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Hybrid monotonic inclusion measures for interval-valued hesitant fuzzy sets

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Abstract. The inclusion measure indicates the degree to which a given partially ordered set is contained in another one. Hybrid monotonic inclusion measure is the rational generalization of inclusion measure. So, the main purpose of this paper is to introduce the hybrid monotonic inclusion measure for interval-valued hesitant fuzzy sets (IVHFSs). Firstly, we present the construction approaches to hybrid monotonic inclusion measures between any two interval-valued hesitant fuzzy elements (IVHFEs) under different order relationships in extended environment. In addition, a new partial order relationship $\leq_{\vee}$ for IVHFEs is also defined, and then some hybrid monotonic inclusion measures under the new partial order $\leq_{\vee}$ for IVHFEs and IVHFSs will be constructed and be proved in detail. Finally, we present some new formulas to calculate the similarity measure and fuzzy entropy derived from the hybrid monotonic inclusion measure for IVHFSs.

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1. Introduction

The fuzzy set theory, which is a generalization of the classical set theory introduced by Zadeh (1965) [28], has been widely studied up to now and successfully applied in various fields. The mathematical framework deals with the incompleteness, inaccuracy and uncertainty of information of real systems due to fuzziness and uncertainty of decision making problems. However, the fuzzy set is characterized by a membership function that depends on that the expert expresses it by an exact numerical value in [0,1], so it can not manage vague information more efficiently. In order to overcome this flaw, many kinds of generalizations of fuzzy sets have been presented. In particular, the interval-valued fuzzy sets were introduced by Gozalczany in 1987 [9] and Turksen [18] represent the membership function by giving for each point not a single value but an interval in 1996. Intuitionistic fuzzy sets, proposed by Atanassov in 1986 [1], are another generalization of fuzzy set, where the set is characterized by both a membership and a non-membership function. A greater extension of the fuzzy sets are the type-2 fuzzy sets, given by Zadeh in 1975 [27], which are three dimensional, where for each point, the membership function is defined over the referential [0,1], which use both primary and secondary membership to provide more degree of freedom and flexibility. However, type-2 fuzzy sets are difficult to work with, so hesitant sets were introduced in 2009 by Torra [17] as an intermediate kind of fuzzy sets. The membership function of a hesitant set assigns a subset of the closed interval [0,1] instead of a fuzzy set to each point. This property makes them more manageable than type-2 fuzzy sets. In fact, these sets have been already introduced by Grattan-Guinness [10] in 1976, with the name of set-valued fuzzy sets. However, Torra provided functional definitions of union and intersection for such sets which were not considered by Grattan-Guinness. Because of a set of possible membership values for an element, the hesitant fuzzy set is more effective in multi-criteria group decision making problem in [2, 7], and especially, in decision making [4, 14]. In addition, different extensions of the hesitant sets have been developed lately. This paper will pay attention to the interval-valued hesitant fuzzy sets, given by Chen et al. [5] in 2013, which have been studied in detail in many literature.

Most of the interval-valued hesitant fuzzy sets are based on the assumption that the IVHFEs for one object in two IVHFSs have same cardinalities usually by adding a number of the biggest, the smallest or a parameterized one of the IVHFE with smaller cardinality. But the mismatching and loss of the transformations are neglected in spite of its simple framework. So in this paper, we will construct a novel partial order $\leq_{\vee}$ for all IVHFEs under difference cardinalities. Due to that the set of all IVHFEs under the partial order $\leq_{\vee}$ is a poset and inclusion measure [12, 15] on a poset indicates the quantitative degree to which a given element of a poset is less than another one, it means that the inclusion measure not only reflects partially ordered relationships between two elements which possess the partial order, but also shows the exact pairwise less than degree between any two elements in a poset. So in this paper, the inclusion measures of IVHFEs and IVHFSs can be studied and constructed in detail under different order relationships.

The inclusion measures, also known as subsethood measures, which are an important mathematical tool for describing the include degree of two sets, have attracted some research's interest. The inclusion measures have been studied mainly by constructive approaches and axiomatic approaches. The constructive approach is suitable for practical applications of inclusion measures, and the axiomatic approach is appropriate for studying the structure of the inclusion measure. In the constructive approach, the inclusion measures were constructed by fuzzy implication operators [8] and conditional probability [13, 23, 24]. Zhang and Leung [23, 24] constructed the inclusion measures by fuzzy possibility, fuzzy measure, conditioned fuzzy measure, conditioned information and so on. In axiomatic approach, a reasonable inclusion measure should satisfy a set of axioms. In the literature, the most accepted axioms have been given for inclusion measure. Firstly, Kitainik [12] proposed four axioms for the inclusion measure according to the crisp inclusion relations properties in 1987. Later, Sinha and Dougherty [15] in 1993 offered nine axioms for inclusion measure, plus three additional ones. Young [22] argued that Sinha and Doughertys axioms are too hard to restrict the applications of the inclusion measures and proposed four simplified axioms for the inclusion measure in 1996. After that, Zhang and Zhang [25] in 2009 proposed the hybrid monotonic inclusion measure which preserves the monotonicity of two variables. As the rational generalization, hybrid monotonicity is necessary for an inclusion measure. As is known, the studies of the hybrid monotonic inclusion measure for IVFSs [16] and HFSs [26] and their applications have been in many literature. However, only few studies on the inclusion measures of IVHFEs and IVHFSs have been presented. Furthermore, the applications of hybrid monotonic inclusion measure for IVHFSs can be seen in many fields, especially in interval-valued hesitant multi-attribute decision making. Furthermore, we can also obtain the similarity measure, fuzzy entropy and distant between IVHFSs based on the hybrid monotonic inclusion measure of IVHFSs. Thus, the hybrid monotonic inclusion measure for IVHFSs is very meaningful and it is worth being studied. So, the main aim of this paper is to research the hybrid monotonic inclusion measure for IVHFEs and IVHFSs under the extended and unextended environments by fuzzy implication operators and prove that these satisfy the axiomatic definition. In addition, similarity measure, distance and fuzzy entropy between IVHFSs in terms of the hybrid monotonic inclusion measure will also be discussed.

