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Interval valued $(\in, \in\vee q)$-fuzzy filters in MTL-algebras

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Abstract. In the present paper, the interval valued $(\in, \in\vee q)$-fuzzy filter theory in MTL-algebras is further studied. Some new concepts of interval valued $(\in, \in\vee q)$-fuzzy prime, ultra, regular, strong and divisible filters are introduced and their properties and equivalent characterizations are investigated. Relationships among the above new concepts and interval valued $(\in, \in\vee q)$-fuzzy Boolean, MV- and G-filters are discussed. It is proved that an interval valued $(\in, \in\vee q)$-fuzzy filter is an interval valued $(\in, \in\vee q)$-fuzzy MV-filter if and only if it is both an interval valued $(\in, \in\vee q)$-fuzzy regular filter and an interval valued $(\in, \in\vee q)$-fuzzy divisible filter.

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Keywords: MTL-algebra, Interval valued $(\in, \in\vee q)$-fuzzy filter, Interval valued $(\in, \in\vee q)$-fuzzy Boolean (MV-, G-, prime, ultra, regular, strong and divisible) filter.

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1. Introduction

To formalize many-valued logic induced by continuous t-norms on the real unit interval $[0, 1]$, in 1998, Hájek [5] introduced a very general many-valued logic, called Basic Logic (BL, for short). It is a well-known result that a t-norm has a residuum if and only if the t-norm is left-continuous, showing that BL is not the most general t-norm-based logic. In fact, a logic weaker than BL, called monoidal t-norm-based logic (MTL, for short), was defined by Esteva and Godo in [4] and proved to be the logic of left-continuous t-norms and their residua. Thus, the MTL is indeed the logic of left-continuous t-norms. In connection with the logic MTL, Esteva and Godo also introduced a new kind of algebra, namely, the MTL-algebra and they studied several basic properties of such an algebra. It is noteworthy that filter theory plays an important role in studying these algebras. From a logical point of view, various different filters correspond to various sets of provable formulas. We notice that Esteva and Godo introduced the concepts of filters and prime filters in MTL-algebras in [4]. Zhang discussed the topological properties of prime filters in MTL-algebras in [11]. Then, Borzooei et al. [1] and Haveshki [6] discussed some other types of filters in MTL-algebras.

On the other hand, the theory of fuzzy sets was first introduced by Zadeh [14] in 1965 and has been applied to many branches in mathematics. By using fuzzy sets, Kim et al. studied the fuzzy filters in MTL-algebras in [8]. As a continuation of the study of fuzzy filters in MTL-algebras, Jun et al. further investigated the properties of fuzzy filters and introduced the notions of fuzzy Boolean filters and fuzzy MV-filters in MTL-algebras [7]. Zhang et al. introduced the concepts of fuzzy ultra filters and fuzzy G-filters in MTL-algebras [12]. Zahiri et al. introduced some types of n-fold filters of MTL-algebras in [13]. Among many theories, Zadeh [15] also introduced the concept of interval valued fuzzy subset by considering the values of the membership functions as the intervals of numbers instead of the numbers only. Recently, Ma et al. [9, 10] applied the ideas of interval valued fuzzy sets to the study of filters in MTL-algebras, introduced the notions of interval valued $(\in, \in\vee q)$-fuzzy filters and interval valued $(\in, \in\vee q)$-fuzzy Boolean, MV- and G-filters and investigated some their properties and relationships.

In this paper, we will further research the properties of interval valued $(\in, \in\vee q)$-fuzzy filters in MTL-algebras. The rest of this article is organized as follows. In Section 2, we review related basic knowledge of MTL-algebras and interval valued fuzzy sets. In Section 3, we give several new characterizations of interval valued $(\in, \in\vee q)$-fuzzy Boolean, MV- and G-filters. In Section 4, Section 5 and Section 6, we introduce some new concepts of interval valued $(\in, \in\vee q)$-fuzzy prime, ultra, regular, strong and divisible filters and investigate their properties and characterizations. At the same time, relationships among the above various different interval valued $(\in, \in\vee q)$-fuzzy filters will be discussed in these sections. Finally, we conclude this paper in Section 7.

2. Preliminaries

Definition 2.1 ([2, 3, 4, 5])**.** A residuated lattice is an algebra $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that for all $x, y, z \in L$,

(R1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice with the greatest element 1 and the least element 0,

(R2) $(L, \otimes, 1)$ is a commutative monoid,

(R3) $(\otimes, \rightarrow)$ is an adjoint pair on L, i.e., $x \otimes y \leqslant z$ if and only if $x \leqslant y \rightarrow z$.

An MTL-algebra is a residuated lattice $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ satisfying the following pre-linearity equation:

(MTL) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

In the sequel, an MTL-algebra $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ will be denoted by L in short.

Lemma 2.2 ([2, 3, 4, 5])**.** *Let L be an MTL-algebra, define $x' = x \rightarrow 0$ for all $x \in L$. Then for all $x, y, z \in L$,*

(M1) $x \leqslant y$ *if and only if* $x \rightarrow y = 1$,

(M2) $x \rightarrow x = 1$ *and* $x \rightarrow 1 = 1$ *and* $1 \rightarrow x = x$,

(M3) $y \leqslant x \rightarrow y$ *and* $x \vee y \leqslant (x \rightarrow y) \rightarrow y$ *and* $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$,

(M4) $x \otimes y \leqslant x \otimes (x \to y) \leqslant x \wedge y \leqslant x \wedge (x \to y) \leqslant x$,
(M5) $(x \to y) \to z \leqslant x \to (y \to z)$ *and* $x \otimes (x \to y) \leqslant y \leqslant x \to (x \otimes y)$,
(M6) $(x \to y) \otimes (y \to z) \leqslant x \to z$ *and* $x \to (y \to z) = (x \otimes y) \to z = y \to (x \to z)$,
(M7) $x \leqslant y$ *implies* $x \otimes z \leqslant y \otimes z$ *and* $z \to x \leqslant z \to y$ *and* $y \to z \leqslant x \to z$,
(M8) $(y \vee z) \otimes x = (y \otimes x) \vee (z \otimes x)$ *and* $x \vee (y \otimes z) \geqslant (x \vee y) \otimes (x \vee z)$,
(M9) $(y \vee z) \to x = (y \to x) \wedge (z \to x)$, *in particular,* $(y \vee z) \to y = z \to y$,
(M10) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, *in particular,* $y \to (y \wedge z) = y \to z$,
(M11) $x \to (y \vee z) \geqslant (x \to y) \vee (x \to z)$ *and* $(y \wedge z) \to x \geqslant (y \to x) \vee (z \to x)$,
(M12) $y \to z \leqslant (x \to y) \to (x \to z) \leqslant x \to (y \to z)$ *and* $x \to y \leqslant (y \to z) \to (x \to z)$,
(M13) $x \otimes (y \to z) \leqslant y \to (x \otimes z) \leqslant (x \otimes y) \to (x \otimes z)$,
(M14) $(y \to x) \otimes ((x \wedge y) \to z) \leqslant (y \to (x \wedge y)) \wedge (y \otimes z)$,
(M15) $(x \to y) \otimes (z \to w) \leqslant (x \vee z) \to (y \vee w)$ *and* $(x \to y) \otimes (z \to w) \leqslant (x \wedge z) \to (y \wedge w)$,
(M16) $(x \to y) \otimes (z \to w) \leqslant (x \otimes z) \to (y \otimes w)$ *and* $(x \to y) \otimes (z \to w) \leqslant (x \to z) \to (y \to w)$,
(M17) $x \to y \leqslant (x \otimes z) \to (y \otimes z)$ *and* $x \leqslant x''$ *and* $x''' = x'$ *and* $x \otimes x' = 0$,
(M18) $x \leqslant y$ *implies* $y' \leqslant x'$ *and* $x'' \leqslant y''$,
(M19) $x \to y \leqslant y' \to x'$ *and* $x \to y' = y \to x' = (x \otimes y)'$,
(M20) $x \leqslant x' \to y$ *and* $x' \otimes y' \leqslant (x \otimes y)'$ *and* $x'' \otimes y'' \leqslant (x \otimes y)''$,
(M21) $x \otimes y = 0$ *if and only if* $x \leqslant y'$,
(M22) $(x \vee y)' = x' \wedge y'$ *and* $(x \wedge y)' = x' \vee y'$,
(M23) $(x \to y')'' = x \to y'$ *and* $x' \to y' = y \to x''$,
(M24) $(x \to y'')'' = x \to y'' = x'' \to y'' \geqslant (x \to y)''$.

By an interval number $\widetilde{a}$, we mean an interval $[a^-, a^+]$, where $0 \leqslant a^- \leqslant a^+ \leqslant 1$. Denote the set of all interval numbers by $D[0,1]$. The interval $[a,a]$ can be identified by the number $a \in [0,1]$.

For the interval numbers $\{\widetilde{a}_i\}_{i=1}^n = \{[a_i^-, a_i^+]\}_{i=1}^n \in D[0,1]$, $n \in \mathbb{N}^*$, we define:
(i) $\mathrm{rmin}\{\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n\} = [\min\{a_1^-, a_2^-, \cdots, a_n^-\}, \min\{a_1^+, a_2^+, \cdots, a_n^+\}]$,
(ii) $\mathrm{rmax}\{\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n\} = [\max\{a_1^-, a_2^-, \cdots, a_n^-\}, \max\{a_1^+, a_2^+, \cdots, a_n^+\}]$.

