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Some fixed fuzzy point results using Hausdorff metric in fuzzy metric spaces

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Abstract. In this paper, we obtain coupled fixed fuzzy point theorem for a fuzzy mapping in Hausdorff fuzzy metric space and using it, we obtain a common fixed fuzzy point for a hybrid fuzzy pair. Also we give an auxiliary example to support our main theorem.

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1. Introduction

The theory of fuzzy sets was introduced by Zadeh [21] in 1965. Since then, to use this concept in topology and analysis, many authors have developed the theory of fuzzy sets and applications. Especially, Deng [3], Erceg [5], Kaleva and Seikkala [11], Kramosil and Michalek [12], Georege and Veeramani [7] have introduced the concept of fuzzy metric space in different ways. Grabiec [8] extended classical fixed point theorems of Banach and Edelstein to complete and compact fuzzy metric spaces. After that, fuzzy metric spaces became one of the most popular area for study of fixed point theory. In recent years, many mathematicians such as [4, 13, 15] etc., established several fixed point theorems in fuzzy metric spaces.

In 2004, Rodríguez-López and Romaguera [14] introduced Hausdorff fuzzy metric on the set of the non-empty compact subsets of a given fuzzy metric space. Later, several authors proved some fixed point theorems for multivalued maps in fuzzy metric spaces. Heilpern [10] first introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy contraction mappings. Many authors extended the results of Heilpern and proved several fixed fuzzy point theorems (see [6, 17, 19, 20]). Recently, Phiangsungnoen et al. [16] have studied fuzzy fixed point theory for fuzzy mappings in Hausdorff fuzzy metric spaces and then, Abbas et al. [1] have obtained fixed fuzzy points of fuzzy mappings in Hausdorff fuzzy metric spaces under generalized contractive conditions.

On the other hand, Bhaskar and Lakshmikantham [2] introduced the notion of a coupled fixed point in partially ordered metric spaces, also discussed some problems of the uniqueness of a coupled fixed point, and applied their results to the problems of the existence and uniqueness of a solution for the periodic boundary value problems.

In this paper, we establish the existence of coupled fixed fuzzy point theorem for fuzzy mappings in Hausdorff fuzzy metric space and give an auxiliary example to support our theorem. As an application, we obtain a common fixed fuzzy point for a hybrid fuzzy pair. Our results can be applied in theoretical computer science.

2. Preliminaries

In the sequel, we need the following.

Definition 2.1 ([18])**.** A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is called a t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative,
(ii) $a * 1 = a$ for every $a \in [0,1]$,
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for $a, b, c, d \in [0,1]$.

If in addition, $*$ is continuous, then $*$ is called a continuous t-norm.

Typical examples of a continuous t-norms are $a*b = \min\{a,b\}$, $a*b = ab/\max\{a,b,\lambda\}$ for $0 < \lambda < 1$, $a * b = ab$, $a * b = \max\{a + b - 1, 0\}$.

Definition 2.2 ([9])**.** For each $a \in [0,1]$, the sequence $\{*^n a\}_{n=1}^{\infty}$ is defined by
$$*^1 a = a \text{ and } *^n a = \left(*^{n-1}a\right) * a.$$
A t-norm $*$ is said to be of H-type, if the sequence of functions $\{*^n a\}_{n=1}^{\infty}$ is equicontinuous at $a = 1$, i.e., for any $\lambda \in (0,1)$, there exist $\delta(\lambda) \in (0,1)$ such that $a > 1 - \delta$ implies $*^n a > 1 - \lambda$, for all $n \in \mathbb{N}$.

The concept of fuzzy metric space is defined by George and Veeramani as follows.

Definition 2.3 ([7])**.** A fuzzy metric space is a triple $(X, M, *)$, where X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0,\infty)$, satisfying the following conditions: for all $x, y, z \in X$, $s, t > 0$,

(GV_1) $M(x, y, t) > 0$,
(GV_2) $M(x, y, t) = 1$ iff $x = y$,
(GV_3) $M(x, y, t) = M(y, x, t)$,
(GV_4) $M(x, y, \cdot) : (0,\infty) \to (0,1]$ is continuous,
(GV_5) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$.