The remainder of this paper is organized as follows. Section 2 gives some preliminary definitions used in this paper. In Section 3, we firstly present the construction approaches to hybrid monotonic inclusion measures between any two IVHFEs under different order relationships in the extended environment. In addition, a new partial order relationship $\leq_\vee$ for IVHFEs is also defined, and then some hybrid monotonic inclusion measures under the new partial order $\leq_\vee$ for IVHFEs and IVHFSs will be constructed and proved in detail. In Section 4, we introduce the axiomatic definitions of the similarity measure, distance and fuzzy entropy for IVHFSs and put forward some new formulas to calculate the similarity measure and fuzzy entropy derived from the hybrid monotonic inclusion measure for IVHFSs. The final section contains the conclusion.

2. Preliminaries

The following recalls necessary concepts and preliminaries required in the sequel of our work.

Definition 2.1. A binary function $T : [0,1]^2 \to [0,1]$ is called a triangular norm, or t-norm for short, if T satisfies the following properties:

(i) $T(a,1) = a$ $\quad (a \in [0,1])$,
(ii) $T(a,b) = T(b,a)$ $\quad (a,b \in [0,1])$,
(iii) $T(T(a,b),c) = T(a,T(b,c))$ $\quad (a,b,c \in [0,1])$,
(iv) $T(a,b) \leq T(c,d)$ $\quad (a \leq c, b \leq d, a,b,c,d \in [0,1])$.

The four main t-norms are as the followings:

the minimum operator: $T_M(a,b) = min(a,b)$,

the algebraic product: $T_P(a,b) = a * b$,
the Łukasiewicz t-norm: $T_L(a,b) = max(0, a+b-1)$,
the drastic product: $T_Z(a,1) = T_Z(1,a) = a$ and $T_Z(a,b) = 0$. otherwise.

Remark 2.2. A t-norm is called a left-continuous t-norm, if T satisfies:

$$T(a, \bigvee_{i\in l} b_i) = \bigvee_{i\in l} T(a, b_i).$$

Definition 2.3. A binary operator $\theta : [0,1]^2 \to [0,1]$ is said to be an implication function if it satisfies:
(i) $\theta(1,0) = 0$,
(ii) $\theta(0,0) = \theta(0,1) = \theta(1,1) = 1$.

Remark 2.4. An implicator θ is called left monotonic iff for all $c \in [0,1]$, if $a \leq b$, then $\theta(a,c) \geq \theta(b,c)$, and an implicator θ is called right monotonic iff for all $a \in [0,1]$, if $b \leq c$, then $\theta(a,b) \leq \theta(a,c)$. If θ is both left monotonic and right monotonic, then it is called hybrid monotonic.

Remark 2.5. An implicator θ is said to be a CP implicator, if it satisfies CP principle (confinement principle), where CP principle means that for all $(a,b) \in [0,1]^2, a \leq b \Longleftrightarrow \theta(a,b) = 1$.

Several classes of implicators have been studied in the literature. We recall here the definition of R-implicator (residual implicator).

Definition 2.6. An implicator is a R-implicator based on a left-continuous t-norm T, if for every $a,b \in [0,1]$, $T(a,b)$ =sup$\{c \in [0,1], T(a,c) \leq b\}$.

Remark 2.7. Every R-implicator is hybrid monotonic and is a CP implicator.

In the next section, we recall the concepts of hesitant fuzzy sets and interval-valued hesitant fuzzy sets. The definition of hesitant fuzzy sets was introduced by Torra recently in [17]. In fact, hesitant fuzzy sets were already introduced in 1976 by Grattan-Guinnedd in [10] with the name of set-valued fuzzy sets. To be easily understood, Xu and Xia (2011) expressed the HFS by a mathematical symbol [21]:

Definition 2.8. Let X be a non-empty set. A hesitant fuzzy set (HFS) A on X is defined in terms of a function $\mu_A(x)$ that returns a subset of [0,1], when it is applied to X, i.e.,

$$A = \{(x, \mu_A(x)) | x \in X\},$$

where $\mu_A(x)$ is a set of some different values in [0,1], representing the possible membership degrees of the element $x \in X$ to A.

In hesitant fuzzy sets, the membership degrees of x are some exact values. However, in reality, the membership degrees of a certain element x to A are not necessary real numbers, they may be a range of values belonging to [0,1]. To deal with such cases, Chen et al. [5] introduced the concept of interval-valued hesitant fuzzy set, which is a generalization of HFS.

Definition 2.9. Let X be a non-empty set, and $D[0,1]$ be the set of all closed subintervals of [0,1]. An interval-valued hesitant fuzzy set (IVHFS) A on X is defined as

$$A = \{(x, \mu_A(x))|x \in X\},$$

where $\mu_A(x)$: $X \to D[0,1]$ denotes all possible interval-valued membership degrees of the element $x \in X$ to A. We will denote the set of all interval-valued hesitant fuzzy sets on X by IVHF(X).

Note that hesitant fuzzy sets are a particular case of interval-valued hesitant fuzzy sets when the interval-values are restricted to be exact values. For convenience, $\mu = \mu_A(x)$ is called an IVHFE and $l_x = l(\mu_A(x))$ is the number of intervals in an IVHFE $\mu_A(x)$. For an IVHFE $\mu_A(x)$, it is necessary to arrange the intervals in $\mu_A(x)$ in an increasing order. To do this, we can employ the score and the accuracy functions for the comparison between two interval numbers [19]. So let $\mu_A^{\sigma(j)}(x)$ stand for the jth largest interval in $\mu_A(x)$, where

$\mu_A^{\sigma(j)}(x) = [\mu_A^{\sigma(j)L}(x), \mu_A^{\sigma(j)U}(x)] \subseteq [0,1]$, $1 \leq j \leq l_x$

are intervals, and

$\mu_A^{\sigma(j)L}(x) =\inf \mu_A^{\sigma(j)}(x)$ $\mu_A^{\sigma(j)U}(x) =\sup \mu_A^{\sigma(j)}(x)$, $1 \leq j \leq l_x$.

For convenience, we denote the IVHFEs which are arranged in an increasing order as $\mu_A = \{\mu_A^1, \mu_A^2, \cdots, \mu_A^{l_x}\}$.

Hesitant fuzzy sets $\mu_A^L(x)$ and $\mu_A^U(x)$ are called a lower hesitant fuzzy set of A and an upper hesitant fuzzy set of A, respectively.