For any interval numbers $\widetilde{a}_1, \widetilde{a}_2 \in D[0,1]$, we define:
(i) $\widetilde{a}_1 \leqslant \widetilde{a}_2$ if and only if $a_1^- \leqslant a_2^-$ and $a_1^+ \leqslant a_2^+$,
(ii) $\widetilde{a}_1 = \widetilde{a}_2$ if and only if $a_1^- = a_2^-$ and $a_1^+ = a_2^+$,
(iii) $\widetilde{a}_1 < \widetilde{a}_2$ if and only if $\widetilde{a}_1 \leqslant \widetilde{a}_2$ and $\widetilde{a}_1 \neq \widetilde{a}_2$,
(iv) $k\widetilde{a} = [ka^-, ka^+]$, whenever $0 \leqslant k \leqslant 1$.
We easily observe that $(D[0,1], \leqslant, \wedge, \vee)$ forms a complete lattice with $0 = [0,0]$ as its least element and $1 = [1,1]$ as its greatest element.

Definition 2.3 ([15]). Let a set X be the fixed domain. An interval valued fuzzy set F on X is an object having the form $F = \{(x, [\mu_F^-(x), \mu_F^+(x)]) | x \in X\}$, where μ_F^- and μ_F^+ are fuzzy sets on X such that $\mu_F^-(x) \leqslant \mu_F^+(x)$, for all $x \in X$.

Now, putting $\widetilde{\mu_F}(x) = [\mu_F^-(x), \mu_F^+(x)]$, we see that $F = \{(x, \widetilde{\mu_F}(x)) | x \in X\}$, where $\widetilde{\mu_F} : X \to D[0,1]$.

Definition 2.4 ([10]). Let L be an MTL-algebra. An interval valued fuzzy set F on L is called an interval valued fuzzy filter of L, if the following conditions are satisfied: for all $x, y \in L$,

(i) $\widetilde{\mu_F}(x \otimes y) \geqslant \mathrm{rmin}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\}$,

(ii) $x \leqslant y$ implies $\widetilde{\mu_F}(x) \leqslant \widetilde{\mu_F}(y)$.

3. Interval valued $(\in, \in\vee q)$-fuzzy (Boolean, MV-, G-) filters

An interval valued fuzzy set $F = \{(x, \widetilde{\mu_F}(x)) | x \in L\}$ on an MTL-algebra L of the form

$$\widetilde{\mu_F}(y) = \begin{cases} \widetilde{t}(\neq [0,0]), & \text{if } y = x \\ [0,0], & \text{if } y \neq x \end{cases} \tag{3.1}$$

is called the fuzzy interval value with support x and denoted by $U(x; \widetilde{t})$.

For a fuzzy interval value $U(x; \widetilde{t})$ and an interval valued fuzzy set F,

(i) $U(x; \widetilde{t})$ is said to belonged to F, written as $U(x; \widetilde{t}) \in F$, if $\widetilde{\mu_F}(x) \geqslant \widetilde{t}$,

(ii) $U(x; \widetilde{t})$ is said to be quasi-coincident with F, written as $U(x; \widetilde{t})qF$, if

$$\widetilde{\mu_F}(x) + \widetilde{t} > [1, 1],$$

(iii) if $U(x; \widetilde{t}) \in F$ or $U(x; \widetilde{t})qF$, then we write $U(x; \widetilde{t}) \in\vee qF$,

(iv) the symbol $\overline{\in\vee q}$ means that $\in\vee q$ does not hold.

In the sequel, we emphasize that every $\widetilde{\mu_F}(x) = [\mu_F^-(x), \mu_F^+(x)]$ satisfies the comparable condition and the following properties:

$$[\mu_F^-(x), \mu_F^+(x)] < [0.5, 0.5] \text{ or } [0.5, 0.5] \leqslant [\mu_F^-(x), \mu_F^+(x)], \text{ for all } x \in L. \tag{3.2}$$

Definition 3.1 ([10]). Let L be an MTL-algebra. An interval valued fuzzy set F on L is called an interval vlued $(\in, \in\vee q)$-fuzzy filter of L, if it satisfies the following conditions: for all interval numbers $\widetilde{t}, \widetilde{r} \in D[0, 1]$ and $x, y \in L$,

(IF1) $U(x; \widetilde{t}) \in F$ and $U(y; \widetilde{r}) \in F$ imply $U(x \otimes y; \mathrm{rmin}\{\widetilde{t}, \widetilde{r}\}) \in\vee qF$,

(IF2) $U(x; \widetilde{r}) \in F$ implies $U(y; \widetilde{r}) \in\vee qF$ with $x \leqslant y$.

The set of all interval valued fuzzy filters of L is denoted by $\mathbf{IFF}(L)$.

Theorem 3.2. *Let L be an MTL-algebra and F an interval valued fuzzy set on L. Consider the following conditions, for all $x, y, z \in L$,*

(IF3) $\widetilde{\mu_F}(x \otimes y) \geqslant rmin\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y), [0.5, 0.5]\}$,

(IF4) $x \leqslant y$ *implies* $\widetilde{\mu_F}(y) \geqslant rmin\{\widetilde{\mu_F}(x), [0.5, 0.5]\}$,

(IF5) $\widetilde{\mu_F}(1) \geqslant rmin\{\widetilde{\mu_F}(x), [0.5, 0.5]\}$,

(IF6) $\widetilde{\mu_F}(y) \geqslant rmin\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(x \to y), [0.5, 0.5]\}$,

(IF7) $x \leqslant y \to z$ *implies* $\widetilde{\mu_F}(z) \geqslant rmin\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y), [0.5, 0.5]\}$,

(IF8) $\widetilde{\mu_F}(x \to z) \geqslant rmin\{\widetilde{\mu_F}(x \to y), \widetilde{\mu_F}(y \to z), [0.5, 0.5]\}$.

Then $F \in \mathbf{IFF}(L) \Longleftrightarrow$(IF3)+(IF4)$\Longleftrightarrow$(IF5)+(IF6)$\Longleftrightarrow$(IF7)$\Longleftrightarrow$(IF5)+(IF8).

Proof. (1) $F \in \mathbf{IFF}(L) \Longleftrightarrow$(IF3)+(IF4)$\Longleftrightarrow$(IF5)+(IF6)$\Longleftrightarrow$(IF7): These can be found in Theorems 3.5, 3.7 and 3.8 of [10].

(2) $F \in \mathbf{IFF}(L) \Longleftrightarrow$(IF5)+(IF8): Assume $F \in \mathbf{IFF}(L)$. Then F satisfies (IF3), (IF4) and (IF5). For any $x, y, z \in L$, since $(x \to y) \otimes (y \to z) \leqslant x \to z$ by (M6), we have that

$$\widetilde{\mu_F}(x \to z) \geqslant \mathrm{rmin}\{\widetilde{\mu_F}((x \to y) \otimes (y \to z)), [0.5, 0.5]\}$$

$$\geqslant \text{rmin}\{\text{rmin}\{\widetilde{\mu_F}(x \to y), \widetilde{\mu_F}(y \to z), [0.5, 0.5]\}, [0.5, 0.5]\}$$
$$= \text{rmin}\{\widetilde{\mu_F}(x \to y), \widetilde{\mu_F}(y \to z), [0.5, 0.5]\},$$

by (IF3) and (IF4). Thus F satisfies (IF8).

Conversely, assume that F satisfies (IF5) and (IF8). In order to show that $F \in$ **IFF**(L), it is sufficient to show F satisfies (IF6). In fact, putting $x = 1$ in (IF8). Then by (M2), we have that

$$\begin{aligned}\widetilde{\mu_F}(z) &= \widetilde{\mu_F}(1 \to z) \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}(1 \to y), \widetilde{\mu_F}(y \to z), [0.5, 0.5]\} \\ &= \text{rmin}\{\widetilde{\mu_F}(y), \widetilde{\mu_F}(y \to z), [0.5, 0.5]\},\end{aligned}$$

for any $y, z \in L$. Thus, (IF6) holds. The proof is completed. □

Remark 3.3. Let L be an MTL-algebra. Then

(1) every interval valued fuzzy filter F of L is an interval valued $(\in, \in\vee q)$-fuzzy filter of L,

(2) let F be an interval valued $(\in, \in\vee q)$-fuzzy filter of L, if $\widetilde{\mu_F}(1) < [0.5, 0.5]$, we easily see that F is an interval valued fuzzy filter of L.

For the above two facts, in what follows, we emphasize that all interval valued $(\in, \in\vee q)$-fuzzy filters of L must satisfy the condition $\widetilde{\mu_F}(1) \geqslant [0.5, 0.5]$.

Definition 3.4 ([10])**.** Let L be an MTL-algebra. An interval valued $(\in, \in\vee q)$-fuzzy filter of L is called an interval valued $(\in, \in\vee q)$-fuzzy Boolean filter of L, if it satisfies the following condition:

(IFB) $\widetilde{\mu_F}(x \vee x') \geqslant [0.5, 0.5]$, for all $x \in L$.

The set of all interval valued $(\in, \in\vee q)$-fuzzy Boolean filters of L is denoted by **IFBF**(L).

Theorem 3.5. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter of L. Then the following conditions are equivalent: for all $x, y, z \in L$,*

(1) $F \in \mathbf{IFBF}(L)$,

(2) $\widetilde{\mu_F}(x \to z) \geqslant rmin\{\widetilde{\mu_F}(x \to (z' \to y)), \widetilde{\mu_F}(y \to z), [0.5, 0.5]\}$,

(3) $\widetilde{\mu_F}(x) \geqslant rmin\{\widetilde{\mu_F}((x \to y) \to x), [0.5, 0.5]\}$,

(4) $\widetilde{\mu_F}(x) \geqslant rmin\{\widetilde{\mu_F}(x' \to x), [0.5, 0.5]\}$,

(5) $\widetilde{\mu_F}(x \vee (x \to y)) \geqslant [0.5, 0.5]$,

(6) $\widetilde{\mu_F}(((x \to y) \to x) \to x) \geqslant [0.5, 0.5]$,

(7) $\widetilde{\mu_F}((x' \to x) \to x) \geqslant [0.5, 0.5]$,

(8) $\widetilde{\mu_F}((((x \vee y) \to z) \to y) \to (x \vee y)) \geqslant [0.5, 0.5]$,

(9) $\widetilde{\mu_F}(((x \vee y)' \to y) \to (x \vee y)) \geqslant [0.5, 0.5]$.

