideal of an ordered $\Gamma-$semiring M and $(x, x) \in M \times M$. Obviously, $\mu \times \mu$ is a $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring $M \times M$. Then

$$\begin{aligned} T(\mu \times \mu(x, x), \mu \times \mu(x, x)) &= T(T(\mu(x), \mu(x)), T(\mu(x), \mu(x))) \\ &= T(\mu(x), \mu(x)) = \mu \times \mu(x, x). \end{aligned}$$

Thus $\mu \times \mu$ is an imaginable. On the other hand,

$$\begin{aligned} \mu \times \mu(x, z) &= T(\mu(x), \mu(z)) \\ &\geq T(T(\mu(x + y), \mu(y)), T(\mu(z + y), \mu(y))) \\ &= T(T(\mu(x + y), \mu(z + y)), T(\mu(y), \mu(y))) \\ &= T(\mu \times \mu(x + y, z + y), \mu \times \mu(y, y)), \end{aligned}$$

for all $(x, z) \in M \times M, y \in M$.
So $\mu \times \mu$ is an imaginable $T-$fuzzy left $k-$ideal of M.

□

Theorem 3.20. *Let μ and γ be $T-$fuzzy left $k-$ideals of an ordered $\Gamma-$semiring M. Then $\mu \times \gamma$ is a $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring $M \times M$.*

Proof. Let μ and γ be $T-$fuzzy left $k-$ideals of an ordered $\Gamma-$semiring M and $(x_1, x_2), (y_1, y_2) \in M \times M, \alpha \in \Gamma$. Then

$$\begin{aligned}
\mu \times \gamma((x_1, x_2) + (y_1, y_2)) &= \mu \times \gamma(x_1 + y_1, x_2 + y_2) \\
&= T(\mu(x_1 + y_1), \gamma(x_2 + y_2)) \\
&\geq T(T(\mu(x_1), \mu(y_1)), T(\gamma(x_2), \gamma(y_2))) \\
&= T(T(\mu(x_1), \gamma(x_2)), T(\mu(y_1), \gamma(y_2))) \\
&= T(\mu \times \gamma(x_1, x_2), \mu \times \gamma(y_1, y_2)), \\
\mu \times \gamma((x_1, x_2)\alpha(y_1, y_2)) &= \mu \times \gamma(x_1\alpha y_1, x_2\alpha y_2) \\
&= T(\mu(x_1\alpha y_1), \gamma(x_2\alpha y_2)) \\
&\geq T(\mu(y_1), \gamma(y_2)) \\
&= \mu \times \gamma(y_1, y_2),
\end{aligned}$$

$$\begin{aligned}
\mu \times \gamma(x, z) &= T(\mu(x), \gamma(z)) \\
&\geq T(T(\mu(x + y), \mu(y)), T(\gamma(z + y), \gamma(y))) \\
&= T(T(\mu(x + y), \gamma(z + y)), T(\mu(y), \gamma(y))) \\
&= T(\mu \times \gamma(x + y, z + y), \mu \times \gamma(y, y)),
\end{aligned}$$

for all $(x, z) \in M \times M, y \in M$.

If $(x_1, x_2) \leq (y_1, y_2)$, then

$$\begin{aligned}
\mu \times \gamma(x_1, x_2) &= T(\mu(x_1), \gamma(x_2)) \\
&\geq T(\mu(y_1), \gamma(y_2)) \\
&= \mu \times \gamma(y_1, y_2).
\end{aligned}$$

Thus $\mu \times \gamma$ is a $T-$fuzzy left $k-$ideal of the ordered $\Gamma-$semiring $M \times M$. □

Corollary 3.21. *Let M_1 and M_2 be ordered $\Gamma-$semirings. If μ_1 and μ_2 are $T-$fuzzy left $k-$ideals of M_1 and M_2 respectively, then $\mu = \mu_1 \times \mu_2$ is a $T-$fuzzy left $k-$ideal of the ordered $\Gamma-$semiring $M_1 \times M_2$.*

Definition 3.18. A fuzzy relation μ on any set X is a fuzzy subset, if $\mu : X \times X \to [0, 1]$.

Definition 3.19. Let γ be a $T-$fuzzy subset on a set M. Then the strongest fuzzy relation μ_γ on M is a fuzzy relation which is defined by

$$\mu_\gamma(x, y) = T(\gamma(x), \gamma(y)), \text{ for all } (x, y) \in M \times M.$$

Theorem 3.22. *Let γ be an imaginable fuzzy subset of an ordered $\Gamma-$semiring M. Then γ is a $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring Mif and only if the strongest fuzzy relation μ_γ on M is an imaginable $T-$fuzzy left $k-$ideal of the ordered $\Gamma-$semiring $M \times M$.*

Proof. Suppose γ is an imaginable $T-$fuzzy left $k-$ideal of an ordered $\Gamma-$semiring. Obviously, μ_γ is a $T-$fuzzy left ideal of M.

$$\begin{aligned}
\mu_\gamma(x_1, x_2) &= T(\gamma(x_1), \gamma(x_2)) \\
&= T(T(\gamma(x_1 + y_1), \gamma(y_1)), T(\gamma(x_2 + y_2), \gamma(y_2))) \\
&= T(T(\gamma(x_1 + y_1), \gamma(x_2 + y_2)), T(\gamma(y_1), \gamma(y_2))) \\
&= T(\mu_\gamma(x_1 + y_1, x_2 + y_2), \mu_\gamma(y_1, y_2)),
\end{aligned}$$

$$\begin{aligned}
T(\mu_\gamma(x_1, x_2), \mu_\gamma(x_1, x_2)) &= T(T(\gamma(x_1), \gamma(x_2)), T(\gamma(x_1), \gamma(x_2))) \\
&= T(T(\gamma(x_1), \gamma(x_1)), T(\gamma(x_2), \gamma(x_2))) \\
&= T(\gamma(x_1), \gamma(x_2)) \\
&= \mu_\gamma(x_1, x_2), \text{ for all } (x_1, x_2), (y_1, y_2) \in M \times M.
\end{aligned}$$