It is known that M is continuous on $X^2 \times (0,\infty)$ and $M(x, y, \cdot)$ is non-decreasing on $(0,\infty)$.

Definition 2.4 ([7])**.** Let $(X, M, *)$ be a fuzzy metric space.

(i) A sequence $\{x_n\}$ is said to be convergent to a point $x \in X$, if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

(ii) A sequence $\{x_n\}$ is said to be a Cauchy sequence, if for each $0 < \varepsilon < 1$ and $t > 0$, there exist $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$.

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

(iv) A subset $A \subseteq X$ is said to be closed, if for each convergent sequence $\{x_n\}$ with $x_n \in A$ and $x_n \to x$, we have $x \in A$.

(v) A subset $A \subseteq X$ is said to be compact, if each sequence in A has a convergent subsequence to a point in A.

Rodríguez-López and Romaguera [14] introduced Hausdorff fuzzy metric of a given fuzzy metric space $(X, M, *)$ on $K_M(X)$, where $K_M(X)$ denotes the family of all non-empty compact subsets of X.

Definition 2.5 ([14])**.** Let $(X, M, *)$ be a fuzzy metric space. For $A, B \in K_M(X)$, $x \in X$ and $t > 0$, define

$$M(x, A, t) = \sup_{a \in A} M(x, a, t)$$

and

$$H_M(A, B, t) = \min\left\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t)\right\}.$$

The 3-tuple $(K_M(X), H_M, *)$ is a fuzzy metric space and the fuzzy metric H_M is called Hausdorff fuzzy metric induced by the fuzzy metric M.

Lemma 2.6. *Let X be a non-empty set and $g : X \to X$. Then there exist a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one.*

Now, we recall some definitions and properties of fixed fuzzy point. Let X be an arbitrary non-empty set. A fuzzy set in X is a function with domain X and values in $[0, 1]$. A fuzzy set A in X is characterized by a mapping $A : X \to [0, 1]$. $A(x)$ is called the grade of membership of x in A.

Let A be a fuzzy set in X. If X endowed with a topology, for $\alpha \in (0, 1]$, the α-level set of A is denoted by $[A]_\alpha$ and is defined as follows:

$$\begin{aligned} [A]_\alpha &= \{x \in X : A(x) \geq \alpha\}, \\ [A]_0 &= \overline{\{x \in X : A(x) > 0\}}, \end{aligned}$$

where $\overline{B}$ denotes the closure of B in X. Corresponding to each $\alpha \in (0, 1]$ and $x \in X$, the fuzzy point $(x)_\alpha$ of X is a fuzzy set $(x)_\alpha : X \to [0, 1]$ given by

$$(x)_\alpha(y) = \begin{cases} \alpha, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.7 ([19])**.** Let X be a non-empty set $I = [0, 1]$ and I^X stands for the collection of all fuzzy sets in X. A mapping $F : X \to I^X$ is said to be fuzzy mapping.

Definition 2.8. A fuzzy point x_α in X is called a fixed point of a fuzzy mapping F, if $x_\alpha \subset Fx$, i.e., $\alpha \leq Fx(x)$ or $x \in [Fx]_\alpha$, i.e., the fixed degree of x in F is at least α.

If $x_1 \subset Fx$, then x is called a fixed point of the fuzzy mapping F.

In 2014, Phiangsungnoen et al. [16] proved the following theorem.

Theorem 2.9. *Let $(X, M, *)$ be a complete fuzzy metric space and $\alpha : X \to (0, 1]$ be a mapping such that $[Tx]_{\alpha(x)}$ is a non-empty compact subset of X for all $x \in X$. Suppose that $T : X \to I^X$ is a fuzzy mapping such that*

$$H_M\left([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}, kt\right) \geq M(x, y, t)$$

for all $t > 0$, where $k \in (0,1)$. If there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha(x_0)}$ such that

$$\lim_{n\to\infty} *_{i=n}^{\infty} M\left(x_0, x_1, th^i\right) = 1$$

for all $t > 0$ and $h > 1$, then T has a fixed fuzzy point.