Proof. (1)⟺(2)⟺(3): It was proved in Theorem 4.8 of [10].

(3)⟺(4): Assume that (3) holds. Putting $y = 0$ in (3). Then we have

$$\widetilde{\mu_F}(x) \geqslant \text{rmin}\{\widetilde{\mu_F}((x \to 0) \to x), [0.5, 0.5]\} = \widetilde{\mu_F}(x) \geqslant \text{rmin}\{\widetilde{\mu_F}(x' \to x), [0.5, 0.5]\}.$$

Thus, (4) holds.

Conversely, assume that (4) holds. Since $0 \leqslant y$, we have $x' = x \to 0 \leqslant x \to y$ by (M7). Thus $(x \to y) \to x \leqslant x' \to x$. By using (IF4), we have

$$\widetilde{\mu_F}(x' \to x) \geqslant \text{rmin}\{\widetilde{\mu_F}((x \to y) \to x), [0.5, 0.5]\}.$$

It follows from (4) that

$$\widetilde{\mu_F}(x) \geqslant \text{rmin}\{\widetilde{\mu_F}(x' \to x), [0.5, 0.5]\}$$

$\geqslant \text{rmin}\{\text{rmin}\{\widetilde{\mu_F}((x \to y) \to x), [0.5, 0.5]\}, [0.5, 0.5]\}$
$= \text{rmin}\{\widetilde{\mu_F}((x \to y) \to x), [0.5, 0.5]\}.$

So, (3) holds.

(1)$\Longrightarrow$(5): Assume that $F \in \mathbf{IFBF}(L)$. Since $x' \leqslant x \to y$, we have $x \vee x' \leqslant x \vee (x \to y)$. Then it follows from (IF4) and (IFB) that

$$\widetilde{\mu_F}(x \vee (x \to y)) \geqslant \text{rmin}\{\widetilde{\mu_F}(x \vee x'), [0.5, 0.5]\} \geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5].$$

Thus, (5) holds.

(5)$\Longrightarrow$(6): Since $x \vee (x \to y) \leqslant ((x \to y) \to x) \to x$ by (M3), by using (IF4) and (5), we have

$$\widetilde{\mu_F}(((x \to y) \to x) \to x) \geqslant \text{rmin}\{\widetilde{\mu_F}(x \vee (x \to y)), [0.5, 0.5]\} \geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5].$$

Then, (6) holds.

(6)$\Longrightarrow$(7): Assume that (6) holds. Putting $y = 0$ in (6). Then we have

$$\widetilde{\mu_F}((x' \to x) \to x) = \widetilde{\mu_F}(((x \to 0) \to x) \to x) \geqslant [0.5, 0.5].$$

Thus, (7) holds.

(7)$\Longrightarrow$(4): Assume that (7) holds. Then by using the condition (IF6), we can get that

$$\begin{aligned}\widetilde{\mu_F}(x) &\geqslant \text{rmin}\{\widetilde{\mu_F}(x' \to x), \widetilde{\mu_F}((x' \to x) \to x), [0.5, 0.5]\}\\ &\geqslant \text{rmin}\{\widetilde{\mu_F}(x' \to x), [0.5, 0.5], [0.5, 0.5]\}\\ &= \text{rmin}\{\widetilde{\mu_F}(x' \to x), [0.5, 0.5]\}.\end{aligned}$$

Thus, (4) holds.

(6)$\Longrightarrow$(8): For all $x, y, z \in L$, by (M1), (M2) and (M6), since

$$\begin{aligned}&(((x \vee y) \to z) \to y) \to (x \vee y)\\ &= (((x \vee y) \to z) \to y) \to ((y \to (x \vee y)) \to (x \vee y))\\ &= ((((x \vee y) \to z) \to y) \otimes (y \to (x \vee y))) \to (x \vee y)\\ &\geqslant (((x \vee y) \to z) \to (x \vee y)) \to (x \vee y),\end{aligned}$$

it follows from (IF4) and (6) that

$$\begin{aligned}&\widetilde{\mu_F}((((x \vee y) \to z) \to y) \to (x \vee y))\\ &\geqslant \text{rmin}\{\widetilde{\mu_F}((((x \vee y) \to z) \to (x \vee y)) \to (x \vee y)), [0.5, 0.5]\}\\ &\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5].\end{aligned}$$

Then, (8) holds.

(8)$\Longrightarrow$(9): Assume that (8) holds. Putting $z = 0$ in (8). Then we have

$$\widetilde{\mu_F}(((x \vee y)' \to y) \to (x \vee y)) = \widetilde{\mu_F}((((x \vee y) \to 0) \to y) \to (x \vee y)) \geqslant [0.5, 0.5].$$

Thus, (9) holds.

(9)$\Longrightarrow$(7): Assume that (9) holds. Putting $x = y$ in (9). Then we have

$$\widetilde{\mu_F}((x' \to x) \to x) = \widetilde{\mu_F}(((x \vee x)' \to x) \to (x \vee x)) \geqslant [0.5, 0.5].$$

Thus, (7) holds. The proof is completed. □

Definition 3.6 ([10]). Let L be an MTL-algebra. An interval valued $(\in, \in\vee q)$-fuzzy filter of L is called an interval valued $(\in, \in\vee q)$-fuzzy MV-filter of L, if it satisfies the following condition:

(IFMV) $\widetilde{\mu_F}(((x \to y) \to y) \to x) \geqslant \text{rmin}\{\widetilde{\mu_F}(y \to x), [0.5, 0.5]\}$, for all $x, y \in L$.

The set of all interval valued $(\in, \in \vee q)$-fuzzy MV-filters of L is denoted by **IFMVF**(L).

Theorem 3.7. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter of L. Then the following conditions are equivalent: for all $x, y, z \in L$,*
(1) $F \in \mathbf{IFMVF}(L)$,
(2)) $\widetilde{\mu_F}(((x \to y) \to y) \to x) \geqslant rmin\{\widetilde{\mu_F}(z \to (y \to x)), \widetilde{\mu_F}(z), [0.5, 0.5]\}$,
(3) $\widetilde{\mu_F}(((x \to y) \to y) \to (x \vee y)) \geqslant [0.5, 0.5]$,
(4) $\widetilde{\mu_F}(((x \to y) \to y) \to ((y \to x) \to x)) \geqslant [0.5, 0.5]$,
(5) $\widetilde{\mu_F}((y \to x) \to (((x \to y) \to y) \to x)) \geqslant [0.5, 0.5]$.

Proof. (1)$\Longleftrightarrow$(2): Assume that $F \in \mathbf{IFMVF}(L)$. Let $x, y, z \in L$. Then by (IFMV) and (IF6), we have

$$\begin{aligned}
&\widetilde{\mu_F}(((x \to y) \to y) \to x) \\
&\geqslant \text{rmin}\{\widetilde{\mu_F}(y \to x), [0.5, 0.5]\} \\
&\geqslant \text{rmin}\{\text{rmin}\{\widetilde{\mu_F}(z), \widetilde{\mu_F}(z \to (y \to x)), [0.5, 0.5]\}, [0.5, 0.5]\} \\
&= \text{rmin}\{\widetilde{\mu_F}(z \to (y \to x)), \widetilde{\mu_F}(z), [0.5, 0.5]\}.
\end{aligned}$$

Thus,(2) holds.

Conversely, assume that (2) holds. Putting $z = 1$ in (2). Then for all $x, y \in L$, we can obtain that

$$\begin{aligned}
&\widetilde{\mu_F}(((x \to y) \to y) \to x) \\
&\geqslant \text{rmin}\{\widetilde{\mu_F}(1 \to (y \to x)), \widetilde{\mu_F}(1), [0.5, 0.5]\} \\
&= \text{rmin}\{\widetilde{\mu_F}(y \to x), \widetilde{\mu_F}(1), [0.5, 0.5]\} \\
&= \text{rmin}\{\widetilde{\mu_F}(y \to x), [0.5, 0.5]\}.
\end{aligned}$$

Thus $F \in \mathbf{IFMVF}(L)$, by Definition 3.6.

(1)$\Longrightarrow$(3): Assume that $F \in \mathbf{IFMVF}(L)$. Then for all $x, y \in L$, it follows from (M9) and (IFMV) that

$$\begin{aligned}
&\widetilde{\mu_F}(((x \to y) \to y) \to (x \vee y)) \\
&= \widetilde{\mu_F}((((x \vee y) \to y) \to y) \to (x \vee y)) \\
&\geqslant \text{rmin}\{\widetilde{\mu_F}(y \to (x \vee y)), [0.5, 0.5]\} \\
&= \text{rmin}\{\widetilde{\mu_F}(1), [0.5, 0.5]\} = [0.5, 0.5].
\end{aligned}$$

Thus, (3) holds.

