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Multi vector space

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Abstract. In the present paper a notion of vector space in multiset settings is introduced. A representation theorem is established. Definitions of balanced, convex and absorbing multisets have been given and their properties are studied. Also the notion of multi basis has been developed.

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1. Introduction

Many fields of modern mathematics have been emerged by violating a basic principle of a given theory only because useful structures could be defined this way. For example, modern non-Euclidean geometries have been emerged by assuming that the Parallel Axiom does not hold. Similarly, in contrast to classical (Cantorian) set theory in which an element cannot appear more than once, a concept of multiset is evolved, which is an unordered collection of objects into a whole in which certain elements are allowed to repeat. The term 'multiset,' as Knuth [16] notes, was first suggested by N. G. deBruijn [4]. From a practical point of view, multisets are very useful structures as they arise in many areas of mathematics and computer science. Some examples of multisets as stated in [22] are as follows: The prime factorization of integers $n > 0$ is a multiset whose elements are primes. Every monic polynomial $f(x)$ over the complex numbers corresponds in a natural way to the multiset of its roots. Other examples of multisets include the zeros and poles of meromorphic functions, invariants of matrices in a canonical form, the invariants of finite abelian groups etc. The terminal string of a non-circular context-free grammar forms a multiset. Processes in an operating system can be thought of as multisets. The mathematical treatment of concurrency involves the use of multisets. In social sciences, multisets can be used to model social structures, etc.

Many authors like Yager [23], Blizard [2, 3], Girish and John [10, 8, 9], Monro [17] etc. have studied on multisets and its applications. More works on multisets and soft multisets can be found in [1, 5, 7, 11, 12, 13, 14, 18, 19, 20, 21, 24]. Vector space structure is one of the most important structures in modern mathematics. Several authors have introduced the notion of vector space in fuzzy sets [15], soft sets [6] etc. Therefore the study of vector spaces in multisets is very natural. We have attempted in this paper for the first time to introduce a notion of vector space in multiset setting and to study some of its properties.

2. Preliminaries

Definition 2.1 ([8])**.** A multiset (mset) M drawn from a set X is represented by a count function $C_M : X \to N$ where N represents the set of non negative integers.

Here $C_M(x)$ is the number of occurrence of the element x in the mset M. The presentation of the mset M drawn from $X = \{x_1, x_2, ..., x_n\}$ will be as $M = \{x_1/m_1, x_2/m_2, ..., x_n/m_n\}$ where m_i is the number of occurrence of the element x_i, $i = 1, 2, ..., n$ in the mset M.

Also here for any positive integer ω, $[X]^\omega$ is the set of all msets whose elements are in X such that no element in the mset occurs more than ω times and it will be referred to as mset spaces. For $M \in [X]^\omega$, $M_n = \{x \in X : C_M(x) \geq n\}, n \in \mathbb{N}$.

The algebraic operations of msets are considered as in [8].

Definition 2.2 ([18])**.** Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. Then

(i) the image of a mset $M \in [X]^\omega$ under the mapping f is denoted by $f(M)$ or $f[M]$, where

$$C_{f(M)}(y) = \begin{cases} \bigvee\limits_{f(x)=y} & C_M(x)\, if\; f^{-1}(y) \neq \phi \\ 0 & \text{otherwise,} \end{cases}$$

(ii) the inverse image of a mset $N \in [Y]^\omega$ under the mapping f is denoted by $f^{-1}(N)$ or $f^{-1}[N]$, where $C_{f^{-1}(N)}(x) = C_N[f(x)]$.

The properties of functions, which are used in this paper, are as in [18].

3. Sums and scalar products of multisets

Throughout the rest of the paper X, Y will denote vector spaces over K (where K is the field of real or complex numbers), f is a linear map from X to Y and msets are taken from $[X]^\omega, [Y]^\omega$.

Definition 3.1. For $A_1, A_2, ..., A_n, B \in [X]^\omega$, define $A_1 + A_2 + ... + A_n$ and $\lambda B (\lambda \in K)$ as:

$C_{A_1+A_2+...+A_n}(x)$
$= \vee\{C_{A_1}(x_1) \wedge C_{A_2}(x_2) \wedge ... \wedge C_{A_n}(x_n) : x_1, x_2, ..., x_n \in X and\; x_1+x_2+...+x_n = x\}$
and

$C_{\lambda B}(y) = \vee\{C_B(x) : \lambda x = y\}$.