Let S be the class of functions $\psi : (0,1] \to (0,1]$ satisfying $\limsup_{n\to\infty} \psi(x_n) < 1$ whenever $\{x_n\} \subseteq (0,1]$ is non-decreasing or $\lim_{n\to\infty} x_n = 1$. Define

$$W_\alpha(X) = \left\{A \in I^X : A_\alpha \text{ is non-empty and compact}\right\}.$$

For $A, B \in W_\alpha(X)$ and $\alpha \in (0,1]$, define

$$\begin{cases} M_\alpha(x,B,t) = \sup_{y\in B_\alpha} M(x,y,t), \\ H_{M_\alpha}(A,B,t) = H_M(A_\alpha, B_\alpha, t). \end{cases}$$

Note that M_α is a non-increasing function of α and H_{M_α} is the Hausdorff fuzzy metric induced by the fuzzy metric M on $W_\alpha(X)$.

Definition 2.10 ([1])**.** Let Y be an arbitrary subset of a fuzzy metric space $(X, M, *)$. A mapping $F : Y \to W_\alpha(X)$ is called a fuzzy mapping over the set Y, i.e., $Fy \in W_\alpha(X)$, for each y in Y.

Lemma 2.11 ([1])**.** *Let $(X, M, *)$ be a fuzzy metric space, $x, y \in X$ and $A, B \in W_\alpha(X)$. Then the following conditions hold:*

(1) *for each $x \in X$, $B \in W_\alpha(X)$ and $t > 0$, there is $(b_0)_\alpha \subset B$ such that*

$$M_\alpha(x,B,t) = M(x, b_0, t),$$

(2) *$M_\alpha(x,A,t) = 1$ implies that $(x)_\alpha \subset A$,*

(3) *for $(y_x)_\alpha \subset B$ with $M(x,y_x,t) = M_\alpha(x,B,t)$, we have $M_\alpha(x,C,t+s) \geq M_\alpha(x,B,t) * M_\alpha(y_x,C,s)$.*

Abbas et al. [1] generalized Theorem 2.9 as follow:

Theorem 2.12 ([1])**.** *Let $(X, M, *)$ be a complete fuzzy metric space and $F : X \to W_\alpha(X)$ a fuzzy mapping. Suppose that, for all $x, y \in X$ and $t > 0$, the following condition holds:*

$$H_{M_\alpha}(Fx, Fy, \psi(M(x,y,t))t) \geq M(x,y,t),$$

where $\psi \in S$. Then F has a fixed fuzzy point.

3. Fixed fuzzy point theorems

In this section we prove a coupled fixed fuzzy point theorem for fuzzy mappings and a common coupled fixed fuzzy point theorem for hybrid fuzzy pair in fuzzy metric spaces. Let $g : X \to X$ be a mapping and $F : X \times X \to W_\alpha(X)$ be a fuzzy mapping and denote $[F(X\times X)]_\alpha = \bigcup_{(x,y)\in X\times X} [F(x,y)]_\alpha$. We call (g, F) a hybrid fuzzy pair.

Definition 3.1. An element $\left((x)_\alpha, (y)_\alpha\right) \in X \times X$ is called

(i) a coupled fixed fuzzy point of the mapping $F : X \times X \to W_\alpha(X)$, if
$(x)_\alpha \subset F(x, y)$, $(y)_\alpha \subset F(y, x)$, i.e., $x \in [F(x, y)]_\alpha$, $y \in [F(y, x)]_\alpha$,

(ii) a coupled coincidence fuzzy point of the mappings $F : X \times X \to W_\alpha(X)$ and $g : X \to X$, if $(gx)_\alpha \subset F(x, y)$, $(gy)_\alpha \subset F(y, x)$, i.e., $gx \in [F(x, y)]_\alpha$, $gy \in [F(y, x)]_\alpha$,

(iii) a common coupled fixed fuzzy point of the mappings $F : X \times X \to W_\alpha(X)$ and $g : X \to X$, if $(x)_\alpha = (gx)_\alpha \subset F(x, y)$, $(y)_\alpha = (gy)_\alpha \subset F(y, x)$, i.e., $x = gx \in [F(x, y)]_\alpha$, $y = gy \in [F(y, x)]_\alpha$.