(3)$\Longrightarrow$(4): Assume that (3) holds. Since $x \vee y \leqslant (y \to x) \to x$, by (M1), we have $(x \vee y) \to ((y \to x) \to x) = 1$. Then it follows (IF8) and (3) that

$$\begin{aligned}
&\widetilde{\mu_F}(((x \to y) \to y) \to ((y \to x) \to x)) \\
&\geqslant \text{rmin}\{\widetilde{\mu_F}(((x \to y) \to y) \to (x \vee y)), \widetilde{\mu_F}((x \vee y) \to ((y \to x) \to x)), [0.5, 0.5]\} \\
&\geqslant \text{rmin}\{[0.5, 0.5], \widetilde{\mu_F}(1), [0.5, 0.5]\} = [0.5, 0.5].
\end{aligned}$$

Thus, (4) holds.

(4)$\Longrightarrow$(5): Assume that (4) holds. Then by (M6), we can obtain that

$$\begin{aligned}
&\widetilde{\mu_F}((y \to x) \to (((x \to y) \to y) \to x)) \\
&= \widetilde{\mu_F}(((x \to y) \to y) \to ((y \to x) \to x)) \\
&\geqslant [0.5, 0.5].
\end{aligned}$$

Thus, (5) holds.

(5)$\Longrightarrow$(1): Assume that (5) holds. Since $F \in \mathbf{IFF}(L)$, by using (IF6), we have that

$$\begin{aligned}
&\widetilde{\mu_F}(((x \to y) \to y) \to x) \\
&\geqslant \text{rmin}\{\widetilde{\mu_F}(y \to x), \widetilde{\mu_F}((y \to x) \to (((x \to y) \to y) \to x)), [0.5, 0.5]\}
\end{aligned}$$

$\geqslant \text{rmin}\{\widetilde{\mu_F}(y \to x), [0.5, 0.5], [0.5, 0.5]\}$
$= \text{rmin}\{\widetilde{\mu_F}(y \to x), [0.5, 0.5]\}.$
Then $F \in \mathbf{IFMVF}(L)$, by Definition 3.6. The proof is completed. □

Definition 3.8 ([10])**.** Let L be an MTL-algebra. An interval valued $(\in, \in\vee q)$-fuzzy filter of L is called an interval valued $(\in, \in\vee q)$-fuzzy G-filter of L, if it satisfies the following condition:

(IFG) $\widetilde{\mu_F}((x \to y) \geqslant \text{rmin}\{\widetilde{\mu_F}((x \otimes x) \to y), [0.5, 0.5]\}$, for all $x, y \in L$.

The set of all interval valued $(\in, \in\vee q)$-fuzzy G-filters of L is denoted by $\mathbf{IFGF}(L)$.

Theorem 3.9. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter of L. Then the following conditions are equivalent: for all $x, y, z \in L$,*

(1) $F \in \mathbf{IFGF}(L)$,
(2) $\widetilde{\mu_F}(x \to y) \geqslant rmin\{\widetilde{\mu_F}(x \to (x \to y)), [0.5, 0.5]\}$,
(3) $\widetilde{\mu_F}(x \to (x \otimes x)) \geqslant [0.5, 0.5]$,
(4) $\widetilde{\mu_F}(x \to z) \geqslant rmin\{\widetilde{\mu_F}(x \to (y \to z)), \widetilde{\mu_F}(x \to y), [0.5, 0.5]\}$,
(5) $\widetilde{\mu_F}((x \wedge (x \to y)) \to y) \geqslant [0.5, 0.5]$,
(6) $\widetilde{\mu_F}((x \wedge y) \to (x \otimes y)) \geqslant [0.5, 0.5]$,
(7) $\widetilde{\mu_F}((x \wedge (x \to y)) \to (x \otimes y)) \geqslant [0.5, 0.5]$,
(8) $\widetilde{\mu_F}((x \otimes (x \to y)) \to (x \otimes y)) \geqslant [0.5, 0.5]$,
(9) $\widetilde{\mu_F}((x \wedge (x \to y)) \to (x \wedge y)) \geqslant [0.5, 0.5]$,
(10) $\widetilde{\mu_F}((x \wedge (x \to y)) \to (x \otimes (x \to y))) \geqslant [0.5, 0.5]$
(11) $\widetilde{\mu_F}((x \wedge (x \to y)) \to (y \wedge (y \to x))) \geqslant [0.5, 0.5]$.

Proof. (1)$\Longleftrightarrow$(2): It is obvious, by using (M6) and (IFG).

(1)$\Longrightarrow$(3): Assume that $F \in \mathbf{IFGF}(L)$. Then by (IFG) and (M2), we have that

$\widetilde{\mu_F}(x \to (x \otimes x)$
$\geqslant \text{rmin}\{\widetilde{\mu_F}((x \otimes x) \to (x \otimes x)), [0.5, 0.5]\}]$
$= \text{rmin}\{\widetilde{\mu_F}(1), [0.5, 0.5]\} = [0.5, 0.5].$

Thus, (3) holds.

(3)$\Longrightarrow$(1): Assume that (3) holds. Since $(x \to (x \otimes x)) \to (x \to y) \geqslant (x \otimes x) \to y$ by (M12), we have $\widetilde{\mu_F}((x \to (x \otimes x)) \to (x \to y)) \geqslant \text{rmin}\{\widetilde{\mu_F}((x \otimes x) \to y), [0.5, 0.5]\}$. Then it follows from (IF6) and (3) that

$\widetilde{\mu_F}(x \to y)$
$\geqslant \text{rmin}\{\widetilde{\mu_F}(x \to (x \otimes x)), \widetilde{\mu_F}((x \to (x \otimes x)) \to (x \to y)), [0.5, 0.5]\}$
$\geqslant \text{rmin}\{\widetilde{\mu_F}(x \to (x \otimes x)), \widetilde{\mu_F}((x \otimes x) \to y), [0.5, 0.5]\}$
$\geqslant \text{rmin}\{[0.5, 0.5], \widetilde{\mu_F}((x \otimes x) \to y), [0.5, 0.5]\}$
$= \text{rmin}\{\widetilde{\mu_F}((x \otimes x) \to y), [0.5, 0.5]\},$

which proves (IFG). Thus $F \in \mathbf{IFGF}(L)$.

(1)$\Longrightarrow$(4): Assume that $F \in \mathbf{IFGF}(L)$. Then by (M6) and (IF8), we have

$\widetilde{\mu_F}((x \otimes x) \to z)$
$= \widetilde{\mu_F}(x \to (x \to z))$
$\geqslant \text{rmin}\{\widetilde{\mu_F}(x \to y), \widetilde{\mu_F}(y \to (x \to z)), [0.5, 0.5]\}.$

Thus it follows from (IFG) that

$\widetilde{\mu_F}(x \to z)$
$\geqslant \text{rmin}\{\widetilde{\mu_F}((x \otimes x) \to z), [0.5, 0.5]\}$
$\geqslant \text{rmin}\{\widetilde{\mu_F}(x \to y), \widetilde{\mu_F}(y \to (x \to z)), [0.5, 0.5]\}$

$= \text{rmin}\{\widetilde{\mu_F}(x \to (y \to z)), \widetilde{\mu_F}(x \to y), [0.5, 0.5]\}$.

So, (4) holds.

(4)$\Longrightarrow$(5): Assume that (4) holds. Since $x \wedge (x \to y) \leqslant x \to y$ and $x \wedge (x \to y) \leqslant x$, by (M1), we have

$$\widetilde{\mu_F}((x \wedge (x \to y)) \to (x \to y)) = \widetilde{\mu_F}(1) \geqslant [0.5, .05]$$

and

$$\widetilde{\mu_F}((x \wedge (x \to y)) \to x) = \widetilde{\mu_F}(1) \geqslant [0.5, .05].$$

Then it follows from (4) that

$$\begin{aligned} &\widetilde{\mu_F}((x \wedge (x \to y)) \to y) \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}((x \wedge (x \to y)) \to (x \to y)), \widetilde{\mu_F}((x \wedge (x \to y)) \to x), [0.5, 0.5]\} \\ &\geqslant [0.5, 0.5]. \end{aligned}$$

Thus, (5) holds.

(5)$\Longrightarrow$(6): Assume that (5) holds. Since $y \leqslant x \to (x \otimes y)$, we have $x \wedge y \leqslant x \wedge (x \to (x \otimes y))$. Then $(x \wedge y) \to (x \otimes y) \geqslant (x \wedge (x \to (x \otimes y))) \to (x \otimes y)$, by (M7). Thus it follows from (IF4) and (5) that

$$\begin{aligned} &\widetilde{\mu_F}((x \wedge y) \to (x \otimes y)) \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}((x \wedge (x \to (x \otimes y))) \to (x \otimes y)), [0.5, 0.5]\} \\ &\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]. \end{aligned}$$

So, (6) holds.

(6)$\Longrightarrow$(7): Since $x \otimes (x \to y) \leqslant x \wedge y$, by (M7), we have

$$(x \wedge (x \to y)) \to (x \otimes (x \to y)) \leqslant (x \wedge (x \to y)) \to (x \wedge y).$$

Then by (IF4) and (6), we have that

$$\begin{aligned} &\widetilde{\mu_F}((x \wedge (x \to y)) \to (x \wedge y)) \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}((x \wedge (x \to y)) \to (x \otimes (x \to y))), [0.5, 0.5]\} \\ &\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]. \end{aligned}$$

Since $((x \wedge (x \to y)) \to (x \wedge y)) \otimes ((x \wedge y) \to (x \otimes y)) \leqslant (x \wedge (x \wedge y)) \to (x \otimes y)$, by (M6), it follows from (IF3), (IF4) and (6) that

$$\begin{aligned} &\widetilde{\mu_F}((x \wedge (x \wedge y)) \to (x \otimes y)) \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}(((x \wedge (x \to y)) \to (x \wedge y)) \otimes ((x \wedge y) \to (x \otimes y))), [0.5, 0.5]\} \\ &\geqslant \text{rmin}\{\text{rmin}\{\widetilde{\mu_F}((x \wedge (x \to y)) \to (x \wedge y)), \widetilde{\mu_F}((x \wedge y) \to (x \otimes y)), [0.5, 0.5]\}, [0.5, 0.5]\} \\ &= \text{rmin}\{\widetilde{\mu_F}((x \wedge (x \to y)) \to (x \wedge y)), \widetilde{\mu_F}((x \wedge y) \to (x \otimes y)), [0.5, 0.5]\} \\ &\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]. \end{aligned}$$

Thus, (7) holds.