The complementary of an IVHFE $\mu_A(x)$, denoted by $\mu_A^c(x)$, is defined as

$\mu_A^c(x) = \{[1 - \mu_A^{\sigma(1)U}(x), 1 - \mu_A^{\sigma(1)L}(x)], [1 - \mu_A^{\sigma(2)U}(x), 1 - \mu_A^{\sigma(2)L}(x)], \cdots, [1 - \mu_A^{\sigma(l_x)U}(x), 1 - \mu_A^{\sigma(l_x)L}(x)]\}$.

3. Hybrid monotonic inclusion measures for IVHFEs and IVHFSs

The inclusion measure describes the extent of one element contained in another one of a partially ordered set. Young introduced a weak axiomatic definition of the inclusion measure in [22]. As the rational generalization of inclusion measure, hybrid monotonicity is necessary for an inclusion measure. So, Zhang et al. in [25] presented the definition of hybrid monotonic inclusion measure by substituting the hybrid monotonicity for fourth condition of Young. It is obvious that the conditions of hybrid monotonic inclusion measure are stricter than Young introduced the inclusion measure. Now, suppose $\leq$ is a partial order on IVHFEs, $A \subseteq B \Leftrightarrow A(x) \leq B(x)$ is a partial order on IVHFSs. Then, the hybrid monotonic inclusion measure for partially ordered sets (IVHFE, $\leq$) is constructed in Definition 3.1.

Definition 3.1. Let μ_1, μ_2 be two IVHFEs, $\leq$ be a partial order on IVHFEs. A function $I_{\leq}(\mu_1, \mu_2) \in [0,1]$ is called the hybrid monotonic inclusion measure between μ_1, μ_2, if $I_{\leq}(\mu_1, \mu_2)$ satisfies the following properties:

(I1) $0 \leq I_{\leq}(\mu_1, \mu_2) \leq 1$,

(I2) if $\mu_1 \leq \mu_2$, then $I_{\leq}(\mu_1, \mu_2) = 1$,

(I3) if $\mu = \{[1,1]\}$,then $I_{\leq}(\mu, \mu^c) = 0$,

(I4) if $\mu_1 \leq \mu_2$, for any IVHFE μ_3, $I_{\leq}(\mu_3, \mu_1) \leq I_{\leq}(\mu_3, \mu_2)$ and $I_{\leq}(\mu_2, \mu_3) \leq I_{\leq}(\mu_1, \mu_3)$.

When the partially ordered set (IVHFEs, $\leq$) is replaced by (IVHFSs, $\subseteq$), then the hybrid monotonic inclusion measure between any two interval-valued hesitant fuzzy sets can be defined in the same way.

In the following part, the some order relationships for IVHFEs and IVHFSs are introduced in Definition 3.2 in the extended environment.

Definition 3.2. Let $\mu_1 = \{\mu_1^1, \mu_1^2, \cdots, \mu_1^l\}$ and $\mu_2 = \{\mu_2^1, \mu_2^2, \cdots, \mu_2^l\}$ be two IVHFEs which have been extended to the same number of values l (the IVHFE with fewer elements should be considered optimistically by repeating its maximum element until it has the same length with another), and let the elements of μ_1 and μ_2 be arranged in increasing order, the partial order relationship under the extended environment can be defined as follows:

$\mu_1 \leq_1 \mu_2$ iff $\mu_1^{iL}(x) \leq \mu_2^{iL}(x), \mu_1^{iU}(x) \leq \mu_2^{iU}(x)$, $i = 1, 2, \cdots l$.

Let $A = \{(x, \mu_A(x))|x \in X\}$ and $B = \{(x, \mu_B(x))|x \in X\}$ be two IVHFSs defined on the reference set X. Then

$A \subseteq_1 B$ iff $\mu_A(x) \leq_1 \mu_B(x)$, $\forall x \in X$.

In [3], Bustince et al. proposed the maxi-min and maxi-max dominance for intervals, which construct linear orders. These orders are also called lexicographical orders. Then, these kind of orders for the two IVHFEs are defined as follows:

Definition 3.3. Let $\mu_1 = \{\mu_1^1, \mu_1^2, \cdots, \mu_1^l\}$ and $\mu_2 = \{\mu_2^1, \mu_2^2, \cdots, \mu_2^l\}$ be two IVHFEs which have been extended to the same number of values l, and let the elements of μ_1 and μ_2 be arranged in increasing order. Then

$\mu_1 \leq_2 \mu_2$ iff $\mu_1^{iL}(x) < \mu_2^{iL}(x)$ or $(\mu_1^{iL}(x) = \mu_2^{iL}(x),\ \mu_1^{iU}(x) \leq \mu_2^{iU}(x))$, $i = 1, 2, \cdots l$.

$\mu_1 \leq_3 \mu_2$ iff $\mu_1^{iU}(x) < \mu_2^{iU}(x)$ or $(\mu_1^{iU}(x) = \mu_2^{iU}(x),\ \mu_1^{iL}(x) \leq \mu_2^{iL}(x))$, $i = 1, 2, \cdots l$.

In [11], Hurwicz presented the method consist in comparing the midpoint of the two intervals. However, it is clear that this does not form an order. Therefore, in 2006, Xu and Yager [20] considered an improved method where they transform it an order. So, this kind of order are also defined for the two IVHFEs.

Definition 3.4. Let $\mu_1 = \{\mu_1^1, \mu_1^2, \cdots, \mu_1^l\}$ and $\mu_2 = \{\mu_2^1, \mu_2^2, \cdots, \mu_2^l\}$ be two IVHFEs which have been extended to the same number of values l, and let the elements of μ_1 and μ_2 be arranged in increasing order. Then

$\mu_1 \leq_4 \mu_2$ iff $\mu_1^{iL}(x) + \mu_1^{iU}(x) < \mu_2^{iL}(x) + \mu_2^{iU}(x)$ or $(\mu_1^{iL}(x) + \mu_1^{iU}(x) = \mu_2^{iL}(x) + \mu_2^{iU}(x)$ and $\mu_1^{iU}(x) - \mu_1^{iL}(x) \leq \mu_2^{iU}(x) - \mu_2^{iL}(x))$, $i = 1, 2, \cdots l$.