Definition 3.2. An element $(x)_\alpha \in X$ is called a common fixed fuzzy point of the mappings $F : X \times X \to W_\alpha(X)$ and $g : X \to X$, if $(x)_\alpha = (gx)_\alpha \subset F(x, x)$, i.e., $x = gx \in [F(x, x)]_\alpha$.

Definition 3.3. The mappings $F : X \times X \to W_\alpha(X)$ and $g : X \to X$ are called weakly fuzzy compatible maps, if $g\left[F(x, y)\right]_\alpha \subseteq \left[F(gx, gy)\right]_\alpha$, $g\left[F(y, x)\right]_\alpha \subseteq \left[F(gy, gx)\right]_\alpha$, whenever $gx \in [F(x, y)]_\alpha$, $gy \in [F(y, x)]_\alpha$.

Definition 3.4. Let $F : X \times X \to W_\alpha(X)$ and $g : X \to X$. The mapping g is called F-weakly commuting at some point $(x, y) \in X \times X$, if $\left(g^2x\right)_\alpha \subset F(gx, gy)$ and $\left(g^2y\right)_\alpha \subset F(gy, gx)$, i.e., $g^2x \in \left[F(gx, gy)\right]_\alpha$ and $g^2y \subset \left[F(gy, gx)\right]_\alpha$.

Theorem 3.5. *Let $(X, M, *)$ be a complete fuzzy metric space such that $*$ is a t-norm of H-type and $a * b \geq ab$ for all $a, b \in [0, 1]$ and M is with $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$. Let $F : X \times X \to W_\alpha(X)$ fuzzy mapping such that*

$$H_{M_\alpha}(F(x, y), F(u, v), kt) \geq [M(x, u, t)]^{1/2} * [M(y, v, t)]^{1/2} \tag{3.1}$$

for all $x, y, u, v \in X$, where $0 < k < 1$. Then F has a fixed fuzzy point.

Proof. Let x_0, y_0 are given points in X. Since $\left[F(x_0, y_0)\right]_\alpha$ and $\left[F(y_0, x_0)\right]_\alpha$ are non-empty we can choose $x_1, y_1 \in X$ such that $x_1 \in \left[F(x_0, y_0)\right]_\alpha$ and $y_1 \in \left[F(y_0, x_0)\right]_\alpha$. Since $\left[F(x_1, y_1)\right]_\alpha$ is compact, from Lemma 2.11, we can choose $x_2 \in \left[F(x_1, y_1)\right]_\alpha$ such that

$$\begin{aligned} M(x_1, x_2, kt) &= \sup_{z\in[F(x_1,y_1)]_\alpha} M(x_1, z, kt) \\ &\geq H_{M_\alpha}\left(F(x_0, y_0), F(x_1, y_1), kt\right) \\ &\geq \left[M(x_0, x_1, t)\right]^{1/2} * \left[M(y_0, y_1, t)\right]^{1/2}. \end{aligned}$$

Similarly, since $\left[F(y_1, x_1)\right]_\alpha$ is compact, from Lemma 2.11, we can choose $y_2 \in \left[F(y_1, x_1)\right]_\alpha$ such that

$$\begin{aligned} M(y_1, y_2, kt) &= \sup_{z\in[F(y_1,x_1)]_\alpha} M(y_1, z, kt) \\ &\geq H_{M_\alpha}\left(F(y_0, x_0), F(y_1, x_1), kt\right) \\ &\geq \left[M(y_0, y_1, t)\right]^{1/2} * \left[M(x_0, x_1, t)\right]^{1/2}. \end{aligned}$$