Suppose $(x_1, x_2), (y_1, y_2) \in M \times M$ and $(x_1, x_2) \leq (y_1, y_2)$. Then $x_1 \leq y_1, x_2 \leq y_2$ and thus $\gamma(x_1) \geq \gamma(y_1), \gamma(x_2) \geq \gamma(y_2)$. So $T(\gamma(x_1), \gamma(x_2)) \geq T(\gamma(y_1), \gamma(y_2))$. Hence

$\mu_\gamma(x_1, x_2) \geq \mu_\gamma(y_1, y_2)$. Therefore μ_γ is an imaginable $T-$fuzzy left $k-$ideal of the ordered $\Gamma-$semiring M,

Conversely, suppose that μ_γ is an imaginable $T-$fuzzy left $k-$ideal of the ordered $\Gamma-$semiring M, $x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}
\gamma(x+y) &= T(\gamma(x+y), \gamma(x+y)) \\
&= \mu_\gamma(x+y, x+y) \\
&= \mu_\gamma((x,x)+(y,y)) \\
&\geq T(\mu_\gamma(x,x), \mu_\gamma(y,y)) \\
&= T(T(\gamma(x),\gamma(x)), T(\gamma(y),\gamma(y))) \\
&= T(T(\gamma(x),\gamma(y)), T(\gamma(x),\gamma(y))) \\
&= T(\mu_\gamma(x,y), \mu_\gamma(x,y)) \\
&= T(\mu_\gamma(x,y) \\
&= T(\gamma(x),\gamma(y)). \\
\gamma(x\alpha y) &= T(\gamma(x\alpha y), \gamma(x\alpha y)) \\
&= \mu_\gamma(x\alpha y, x\alpha y) \\
&= \mu_\gamma((x,x)\alpha(y,y)) \\
&\geq \mu_\gamma(y,y) \\
&= T(\gamma(y),\gamma(y)) = \gamma(y) \\
\gamma(x) &= T(\gamma(x),\gamma(x)) \\
&= \mu_\gamma(x,x) \\
&\geq T(\mu_\gamma(x+y,x+y), \mu_\gamma(y,y)) \\
&= T(T(\gamma(x+y),\gamma(x+y)), T(\gamma(y),\gamma(y))) \\
&= T(\gamma(x+y),\gamma(y)).
\end{aligned}$$

Suppose $x, y \in M$ and $x \leq y$. Then $(x,x) \leq (y,y)$. Thus $\mu_\gamma(x,x) \geq \mu_\gamma(y,y)$. So $T(\gamma(x),\gamma(x)) \geq T(\gamma(y),\gamma(y))$. Hence $\gamma(x) \geq \gamma(y)$. Therefore γ is an imaginable $T-$fuzzy left $k-$ideal of the ordered $\Gamma-$semiring M, □

4. $T-$fuzzy quasi ideal, $T-$fuzzy bi-ideal and $T-$fuzzy interior ideal

In this section, we introduce the notion of $T-$fuzzy quasi ideal, $T-$fuzzy bi-ideal and $T-$fuzzy interior ideal in an ordered $\Gamma-$semiring. We characterize the regular ordered $\Gamma-$semiring in terms of $T-$fuzzy quasi ideal, $T-$fuzzy bi -ideal and $T-$fuzzy interior ideal. We prove that if μ is a $T-$fuzzy interior ideal of simple ordered $\Gamma-$semiring then μ is a constant function and also we establish that $T-$fuzzy ideal, $T-$fuzzy bi-ideal, $T-$fuzzy quasi ideal and $T-$fuzzy interior ideal are equivalent in a regular ordered $\Gamma-$semiring.

Definition 4.1. A fuzzy subset μ of an ordered $\Gamma-$semiring M is called a fuzzy bi-ideal, if it satisfies the following conditions:

(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y\beta z) \geq \min\{\mu(x), \mu(z)\}$,
(iii) if $x \leq y$, then $\mu(x) \geq \mu(y)$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 4.2. A fuzzy subset μ of an ordered $\Gamma-$semiring M is called a $T-$fuzzy bi-ideal, if it satisfies the following conditions:

(i) $\mu(x+y) \geq T(\mu(x), \mu(y))$,

(ii) $\mu(x\alpha y\beta z) \geq T(\mu(x), \mu(z))$,
(iii) if $x \leq y$, then $\mu(x) \geq \mu(y)$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 4.3. A fuzzy subset μ of an ordered $\Gamma-$semiring M is called a fuzzy quasi ideal, if it satisfies the following conditions:

(i) $\mu(x+y) \geq \min(\mu(x), \mu(y))$, (ii)$\mu \circ \chi_M \wedge \chi_M \circ \mu \subseteq \mu$
(iii) if $x \leq y$, then $\mu(x) \geq \mu(y)$, for all $x, y \in M$.

Definition 4.4. A fuzzy subset μ of an ordered $\Gamma-$semiring M is called a $T-$fuzzy quasi ideal, if it satisfies the following conditions:

(i) $\mu(x+y) \geq T(\mu(x), \mu(y))$, (ii)$\mu \circ \chi_M \wedge \chi_M \circ \mu \subseteq \mu$,
(iii) if$x \leq y$, then $\mu(x) \geq \mu(y)$, for all $x, y \in M$.