Similarly, let $A = \{(x, \mu_A(x))|x \in X\}$ and $B = \{(x, \mu_B(x))|x \in X\}$ be two IVHFSs defined on the reference set X. Then

$A \subseteq_2 B$ iff $\mu_A(x) \leq_2 \mu_B(x)$,$\forall x \in X$,

$A \subseteq_3 B$ iff $\mu_A(x) \leq_3 \mu_B(x)$,$\forall x \in X$,

$A \subseteq_4 B$ iff $\mu_A(x) \leq_4 \mu_B(x)$,$\forall x \in X$.

Remark 3.5. It is obvious that $\leq_1$ is a partial order, which based on the lattice order in intervals. It is the thoughest one but, regardless of the adopted point of view, it can be understand as the most intuitive one. $\leq_2$, $\leq_3$ and $\leq_4$ are linear orders, they base on the bounds comparison and midpoint comparison in intervals.

Now, we present the constructive approaches to hybrid monotonic inclusion measures between any two elements in (IVHFE, $\leq$) and (IVHFS, $\subseteq$) in extended environment, respectively.

Theorem 3.6. *Let θ be an implicastor which satisfies hybrid monotonic and CP principle, and $\mu_1 = \{\mu_1^1, \mu_1^2, \cdots, \mu_1^l\}$, $\mu_2 = \{\mu_2^1, \mu_2^2, \cdots, \mu_2^l\}$ and $\mu_3 = \{\mu_3^1, \mu_3^2, \cdots, \mu_3^l\}$ be three IVHFEs. Then the following functions are hybrid monotonic inclusion measures for IVHFEs under the partial order relationship $\leq_1$:*

(1) $I_{\leq_1 1}(\mu_1, \mu_2) = \bigwedge\limits_{i=1}^{l} [\frac{1}{2}(\theta(\mu_1^{iL}, \mu_2^{iL}) + \theta(\mu_1^{iU}, \mu_2^{iU}))]$,

(2) $I_{\leq_1 2}(\mu_1, \mu_2) = \frac{1}{2}\sum\limits_{i=1}^{l} \lambda_i[\theta(\mu_1^{iL}, \mu_2^{iL}) + \theta(\mu_1^{iU}, \mu_2^{iU})]$,

for all $x \in X$, where λ_i is positive real number satisfying $\sum\limits_{i=1}^{n} \lambda_i = 1$.

The hybrid monotonic inclusion measure between any two IVHFEs can also be constructed under the other order relationships :

1. the lower hybrid monotonic inclusion measures for IVHFEs under the order relationship $\leq_2$:

$$I_{\leq_2 1}(\mu_1, \mu_2) = \bigwedge_{i=1}^{l} \theta(\mu_1^{iL}, \mu_2^{iL}),$$

$$I_{\leq_2 2}(\mu_1, \mu_2) = \sum_{i=1}^{l} \lambda_i\theta(\mu_1^{iL}, \mu_2^{iL}),$$

where λ_i is positive real number satisfying $\sum\limits_{i=1}^{n} \lambda_i = 1$.

2. the upper hybrid monotonic inclusion measures for IVHFEs under the order relationship $\leq_3$:

$$I_{\leq_3 1}(\mu_1, \mu_2) = \bigwedge_{i=1}^{l} \theta(\mu_1^{iU}, \mu_2^{iU}),$$

$$I_{\leq_3 2}(\mu_1, \mu_2) = \sum_{i=1}^{l} \lambda_i\theta(\mu_1^{iU}, \mu_2^{iU}),$$

where λ_i is positive real number satisfying $\sum\limits_{i=1}^{n} \lambda_i = 1$.

3. the midpoint hybrid monotonic inclusion measures for IVHFEs under the order relationship $\leq_4$:

$$I_{\leq_4 1}(\mu_1, \mu_2) = \bigwedge_{i=1}^{l} \theta[\tfrac{1}{2}(\mu_1^{iL} + \mu_1^{iU}), \tfrac{1}{2}(\mu_2^{iL} + \mu_2^{iU})],$$

$$I_{\leq_4 2}(\mu_1, \mu_2) = \sum_{i=1}^{l} \lambda_i\theta[\tfrac{1}{2}(\mu_1^{iL} + \mu_1^{iU}), \tfrac{1}{2}(\mu_2^{iL} + \mu_2^{iU})],$$

where λ_i is positive real number satisfying $\sum\limits_{i=1}^{n} \lambda_i = 1$.

Up to now, we have discussed the hybrid monotonic inclusion measure of IVHFEs in the extended environment. In the next section, we will define a new partial order $\leq_\vee$ between any two IVHFEs and discuss the hybrid monotonic inclusion measure on the partially ordered set.

Definition 3.7. Let $\mu_1 = \{\mu_1^1, \mu_1^2, \cdots, \mu_1^{l_1}\}$ and $\mu_2 = \{\mu_2^1, \mu_2^2, \cdots, \mu_2^{l_2}\}$ be two IVHFEs, an order $\leq_\vee$ between μ_1 and μ_2 is defined as follows:

$$\mu_1 \leq_\vee \mu_2 \text{ iff} \begin{cases} \mu_1^{iL} \leq \mu_2^{iL}, \mu_1^{iU} \leq \mu_2^{iU}, & i = 1, 2, \cdots, l_1, l_1 \leq l_2, \\ \mu_1^{(l_1-l_2+i)L} \leq \mu_2^{iL}, \mu_1^{(l_1-l_2+i)U} \leq \mu_2^{iU}, & i = 1, 2, \cdots, l_2, l_1 > l_2. \end{cases}$$

Theorem 3.8. *The order $\leq_\vee$ is a partial order between IVHFEs μ_1 and μ_2.*

Proof. (1). It is obvious that $\leq_\vee$ satisfies reflexivity.

(2). Suppose that $\mu_1 \leq_\vee \mu_2$ and $\mu_2 \leq_\vee \mu_1$.