Lemma 3.2. *Let $\lambda \in K$ and $B \in [X]^\omega$. Then*
(1) *for $\lambda \neq 0$, $C_{\lambda B}(y) = C_B(\lambda^{-1} y), \forall y \in X$,*

$$\textit{for } \lambda = 0,\ C_{\lambda B}(y) = \begin{cases} 0, & y \neq 0, \\ \underset{x \in X}{sup}\, C_B(x), & y = 0 \end{cases},$$

(2) *for all scalars* $\lambda \in K$ *and for all* $x \in X$*, we have* $C_{\lambda B}(\lambda x) \geq C_B(x)$.

Proposition 3.3. *For* A, B *in* $[X]^\omega$ *and for* $\lambda \in K$*,*
(1) $f(A + B) = f(A) + f(B)$,
(2) $f(\lambda A) = \lambda f(A)$.

Proof. (1) Let $M = f(X)$, $w \in Y$, $m = C_{f(A+B)}(w), n = C_{f(A)+f(B)}(w)$.

In case $w \notin M, m = 0$. Also, $x, y \in Y, x + y = w$ implies that not both x, y belong to M and then $n = 0$. Let $w \in M$. Given $\epsilon > 0$, there exists $z \in X$, with $f(z) = w$, such that

$$C_{A+B}(z) > m - \epsilon.$$

Thus there exist $x, y \in X$, with $x + y = z$, such that $min\{C_A(x), C_B(y)\} > m - \epsilon$. Since $f(x) + f(y) = f(z) = w$, we have

$$n \geq min\{C_{f(A)}(f(x)), C_{f(B)}(f(y))\} \geq min\{C_A(x), C_B(y)\} > m - \epsilon.$$

Since $\epsilon > 0$, is arbitrary, we get $n \geq m$.

Again for $n > \epsilon > 0$, there exist $z_1, z_2 \in Y$ with $z_1 + z_2 = w$, such that

$$n - \epsilon < min\{C_{f(A)}(z_1), C_{f(B)}(z_2)\}.$$

So, there are $x_1, x_2 \in X$, with $f(x_1) = z_1$ and $f(x_2) = z_2$, such that

$$n - \epsilon < min\{C_A(x_1), C_B(x_2)\}.$$

Since $f(x_1 + x_2) = f(x_1) + f(x_2) = z_1 + z_2 = w$, we get $m > n - \epsilon$. Since $\epsilon > 0$ is arbitrary, $m \geq n$. This proves (1).

(2) Let $w \in Y$, $c = C_{\lambda f(A)}(w)$ and $d = C_{f(\lambda A)}(w)$.

If $w \notin M$, then $c = d = 0$.

Suppose that $w \in M$.

If $\lambda \neq 0$, then $c = C_{f(A)}(\lambda^{-1}w) = \underset{f(x)=\lambda^{-1}w}{sup} C_A(x)$

$$= \underset{f(\lambda x)=w}{sup} C_{\lambda A}(\lambda x) = \underset{f(y)=w}{sup} C_{\lambda A}(y) = d.$$

Next assume that $\lambda = 0$. If $w \neq \theta_Y$, then $c = 0$. Also $d = \underset{f(x)=w}{sup} C_{0A}(x) = 0$, when $f(x) = w \neq \theta_Y$, $x \neq \theta_X$.

For $w = \theta_Y$, we have

$$c = \underset{x \in Y}{sup}\, C_{f(A)}(x) = \underset{y \in X}{sup}\, C_A(y)$$

and

$$d = \underset{f(x)=\theta_Y}{sup} C_{0A}(x) = C_{0A}(\theta_X) = \underset{y \in X}{sup}\, C_A(y).$$

This completes the proof. □

Corollary 3.4. $\lambda(A + B) = \lambda A + \lambda B$ *for all* A, B *in* $[X]^\omega$ *and* $\lambda \in K$.

Proposition 3.5. *Let* $A, A_1, ..., A_n \in [X]^\omega$ *and* $\lambda_1, ..., \lambda_n \in K$*. Then the following assertions are equivalent:*
(1) $\lambda_1 A_1 + \lambda_2 A_2 + ... + \lambda_n A_n \subseteq A$,

(2) *For all* $x_1, x_2, ..., x_n$ *in* X*, we have*

$$C_A(\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n) \geq min\{C_{A_1}(x_1), ..., C_{A_n}(x_n)\}.$$

Proof. (1)⇒(2) is immediate.