Definition 4.5. A fuzzy subset μ of an ordered $\Gamma-$semiring M is called a fuzzy interior ideal, if it satisfies the following conditions:

(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x\alpha y\beta z) \geq \mu(y)$,
(iii) if $x \leq y$, then $\mu(x) \geq \mu(y)$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 4.6. A fuzzy subset μ of an ordered $\Gamma-$semiring M is called a $T-$fuzzy interior ideal, if it satisfies the following conditions:

(i) $\mu(x+y) \geq T(\mu(x), \mu(y))$,
(ii) $\mu(x\alpha y\beta z) \geq \mu(y)$,
(iii) if $x \leq y$, then $\mu(x) \geq \mu(y)$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$,

Definition 4.7. An ordered $\Gamma-$semiring M is called a regular, if for each $a \in M$, there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$.

Example 4.8. Let M be the additive abelian semigroup of all 2×2 matrices, where elements belong to $N \cup \{0\}$ and Γ be the additive abelian semigroup of all 2×2 matrices, whose elements belong to N and ternary operation is defined as $M \times \Gamma \times M \rightarrow M$ by $(x, \alpha, y) \rightarrow x\alpha y$ using usual matrix multiplication for all $x, y \in M$ and $\alpha \in \Gamma$. Let $A = (a_{ij}), B = (b_{ij}) \in M$, we define $A \leq B \Leftrightarrow a_{ij} \leq b_{ij}$, for all i, j. Then M is ordered $\Gamma-$semiring.

Let the set $B = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a, b \in N \cup \{0\} \right\}$ and $Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| a, b \in N \cup \{0\} \right\}$.
Then B is the bi-ideal of an ordered $\Gamma-$semiring M, B is not an ideal of M. Q is an interior ideal of M but neither a left ideal nor a right ideal of M.

$$\text{Define } \mu : M \rightarrow [0,1] \text{ by } \mu(A) = \begin{cases} 0.3, & \text{if } A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \\ 0.2, & \text{if } A = \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix}, \\ 0.1, & \text{otherewise.} \end{cases}$$

Where $a, b \in N \cup \{0\}$. Then μ is a $T-$fuzzy right ideal, $T-$fuzzy quasi ideal, $T-$fuzzy bi-ideal with respect to $t-$norm T_p.

Example 4.9. Let $M = \{0, 1, 2, 3\}$ and $\Gamma = \{0, 1, 2, 3\}$. Then $(M, +), (\Gamma, +)$ are semigroups, when $+$ is defined as $a + b = \max\{a, b\}$ for all $\alpha \in \Gamma$, ternary operation is defined as $\begin{array}{c|cccc} \alpha & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 \\ 2 & 0 & 2 & 1 & 0 \\ 3 & 0 & 3 & 0 & 0 \end{array}$.

Then M is an ordered $\Gamma-$semiring with respect to usual ordering.

Define fuzzy subset μ of M as $\mu(x) = \begin{cases} 0.8 & \text{if } x = 0 \\ 0.4 & \text{if } x = 1 \\ 0.2 & \text{if } x = 2 \\ 0.1 & \text{if } x = 3. \end{cases}$

Let $T : [0, 1] \times [0, 1] \to [0, 1]$ be defined by $T(a, b) = ab$. Then T is a norm. Thus μ is a $T-$fuzzy interior ideal but μ is not a fuzzy interior ideal and μ is not a $T-$fuzzy ideal but μ is a $T-$fuzzy quasi ideal.

Theorem 4.1. *A fuzzy subset μ is a $T-$fuzzy quasi ideal of an ordered $\Gamma-$semiring M if and only if $\mu(x) \geq T(\sup\limits_{(y,z)\in M_x} \mu(y), \sup\limits_{(y,z)\in M_x} \mu(z))$ for all $x \in M$.*

Proof. Let μ be a $T-$fuzzy quasi ideal of an ordered $\Gamma-$semiring M. By definition of $T-$fuzzy quasi ideal, we have

$$\begin{aligned} &\mu(x) \geq \mu \circ \chi_M \wedge \chi_M \circ \mu(x) \\ \Leftrightarrow\ &\mu(x) \geq T(\mu \circ \chi_M(x), \chi_M \circ \mu(x)) \\ \Leftrightarrow\ &\mu(x) \geq T(\sup_{(y,z)\in M_x} \{T(\mu(y), \chi_M(z)\}, \sup_{(y,z)\in M_x} \{T(\chi_M(y), \mu(z))\}) \\ \Leftrightarrow\ &\mu(x) \geq T(\sup_{(y,z)\in M_x} \mu(y), \sup_{(y,z)\in M_x} \mu(z)), \end{aligned}$$

for all $x \in M$. □

Corollary 4.2. *A fuzzy subset μ is a fuzzy quasi ideal of an ordered $\Gamma-$semiring M if and only if $\mu(x) \geq min\{\sup\limits_{(y,z)\in M_x} \mu(y), \sup\limits_{(y,z)\in M_x} \mu(z)\}$, for all $x \in M$.*

Theorem 4.3. *Every fuzzy quasi ideal of an ordered $\Gamma-$semiring M is a $T-$fuzzy quasi-ideal.*

Proof. Suppose μ is a fuzzy quasi ideal of an ordered $\Gamma-$semiring M. By Corollary 4.2,

$$\begin{aligned} \mu(x) &\geq min\{\sup_{(y,z)\in M_x} \mu(y), \sup_{(y,z)\in M_x} \mu(z)\} \\ &\geq T(\sup_{(y,z)\in M_x} \mu(y), \sup_{(y,z)\in M_x} \mu(z)). \end{aligned}$$

Then, by Theorem 4.1, μ is a $T-$fuzzy quasi ideal of the ordered $\Gamma-$semiring M. □