(i) Suppose that $l_1 \leq l_2$. Then

$$\mu_1^{iL} \leq \mu_2^{iL}, \mu_1^{iU} \leq \mu_2^{iU}, i = 1, 2, \cdots, l_1$$

and

$$\mu_2^{(l_2-l_1+i)L} \leq \mu_1^{iL}, \mu_2^{(l_2-l_1+i)U} \leq \mu_1^{iU}, i = 1, 2, \cdots, l_1.$$

Because of the increasing order, we can get

$\mu_2^{iL} \leq \mu_2^{(l_2-l_1+i)L} \leq \mu_1^{iL} \leq \mu_2^{iL}$, $\mu_2^{iU} \leq \mu_2^{(l_2-l_1+i)U} \leq \mu_1^{iU} \leq \mu_2^{iU}$, $i = 1, 2, \cdots, l_1$.

Thus $l_1 = l_2$, $\mu_1^{iL} = \mu_2^{iL}$, $\mu_1^{iU} = \mu_2^{iU}$. So $\mu_1 = \mu_2$.

(ii) In the same way, we can prove $\mu_1 = \mu_2$, if $l_1 > l_2$.

We conclude that $\leq_\vee$ satisfies antisymmetry.

(3). Now we prove that $\leq_\vee$ is transitive, namely, if $\mu_1 \leq_\vee \mu_2$ and $\mu_2 \leq_\vee \mu_3$ for any three IVHFEs μ_1, μ_2 and μ_3, we need to testify $\mu_1 \leq_\vee \mu_3$.

(i) If $l_1 \leq l_2$, $\mu_1 \leq_\vee \mu_2$, then $\mu_1^{iL} \leq \mu_2^{iL}, \mu_1^{iU} \leq \mu_2^{iU}, i = 1, 2, \cdots, l_1$ based on the definition of $\leq_\vee$.

If $l_2 \leq l_3$, then we have $\mu_2^{iL} \leq \mu_3^{iL}, \mu_2^{iU} \leq \mu_3^{iU}, i = 1, 2, \cdots, l_2$. Thus we can get $\mu_1^{iL} \leq \mu_3^{iL}, \mu_1^{iU} \leq \mu_3^{iU}, i = 1, 2, \cdots, l_1$. So we conclude that $\mu_1 \leq_\vee \mu_3$.

If $l_2 > l_3$, then we have $\mu_2^{(l_2-l_3+i)L} \leq \mu_3^{iL}, \mu_2^{(l_2-l_3+i)U} \leq \mu_3^{iU}, i = 1, 2, \cdots, l_3$. We suppose that $l_1 \leq l_2$, $l_1 \leq l_3$. Then

$\mu_1^{iL} \leq \mu_2^{iL} \leq \mu_2^{(l_2-l_3+i)L} \leq \mu_3^{iL}$, $\mu_1^{iU} \leq \mu_2^{iU} \leq \mu_2^{(l_2-l_3+i)U} \leq \mu_3^{iU}$, $i = 1, 2, \cdots, l_1$.

Thus $\mu_1 \leq_\vee \mu_3$. Otherwise, $l_1 \leq l_2$, $l_1 > l_3$. So

$$\mu_1^{(l_1-l_3+i)L} \leq \mu_2^{(l_1-l_3+i)L} \leq \mu_2^{(l_2-l_3+i)L} \leq \mu_3^{iL},\ \mu_1^{(l_1-l_3+i)U} \leq \mu_2^{(l_1-l_3+i)U} \leq \mu_2^{(l_2-l_3+i)U} \leq \mu_3^{iU},\ i = 1, 2, \cdots, l_3.$$

Hence we can conclude that $\mu_1 \leq_\vee \mu_3$.

(ii) If $l_1 > l_2$, we can prove that $\leq_\vee$ is transitive, by the similar way. Thus, $\leq_\vee$ satisfies transitivity.

Therefore, $\leq_\vee$ is a partial order. □

Remark 3.9. By analyzing the above two partial order relationships $\leq_1$ and $\leq_\vee$, it is obvious that $\leq_\vee$ is a special case of $\leq_1$. Namely, if the two IVHFEs satisfy the partial order $\leq_\vee$, they must satisfy the partial order $\leq_1$, but if they satisfy the partial order $\leq_1$, they may not satisfy the partial order $\leq_\vee$. For example, $\mu_1 = \{[0.1, 0, 2], [0.3, 0.4], [0.5, 0.6], [0.7, 0.8]\}$ and $\mu_2 = \{[0.2, 0.3], [0.4, 0.5], [0.8, 0.9]\}$, μ_2 will be extended via add $[0.8, 0.9]$, then $\mu_1 \leq_1 \mu_2$, but $\mu_1 \leq_\vee \mu_2$ is not hold.

For any two IVHFSs A and B, $A \subseteq_\vee B$ iff $\mu_A(x) \leq \mu_B(x)$, $\forall x \in X$. The ordered set is denoted as (IVHFS, $\subseteq_\vee$). It is obvious that (IVHFS, $\subseteq_\vee$) is a partially ordered set by Theorem 3.2.

In the following part, the hybrid monotonic inclusion measure for IVHFEs under the partial order $\leq_\vee$ will be proposed.

Theorem 3.10. *Let θ be an implicator which satisfies hybrid monotonic and CP principle, and μ_1 and μ_2 be two IVHFEs. Then $I_{\leq_\vee}(\mu_1, \mu_2)$ is a hybrid monotonic inclusion measure for IVHFEs under the partial order $\leq_\vee$:*

$$(1)\ I_{\leq_{\vee 1}}(\mu_1,\mu_2)=\begin{cases}\bigwedge_{i=1}^{l_1}[\frac{1}{2}(\theta(\mu_1^{iL},\mu_2^{iL})+\theta(\mu_1^{iU},\mu_2^{iU}))], \\ \qquad i=1,2,\cdots,l_1, if\ l_1\leq l_2, \\ \bigwedge_{i=1}^{l_2}[\frac{1}{2}(\theta(\mu_1^{(l_1-l_2+i)L},\mu_2^{iL})+\theta(\mu_1^{(l_1-l_2+i)U},\mu_2^{iU}))], \\ \qquad i=1,2,\cdots,l_2, else.\end{cases}$$

$$(2)\ I_{\leq_{\vee 2}}(\mu_1,\mu_2)=\begin{cases}\frac{1}{2l_1}\sum_{i=1}^{l_1}[\theta(\mu_1^{iL},\mu_2^{iL})+\theta(\mu_1^{iU},\mu_2^{iU})], \\ \qquad i=1,2,\cdots,l_1, if\ l_1\leq l_2, \\ \frac{1}{2l_2}\sum_{i=1}^{l_2}[\theta(\mu_1^{(l_1-l_2+i)L},\mu_2^{iL})+\theta(\mu_1^{(l_1-l_2+i)U},\mu_2^{iU})], \\ \qquad i=1,2,\cdots,l_2, else.\end{cases}$$