(2)⇒(1): By rearranging the order if necessary, we may assume that $\lambda_i \neq 0$ for $i = 1, 2, ..., k$, and $\lambda_i = 0$ for $k \leq i \leq n$. Let $x_1, x_2, ..., x_k \in X$. For all $y_1, y_2, ..., y_{n-k} \in X$, we have

$$C_A(\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_k x_k)$$
$$\geq min\{C_{A_1}(x_1), ..., C_{A_k}(x_k), C_{A_{k+1}}(y_1), ..., C_{A_n}(y_{n-k})\}.$$

Since $C_{0A_j}(\theta) = \underset{y \in X}{sup}\, C_{A_j}(y)$, we get

$$C_A(\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_k x_k)$$
$$\geq min\{C_{A_1}(x_1), ..., C_{A_k}(x_k), C_{0A_{k+1}}(\theta), ..., C_{0A_n}(\theta)\}.$$

On the other hand,

$$C_{\lambda_1 A_1 + \lambda_2 A_2 + ... + \lambda_n A_n}(z)$$
$$= \sup_{x_1 + x_2 + ... + x_n = z} [min\{C_{\lambda_1 A_1}(x_1), ..., C_{\lambda_n A_n}(x_n)\}]$$
$$= \sup_{x_1 + x_2 + ... + x_n = z} [min\{C_{\lambda_1 A_1}(x_1), ..., C_{\lambda_n A_n}(x_n), C_{0A_{k+1}}(x_{k+1}), ..., C_{0A_n}(x_n)\}]$$
$$= \sup_{x_1 + x_2 + ... + x_k = z} [min\{C_{A_1}(\lambda_1^{-1} x_1), ..., C_{A_k}(\lambda_k^{-1} x_k), C_{0A_{k+1}}(\theta), ..., C_{0A_n}(\theta)\}]$$

[Since $C_{0A_i}(x_i) = 0$, if $x_i \neq \theta$, $i = k+1, ..., n$]

$$\leq \sup_{x_1 + x_2 + ... + x_k = z} C_A(\lambda_1 \lambda_1^{-1} x_1 + ... + \lambda_k \lambda_k^{-1} x_k) = C_A(z).$$ □

Lemma 3.6. *Let* $A, B \in [X]^\omega$*. Then*

(1) $A + 0B \subseteq A$,

(2) $A + 0B = A$ *iff* $\underset{x \in X}{sup}\, C_A(x) \leq \underset{x \in X}{sup}\, C_B(x)$.

Proof. (1) $C_A(x + 0y) = C_A(x) \geq min\{C_A(x), C_B(y)\}$. Then (1) follows from Proposition 3.5.

(2) Suppose that $sup\, C_A(x) \leq sup\, C_B(x) = C_{0B}(\theta)$. Then

$$C_{A+0B}(z) = \sup_{x+y=z} [min\{C_A(x), C_{0B}(y)\}] = min\{C_A(z), C_{0B}(\theta)\} = C_A(z).$$

On the other hand, if $C_A(z) > sup\, C_B(x) = C_{0B}(\theta)$ for some z, then

$$C_{A+0B}(z) = min\{C_A(z), C_B(\theta)\} < C_A(z)$$

Thus $A + 0B \neq A$. □

4. Multi vector space

Definition 4.1. A multiset V in $[X]^\omega$ is said to be a multi vector space or multi linear space (in short, mvector space) over the linear space X, if

(i) $V + V \subseteq V$,

(ii) $\lambda V \subseteq V$, for every scalar λ.

We denote the set of all mvector spaces over a vector space X by $MV(X)$.

Lemma 4.2. *Let* V *be a multiset in* $[X]^\omega$*. Then, the followings are equivalent:*

(1) V *is a multi vector space over* X*.*,

(2) *for all* $k, m \in K$*, we have* $kV + mV \subseteq V$,

(3) *for all* $k, m \in K$ *and for all* $x, y \in X$, *we have* $C_V(kx+my) \geq min\{C_V(x), C_V(y)\}$.