Theorem 4.4. *If μ is a $T-$fuzzy quasi ideal of a regular ordered $\Gamma-$semiring M, then μ is a $T-$fuzzy ideal of M.*

Proof. Suppose μ is a $T-$fuzzy quasi ideal of a regular ordered $\Gamma-$semiring M, $x, y \in M$ and $\alpha \in \Gamma$. Then there exist $\gamma, \delta \in \Gamma, z \in M$ such that $x\alpha y \leq (x\alpha y)\gamma z\delta(x\alpha y)$. Since μ is a $T-$fuzzy quasi ideal, $\mu(x\alpha y) \geq \mu \circ \chi_M \wedge \chi_M \circ \mu(x\alpha y)$. Thus $\mu(x\alpha y) \geq T(\mu \circ \chi_M(x\alpha y), \chi_M \circ \mu(x\alpha y))$. So

$$\mu(x\alpha y) \geq T(\sup_{(x,y\gamma z\delta x\alpha y)\in M_{x\alpha y}} \{T(\mu(x), \chi_M(y\gamma z\delta x\alpha y)\},$$

$$\sup_{(x,y\gamma z\delta x\alpha y)\in M_{x\alpha y}} \{T(\chi_M(x\alpha y\gamma z\delta x),\mu(y))\}$$
$$= T\big(\sup_{(x,y\gamma z\delta x\alpha y)\in M_{x\alpha y}} \{T(\mu(x),1)\}, \sup_{(x,y\gamma z\delta x\alpha y)\in M_{x\alpha y}} \{T(1,\mu(y))\ \}\big)$$
$$\geq T(\mu(x),\mu(y)).$$

Hence μ is a $T-$fuzzy ideal of M. □

Theorem 4.5. *Every fuzzy right ideal of an ordered $\Gamma-$semiring M is a $T-$fuzzy quasi ideal of M.*

Proof. Let μ be a fuzzy right ideal of an ordered $\Gamma-$semiring $M, x \in M$ and $x \leq u\alpha v, u, v \in M, \alpha \in \Gamma$. Then

$$\begin{aligned}\mu(x) &\geq \mu(u\alpha v)\\ &\geq \mu(u)\\ &= \min\{\mu(u),\chi_M(v)\}\\ &\geq T(\mu(u),\chi_M(v)).\end{aligned}$$

Thus $\mu(x) \geq \sup_{(y,z)\in M_x} \{T(\mu(u),\chi_M(v))\} = \mu\circ\chi_M(x)$.

Similarly we can prove $\mu(x) \geq \chi_M\circ\mu(x)$.

$$\begin{aligned}\text{Therefore}\quad \mu(x) &\geq \min\{\mu\circ\chi_M(x),\chi_M\circ\mu(x)\}\\ &\geq T(\mu\circ\chi_M(x),\chi_M\circ\mu(x))\\ &= \chi_M\circ\mu\wedge\mu\circ\chi_M(x).\end{aligned}$$

So μ is a $T-$fuzzy quasi ideal of M. □

Corollary 4.6. *Every fuzzy left ideal of an ordered $\Gamma-$semiring M is a $T-$fuzzy quasi ideal of M.*

Corollary 4.7. *If μ is a $T-$fuzzy ideal of an ordered $\Gamma-$semiring M, then μ is a $T-$fuzzy quasi ideal of M.*

Theorem 4.8. *Let μ and γ be $T-$fuzzy right ideal and $T-$fuzzy left ideal respectively of an ordered $\Gamma-$semiring M. Then $\mu\wedge\gamma$ is a $T-$fuzzy quasi ideal of an ordered $\Gamma-$semiring M.*

Proof. Let μ and γ be $T-$fuzzy right ideal and $T-$fuzzy left ideal respectively of an ordered $\Gamma-$semiring M, $x, y, a, b \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned}\mu\wedge\gamma(x+y) &= T(\mu(x+y),\gamma(x+y))\\ &\geq T(T(\mu(x),\mu(y)),T(\gamma(x),\gamma(y)))\\ &= T(T(\mu(x),\gamma(x)),T(\mu(y),\gamma(y)))\\ &= T(\mu\wedge\gamma(x),\mu\wedge\gamma(y)),\\ \mu\wedge\gamma\circ\chi_M(x) &= \sup_{(a,b)\in M_x}\{T(\mu\wedge\gamma(a),\chi_M(b))\}\\ &= \sup_{(a,b)\in M_x}\{T(\mu\wedge\gamma(a),1)\}\\ &\leq \sup_{(a,b)\in M_x}\{T(\mu\wedge\gamma(a\alpha b),1)\}\\ &\leq T(\mu\wedge\gamma(x),1)\\ &= \mu\wedge\gamma(x).\end{aligned}$$

Similarly, we can prove, $\chi_M\circ\mu\wedge\gamma(x) \leq \mu\wedge\gamma(x)$. Thus

$$\mu\wedge\gamma\circ\chi_M\wedge\chi_M\circ\mu\wedge\gamma(x) \leq \mu\wedge\gamma(x).$$

So $\mu\wedge\gamma$ is a $T-$fuzzy quasi ideal of ordered $\Gamma-$semiring M. □

Corollary 4.9. *If μ and γ are $T-$fuzzy quasi ideals of an ordered $\Gamma-$semiring M then $\mu \wedge \gamma$ is a $T-$fuzzy quasi ideal of M.*

Theorem 4.10. *Every fuzzy bi-ideal of an ordered $\Gamma-$semiring M is a $T-$fuzzy bi-ideal of M.*