$$(3)\ I_{\leq_{\vee 3}}(\mu_1,\mu_2)=\begin{cases}\frac{\sum_{i=1}^{l_1}\lambda_i[\theta(\mu_1^{iL},\mu_2^{iL})+\theta(\mu_1^{iU},\mu_2^{iU})]}{2\sum_{i=1}^{l_1}\lambda_i}, \\ \qquad \lambda_i>0, i=1,2,\cdots,l_1, if\ l_1\leq l_2, \\ \frac{\sum_{i=1}^{l_2}\lambda_i[\theta(\mu_1^{(l_1-l_2+i)L},\mu_2^{iL})+\theta(\mu_1^{(l_1-l_2+i)U},\mu_2^{iU})]}{2\sum_{i=1}^{l_2}\lambda_i}, \\ \qquad \lambda_i>0, i=1,2,\cdots,l_2, else.\end{cases}$$

Proof. We only prove that $I_{\leq_{\vee 1}}(\mu_1,\mu_2)$ satisfies the conditions (I1), (I2), (I3) and (I4) in Definition 3.1. Then it is a hybrid monotonic inclusion measure. The others can be proved in a similar way. It is obvious that $I_{\leq_{\vee 1}}(\mu_1,\mu_2)$ satisfies (I1), (I2) and (I3). Now we prove that $I_{\leq_{\vee 1}}(\mu_1,\mu_2)$ satisfies (I4), namely monotonicity.

Suppose that $\mu_1\leq_{\vee}\mu_2$, for all μ_3.

Case 1: If $l_1\leq l_2$, then $\mu_1^{iL}\leq\mu_2^{iL},\mu_1^{iU}\leq\mu_2^{iU}, i=1,2,\cdots,l_1$, for $l_1\leq l_2\leq l_3$, and $l_3\leq l_1\leq l_2$. Thus we can obtain directly that $I_{\leq_{\vee 1}}(\mu_3,\mu_1)\leq I_{\leq_{\vee 1}}(\mu_3,\mu_2)$ and $I_{\leq_{\vee 1}}(\mu_2,\mu_3)\leq I_{\leq_{\vee 1}}(\mu_1,\mu_3)$ by the definition of implicator and the increasing order of IVHFEs. If $l_1\leq l_3\leq l_2$, then we have

$$\begin{aligned}I_{\leq_{\vee 1}}(\mu_3,\mu_2)&=\bigwedge_{i=1}^{l_3}[\tfrac{1}{2}(\theta(\mu_3^{iL},\mu_2^{iL})+\theta(\mu_3^{iU},\mu_2^{iU}))]\\ &\geq\tfrac{1}{2}[\theta(\mu_3^{(l_3-l_1+1)L},\mu_2^{1L})+\theta(\mu_3^{(l_3-l_1+1)U},\mu_2^{1U})]\wedge\cdots\wedge\tfrac{1}{2}[\theta(\mu_3^{l_3L},\mu_2^{l_1L})+\\ &\quad\theta(\mu_3^{l_3U},\mu_2^{l_1U})]\wedge\tfrac{1}{2}[\theta(\mu_3^{l_3L},\mu_2^{(l_1+1)L})+\theta(\mu_3^{l_3U},\mu_2^{(l_1+1)U})]\wedge\cdots\\ &\quad\wedge\tfrac{1}{2}[\theta(\mu_3^{l_3L},\mu_2^{l_3L})+\theta(\mu_3^{l_3U},\mu_2^{l_3U})]\\ &\geq\tfrac{1}{2}[\theta(\mu_3^{(l_3-l_1+1)L},\mu_2^{1L})+\theta(\mu_3^{(l_3-l_1+1)U},\mu_2^{1U})]\wedge\cdots\wedge\tfrac{1}{2}[\theta(\mu_3^{l_3L},\mu_2^{l_1L})+\\ &\quad\theta(\mu_3^{l_3U},\mu_2^{l_1U})]\\ &=\bigwedge_{i=1}^{l_1}[\tfrac{1}{2}(\theta(\mu_3^{(l_3-l_1+i)L},\mu_1^{iL})+\theta(\mu_3^{(l_3-l_1+i)U},\mu_1^{iU}))]\\ &=I_{\leq_{\vee 1}}(\mu_3,\mu_1),\end{aligned}$$

$$I_{\leq_{\vee 1}}(\mu_2,\mu_3)=\bigwedge_{i=1}^{l_3}[\tfrac{1}{2}(\theta(\mu_2^{(l_2-l_3+i)L},\mu_3^{iL})+\theta(\mu_2^{(l_2-l_3+i)U},\mu_3^{iU}))]$$

$\leq \bigwedge_{i=1}^{l_1} [\frac{1}{2}(\theta(\mu_2^{(l_2-l_3+i)L}, \mu_3^{iL}) + \theta(\mu_2^{(l_2-l_3+i)U}, \mu_3^{iU}))]$
$\leq \bigwedge_{i=1}^{l_1} [\frac{1}{2}(\theta(\mu_2^{iL}, \mu_3^{iL}) + \theta(\mu_2^{iU}, \mu_3^{iU}))]$
$\leq \bigwedge_{i=1}^{l_1} [\frac{1}{2}(\theta(\mu_1^{iL}, \mu_3^{iL}) + \theta(\mu_1^{iU}, \mu_3^{iU}))]$
$= I_{\leq_{\vee 1}}(\mu_1, \mu_3)$.