(7)$\Longrightarrow$(8): Assume that (7) holds. Since $x \otimes (x \to y) \leqslant x \wedge (x \to y)$, we have $(x \otimes (x \to y)) \to (x \otimes y) \geqslant (x \wedge (x \to y)) \to (x \otimes y)$, by (M7). It follows from (IF4) and (7) that

$$\begin{aligned} &\widetilde{\mu_F}((x \otimes (x \to y)) \to (x \otimes y)) \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}((x \wedge (x \to y)) \to (x \otimes y)), [0.5, 0.5]\} \\ &\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]. \end{aligned}$$

Thus, (8) holds.

(8)$\Longrightarrow$(3): Assume that (8) holds. By putting $y = x$ in (8), we have that

$$\begin{aligned} [0.5, 0.5] &\leqslant \widetilde{\mu_F}((x \otimes (x \to x)) \to (x \otimes x)) \\ &= \widetilde{\mu_F}((x \otimes 1) \to (x \otimes x)) = \widetilde{\mu_F}(x \to (1 \to (x \otimes x))) \\ &= \widetilde{\mu_F}(x \to (x \otimes x)). \end{aligned}$$

So, (3) holds.

(7)$\Longrightarrow$(9)$\Longrightarrow$(10)$\Longrightarrow$(11): Assume that (7) holds. Since

$$x \otimes y \leqslant x \otimes (x \to y) \leqslant x \wedge y \leqslant y \wedge (y \to x),$$

by (M7), we have

$$\begin{aligned}
&(x \wedge (x \to y)) \to (x \otimes y) \\
\leqslant\ &(x \wedge (x \to y)) \to (x \otimes (x \to y)) \\
\leqslant\ &(x \wedge (x \to y)) \to (x \wedge y) \\
\leqslant\ &(x \wedge (x \to y)) \to (y \wedge (y \to x)).
\end{aligned}$$

Then it follows from (IF4) and (7) that

$$\begin{aligned}
&\widetilde{\mu_F}((x \wedge (x \to y)) \to (x \otimes (x \to y))) \\
\geqslant\ &\mathrm{rmin}\{\widetilde{\mu_F}((x \wedge (x \to y)) \to (x \otimes y)), [0.5, 0.5]\} \\
\geqslant\ &\mathrm{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5].
\end{aligned}$$

Thus, (9) holds.

Next, by using (IF4) and (9), we have

$$\begin{aligned}
&\widetilde{\mu_F}((x \wedge (x \to y)) \to (x \wedge y)) \\
\geqslant\ &\mathrm{rmin}\{\widetilde{\mu_F}((x \wedge (x \to y)) \to (x \otimes (x \to y))), [0.5, 0.5]\} \\
\geqslant\ &\mathrm{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5].
\end{aligned}$$

So, (10) also holds.

Finally, by using (IF4) and (10), we have

$$\begin{aligned}
&\widetilde{\mu_F}((x \wedge (x \to y)) \to (y \wedge (y \to x))) \\
\geqslant\ &\mathrm{rmin}\{\widetilde{\mu_F}((x \wedge (x \to y)) \to (x \wedge y)), [0.5, 0.5]\} \\
\geqslant\ &\mathrm{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5].
\end{aligned}$$

Hence, (11) holds.

(11)$\Longrightarrow$(6): Since $(x \otimes y) \wedge ((x \otimes y) \to x) \leqslant x \otimes y$, by (M7), we have

$$\begin{aligned}
&(x \wedge (x \to (x \otimes y))) \to ((x \otimes y) \wedge ((x \otimes y) \to x)) \\
\leqslant\ &(x \wedge (x \to (x \otimes y))) \to (x \otimes y).
\end{aligned}$$

Then it follows from (IF4) and (11) that

$$\begin{aligned}
&\widetilde{\mu_F}((x \wedge (x \to (x \otimes y))) \to (x \otimes y)) \\
\geqslant\ &\mathrm{rmin}\{\widetilde{\mu_F}((x \wedge (x \to (x \otimes y))) \to ((x \otimes y) \wedge ((x \otimes y) \to x))), [0.5, 0.5]\} \\
\geqslant\ &\mathrm{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5].
\end{aligned}$$

Since $y \leqslant x \to (x \otimes y)$, we have $x \wedge y \leqslant x \wedge (x \to (x \otimes y))$. Thus

$$(x \wedge y) \to (x \otimes y) \geqslant (x \wedge (x \to (x \otimes y))) \to (x \otimes y).$$

So, by using (IF4), we can obtain that

$$\begin{aligned}
&\widetilde{\mu_F}((x \wedge y) \to (x \otimes y)) \\
\geqslant\ &\mathrm{rmin}\{\widetilde{\mu_F}((x \wedge (x \to (x \otimes y))) \to (x \otimes y)), [0.5, 0.5]\} \\
\geqslant\ &\mathrm{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5].
\end{aligned}$$

Hence (6) holds. The proof is finished. □

Theorem 3.10 ([10]). *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter. Then $F \in$ **IFBF**(L) if and only if $F \in$ **IFGF**(L) and $F \in$ **IFMVF**(L).*

Example 3.11. Let lattice $L = \{0, a, b, 1\}$ and $0 < a < b < 1$. Define $\otimes$ and $\to$ as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Define $\wedge = \min$ and $\vee = \max$ on L, respectively. Then $(L, \wedge, \vee, \otimes, \to, 0, 1)$ is an MTL-algebra. Define an interval valued fuzzy set F on L by $\widetilde{\mu_F}(0) = \widetilde{\mu_F}(a) = \widetilde{\mu_F}(b) = [0.3, 0.4], \widetilde{\mu_F}(1) = [0.8, 0.9]$. Then $F \in \mathbf{IFMVF}(L)$, but $F \notin \mathbf{IFGF}(L)$, because $\widetilde{\mu_F}(b \to a) = \widetilde{\mu_F}(b) = [0.3, 0.4] < [0.5, 0.5] = \mathrm{rmin}\{\widetilde{\mu_F}((b \otimes b) \to a), [0.5, 0.5]\}$.

Example 3.12. Let lattice $L = \{0, a, b, 1\}$ and $0 < a < b < 1$. Define $\otimes$ and $\to$ as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Define $\wedge = \min$ and $\vee = \max$ on L, respectively. Then $(L, \wedge, \vee, \otimes, \to, 0, 1)$ is an MTL-algebra. Define an interval valued fuzzy set F on L by $\widetilde{\mu_F}(0) = \widetilde{\mu_F}(a) = \widetilde{\mu_F}(b) = [0.3, 0.4], \widetilde{\mu_F}(1) = [0.8, 0.9]$. Then $F \in \mathbf{IFGF}(L)$, but $F \notin \mathbf{IFMVF}(L)$, because $\widetilde{\mu_F}(((b \to a) \to a) \to b) = \widetilde{\mu_F}(b) = [0.3, 0.4] < [0.5, 0.5] = \mathrm{rmin}\{\widetilde{\mu_F}(a \to b), [0.5, 0.5]\}$.

4. Interval valued $(\in, \in\vee q)$-fuzzy prime and ultra filters

Definition 4.1. Let L be an MTL-algebra. An interval valued $(\in, \in\vee q)$-fuzzy filter of L is called an interval valued $(\in, \in\vee q)$-fuzzy prime filter of L, if it satisfies the following condition:

(IFP) $\mathrm{rmax}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\} \geqslant \mathrm{rmin}\{\widetilde{\mu_F}(x \vee y), [0.5, 0.5]\}$, for all $x, y \in L$.

The set of all interval valued $(\in, \in\vee q)$-fuzzy prime filters of L is denoted by $\mathbf{IFPF}(L)$.

Example 4.2. Let lattice $L = \{0, a, b, 1\}$ and $0 < a < b < 1$. Define $\otimes$ and $\to$ as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Define $\wedge = \min$ and $\vee = \max$ on L, respectively. Then $(L, \wedge, \vee, \otimes, \to, 0, 1)$ is an MTL-algebra. Define an interval valued fuzzy set F on L by $\widetilde{\mu_F}(0) = \widetilde{\mu_F}(a) = [0.3, 0.4], \widetilde{\mu_F}(b) = [0.5, 0.6], \widetilde{\mu_F}(1) = [0.8, 0.9]$. Then, one can easily verify that $F \in \mathbf{IFPF}(L)$.

Theorem 4.3. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter of L. Then the following conditions are equivalent: for all $x, y \in L$,*

(1) $F \in \mathbf{IFPF}(L)$,

(2) $\widetilde{\mu_F}(x \vee y) \geqslant [0.5, 0.5]$ *implies* $\widetilde{\mu_F}(x) \geqslant [0.5, 0.5]$ *or* $\widetilde{\mu_F}(y) \geqslant [0.5, 0.5]$,

(3) $\widetilde{\mu_F}(x \to y) \geqslant [0.5, 0.5]$ *or* $\widetilde{\mu_F}(y \to x) \geqslant [0.5, 0.5]$.