Continuing this way, we can obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$x_{n+1} \in \left[F(x_n, y_n)\right]_\alpha, \; y_{n+1} \in \left[F(y_n, x_n)\right]_\alpha \text{ for all } n \geq 0$$

and

$$M(x_n, x_{n+1}, kt) \geq \left[M(x_{n-1}, x_n, t)\right]^{1/2} * \left[M(y_{n-1}, y_n, t)\right]^{1/2}, \tag{3.2}$$

$$M\left(y_n, y_{n+1}, kt\right) \geq \left[M\left(y_{n-1}, y_n, t\right)\right]^{1/2} * \left[M(x_{n-1}, x_n, t)\right]^{1/2}. \tag{3.3}$$

We prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

Let $\delta_n\left(t\right) = \left[M\left(x_n, x_{n+1}, t\right)\right]^{1/2} * \left[M(y_n, y_{n+1}, t)\right]^{1/2}$. Then, by operating t-norm $*$ on (3.2) and (3.3), from $a * b \geq ab$ we obtain $\delta_n\left(kt\right) \geq \delta_{n-1}\left(t\right)$. This implies that

$$\delta_n\left(t\right) \geq \delta_{n-1}\left(\frac{t}{k}\right) \geq \cdots \geq \delta_0\left(\frac{t}{k^n}\right).$$

Since $\lim_{n\to\infty} \delta_0\left(\frac{t}{k^n}\right) = 1$, for all $t > 0$, we have

$$\lim_{n\to\infty} \delta_n\left(t\right) = 1 \text{ for all } t > 0. \tag{3.4}$$

We show that, for any $p \in \mathbb{N}$

$$M\left(x_n, x_{n+p}, t\right) \geq *^p \delta_{n-1}\left(t - kt\right) \text{ and } M\left(y_n, y_{n+p}, t\right) \geq *^p \delta_{n-1}\left(t - kt\right) \tag{3.5}$$

It is obvious for $p = 1$. Assume that (3.5) holds for some p. By (3.2), we have $M\left(x_n, x_{n+1}, t\right) \geq M\left(x_n, x_{n+1}, kt\right) \geq \delta_{n-1}\left(t\right)$, and so $M\left(x_n, x_{n+1}, t - kt\right) \geq \delta_{n-1}\left(t - kt\right)$. From (3.1), (3.5) and $a * b \geq ab$, we have

$$M\left(x_{n+1}, x_{n+p+1}, kt\right) \geq \left[M\left(x_n, x_{n+p}, t\right)\right]^{1/2} * \left[M\left(y_n, y_{n+p}, t\right)\right]^{1/2} \geq *^p \delta_{n-1}\left(t - kt\right).$$

Then by the (GV_5), we have

$$\begin{aligned} M\left(x_n, x_{n+p+1}, t\right) &= M\left(x_n, x_{n+p+1}, t - kt + kt\right) \\ &\geq M\left(x_n, x_{n+1}, t - kt\right) * M\left(x_{n+1}, x_{n+p+1}, kt\right) \\ &\geq \delta_{n-1}\left(t - kt\right) * \left(*^p \delta_{n-1}\left(t - kt\right)\right) \\ &= *^{p+1} \delta_{n-1}\left(t - kt\right). \end{aligned}$$

Similarly, we have $M\left(y_n, y_{n+p+1}, t\right) \geq *^{p+1} \delta_{n-1}\left(t - kt\right)$. Then (3.5) holds, for all p. Suppose that $t > 0$ and $\varepsilon \in (0, 1]$ are given. By hypothesis, $*$ is a t-norm of H-type, there exist $\eta > 0$ such that

$$*^p\left(s\right) > 1 - \varepsilon, \text{ for all } s \in (1 - \eta, 1] \text{ and for all p.} \tag{3.6}$$