Case 2: If $l_1 > l_2$, then $\mu_1^{(l_1-l_2+i)L} \leq \mu_2^{iL}, \mu_1^{(l_1-l_2+i)U} \leq \mu_2^{iU}, i = 1, 2, \cdots, l_1$, for $l_3 \geq l_1 > l_2$, and $l_1 > l_2 \geq l_3$. Thus we can obtain directly that $I_{\leq_{\vee 1}}(\mu_3, \mu_1) \leq I_{\leq_{\vee 1}}(\mu_3, \mu_2)$ and $I_{\leq_{\vee 1}}(\mu_2, \mu_3) \leq I_{\leq_{\vee 1}}(\mu_1, \mu_3)$ by the definition of implicator and the increasing order of IVHFEs. If $l_1 \geq l_3 > l_2$, then we have

$$\begin{aligned}
I_{\leq_{\vee 1}}(\mu_1, \mu_3) &= \bigwedge_{i=1}^{l_3} [\tfrac{1}{2}(\theta(\mu_1^{(l_1-l_3+i)L}, \mu_3^{iL}) + \theta(\mu_1^{(l_1-l_3+i)U}, \mu_3^{iU}))] \\
&\geq \tfrac{1}{2}[\theta(\mu_1^{(l_1-l_2+1)L}, \mu_3^{1L}) + \theta(\mu_1^{(l_1-l_2+1)U}, \mu_3^{1U})] \wedge \cdots \wedge \tfrac{1}{2}[\theta(\mu_1^{l_1L}, \mu_3^{l_2L}) + \\
&\quad \theta(\mu_1^{l_1U}, \mu_3^{l_2U})] \wedge \tfrac{1}{2}[\theta(\mu_1^{l_1L}, \mu_3^{(l_2+1)L}) + \theta(\mu_1^{l_1U}, \mu_3^{(l_2+1)U})] \wedge \cdots \\
&\quad \wedge \tfrac{1}{2}[\theta(\mu_1^{l_1L}, \mu_3^{l_3L}) + \theta(\mu_1^{l_1U}, \mu_3^{l_3U})] \\
&\geq \tfrac{1}{2}[\theta(\mu_2^{1L}, \mu_3^{1L}) + \theta(\mu_2^{1U}, \mu_3^{1U})] \wedge \cdots \wedge \tfrac{1}{2}[\theta(\mu_2^{l_2L}, \mu_3^{l_2L}) + \theta(\mu_2^{l_2U}, \mu_3^{l_2U})] \\
&= I_{\leq_{\vee 1}}(\mu_2, \mu_3), \\
I_{\leq_{\vee 1}}(\mu_3, \mu_1) &= \bigwedge_{i=1}^{l_3} [\tfrac{1}{2}(\theta(\mu_3^{iL}, \mu_1^{iL}) + \theta(\mu_3^{iU}, \mu_1^{iU}))] \\
&\leq \tfrac{1}{2}[\theta(\mu_3^{(l_3-l_2+1)L}, \mu_1^{(l_3-l_2+1)L}) + \theta(\mu_3^{(l_3-l_2+1)U}, \mu_1^{(l_3-l_2+1)U})] \wedge \cdots \\
&\qquad \wedge \tfrac{1}{2}[\theta(\mu_3^{l_3L}, \mu_1^{l_3L}) + \theta(\mu_3^{l_3U}, \mu_1^{l_3U})] \\
&\leq \tfrac{1}{2}[\theta(\mu_3^{(l_3-l_2+1)L}, \mu_1^{(l_1-l_2+1)L}) + \theta(\mu_3^{(l_3-l_2+1)U}, \mu_1^{(l_1-l_2+1)U})] \wedge \cdots \\
&\qquad \wedge \tfrac{1}{2}[\theta(\mu_3^{l_3L}, \mu_1^{l_1L}) + \theta(\mu_3^{l_3U}, \mu_1^{l_1U})] \\
&\leq \tfrac{1}{2}[\theta(\mu_3^{(l_3-l_2+1)L}, \mu_2^{1L}) + \theta(\mu_3^{(l_3-l_2+1)U}, \mu_2^{1U})] \wedge \cdots \\
&\qquad \wedge \tfrac{1}{2}[\theta(\mu_3^{l_3L}, \mu_2^{l_2L}) + \theta(\mu_3^{l_3U}, \mu_2^{l_2U})] \\
&= I_{\leq_{\vee 1}}(\mu_3, \mu_2).
\end{aligned}$$

So, $I_{\leq_{\vee 1}}(\mu_1, \mu_2)$ is a hybrid monotonic inclusion measure. □

The hybrid monotonic inclusion measure can also be constructed from existing hybrid monotonic inclusion measure.

Theorem 3.11. *Let $I'(\mu_1, \mu_2)$ and $I''(\mu_1, \mu_2)$ be hybrid monotonic inclusion measures for IVHFEs μ_1 and μ_2 via $\leq_\vee$. Then the following formulae are hybrid monotonic inclusion measures on (IVHFEs,$\leq_\vee$):*

(1) $I_{\leq_{\vee 4}}(\mu_1, \mu_2) = T(I'(\mu_1, \mu_2), I''(\mu_1, \mu_2))$, *$T$ is a t-norm.*
(2) $I_{\leq_{\vee 5}}(\mu_1, \mu_2) = T(I'(\mu_1, \mu_2), I'(\mu_2^c, \mu_1^c))$, *$T$ is a t-norm.*
(3) $I_{\leq_{\vee 6}}(\mu_1, \mu_2) = \alpha I'(\mu_1, \mu_2) + \beta I''(\mu_1, \mu_2)$, $\alpha + \beta = 1$.

Up to now, we have discussed the hybrid monotonic inclusion measure on (IVHFEs, $\leq_\vee$). Hybrid monotonic inclusion measure on (IVHFSs, $\subseteq_\vee$) can be constructed similarly by aggregation of all hybrid monotonic inclusion measures of IVHFEs, respectively. Now we present the constructive approaches to hybrid monotonic inclusion measure on (IVHFSs, $\subseteq_\vee$).

Theorem 3.12. *Let A, B be two IVHFSs on X and θ be an implicator which satisfies hybrid monotonic and CP principle. Then for $x_i \in X$, $|X| = n$, the hybrid monotonic inclusion measures $I(A, B)$ can be defined as follows:*

(1) $I_1(A,B) = \frac{1}{n}\sum_{i=1}^{n} I(\mu_A(x_i), \mu_B(x_i))$.

(2) $I_2(A,B) = \bigwedge_{i=1}^{n} I(\mu_A(x_i), \mu_B(x_i))$.

(3) $I_3(A,B) = \frac{\sum_{i=1}^{n} \alpha_i I(\mu_A(x_i),\mu_B(x_i))}{\sum_{i=1}^{n} \alpha_i}$, $\alpha_i > 0$.