Proof. (1)$\Longrightarrow$(2): Suppose that (1) holds and $\widetilde{\mu_F}(x \vee y) \geqslant [0.5, 0.5]$, for all $x, y \in L$. Then, by using (IFP), we have

$$\begin{aligned} &\operatorname{rmax}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\} \\ \geqslant\ &\operatorname{rmin}\{\widetilde{\mu_F}(x \vee y), [0.5, 0.5]\} \\ \geqslant\ &\operatorname{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]. \end{aligned}$$

Thus it follows that $\widetilde{\mu_F}(x) \geqslant [0.5, 0.5]$ or $\widetilde{\mu_F}(y) \geqslant [0.5, 0.5]$. So, (1) holds.

(2)$\Longrightarrow$(3): Suppose that (2) holds. Since $(x \to y) \vee (y \to x) = 1$, we have $\widetilde{\mu_F}((x \to y) \vee (y \to x)) = \widetilde{\mu_F}(1) \geqslant [0.5, 0.5]$. Thus it follows from (2) that

$$\widetilde{\mu_F}(x \to y) \geqslant [0.5, 0.5] \text{ or } \widetilde{\mu_F}(y \to x) \geqslant [0.5, 0.5].$$

So, (3) holds.

(3)$\Longrightarrow$(1): Suppose that (3) holds. Since $(x \vee y) \to y = x \to y$ and $(x \vee y) \to x = y \to x$, by (M9), we have that $\widetilde{\mu_F}((x \vee y) \to y) = \widetilde{\mu_F}(x \to y) \geqslant [0.5, 0.5]$ or $\widetilde{\mu_F}((x \vee y) \to x) = \widetilde{\mu_F}(y \to x) \geqslant [0.5, 0.5]$ by using (3). Then it follows from (IF6) that $\widetilde{\mu_F}(y) \geqslant \operatorname{rmin}\{\widetilde{\mu_F}(x \vee y), \widetilde{\mu_F}((x \vee y) \to y), [0.5, 0.5]\} \geqslant \operatorname{rmin}\{\widetilde{\mu_F}(x \vee y), [0.5, 0.5]\}$ or $\widetilde{\mu_F}(x) \geqslant \operatorname{rmin}\{\widetilde{\mu_F}(x \vee y), \widetilde{\mu_F}((x \vee y) \to x), [0.5, 0.5]\} \geqslant \operatorname{rmin}\{\widetilde{\mu_F}(x \vee y), [0.5, 0.5]\}$. These show that $\operatorname{rmax}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\} \geqslant \operatorname{rmin}\{\widetilde{\mu_F}(x \vee y), [0.5, 0.5]\}$, i.e., (IFP) holds. Thus, $F \in \mathbf{IFPF}(L)$ by Definition 4.1. The proof is completed. □

Definition 4.4. Let L be an MTL-algebra. An interval valued $(\in, \in\vee q)$-fuzzy filter of L is called an interval valued $(\in, \in\vee q)$-fuzzy ultra filter of L, if it satisfies the following condition:

(IFU) $\widetilde{\mu_F}(x) \geqslant [0.5, 0.5]$ or $\widetilde{\mu_F}(x') \geqslant [0.5, 0.5]$, for all $x \in L$.

The set of all interval valued $(\in, \in\vee q)$-fuzzy ultra filters of L is denoted by $\mathbf{IFUF}(L)$.

Example 4.5. Let $L = \{0, a, b, c, d, 1\}$, the Hasse diagram of L be defined as Fig. 1, and the operators $\otimes$ and $\to$ of L is defined as follows:

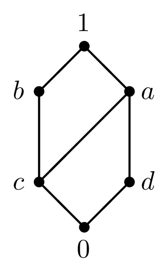


Fig. 1 Hasse Diagram of L

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	d	c	0	d	a
b	0	c	b	c	0	b
c	0	0	c	0	0	c
d	0	d	0	0	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	b	a	1
b	d	a	1	a	d	1
c	a	1	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Then $(L, \wedge, \vee, \otimes, \to, 0, 1)$ is an MTL-algebra. Define an interval valued fuzzy set F on L by $\widetilde{\mu_F}(0) = \widetilde{\mu_F}(b) = \widetilde{\mu_F}(c) = [0.3, 0.4], \widetilde{\mu_F}(a) = \widetilde{\mu_F}(d) = \widetilde{\mu_F}(1) = [0.8, 0.9]$. Then, one can easily verify that $F \in \mathbf{IFUF}(L)$.

Theorem 4.6. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter of L. Then $F \in \mathbf{IFUF}(L)$ if and only if $\widetilde{\mu_F}(x) < [0.5, 0.5]$ and $\widetilde{\mu_F}(y) < [0.5, 0.5]$ imply $\widetilde{\mu_F}(x \to y) \geqslant [0.5, 0.5]$ and $\widetilde{\mu_F}(y \to x) \geqslant [0.5, 0.5]$ for all $x, y \in L$.*

Proof. $\Longrightarrow$: Assume that $F \in \mathbf{IFUF}(L)$ and $\widetilde{\mu_F}(x) < [0.5, 0.5]$, $\widetilde{\mu_F}(y) < [0.5, 0.5]$. Then $\widetilde{\mu_F}(x') \geqslant [0.5, 0.5]$ and $\widetilde{\mu_F}(y') \geqslant [0.5, 0.5]$ follows from (IFU). Since $0 \leqslant y$, we have $x' = x \to 0 \leqslant x \to y$, by (M7). Thus by (IFU) and $\widetilde{\mu_F}(x') \geqslant [0.5, 0.5]$, we can obtain that

$\widetilde{\mu_F}(x \to y) \geqslant \mathrm{rmin}\{\widetilde{\mu_F}(x'), [0.5, 0.5]\}$
$\geqslant \mathrm{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]$.

Similarly, we can prove $\widetilde{\mu_F}(y \to x) \geqslant [0.5, 0.5]$ form $\widetilde{\mu_F}(y') \geqslant [0.5, 0.5]$. So, the necessity is proved.

$\Longleftarrow$: Let $\widetilde{\mu_F}(x) < [0.5, 0.5]$ and $\widetilde{\mu_F}(y) < [0.5, 0.5]$ imply $\widetilde{\mu_F}(x \to y) \geqslant [0.5, 0.5]$ and $\widetilde{\mu_F}(y \to x) \geqslant [0.5, 0.5]$, for all $x, y \in L$. Assume that $\widetilde{\mu_F}(x) < [0.5, 0.5]$. Since $0 \leqslant x$, using (IF4), we get $\widetilde{\mu_F}(x) \geqslant \mathrm{rmin}\{\widetilde{\mu_F}(0), [0.5, 0.5]\}$. If $\widetilde{\mu_F}(0) \geqslant [0.5, 0.5]$, then $\widetilde{\mu_F}(x) \geqslant [0.5, 0.5]$. This is a contradiction. Thus, we get that $\widetilde{\mu_F}(0) < [0.5, 0.5]$. So it follows from the known condition that $\widetilde{\mu_F}(x') = \widetilde{\mu_F}(x \to 0) \geqslant [0.5, 0.5]$, that is, (IFU) holds. Hence, $F \in \mathbf{IFUF}(L)$, by Definition 4.4. □

Theorem 4.7. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter of L. Then $F \in \mathbf{IFUF}(L)$ if and only if $F \in \mathbf{IFPF}(L)$ and $F \in \mathbf{IFBF}(L)$.*

Proof. $\Longleftarrow$: Assume that $F \in \mathbf{IFPF}(L)$ and $F \in \mathbf{IFBF}(L)$. For all $x \in L$, by using (IFP) and (IFB), we have

$\mathrm{rmax}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(x')\}$
$\geqslant \mathrm{rmin}\{\widetilde{\mu_F}(x \vee x'), [0.5, 0.5]\}$
$\geqslant \mathrm{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]$.c Then it follows that $\widetilde{\mu_F}(x) \geqslant [0.5, 0.5]$ or $\widetilde{\mu_F}(x') \geqslant [0.5, 0.5]$, that is, (IFU) holds. Thus, $F \in \mathbf{IFUF}(L)$, by Definition 4.4.

$\Longrightarrow$: Assume that $F \in \mathbf{IFUF}(L)$. For all $x \in L$, since $x \leqslant x \vee x'$ and $x' \leqslant x \vee x'$, By using (IF4), we have $\widetilde{\mu_F}(x \vee x') \geqslant \mathrm{rmin}\{\widetilde{\mu_F}(x), [0.5, 0.5]\}$ and $\widetilde{\mu_F}(x \vee x') \geqslant \mathrm{rmin}\{\widetilde{\mu_F}(x'), [0.5, 0.5]\}$. According to (IFU), we have $\widetilde{\mu_F}(x) \geqslant [0.5, 0.5]$ or $\widetilde{\mu_F}(x') \geqslant [0.5, 0.5]$. So, $\widetilde{\mu_F}(x \vee x') \geqslant [0.5, 0.5]$. This means that $F \in \mathbf{IFBF}(L)$, by Definition 3.4.

For all $x, y \in L$, since $x \vee y \leqslant (x \to y) \to y$, by using (IF4), we have

$$\widetilde{\mu_F}((x \to y) \to y) \geqslant \mathrm{rmin}\{\widetilde{\mu_F}(x \vee y), [0.5, 0.5]\}.$$

From $0 \leqslant y$ and (M7), we get

$$x \to 0 \leqslant x \to y$$

and

$$(x \to y) \to y \leqslant (x \to 0) \to y = x' \to y.$$

Then,

$\widetilde{\mu_F}(x' \to y)$
$\geqslant \mathrm{rmin}\{\widetilde{\mu_F}((x \to y) \to y), [0.5, 0.5]\}$
$\geqslant \mathrm{rmin}\{\mathrm{rmin}\{\widetilde{\mu_F}(x \vee y), [0.5, 0.5]\}, [0.5, 0.5]\}$
$= \mathrm{rmin}\{\widetilde{\mu_F}(x \vee y), [0.5, 0.5]\}$.