Proposition 4.3. *$V \in MV(X)$ and $W \in MV(Y)$ implies that $f(V) \in MV(Y)$ and $f^{-1}(W) \in MV(X)$.*

Proof. Let $V \in MV(X)$. Then for $k, m \in K$, $kV + mV \subset V$. Thus,
$kf(V) + mf(V) = f(kV + mV)$ [By Proposition 3.3] $\subseteq f(V)$,
which shows that $f(V) \in MV(Y)$.
Also if $W \in MV(Y)$, then for any scalar k, m,
$C_{f^{-1}(W)}(kx + my) = C_W(f(kx + my)) = C_W(kf(x) + mf(y))$
$\geq min\{C_W(f(x)), C_W(f(y))\}$ [By Lemma 4.2]
$= min\{C_{f^{-1}(W)}(x), C_{f^{-1}(W)}(y)\}$.
Thus $f^{-1}(W) \in MV(X)$, by Lemma 4.2. □

Proposition 4.4. *If $V, W \in MV(X)$ and $k \in K$, then $V + W$, $kV \in MV(X)$.*

Proof. Let $x, y \in X$ and $k, m \in K$. Then

$$C_{V+W}(kx + my) = \bigvee_{z_1+z_2=kx+my} \{C_V(z_1) + C_W(z_2)\}.$$

Now if $x_1 + x_2 = x$ and $y_1 + y_2 = y$, for x_1, x_2, $y_1, y_2 \in X$, then

$$(kx_1 + my_1) + (kx_2 + my_2) = kx + my.$$

Thus,

$$\begin{aligned}
&C_{V+W}(kx + my)\\
&\geq \bigvee_{x_1+x_2=x, y_1+y_2=y} \{C_V(kx_1 + my_1) \wedge C_W(kx_2 + my_2)\}\\
&\geq \bigvee_{x_1+x_2=x, y_1+y_2=y} \{C_V(x_1) \wedge C_V(y_1) \wedge C_W(x_2) \wedge C_W(y_2)\} \text{ [As } V, W \in MV(X) \text{]}\\
&= \bigvee_{x_1+x_2=x, y_1+y_2=y} \{C_V(x_1) \wedge C_W(x_2) \wedge C_V(y_1) \wedge C_W(y_2)\}\\
&= \left[\bigvee_{x_1+x_2=x} \{C_V(x_1) \wedge C_W(x_2)\}\right] \wedge \left[\bigvee_{y_1+y_2=y} \{C_V(y_1) \wedge C_W(y_2)\}\right]\\
&= C_{V+W}(x) \wedge C_{V+W}(y).
\end{aligned}$$

So $V + W \in MV(X)$.
Again, $kV \in MV(X)$ follows from Proposition 3.3. □

Proposition 4.5. *If $V_i \in MV(X), i \in I$, then $\bigcap_{i \in I} V_i \in MV(X)$.*

Proposition 4.6. *Let $V \in MV(X)$. Then $C_V(\theta) \geq C_V(x), \forall x \in X$.*

Proposition 4.7. *Let $V \in MV(X)$. Then*
(1) *for $n \in \mathbb{N}$, V_n is either empty or a subspace of X,*
(2) *$V^* = \{x \in X; C_V(x) = C_V(0)\}$ and $V_* = \{x \in X; C_V(x) > 0\}$ are subspaces of X.*

Proposition 4.8. *For any two $V_1, V_2 \in MV(X)$ and any $n \in \{0, 1, 2, ..., \omega\}$,*
(1) $(V_1 \cap V_2)_n = (V_1)_n \cap (V_2)_n$,
(2) $(V_1 + V_2)_n = (V_1)_n + (V_2)_n$.

Proposition 4.9. *Let $V \in MV(X)$ with $dim X = m$. Then the range of C_V contains at most $m + 1$ points of $\{0, 1, 2, ..., \omega\}$.*

Proof. If possible suppose $x_0, x_1, ..., x_m \in X \setminus \{\theta\}$ such that

$$C_V(x_0) < C_V(x_1) < ... < C_V(x_m).$$

Then $x_0 \notin vct\{x_1, x_2, ..., x_m\}$ ($vct\{x_1, x_2, ..., x_m\}$ is the vector space spanned by $\{x_1, x_2, ..., x_m\}$). Otherwise there exist $a_1, a_2, ..., a_m \in K$ such that $x_0 = \sum_{i=1}^{m} a_i x_i$ and by Lemma 4.2, $C_V(x_0) \geq C_V(x_1)$, which is impossible. Analogously $x_1 \notin vct\{x_2, ..., x_m\}, ..., x_{m-1} \notin vct\{x_m\}$. Since all $x_i \neq \theta$, we have
$dim(vct\{x_0, x_1, ..., x_m\}) = 1 + dim(vct\{x_1, x_2, ..., x_m\}) = m + dim(vct\{x_m\}) = m+1$. This is impossible, since $dim\ X = m$. Consequently the range of C_V is a subset of $\{0, 1, 2, ..., \omega\}$ with at most $m+1$ points of which m values are attained at points of $X \setminus \{\theta\}$ and the maximum one is attained at θ. □