By (3.4), there exist n_0 such that $\delta_{n-1}\left(t - kt\right) > 1 - \eta$, for all $n \geq n_0$. Thus, from (3.5) and (3.6), we get $M\left(x_n, x_{n+p}, t\right) > 1 - \varepsilon$ and $M\left(y_n, y_{n+p}, t\right) > 1 - \varepsilon$, for all $n \geq n_0$ and for all p. So $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is complete, $x, y \in X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Now we show that $x \in \left[F\left(x, y\right)\right]_\alpha$ and $y \in \left[F\left(y, x\right)\right]_\alpha$. We write $x = x_n$, $y = y_n$, $u = x$, $v = y$ in (3.1)

$$H_{M_\alpha}(F(x_n, y_n), F(x, y), kt) \geq \left[M(x_n, x, t)\right]^{1/2} * \left[M(y_n, y, t)\right]^{1/2}.$$

On taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} H_{M_\alpha}(F(x_n, y_n), F(x, y), kt) = 1$$

which implies that

$$\lim_{n\to\infty} \sup_{x' \in [F(x,y)]_\alpha} M(x_{n+1}, x', t) = 1.$$

Hence there exist a sequence $\{x'_n\}$ with $x'_n \in \left[F(x, y)\right]_\alpha$ such that

$$\lim_{n\to\infty} M\left(x_n, x'_n, t\right) = 1,$$

for each $t > 0$. From (GV_5) , we have

$$M\left(x_n', x, t\right) \geq M\left(x_n', x_n, t/2\right) * M\left(x_n, x, t/2\right)$$

and let $n \to \infty$, we obtain

$$\lim_{n \to \infty} M\left(x_n', x, t\right) = 1,$$

i.e., $\lim_{n\to\infty} x_n' = x$. Since $[F(x,y)]_\alpha$ is compact, $x \in [F(x,y)]_\alpha$. By the same way, we can show that $y \in [F(y,x)]_\alpha$. Therefore (x,y) is a coupled fixed fuzzy point of F. □

Example 3.6. Let $X = \{1,2,3\}$, $a * b = ab$, for all $a, b \in [0,1]$ and

$$M(x,y,t) = \begin{cases} \frac{x}{y}, & x \leq y \\ \frac{y}{x}, & y \leq x, \end{cases}$$

for all $x, y \in X$, $t > 0$.

Let $\alpha \in \left(0, \frac{1}{3}\right)$, define a fuzzy mapping $F : X \times X \to W_\alpha(X)$ as follows:

· for $x = 1$, $y \in X$,

$$F(x,y)(z) = \begin{cases} \alpha, & z = 1 \\ 0, & z = 2 \\ \frac{\alpha}{2}, & z = 3 \end{cases}$$

· for $x = 2$, $y \in X$,

$$F(x,y)(z) = \begin{cases} \frac{\alpha}{3}, & z = 1 \\ \alpha, & z = 2 \\ 0, & z = 3 \end{cases}$$

· for $x = 3$, $y \in X$,

$$F(x,y)(z) = \begin{cases} 2\alpha, & z = 1 \\ \frac{\alpha}{3}, & z = 2 \\ 0, & z = 3. \end{cases}$$

Note that $[F(x,y)]_\alpha = \{1\}$, for $x = 1$, $y \in X$; $[F(x,y)]_\alpha = \{2\}$, for $x = 2$, $y \in X$; $[F(x,y)]_\alpha = \{1\}$ for $x = 3$, $y \in X$. Thus $x = u = 1$, $y, v \in X$, $x = u = 2$, $y, v \in X$, and $x = u = 3$, $y, v \in X$, we have

$$H_{M_\alpha}(F(x,y), F(u,v), kt) = 1.$$

Also we consider the following cases:

Case 1: If $x = 1$, $u = 2$, $y, v \in X$, we obtain

$$H_{M_\alpha}(F(x,y), F(u,v), kt) = \frac{1}{2} \geq [M(x,u,t)]^{1/2} * [M(y,v,t)]^{1/2}.$$