If $\widetilde{\mu_F}(x) \geqslant [0.5, 0.5]$, then

$\mathrm{rmax}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\} \geqslant \widetilde{\mu_F}(x) \geqslant [0.5, 0.5]$
$\geqslant \mathrm{rmin}\{\widetilde{\mu_F}(x \vee y), [0.5, 0.5]\}$.

If $\widetilde{\mu_F}(x) < [0.5, 0.5]$, then $\widetilde{\mu_F}(x') \geqslant [0.5, 0.5]$, by (IFU). Thus,

$\widetilde{\mu_F}(y)$
$\geqslant \mathrm{rmin}\{\widetilde{\mu_F}(x'), \widetilde{\mu_F}(x' \to y), [0.5, 0.5]\}$
$\geqslant \mathrm{rmin}\{\widetilde{\mu_F}(x' \to y), [0.5, 0.5]\}$
$\geqslant \mathrm{rmin}\{\mathrm{rmin}\{\widetilde{\mu_F}(x \vee y), [0.5, 0.5]\}, [0.5, 0.5]\}$

$= \text{rmin}\{\widetilde{\mu_F}(x \vee y), [0.5, 0.5]\}$.
So, $\text{rmax}\{\widetilde{\mu_F}(x), \widetilde{\mu_F}(y)\} \geqslant \widetilde{\mu_F}(y) \geqslant \text{rmin}\{\widetilde{\mu_F}(x \vee y), [0.5, 0.5]\}$. These mean that (IFP) holds. Hence $F \in \mathbf{IFPF}(L)$, by Definition 4.1. The proof is completed. □

5. Interval valued $(\in, \in\vee q)$-fuzzy regular and strong filters

Definition 5.1. Let L be an MTL-algebra. An interval valued $(\in, \in\vee q)$-fuzzy filter of L is called an interval valued $(\in, \in\vee q)$-fuzzy regular filter of L, if it satisfies the following condition:

(IFR) $\widetilde{\mu_F}(x'' \to x) \geqslant [0.5, 0.5]$, for all $x \in L$.

The set of all interval valued $(\in, \in\vee q)$-fuzzy regular filters of L is denoted by $\mathbf{IFRF}(L)$.

Theorem 5.2. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter of L. Then the following conditions are equivalent: for all $x, y \in L$,*

(1) $F \in \mathbf{IFRF}(L)$,
(2) $\widetilde{\mu_F}((y' \to x') \to (x \to y)) \geqslant [0.5, 0.5]$,
(3) $\widetilde{\mu_F}(y \to x) \geqslant rmin\{\widetilde{\mu_F}(x' \to y'), [0.5, 0.5]\}$,
(4) $\widetilde{\mu_F}(y' \to x) \geqslant rmin\{\widetilde{\mu_F}(x' \to y), [0.5, 0.5]\}$.

Proof. (1)$\Longrightarrow$(2): Assume that $F \in \mathbf{IFRF}(L)$. For all $x, y \in L$, since $y \leqslant y''$, by using (M7) and (M12), we have $x' \to y' \leqslant (y' \to 0) \to (x' \to 0) = y'' \to x'' \leqslant y \to x''$. Then $(x' \to y') \to (y \to x) \geqslant (y \to x'') \to (y \to x) \geqslant x'' \to x$. Thus it follows from (IF4) and (IFR) that

$\widetilde{\mu_F}((y' \to x') \to (x \vee y))$
$\geqslant \text{rmin}\{\widetilde{\mu_F}(x'' \to x), [0.5, 0.5]\}$
$\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]$.

Thus, (2) holds.

(2)$\Longrightarrow$(3): Assume that (2) holds. For all $x, y \in L$, by using (IF6) and (2), we can obtain that

$\widetilde{\mu_F}(y)\widetilde{\mu_F}(y \to x)$
$\geqslant \text{rmin}\{\widetilde{\mu_F}(x' \to y'), \widetilde{\mu_F}((y' \to x') \to (x \to y)), [0.5, 0.5]\}$
$\geqslant \text{rmin}\{\widetilde{\mu_F}(x' \to y'), [0.5, 0.5], [0.5, 0.5]\}$
$= \text{rmin}\{\widetilde{\mu_F}(x' \to y'), [0.5, 0.5]\}$.

Thus, (3) holds.

(3)$\Longrightarrow$(4): Assume (3) holds. For all $x, y \in L$, since $y \leqslant y''$, we have $x' \to y \leqslant x' \to y''$, by (M7). Then, $\widetilde{\mu_F}(x' \to y'') \geqslant \text{rmin}\{\widetilde{\mu_F}(x' \to y), [0.5, 0.5]\}$. Thus it follows from (3) that

$\widetilde{\mu_F}(y' \to x)$
$\geqslant \text{rmin}\{\widetilde{\mu_F}(x' \to y''), [0.5, 0.5]\}$
$\geqslant \text{rmin}\{\text{rmin}\{\widetilde{\mu_F}(x' \to y), [0.5, 0.5]\}, [0.5, 0.5]\}$
$= \text{rmin}\{\widetilde{\mu_F}(x' \to y), [0.5, 0.5]\}$.

So, (4) holds.

(4)$\Longrightarrow$(1): Assume that (4) holds. For all $x \in L$, since $x' \to x' = 1$, by using (4), we have that

$\widetilde{\mu_F}(x'' \to x)$
$\geqslant \text{rmin}\{\widetilde{\mu_F}(x' \to x'), [0.5, 0.5]\}$

$= \text{rmin}\{\widetilde{\mu_F}(1), [0.5, 0.5]\} = [0.5, 0.5]$.
Then (IFR) holds. Thus $F \in \mathbf{IFRF}(L)$, by Definition 5.1. The proof is completed. □

Theorem 5.3. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter. Then $F \in \mathbf{IFMVF}(L)$ implies $F \in \mathbf{IFRF}(L)$. But the converse does not always hold.*

Proof. Assume that $F \in \mathbf{IFMVF}(L)$. Then it follows from (IFMV) that

$$\begin{aligned} &\widetilde{\mu_F}(x'' \to x) \\ =\ &\widetilde{\mu_F}(((x \to 0) \to 0) \to x) \\ \geqslant\ &\text{rmin}\{\widetilde{\mu_F}(0 \to x), [0.5, 0.5]\} \\ =\ &\text{rmin}\{\widetilde{\mu_F}(1), [0.5, 0.5]\} = [0.5, 0.5]. \end{aligned}$$

Then (IFR) holds. Thus $F \in \mathbf{IFRF}(L)$, by Definition 5.1.

Example 5.4 shows that the converse does not always hold. The proof is completed. □

Example 5.4. Let lattice $L = \{0, a, b, c, 1\}$ and $0 < a < b < c < 1$. Define $\otimes$ and $\to$ as follows:

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Define $\wedge = \min$ and $\vee = \max$ on L, respectively. Then $(L, \wedge, \vee, \otimes, \to, 0, 1)$ is an MTL-algebra. Define an interval valued fuzzy set F on L by $\widetilde{\mu_F}(0) = \widetilde{\mu_F}(a) = \widetilde{\mu_F}(b) = [0.3, 0.4], \widetilde{\mu_F}(c) = \widetilde{\mu_F}(1) = [0.8, 0.9]$. Then $F \in \mathbf{IFRF}(L)$, but $F \notin \mathbf{IFMVF}(L)$, because $\widetilde{\mu_F}(((b \to a) \to a) \to b) = [0.3, 0.4] < [0.5, .05] = \text{rmin}\{\widetilde{\mu_F}(a \to b), [0.5, 0.5]\}$.

Theorem 5.5. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter of L. Then $F \in \mathbf{IFBF}(L)$ if and only if $F \in \mathbf{IFRF}(L)$ and $F \in \mathbf{IFGF}(L)$.*

Proof. $\Longrightarrow$: Assume that $F \in \mathbf{IFBF}(L)$. It follows from Theorem 3.10 and Theorem 5.3 that $F \in \mathbf{IFRF}(L)$ and $F \in \mathbf{IFGF}(L)$.

$\Longleftarrow$: Assume that $F \in \mathbf{IFRF}(L)$ and $F \in \mathbf{IFGF}(L)$. For all $x \in L$, since $x \leqslant x''$, we have $x' \to x \leqslant x' \to x''$, by (M7). Then it follows from (IF4) that $\widetilde{\mu_F}(x' \to x'') \geqslant \text{rmin}\{\widetilde{\mu_F}(x' \to x), [0.5, 0.5]\}$. Since $F \in \mathbf{IFGF}(L)$, by Theorem 3.9(2), we obtain

$$\begin{aligned} \widetilde{\mu_F}(x'') &= \widetilde{\mu_F}(x' \to 0) \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}(x' \to (x' \to 0)), [0.5, 0.5]\} \\ &= \text{rmin}\{\widetilde{\mu_F}(x' \to x''), [0.5, 0.5]\}. \end{aligned}$$

Thus, by $F \in \mathbf{IFRF}(L)$ and (IF6), we have

$$\begin{aligned} \widetilde{\mu_F}(x) &\geqslant \text{rmin}\{\widetilde{\mu_F}(x'' \to x), \widetilde{\mu_F}(x''), [0.5, 0.5]\} \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}(x''), [0.5, 0.5]\} \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}(x' \to x''), [0.5, 0.5]\} \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}(x' \to x), [0.5, 0.5]\}. \end{aligned}$$

So, $F \in \mathbf{IFBF}(L)$, by Theorem 3.5 (4). The proof is completed. □

Remark 5.6. Let L be the MTL-algebra given in Example 3.12. Define an interval valued fuzzy set F on L by $\widetilde{\mu_F}(0) = \widetilde{\mu_F}(a) = [0.3, 0.4], \widetilde{\mu_F}(b) = \widetilde{\mu_F}(1) = [0.8, 0.9]$. Then $F \in \mathbf{IFGF}(L)$, but $F \notin \mathbf{IFRF}(L)$, because $\widetilde{\mu_F}(a'' \to a) = \widetilde{\mu_F}(a) = [0.3, 0.4] < [0.5, .05]$.