Proposition 4.10. *(Representation Theorem) Let $V \in MV(X)$ with $dim\ X = m$ and range of $C_V = \{n_0, n_1, ..., n_k\}$, $k \leq m$, $n_0 = C_V(\theta)$ and $\omega \geq n_0 > n_1 > ... > n_k \geq 0$. Then there exists a nested collection of subspaces of X as*
$\{\theta\} \subseteq V_{n_0} \subset V_{n_1} \subset .... \subset V_{n_k} = X$ such that $V = n_0 V_{n_0} \cup n_1 V_{n_1} \cup ..... \cup n_k V_{n_k}$. Also,
(1) *if $n, m \in (n_{i+1}, n_i]$, then $V_n = V_m = V_{n_i}$,*
(2) *if $n \in (n_{i+1}, n_i]$ and $m \in (n_i, n_{i-1}]$, then $V_n \supset V_m$.*

Proof. From Proposition 4.7, $V_{n_i} = \{x \in X : C_V(x) \geq n_i\}$ are subspaces of X, for $i = 0, 1, 2, ..., k$. As $n_i > n_{i+1}$, for $i = 0, 1, ..., k-1$, we have a nested collection of subspaces of X as

$$\{\theta\} \subseteq V_{n_0} \subset V_{n_1} \subset .... \subset V_{n_k} = X.$$

Now we show that $V = n_0 V_{n_0} \cup n_1 V_{n_1} \cup ..... \cup n_k V_{n_k}$. Let $x \in X$ and $C_V(x) = n_j$. Then $x \in V_{n_j}$ and $x \notin V_{n_l}$, for $l < j$. Thus

$$\begin{aligned} & C_{n_0 V_{n_0} \cup n_1 V_{n_1} \cup ..... \cup n_k V_{n_k}}(x) \\ &= C_{n_0 V_{n_0}}(x) \vee C_{n_1 V_{n_1}}(x) \vee .... \vee C_{n_k V_{n_k}}(x) \\ &= n_j \vee .... \vee n_k = n_j. \end{aligned}$$

(1) Let $n \in (n_{i+1}, n_i]$. Then obviously $V_{n_i} \subseteq V_n$. Next let $x \in V_n$. Then $C_V(x) \geq n > n_{i+1}$. This implies that $C_V(x) \geq n_i$. Thus $x \in V_{n_i}$. So $V_n \subseteq V_{n_i}$. Hence $V_n = V_{n_i}$. Similarly, $V_m = V_{n_i}$. Therefore (1) holds.

(2) is straightforward. □

Example 4.11. Let $X = \mathbb{R}^2, \omega = 4$ and V be defined as

$$\begin{aligned} C_V(x) &= 2,\ if\ x \neq \theta \\ &= 4,\ if\ x = \theta. \end{aligned}$$

Then $V = 4V_4 \cup 2V_2$ is a decomposition V.

5. Multi bases of a multi vector space

Definition 5.1. Let X be a finite dimensional vector space with $dim\ X = m$ and $V \in MV(X)$. Consider Proposition 4.10. Let B_{n_i} be a basis of $V_{n_i}, i = 0, 1, ..., k$ such that

$$B_{n_0} \subset B_{n_1} \subset B_{n_2} \subset ... \subset B_{n_k} \tag{5.1}$$

Define a multi subset β of X by

$$\begin{aligned} C_\beta(x) &= \vee\{n_i : x \in B_{n_i}\} \\ &= 0, \text{ otherwise.} \end{aligned}$$

Then β is called a multi basis of V corresponding to (5.1).