Proof. Let μ be a fuzzy bi-ideal of an ordered $\Gamma-$semiring M and $x, y, z \in M, \alpha, \beta \in \Gamma$. Then $\mu(x+y) \geq \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$
and

$$\mu(x\alpha y\beta z) \geq \min\{\mu(x), \mu(z)\} \geq T(\mu(x), \mu(z)).$$

Thus μ is a $T-$fuzzy bi-ideal of M. □

Theorem 4.11. *Every $T-$fuzzy ideal of an ordered $\Gamma-$semiring M is a $T-$fuzzy bi-ideal of ordered $\Gamma-$semiring M.*

Proof. Let μ be a $T-$fuzzy ideal of an ordered $\Gamma-$semiring M and $x, y, z \in M, \alpha, \beta \in \Gamma$. Then $\mu(x\alpha y\beta z) \geq T(\mu(x), \mu(y\beta z)) \geq T(\mu(x), \mu(z))$. Thus μ is a $T-$fuzzy bi-ideal of M. □

Theorem 4.12. *Every $T-$fuzzy quasi ideal of an ordered $\Gamma-$semiring M is a $T-$fuzzy bi-ideal of ordered $\Gamma-$semiring.*

Proof. Let μ be a $T-$fuzzy quasi ideal of an ordered $\Gamma-$semirng $M, x, y \in M$ and $\alpha, \beta \in \Gamma$. Then $\mu(x\alpha y\beta z) \geq \chi_M \circ \mu \wedge \mu \circ \chi_M(x\alpha y\beta z)$. Thus

$$\begin{aligned}
\mu(x\alpha y\beta z) &\geq T\big(\chi_M \circ \mu(x\alpha y\beta z), \mu \circ \chi_M(x\alpha y\beta z)\big) \\
&= T\big(\sup_{(a,b)\in M_{x\alpha y\beta z}} \{T(\chi_M(a), \mu(b))\}, \sup_{(a,b)\in M_{x\alpha y\beta z}} \{T(\mu(a), \chi_M(b))\big)\} \\
&\geq T\big(T(\chi_M(x\alpha y), \mu(z)), T(\mu(x), \chi_M(y\beta z))\big) \\
&= T(T(1, \mu(z)), T(\mu(x), 1)\big) \\
&= T(\mu(z), \mu(x)) \\
&= T(\mu(x), \mu(z)).
\end{aligned}$$

So μ is a $T-$fuzzy bi-ideal of M. □

Theorem 4.13. *If μ and λ are $T-$fuzzy bi-ideals of an ordered $\Gamma-$semiring M, then $\mu \wedge \lambda$ is a $T-$fuzzy bi-ideal of M.*

Proof. Let μ and λ be $T-$fuzzy bi-ideals of an ordered $\Gamma-$semiring M and $x, y \in M, \alpha \in \Gamma$. Then

$$\begin{aligned}
\mu \wedge \lambda(x+y) &= T(\mu(x+y), \lambda(x+y)) \\
&\geq T(T(\mu(x), \mu(y)), T(\lambda(x), \lambda(y))) \\
&= T(T(\mu(x), \lambda(x)), T(\mu(y), \lambda(y))) \\
&= T(\mu \wedge \lambda(x), \mu \wedge \lambda(y)).
\end{aligned}$$

Let $x, y, z \in M, \alpha, \beta \in \Gamma$. Then

$$\begin{aligned}
\mu \wedge \lambda(x\alpha y\beta z) &= T(\mu(x\alpha y\beta z), \lambda(x\alpha y\beta z)) \\
&\geq T(T(\mu(x), \mu(z)), T(\lambda(x), \lambda(z))) \\
&= T(T(\mu(x), \lambda(x)), T(\mu(z), \lambda(z))) \\
&= T(\mu \wedge \lambda(x), \mu \wedge \lambda(z)).
\end{aligned}$$

Let $x, y \in M$ and $x \leq y$. Then $\mu(x) \geq \mu(y), \lambda(x) \geq \lambda(y)$. Thus $T(\mu(x), \lambda(x)) \geq T(\mu(y), \lambda(y))$. So $\mu \wedge \lambda(x) \geq \mu \wedge \lambda(y)$. Hence $\mu \wedge \lambda$ is a $T-$fuzzy bi-ideal of M. □

Theorem 4.14. *If μ and λ are $T-$fuzzy right ideal and $T-$fuzzy left ideal respectively of an ordered $\Gamma-$semiring M, then $\mu \wedge \lambda$ is a $T-$fuzzy bi-ideal of M.*

Proof. By Theorem 4.8, $\mu \wedge \lambda$ is a $T-$fuzzy quasi ideal of M. By Theorem 4.12, $\mu \wedge \lambda$ is a $T-$fuzzy bi-ideal of M. □

Theorem 4.15. *Let M be a regular ordered $\Gamma-$semiring. If μ is an imaginable $T-$fuzzy bi-ideal of M then μ is a $T-$fuzzy quasi ideal of M.*

Proof. Let μ be an imaginable $T-$fuzzy bi-ideal of a regular ordered $\Gamma-$semiring M and $x \in M$. Suppose $\mu \circ \chi_M(x) \leq \mu(x)$. Then

$$\begin{aligned}\mu(x) &\geq \mu \circ \chi_M(x) \geq \min\{\chi_M \circ \mu(x), \mu \circ \chi_M(x)\}\\ &\geq T\{\chi_M \circ \mu(x), \mu \circ \chi_M(x)\}\\ &= \chi_M \circ \mu \wedge \mu \circ \chi_M(x).\end{aligned}$$

Suppose $\mu \circ \chi_M(x) > \mu(x)$. Then $\sup\limits_{(u,v)\in M_x} \{T(\mu(u), \chi_M(v))\} > \mu(x)$. Thus $T(\mu(u), \chi_M(v)) > \mu(x)$. So $\mu(u) > \mu(x)$.