Proof. This theorem can be proved directly. □

4. Similarity measure, distance and fuzzy entropy between IVHFSs in terms of the hybrid monotonic inclusion measure

The distance, the similarity measure and entropy for IVHFSs have been obtained under the extended environment in [7]. In this section, we construct the similarity measure, distance and fuzzy entropy for IVHFSs based on the hybrid monotonic inclusion measure under the unextended environment. First, the similarity measure and distance of IVHFSs are defined as given in Definition 4.1 and Definition 4.2 according to [7] and fuzzy entropy of IVHFSs are defined as given in Definition 4.3 according to [6].

Definition 4.1. Let A, B and C be IVHFSs. Then $S(A, B)$ is called a similarity measure for IVHFSs, if it possesses the following properties:
(S1) $0 \leq S(A, B) \leq 1$,
(S2) $S(A, B) = S_(B, A)$,
(S3) if $A = \{\overline{[0, 0]}\}$ or $A = \{\overline{[1, 1]}\}$, then $S(A, A^c) = 0$,
(S4) if $A = B$, then $S(A, B) = 1$,
(S5) if $A \subseteq_\vee B \subseteq_\vee C$, then $S(A, C) \leq S(A, B)$, $S(A, C) \leq S(B, C)$,
where $A = \{\overline{[0, 0]}\}$ denotes empty IVHFS which its any IVHFE $\mu_A(x) = \{[0, 0]\}$ and $A = \{\overline{[1, 1]}\}$ denotes full IVHFS which its any IVHFE $\mu_A(x) = \{[1, 1]\}$.

Definition 4.2. Let A, B and C be IVHFSs. Then $D(A, B)$ is called a distance measure for IVHFSs, if it possesses the following properties:
(d1) $0 \leq D_(A, B) \leq 1$,
(d2) $D(A, B) = D(B, A)$,
(d3) if $A = \{\overline{[0, 0]}\}$ or $A = \{\overline{[1, 1]}\}$, then $D(A, A^c) = 1$,
(d4) if $A = B$, then $D(A, B) = 0$,
(d5) if $A \subseteq_\vee B \subseteq_\vee C$, then $D(A, B) \leq D(A, C)$, $D(B, C) \leq D(A, C)$.

Definition 4.3. Let A and B be IVHFSs. Then E is called a fuzzy entropy for IVHFSs if it possesses the following properties:
(E1) $0 \leq E(A) \leq 1$,
(E2) if $A = \{\overline{[0, 0]}\}$ or $A = \{\overline{[1, 1]}\}$, then $E(A) = 0$,
(E3) if $A = \{\overline{[\frac{1}{2}, \frac{1}{2}]}\}$, then $E(A) = 1$,
(E4) $E(A) = E(A^c)$,

(E5) if $B \subseteq_\vee A$, when $\{\overline{[\frac{1}{2}, \frac{1}{2}]}\} \subseteq_\vee B$ and $A \subseteq_\vee B \subseteq_\vee \{\overline{[\frac{1}{2}, \frac{1}{2}]}\}$, then $E(A) \leq E(B)$, where $A = \{\overline{[\frac{1}{2}, \frac{1}{2}]}\}$ denotes its any IVHFE $\mu_A(x) = \{[\frac{1}{2}, \frac{1}{2}]\}$.

Now, we construct the similarity measure for IVHFSs by the following formulae.

Theorem 4.4. *Let A and B be IVHFSs, $I_{\leq_\vee}$ be a hybrid monotonic inclusion measure. Then the following functions are similarity measures:*
(1) $S_1(A, B) = I_{\leq_\vee}(A \bigcup B, A \bigcap B)$,
(2) $S_2(A, B) = T(I_{\leq_\vee}(A, B), I_{\leq_\vee}(B, A)$, *where T is a t-norm.*

Proof. It is easily proven from the definition of hybrid monotonic inclusion measure and the properties of t-norm. □

The distance measure between two IVHFSs is the measure that describes the difference between them. It is easy to see that the distance measure is complementary to similarity measure. It is referred to in [7] for details. So it can be obtained straightforwardly by the similarity measure. Now the fuzzy entropy is constructed by the similarity measure of the aforementioned presents.

Theorem 4.5. *Let $S(A, B)$ be a similarity measure between IVHFSs A and B. Then fuzzy entropy for IVHFSs A is defined as follows:*
(1) $E_1(A) = S(A \bigcap A^c, A \bigcup A^c)$,
(2) $E_2(A) = S(A, A^c)$.

Proof. We only prove that E_1 satisfies the conditions (E1)-(E5) in Definition 4.3, then it is a fuzzy entropy, E_2 can be proved in a similar way. It is obvious that E_1 satisfies (E1), (E2) and (E3). Now we prove that E_1 satisfies (E4) and (E5).

(E4) $E_1(A) = S(A \bigcap A^c, A \bigcup A^c) = S(A^c \bigcap A, A^c \bigcup A) = E_1(A^c)$.

(E5) Case 1: if $A \subseteq_\vee B$, when $B \subseteq_\vee \{\overline{[\frac{1}{2}, \frac{1}{2}]}\}$, then $A \subseteq_\vee B \subseteq_\vee B^c \subseteq_\vee A^c$. Thus $A \bigcap A^c \subseteq_\vee B \bigcap B^c \subseteq_\vee B \bigcup B^c \subseteq_\vee A \bigcup A^c$. So,

$E_1(A) = S(A \bigcap A^c, A \bigcup A^c) \leq S(B \bigcap B^c, A \bigcup A^c) \leq S(B \bigcap B^c, B \bigcup B^c) = E_1(B)$.

Case 2: if $A \subseteq_\vee B$, when $\{\overline{[\frac{1}{2}, \frac{1}{2}]}\} \subseteq_\vee B$, then it can be proved by the similarly way. □

5. Conclusions

In this paper, we presented some constructive approaches for hybrid monotonic inclusion measure between any two IVHFEs under some different order relationships in the extended environment. Then a novel partial order for IVHFEs have been defined and a series of hybrid monotonic inclusion measures for IVHFEs and IVHFSs have been proposed. Finally, similarity measure, distance and fuzzy entropy for IVHFSs have been constructed by the hybrid monotonic inclusion measure.

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