Example 5.2. Let $X = \mathbb{R}^2$ and $\omega = 6$. Define a multi vector space V by $C_V : X \to N$ by

$$\begin{aligned} C_V(x) &= 6, \; if \; x \in \{(a, 0) : a \in \mathbb{R}\} \\ &= 1, \text{ otherwise.} \end{aligned}$$

Then we have $\{\theta\} \subset V_6 \subset V_1 = \mathbb{R}^2$. Let $e_1 = (1, 0)$, $e_2 = (0, 1)$, $B_6 = \{e_1\}$ and $B_1 = \{e_1, e_2\}$. Then β is a multi basis of V where $C_\beta(x)$ is defined by:

$$C_\beta(x) = \begin{cases} 6, & if \; x = (1, 0) \\ 1, & if \; x = (0, 1) \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 5.3. *Let β be a multi basis of V obtained by (5.1). Then*
(1) *if $n, m \in (n_{i+1}, n_i]$, then $\beta_n = \beta_m = B_{n_i}$,*
(2) *if $n \in (n_{i+1}, n_i]$ and $m \in (n_i, n_{i-1}]$, then $\beta_n \supset \beta_m$,*
(3) *β_n is a basis of V_n, for all $n \in \{1, 2, ..., \omega\}$.*

6. Convex, balanced and absorbing multisets

Definition 6.1. A multiset M in $[X]^\omega$ is said to be:
(i) convex, if $\lambda M + (1 - \lambda)M \subseteq M$, for all $\lambda \in [0, 1]$,
(ii) balanced, if $\lambda M \subseteq M$, for all scalars λ with $|\lambda| \leq 1$,
(iii) absorbing, if for each $x \in X$, $C_{\underset{k>0}{\cup} kM}(x) = \omega$,
(iv) absolutely convex, if it is both convex and balanced.

Proposition 6.2. *Let $M \in [X]^\omega$. Then the followings assertions are equivalent:*
(1) *M is convex (balanced),*
(2) *$C_M(\lambda x + (1 - \lambda)y) \geq min\{C_M(x), C_M(y)\}$, for all $x, y \in X$ and all $\lambda \in [0, 1]$ ($C_M(\lambda x) \geq C_M(x)$, for all $|\lambda| \leq 1$),*
(3) *For each $n \in \{1, 2, ..., \omega\}$, M_n is convex (balanced) in X.*

Proof. (1) $\Leftrightarrow$ (2) is immediate.

(2)$\Leftrightarrow$(3): We only prove the convex case. The proof for the balanced case is similar. Let $C_M(\lambda x + (1 - \lambda)y) \geq min\{C_M(x), C_M(y)\}$, for all $x, y \in X$ and all $\lambda \in [0, 1]$. If $M_n \neq \phi$, take $x, y \in M_n$. Then $C_M(x)$, $C_M(y) \geq n$. Thus

$$C_M(\lambda x + (1 - \lambda)y) \geq min\{C_M(x), C_M(y)\} \geq n.$$

So $\lambda x + (1 - \lambda)y \in M_n$, for all $\lambda \in [0, 1]$. Hence M_n is convex in X.

Conversely, assume that the sets M_n, $n \in \{1, 2, ..., \omega\}$ are convex in X. Let $x, y \in X$ and $min\{C_M(x), C_M(y)\} = n_0$. If $n_0 = 0$, then obviously,

$$C_M(\lambda x + (1 - \lambda)y) \geq min\{C_M(x), C_M(y)\}.$$

If $n_0 \neq 0$, then $C_M(x), C_M(y) \geq n_0$. Thus $x, y \in M_{n_0}$. By convexity of M_{n_0}, $\lambda x + (1-\lambda)y \in M_{n_0}$, for all $\lambda \in [0,1]$. So $C_M(\lambda x+(1-\lambda)y) \geq n_0 = min\{C_M(x), C_M(y)\}$. Hence (2) holds. □

Proposition 6.3. *Let $M \in [X]^\omega$. Then the followings are equivalent:*
(1) *M is absolutely convex,*
(2) *$\lambda M + \mu M \subseteq M$, for all scalars λ, μ with $\mid \lambda \mid + \mid \mu \mid \leq 1$,*
(3) *$C_M(\lambda x + \mu y) \geq min\{C_M(x), C_M(y)\}$, for all $x, y \in X$ and all scalars λ, μ with $\mid \lambda \mid + \mid \mu \mid \leq 1$.*
(4) *For each $n \in \{0, 1, 2, ..., \omega\}$, the ordinary set $M_n = \{x \in X : C_M(x) \geq n\}$ is absolutely convex.*

Proof. (1)⇔(2): Let M be absolutely convex and choose scalars λ, μ with $\mid \lambda \mid + \mid \mu \mid \leq 1$.

If $\lambda = 0$ or $\mu = 0$, then evidently, $\lambda M + \mu M \subseteq M$ (as M is balanced).