As M is a regular ordered $\Gamma-$semiring, there exist $m \in M, \alpha, \beta \in \Gamma$ such that $x \leq x\alpha m\beta x \leq u\delta v\alpha m\beta x$. Since μ is an imaginable bi-ideal, $\mu(x) \geq T(\mu(u), \mu(x))$. Then $T(\mu(x), \mu(x)) \geq T(\mu(u), \mu(x))$. This is a contradiction to $\mu(u) > \mu(x)$. Thus $\mu \circ \chi_M(x) \leq \mu(x)$.

Similarly, we can prove $\chi_M \circ \mu(x) \leq \mu(x)$. So $\mu \circ \chi_M \wedge \chi_M \circ \mu \leq \mu$. Hence μ is a $T-$fuzzy quasi ideal of M. □

Theorem 4.16. *Let M be an ordered $\Gamma-$semiring. Then M is a regular ordered $\Gamma-$semiring if and only if $B = B\Gamma M\Gamma B$ for every bi-ideal B of M.*

Proof. Suppose M is a regular ordered $\Gamma-$semiring . Let B be a bi-ideal of M and $x \in B$. Since M is regular, there exist $\alpha, \beta \in \Gamma, y \in M$ such that $x \leq x\alpha y\beta x \in B\Gamma M\Gamma B$. Then $B \subseteq B\Gamma M\Gamma B$. Since B is a bi-ideal of an ordered $\Gamma-$semiring M, we have $B\Gamma M\Gamma B \subseteq B$. Thus $B\Gamma M\Gamma B = B$.

Conversely, suppose that $B\Gamma M\Gamma B = B$ for every bi-ideal B of an ordered $\Gamma-$semiring M. Let R and L be any right and left ideal of M respectively. Obviously, $R \cap L$ is a quasi ideal. Also, $R\cap L = R\cap L\Gamma M\Gamma R\cap L \subseteq R\Gamma M\Gamma L \subseteq R\Gamma L \subseteq R\cap L$ which implies that $R \cap L = R\Gamma L$. So, by Theorem 3.8, M is a regular ordered $\Gamma-$semiring. □

Since every quasi ideal is a bi-ideal, proof of the following theorem follows from the Theorem 4.16 .

Theorem 4.17. *Let M be an ordered $\Gamma-$semiring. Then M is a regular ordered $\Gamma-$semiring if and only if $Q = Q\Gamma M\Gamma Q$ for every quasi ideal Q of ordered $\Gamma-$semiring.*

Theorem 4.18. *Let M be an ordered $\Gamma-$semiring. If $\mu \subseteq \mu \circ \chi_M \circ \mu$ for every $T-$fuzzy quasi ideal μ of an ordered $\Gamma-$semiring M, then M is a regular ordered $\Gamma-$semiring.*

Proof. Let Q be a quasi ideal of an ordered $\Gamma-$semiring M. Then χ_Q is a $T-$fuzzy quasi ideal and thus $\chi_Q \subseteq \chi_Q \circ \chi_M \circ \chi_Q = \chi_{Q\Gamma M\Gamma Q}$. So $Q \subseteq Q\Gamma M\Gamma Q$. Since Q is a quasi ideal of M, we have $Q\Gamma M\Gamma Q \subseteq Q\Gamma M \cap M\Gamma Q \subseteq Q$. Hence, $Q\Gamma M\Gamma Q = Q$. Therefore by Theorem 4.17, M is a regular ordered $\Gamma-$semiring. □

Corollary 4.19. *Let M be an ordered $\Gamma-$semiring. If $\mu \subseteq \mu \circ \chi_M \circ \mu$ for every $T-$fuzzy bi-ideal μ of an ordered $\Gamma-$semiring M, then M is a regular ordered $\Gamma-$semiring.*

Theorem 4.20. *Let M be an ordered $\Gamma-$semiring. Then M is a regular if and only if $\mu \subseteq \mu \circ \chi_M \circ \mu$, for all $T-$fuzzy bi-ideals μ of M.*

Proof. Suppose μ is a $T-$fuzzy bi-ideal of a regular ordered $\Gamma-$semiring M and $x \in M$. Then there exist $\alpha, \beta \in \Gamma, a \in M$ such that $x \leq x\alpha a\beta x$ and

$$\begin{aligned} \mu \circ \chi_M \circ \mu(x) &= \sup_{(x\alpha a, x) \in M_x} \{T(\mu \circ \chi_M(x\alpha a), \mu(x))\} \\ &\geq T(\mu \circ \chi_M(x\alpha a), \mu(x)) \\ &= T(\sup_{(c,d) \in M_{x\alpha a}} \{T(\mu(c), \chi_M(d)), \mu(x))\} \\ &\geq T(\mu(x), \chi_M(a)), \mu(x)) \\ &= T(T(\mu(x), 1), \mu(x)) \\ &= T(\mu(x), \mu(x)) \\ &\geq \mu(x\alpha a\beta x) \\ &\geq \mu(x), \text{ for all } x \in M. \end{aligned}$$

Thus $\mu \subseteq \mu \circ \chi_M \circ \mu$. Converse follows from Corollary 4.19 □

Corollary 4.21. *If M is a regular ordered $\Gamma-$semiring, then $\mu \subseteq \mu \circ \chi_M \circ \mu$ for all $T-$fuzzy quasi ideals μ of M.*

Theorem 4.22. *Let M be an ordered $\Gamma-$semiring. Then M is a regular ordered $\Gamma-$semiring if and only if $\mu \wedge \gamma \subseteq \mu \circ \gamma \circ \mu$ for every $T-$fuzzy bi-ideal μ and every $T-$fuzzy ideal γ of M.*