Case 2 : If $x = 1$, $u = 3$, $y, v \in X$, we obtain

$$H_{M_\alpha}(F(x,y), F(u,v), kt) = \frac{1}{3} \geq [M(x,u,t)]^{1/2} * [M(y,v,t)]^{1/2}.$$

Case 3: If $x = 2$, $u = 3$, $y, v \in X$, we obtain

$$H_{M_\alpha}(F(x,y), F(u,v), kt) = \frac{1}{2} \geq [M(x,u,t)]^{1/2} * [M(y,v,t)]^{1/2}.$$

Consequently, for each $k \in (0,1)$, $x, y, u, v \in X$ and $t > 0$, we have

$$H_{M_\alpha}(F(x,y), F(u,v), kt) \geq [M(x,u,t)]^{1/2} * [M(y,v,t)]^{1/2}.$$

Thus all conditions of Theorem 3.5 are satisfied. So $(1, 2)$ is a coupled fixed fuzzy point of F.

As an application of the Theorem 3.5, we obtain a coupled coincidence and common fixed fuzzy point theorem for a hybrid fuzzy pair. We denote the set of coupled coincidence points of mappings F and g by $C_\alpha(F, g)$. Note that if $(x, y) \in C_\alpha(F, g)$, then (y, x) is also in $C_\alpha(F, g)$.

Theorem 3.7. *Let $(X, M, *)$ be a complete fuzzy metric space such that $*$ is a t-norm of H-type and $a * b \geq ab$, for all $a, b \in [0, 1]$ and M is with $\lim_{t\to\infty} M(x, y, t) = 1$, for all $x, y \in X$ and (g, F) a hybrid fuzzy pair such that $[F(X \times X)]_\alpha \subset g(X)$. Suppose that, for all $x, y \in X$, $0 < k < 1$ and $t > 0$, the following condition holds:*

$$H_{M_\alpha}(F(x, y), F(u, v), kt) \geq [M(gx, gu, t)]^{1/2} * [M(gy, gv, t)]^{1/2}. \tag{3.7}$$

Then F and g have a coupled coincidence fuzzy point. Moreover F and g have a coupled common fixed point if one of the following conditions holds:

(1) F and g are weakly fuzzy compatible, $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$, for some $(x, y) \in C_\alpha(F, g)$, $u, v \in X$, and g is continuous at uand v,

(2) g is F-weakly commuting, for some$(x, y) \in C_\alpha(F, g)$, gx and gy are fixed points of g, i.e,, $g^2x = gx$ and $g^2y = gy$,

(3) g is continuous at x, y, for some $(x, y) \in C_\alpha(F, g)$ and for some $u, v \in X$, $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$.

Proof. By Lemma 2.6, there exist $E \subseteq X$ such that $g : E \to X$ is one-to-one and $g(E) = g(X)$. Now, define a mapping $\Lambda : g(E) \times g(E) \to W_\alpha(X)$ by

$$\Lambda(gx, gy) = F(x, y) \text{ for all } gx, gy \in g(E).$$

Since g is one-one on E, Λ is well defined. Since

$$\begin{aligned} H_{M_\alpha}(\Lambda(gx, gy), \Lambda(gu, gv), kt) &= H_{M_\alpha}(F(x, y), F(u, v), kt) \\ &\geq [M(gx, gu, t)]^{1/2} * [M(gy, gv, t)]^{1/2}, \end{aligned}$$

the mapping Λ satisfies (3.1) and thus all conditions of Theorem 3.5. By using Theorem 3.5 with a mapping Λ, it follows that Λ has a coupled fixed fuzzy point $(u, v) \in g(E) \times g(E)$ such that $u \in [\Lambda(u, v)]_\alpha$ and $v \in [\Lambda(v, u)]_\alpha$. Since $[F(X \times X)]_\alpha \subset g(X)$, there exist $u', v' \in X$ such that $(gu')_\alpha = (u)_\alpha$ and $(gv')_\alpha = (v)_\alpha$. So $(gu')_\alpha \subset \Lambda(gu', gv') = F(u', v')$ and $(gv')_\alpha \subset \Lambda(gv', gu') = F(v', u')$, i.e., $gu' \in [\Lambda(gu', gv')]_\alpha = [F(u', v')]_\alpha$ and $gv' \in [\Lambda(gv', gu')]_\alpha = [F(v', u')]_\alpha$. This implies that $(u', v') \in X \times X$ is a coupled coincidence fuzzy point of the mappings F and g. Hence $C_\alpha(F, g)$ is nonempty.