If $\lambda \neq 0$ and $\mu \neq 0$, then $\frac{\lambda}{|\lambda|}M \subseteq M$ and $\frac{\mu}{|\mu|}M \subseteq M$ (as M is balanced) and $\frac{|\lambda|}{|\lambda|+|\mu|} + \frac{|\mu|}{|\lambda|+|\mu|} = 1$. Thus

$$\lambda M + \mu M = (\mid \mu \mid + \mid \lambda \mid)\left\{\frac{\mid \lambda \mid}{\mid \lambda \mid + \mid \mu \mid}\left(\frac{\lambda}{\mid \lambda \mid}M\right) + \frac{\mid \mu \mid}{\mid \lambda \mid + \mid \mu \mid}\left(\frac{\mu}{\mid \mu \mid}M\right)\right\} \subseteq M.$$

Conversely, let the condition hold for a multiset M in $[X]^\omega$. Then choosing $\mu = 0$, we find that M is balanced and choosing $\lambda > 0, \mu > 0$ and $\lambda + \mu = 1$, we find that M is convex. Thus M is absolutely convex.

Proofs of (2) ⇔ (3) and (1) ⇔ (4) are immediate. □

Proposition 6.4. *$M \in [X]^\omega$ is absorbing iff M_n is absorbing, for each $n \in \{1, 2, ..., \omega\}$.*

Proof. Suppose M is absorbing. Then for $x \in X$, $C_{\underset{k>0}{\cup} kM}(x) = \omega$. Hence $\underset{k>0}{Sup}\, C_{kM}(x) = \underset{k>0}{Sup}\, C_M(k^{-1}x) = \omega$. Then $C_M(k^{-1}x) = \omega$, for some $k > 0$. Thus, for each $n \in \{1, 2, ..., \omega\}$, $k^{-1}x \in M_n$, i.e., M_n is absorbing.

Conversely, suppose that for each $n \in \{1, 2, ..., \omega\}$, M_n is absorbing. Then for $x \in X, n \in \{1, 2, ..., \omega\}$, there exists $k_n > 0$, such that $k_n^{-1}x \in M_n$. Thus

$\underset{k>0}{Sup}\, C_M(k^{-1}x) = \omega$, i.e., $\underset{k>0}{Sup}\, C_{kM}(x) = C_{\underset{k>0}{\cup} kM}(x) = \omega$.

So M is absorbing. □

Proposition 6.5. *Let $M, M' \in [X]^\omega$ and $N \in [Y]^\omega$.*
(1) *If M is a convex (balanced), $f(M)$ is a convex (balanced) multiset in $[Y]^\omega$.*
(2) *If N is a convex (balanced or absorbing), $f^{-1}(N)$ is a convex (balanced or absorbing) in $[X]^\omega$.*
(3) *If M, M' are convex (balanced), then $M + M' \in [X]^\omega$ is convex (balanced).*

Proposition 6.6. *If $\{M_i \in [X]^\omega, i \in I\}$ is convex (balanced), then $\underset{i \in I}{\cap} M_i$ is also so.*

Definition 6.7. Let M be a multiset in $[X]^\omega$. The convex (balanced) hull of M is the intersection of all convex (balanced) sets in $[X]^\omega$ which contains M.

Proposition 6.8. *Let $M \in [X]^\omega$. Then the balanced hull of M is the multiset $\underset{|\lambda| \leq 1}{\cup} \lambda M$.*

Proof. The multiset $N = \underset{|\lambda|\leq 1}{\cup} \lambda M$ is contained in any balanced multiset which contains M. Since $N \supseteq M$, it suffices to show that N is balanced. Let $a \in K$, $|\, a \,|\leq 1$ and $x \in X$. Then

$$\begin{aligned} C_N(x) = \underset{|\lambda|\leq 1}{sup}\, C_{\lambda M}(x) &\leq \underset{|\lambda|\leq 1}{sup}\, C_{a\lambda M}(ax) \\ &\leq \underset{|a\lambda|\leq 1}{sup}\, C_{a\lambda M}(ax) \\ &\leq \underset{|\lambda|\leq 1}{sup}\, C_{\lambda M}(ax) = C_N(ax). \end{aligned}$$

Thus $aN \subseteq N$, by Proposition 3.5. So N is balanced. □

7. Conclusion

In our future study, we have a plan to develop further properties of multi vector space. Introduction of the concept of multi topological vector space is also another future plan.

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