Proof. Suppose μ is a $T-$fuzzy bi-ideal and γ is a $T-$fuzzy ideal of a regular ordered $\Gamma-$semiring M and $x \in M$. Then there exist $\alpha, \beta \in \Gamma, y \in M$ such that $x \leq x\alpha y\beta x$. Thus $x\alpha y \leq x\alpha y\beta x\alpha y$. So

$$\begin{aligned} \mu \circ \gamma \circ \mu(x) &= \sup_{(y,z) \in M_x} \{T(\mu \circ \gamma(y), \mu(z))\} \\ &\geq T(\mu \circ \gamma(x\alpha y), \mu(x)) \\ &= T(\sup_{(a,b) \in M_{a\delta b}} \{T(\mu(a), \gamma(b))\}, \mu(x)) \\ &\geq T(T(\mu(x), \gamma(y\beta x\alpha y)), \mu(x)) \\ &\geq T(T(\mu(x), \gamma(x)), \mu(x)), \text{ since } \gamma \text{ is a } T-\text{fuzzy ideal} \\ &= T(\mu(x), \gamma(x)) \\ &= \mu \wedge \gamma(x). \end{aligned}$$

Hence $\mu \circ \gamma \circ \mu(x) \geq \mu \wedge \gamma(x)$.

Conversely, suppose that the condition holds. Let μ and χ_M be a $T-$fuzzy bi-ideal and fuzzy ideal of an ordered $\Gamma-$semiring M respectively. Then $\mu \wedge \chi_M \subseteq \mu \circ \chi_M \circ \mu$ and $\mu = \mu \wedge \chi_M$. By Corollary 4.19, M is a regular ordered $\Gamma-$semiring. □

Theorem 4.23. *Every fuzzy interior ideal of ordered $\Gamma-$semiring M is a T- fuzzy interior ideal of M.*

Proof. Let μ be a fuzzy interior ideal of an ordered $\Gamma-$srmiring M. Since $\min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y)\}, \mu$ is a $T-$fuzzy interior ideal of M. □

Converses of Corollary 4.8 and Theorem 4.24 are not true in general, which is shown in the Example 4.2.

Theorem 4.24. *Let A be an interior ideal of an ordered $\Gamma-$semiring M. Then for every $t \in (0,1]$, there exists a $T-$fuzzy interior ideal μ of M such that $\mu_t = A$.*

Proof. Let A be an interior ideal of an ordered $\Gamma-$semiring M and μ be a fuzzy subset of M is defined by $\mu(x) = \begin{cases} t & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ where $t \in (0,1]$.
Suppose $x, y \in A, \alpha \in \Gamma$. Then $x+y, x\alpha y \in A$. Thus $\mu(x+y) = t \geq T(\mu(x), \mu(y))$ and $\mu(x\alpha y) = t \geq T(\mu(x), \mu(y))$. So μ is a fuzzy $\Gamma-$sub semiring of M.

Let $x, y \in M, \alpha, \beta \in \Gamma$ and $a \in A$. Then $x\alpha a\beta y \in A \Rightarrow \mu(x\alpha a\beta y) = t = \mu(a)$. If $a \notin A$, then $\mu(a) = 0$. Thus $\mu(x\alpha a\beta y) \geq 0 = \mu(a)$. Suppose $x \leq y$ and $x, y \in A$. Then $\mu(x) = t = \mu(y)$.

Case(i): If $x \notin A, y \in A$, then $\mu(x) = 0, \mu(y) = t$. Thus $\mu(y) \geq \mu(x)$.

Case(ii): If $x \notin A, y \notin A$, then $\mu(x) = 0 = \mu(y)$. Thus μ is a $T-$fuzzy interior ideal of Msuch that $\mu_t = A$. □

Theorem 4.25. *Every $T-$fuzzy ideal of an ordered $\Gamma-$semiring M is a $T-$fuzzy interior ideal.*

Proof. Suppose μ is a $T-$fuzzy ideal of an ordered $\Gamma-$semiring M. Then

$$\mu(a\alpha x\beta b) \geq \mu(a\alpha x) \geq \mu(x), \text{ for all } a, x, b \in M, \alpha \in \Gamma.$$

Thus μ is a $T-$fuzzy interior ideal of Ml. □

Theorem 4.26. *Let M be a regular ordered $\Gamma-$semiring. If μ is a $T-$fuzzy interior ideal of M, then μ is a $T-$fuzzy ideal of M.*

Proof. Suppose μ is a $T-$fuzzy interior ideal of a regular ordered $\Gamma-$semiring M. Since μ is a $T-$fuzzy interior ideal, $\mu(a\alpha b) \geq \mu(a\alpha b\beta(x\delta a\alpha b)) \geq \mu(b)$. Then $\mu(a\alpha b) \geq \min\{\mu(a), \mu(b)\}$. Thus $\mu(a\alpha b) \geq T(\mu(a), \mu(b))$. So μ is a $T-$fuzzy ideal of M. □

Theorem 4.27. *If μ is an imaginable $T-$fuzzy interior ideal of an ordered $\Gamma-$semiring M then every non empty level subset μ_t of μ is an interior ideal of M for every $t \in [0,1]$, when $\mu_t \neq \varnothing$.*

Proof. Suppose that $\mu_t \neq \varnothing, t \in [0,1]$ and $a, b \in \mu_t, \alpha \in \Gamma$. Then $\mu(a) \geq t, \mu(b) \geq t$. Thus $\mu(a+b) \geq T(\mu(a), \mu(b)) \geq T(t,t) = t$. So $a+b \in \mu_t$. Also,

$$\mu(a\alpha b) \geq T(\mu(a), \mu(b)) \geq T(t,t) = t.$$

Hence $a\alpha b \in \mu_t$.