Suppose that (1) holds. Then for some $(x, y) \in C_\alpha(F, g)$,

$$\lim_{n\to\infty} g^n x = u \text{ and } \lim_{n\to\infty} g^n y = v,$$

where $u, v \in X$. Since g is continuous at u and v, we have that $gu = u$ and $gv = v$. As F and g are weakly fuzzy compatible $(g^n x, g^n y) \in C_\alpha(F, g)$, for all $n \geq 1$,i.e., for all $n \geq 1$, we have $(g^n x)_\alpha \subset F(g^{n-1}x, g^{n-1}y)$ and $(g^n y)_\alpha \subset F(g^{n-1}y, g^{n-1}x)$.

Using (3.7), we obtain

$$\begin{aligned} M_\alpha\left(gu, F\left(u,v\right),t\right) &\geq M\left(gu, g^n x, \frac{t}{2}\right) * M_\alpha\left(g^n x, F\left(u,v\right), \frac{t}{2}\right) \\ &\geq M\left(gu, g^n x, \frac{t}{2}\right) * H_{M_\alpha}\left(F\left(g^{n-1}x, g^{n-1}y\right), F\left(u,v\right), \frac{t}{2}\right) \\ &\geq M\left(gu, g^n x, \frac{t}{2}\right) * H_{M_\alpha}\left(F\left(g^{n-1}x, g^{n-1}y\right), F\left(u,v\right), k\frac{t}{2}\right) \\ &\geq M\left(gu, g^n x, \frac{t}{2}\right) * \left[M\left(g^n x, gu, \frac{t}{2}\right)\right]^{1/2} * \left[M\left(g^n y, gv, \frac{t}{2}\right)\right]^{1/2}. \end{aligned}$$

On taking limit as $n \to \infty$, we have

$$M_\alpha\left(gu, F\left(u,v\right),t\right) \geq 1 * 1 * 1.$$

This implies that $gu \in \left[F\left(u,v\right)\right]_\alpha$ and $\left(u\right)_\alpha = \left(gu\right)_\alpha \subset F\left(u,v\right)$. By the same way, we obtain $gv \in \left[F\left(v,u\right)\right]_\alpha$ and $\left(v\right)_\alpha = \left(gv\right)_\alpha \subset F\left(v,u\right)$. Therefore $\left(u,v\right)$ is a common fixed fuzzy point of F and g.

Now suppose that (2) holds. If for some $\left(x,y\right) \in C_\alpha\left(F,g\right)$, g is F-weakly commuting and $g^2 x = gx$ and $g^2 y = gy$, then

$$gx = gx^2 \in \left[F\left(gx,gy\right)\right]_\alpha \text{ and } gy = gy^2 \in \left[F\left(gy,gx\right)\right]_\alpha.$$

Thus, $\left(gx,gy\right)$ is a coupled common fixed point of F and g.

If (3) holds, then by the continuity of g at x,y, we get $x = gx \in \left[F\left(x,y\right)\right]_\alpha$ and $y = gy \in \left[F\left(y,x\right)\right]_\alpha$. Thus $\left(x,y\right)$ is a coupled common fixed point of F and g. □

4. Conclusions

In this paper, we obtain a coupled fixed fuzzy point theorem for a fuzzy mapping in fuzzy metric space and using it, we obtain a common fixed fuzzy point for a hybrid fuzzy pair. Our results can be applied in theoretical computer science.

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