Let $x, y \in M, a \in \mu_t, \alpha, \beta \in \Gamma$. Then $\mu(x\alpha a\beta y) \geq \mu(a) \geq t$ and thus $x\alpha a\beta y \in \mu_t$. So μ_t is an interior ideal of M. □

Theorem 4.28. *If μ is a $T-$fuzzy interior ideal of a simple ordered $\Gamma-$semiring M, then μ is a constant function.*

Proof. Let μ be a $T-$fuzzy interior ideal of a simple ordered $\Gamma-$semiring M and $x \in M$. Let $\big(M\Gamma x\Gamma M\big] = \{t \in M \mid t \leq y,\ y \in M\Gamma x\Gamma M\}$. Then $\big(M\Gamma x\Gamma M\big]$ is an ideal of a simple ordered $\Gamma-$semiring M. Thus $M = \big(M\Gamma x\Gamma M\big]$.

Suppose $a \in M$. Then $a \in \big(M\Gamma x\Gamma M\big]$. Thus $a \leq c\alpha x\beta d,\quad c, d \in M, \alpha, \beta \in \Gamma$. So $\mu(a) \geq \mu(c\alpha x\beta d) \geq \mu(x)$.

Similarly, we can prove $\mu(x) \geq \mu(a)$. Hence $\mu(x) = \mu(a)$. Therefore the Theorem holds. □

Theorem 4.29. *Let M be a regular ordered $\Gamma-$semiring. If μ is a $T-$fuzzy bi-ideal of M, then μ is a $T-$fuzzy interior ideal of M.*

Proof. Let μ be a $T-$fuzzy bi-ideal of a regular ordered $\Gamma-$semiring $M, a, x \in M$, and $\alpha \in \Gamma$. Since M is a regular and $a\alpha x \in M$, there exist $\gamma, \delta \in M$ and $y \in M$ such that $a\alpha x \leq a\alpha x\gamma y\delta a\alpha x$. Then $\mu(a\alpha x) \geq \mu(a\alpha x\gamma y\delta a\alpha x)$. Thus $\mu(a\alpha x) \geq T(\mu(a), \mu(x))$. So μ is a $T-$fuzzy ideal of a regular ordered $\Gamma-$semiring M. By Theorem 4.25, μ is a $T-$fuzzy interior ideal of M. Hence every $T-$fuzzy bi-ideal of a regular ordered $\Gamma-$semiring M is a $T-$fuzzy interior ideal of M. □

Definition 4.10. An ordered $\Gamma-$semiring M is called a $T-$fuzzy simple, if every imaginable $T-$fuzzy ideal is a constant function.

Theorem 4.30. *In an ordered $\Gamma-$semiring M, the following are equivalent:*
(1) *M is a simple $\Gamma-$semiring.*
(2) *M is a $T-$fuzzy simple.*
(3) *Every $T-$fuzzy interior ideal of M is a constant function.*

Proof. Let M be an ordered $\Gamma-$semiring.

(1) $\Rightarrow$ (2): Suppose M is a simple ordered $\Gamma-$semiring, μ is an imaginable $T-$fuzzy ideal of M and $a, b \in M$. By Corollary 3.5, I_a is an ideal of M. Since M is simple, $I_a = M$. Then $b \in I_a$. Thus $\mu(b) \geq \mu(a)$. By symmetry, we have $\mu(b) \leq \mu(a)$. So $\mu(b) = \mu(a)$. Hence μ is a constant function.

(2) $\Rightarrow$ (3): Suppose M is a $T-$fuzzy simple. By Theorem 4.27, every $T-$fuzzy ideal is a $T-$fuzzy interior ideal. Then, By Theorem 4.30, every $T-$fuzzy interior ideal is a constant function.

(3) $\Rightarrow$ (1): Suppose every $T-$fuzzy interior ideal of M is a constant function and I is a proper ideal of an ordered $\Gamma-$semiring M. Then by Theorem 3.7, χ_I is a $T-$fuzzy ideal. Thus, by Theorem 4.27, χ_I is a $T-$fuzzy interior ideal of an ordered $\Gamma-$semiring M. So χ_I is a constant function. Hence $I = M$. Therefore M is a simple ordered $\Gamma-$semiring M. □

Theorem 4.31. *Let M be a regular ordered $\Gamma-$semiring. Then the following fuzzy ideals are equivalent:*
(1) *μ is a $T-$fuzzy bi-ideal.*
(2) *μ is a $T-$fuzzy interior ideal.*
(3) *μ is a $T-$fuzzy ideal.*
(4) *μ is a $T-$fuzzy quasi ideal.*

Proof. Let M be a regular ordered $\Gamma-$semiring. By Theorem 4.31, (1) implies (2). By Theorem 4.26, (2) implies (3) . By Corollary 4.7, (3) implies (4). By Theorem 4.14, (4) implies (1). Then the Theorem holds. □

5. Conclusion

In this paper, we introduced the notion of $T-$fuzzy ideal, $T-$fuzzy left (right) $k-$ideal, $T-$fuzzy interior ideal, $T-$fuzzy quasi ideal, $T-$fuzzy bi-ideal in an ordered $\Gamma-$semiring. We studied some of their properties and characterized the regular ordered $\Gamma-$semiring and simple13 ordered $\Gamma-$semiring in terms of $T-$fuzzy left (right) ideal, $T-$fuzzy quasi ideal, $T-$fuzzy bi-ideal and $T-$fuzzy interior ideal and studied properties and relations between them.

In continuation of this paper, we propose to introduce the notion of a bi-quasi ideal of ordered semiring and study the characterization of ordered semiring in terms of fuzzy bi- quasi ideal.

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