Example 5.7. Let lattice $L = \{0, a, b, 1\}$ and $0 < a < b < 1$. Define $\otimes$ and $\to$ as follows:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	a	1	1
1	0	a	b	1

Define $\wedge = \min$ and $\vee = \max$ on L, respectively. Then $(L, \wedge, \vee, \otimes, \to, 0, 1)$ is an MTL-algebra. Define an interval valued fuzzy set F on L by $\widetilde{\mu_F}(0) = \widetilde{\mu_F}(a) = \widetilde{\mu_F}(b) = [0.3, 0.4], \widetilde{\mu_F}(1) = [0.8, 0.9]$. Then $F \in \mathbf{IFRF}(L)$, but $F \notin \mathbf{IFGF}(L)$, because $\widetilde{\mu_F}(a \to 0) = \widetilde{\mu_F}(b) = [0.3, 0.4] < [0.5, .05] = \text{rmin}\{\widetilde{\mu_F}((a \otimes a) \to 0), [0.5, 0.5]\}$.

Definition 5.8. Let L be an MTL-algebra. An interval valued $(\in, \in\vee q)$-fuzzy filter of L is called an interval valued $(\in, \in\vee q)$-fuzzy strong filter of L, if it satisfies the following condition:

(IFS) $\widetilde{\mu_F}((x'' \to x)'') \geqslant [0.5, 0.5]$, for all $x \in L$.

The set of all interval valued $(\in, \in\vee q)$-fuzzy strong filters of L is denoted by $\mathbf{IFSF}(L)$.

Theorem 5.9. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter of L. Then the following conditions are equivalent: for all $x, y \in L$,*

(1) $F \in \mathbf{IFSF}(L)$,
(2) $\widetilde{\mu_F}((y \to x'') \to (y \to x)'') \geqslant [0.5, 0.5]$,
(3) $\widetilde{\mu_F}((x \to y) \to (x'' \to y)'') \geqslant [0.5, 0.5]$,
(4) $\widetilde{\mu_F}((x' \to y) \to (y' \to x)'') \geqslant [0.5, 0.5]$.

Proof. (1)$\Longrightarrow$(2): Assume that $F \in \mathbf{IFSF}(L)$. For all $x, y \in L$, since $(x'' \to x)'' \leqslant ((y \to x'') \to (y \to x))'' \leqslant ((y \to x'') \to (y \to x)'')'' = (y \to x'') \to (y \to x)''$ by (M7), (M12) and (M24), it follows from (IF4) and (IFS) that

$$\begin{aligned} &\widetilde{\mu_F}((y \to x'') \to (y \to x)'') \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}((x'' \to x)''), [0.5, 0.5]\} \\ &\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]. \end{aligned}$$

Then, (2) holds.

(2)$\Longrightarrow$(1): Assume that (2) holds. By putting $y = x''$ in (2), we have

$$[0.5, 0.5] \leqslant \widetilde{\mu_F}((x'' \to x'') \to (x'' \to x)'') = \widetilde{\mu_F}((x'' \to x)'').$$

Then, (IFS) holds. Thus $F \in \mathbf{IFSF}(L)$, by Definition 5.8.

(1)$\Longrightarrow$(3): Assume that $F \in \mathbf{IFSF}(L)$. For all $x, y \in L$, since $(x'' \to x)'' \leqslant ((x \to y) \to (x'' \to y))'' \leqslant ((x \to y) \to (x'' \to y)'')'' = (x \to y) \to (x'' \to y)''$, by (M7), (M12) and (M24), it follows from (IF4) and (IFS) that

$$\begin{aligned} &\widetilde{\mu_F}((x \to y) \to (x'' \to y)'') \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}((x'' \to x)''), [0.5, 0.5]\} \end{aligned}$$

$\geqslant \mathrm{rmin}\{[0.5,0.5],[0.5,0.5]\} = [0.5,0.5]$.
Thus, (3) holds.

(3)$\Longrightarrow$(1): Assume that (3) holds. By putting $y = x$ in (3), we have

$$[0.5,0.5] \leqslant \widetilde{\mu_F}((x \to x) \to (x'' \to x)'') = \widetilde{\mu_F}((x'' \to x)'').$$

Then, (IFS) holds. Thus $F \in \mathbf{IFSF}(L)$, by Definition 5.8.

(1)$\Longrightarrow$(4): Assume that $F \in \mathbf{IFSF}(L)$. For all $x, y \in L$, since

$$\begin{aligned}(x'' \to x)'' &= ((x' \to 0) \to x)'' \\ &\leqslant (((x' \to y) \otimes (y \to 0)) \to x)'' \\ &= (((x' \to y) \otimes y') \to x)'' \\ &= ((x' \to y) \to (y' \to x))'' \\ &\leqslant ((x' \to y) \to (y' \to x)'')'' \\ &= (x' \to y) \to (y' \to x)'',\end{aligned}$$

by (M6), (M7), (M12) and (M24), it follows from (IF4) and (IFS) that

$$\begin{aligned}&\widetilde{\mu_F}((x' \to y) \to (y' \to x)'') \\ &\geqslant \mathrm{rmin}\{\widetilde{\mu_F}((x'' \to x)''), [0.5,0.5]\} \\ &\geqslant \mathrm{rmin}\{[0.5,0.5],[0.5,0.5]\} = [0.5,0.5].\end{aligned}$$

Then, (4) holds.

(4)$\Longrightarrow$(1): Assume that (4) holds. By putting $y = x'$ in (4), we have

$$[0.5,0.5] \leqslant \widetilde{\mu_F}((x' \to x') \to (x'' \to x)'') = \widetilde{\mu_F}((x'' \to x)'').$$

Thus, (IFS) holds. So $F \in \mathbf{IFSF}(L)$, by Definition 5.8. □

Theorem 5.10. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter. Then $F \in \mathbf{IFRF}(L)$ implies $F \in \mathbf{IFSF}(L)$. But the converse does not always hold.*

Proof. Assume that $F \in \mathbf{IFRF}(L)$. Since $x'' \to x \leqslant (x'' \to x)''$, for all $x \in L$, by using (IF4) and (IFR), we have that

$$\widetilde{\mu_F}((x'' \to x)'') \geqslant \mathrm{rmin}\{\widetilde{\mu_F}(x'' \to x), [0.5,0.5]\} \geqslant \mathrm{rmin}\{[0.5,0.5],[0.5,0.5]\} = [0.5,0.5].$$

Then (IFS) holds. Thus $F \in \mathbf{IFSF}(L)$, by Definition 5.8.

Example 5.11 shows that the converse does not always hold. The proof is completed. □

Example 5.11. Let lattice $L = \{0, a, b, c, 1\}$ and $0 < a < b < c < 1$. Define $\otimes$ and $\to$ as follows:

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	a	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

Define $\wedge = \min$ and $\vee = \max$ on L, respectively. Then $(L, \wedge, \vee, \otimes, \to, 0, 1)$ is an MTL-algebra. Define an interval valued fuzzy set F on L by $\widetilde{\mu_F}(0) = \widetilde{\mu_F}(a) = \widetilde{\mu_F}(b) = \widetilde{\mu_F}(c) = [0.3,0.4], \widetilde{\mu_F}(1) = [0.8,0.9]$. Then $F \in \mathbf{IFSF}(L)$, but $F \notin \mathbf{IFRF}(L)$, because $\widetilde{\mu_F}(a'' \to a) = \widetilde{\mu_F}(a) = [0.3,0.4] < [0.5,.05]$.

6. Interval valued $(\in, \in\vee q)$-fuzzy divisible filters

Definition 6.1. Let L be an MTL-algebra. An interval valued $(\in, \in\vee q)$-fuzzy filter of L is called an interval valued $(\in, \in\vee q)$-fuzzy divisible filter of L, if it satisfies the following condition:

(IFD) $\widetilde{\mu_F}((x \wedge y) \to (x \otimes (x \to y))) \geqslant [0.5, 0.5]$, for all $x, y \in L$.

The set of all interval valued $(\in, \in\vee q)$-fuzzy divisible filters of L is denoted by **IFDF**(L).

Theorem 6.2. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter of L. Then the following conditions are equivalent: for all $x, y, z \in L$,*

(1) $F \in \mathbf{IFDF}(L)$,

(2) $\widetilde{\mu_F}((x \to (y \wedge z)) \to ((x \to y) \otimes ((x \wedge y) \to z))) \geqslant [0.5, 0.5]$,

(3) $\widetilde{\mu_F}((x \otimes (y \to x)) \to (x \otimes (x \wedge y))) \geqslant [0.5, 0.5]$.

Proof. (1)$\Longrightarrow$(2): Assume that $F \in \mathbf{IFDF}(L)$. Then, for all $x, y, z \in L$, by (M10) and (IFD), we have

$$\begin{aligned}&\widetilde{\mu_F}((x \to (y \wedge z)) \to ((x \to y) \otimes ((x \to y) \to (x \to z))))\\ =&\widetilde{\mu_F}(((x \to y) \wedge (x \to z)) \to ((x \to y) \otimes ((x \to y) \to (x \to z))))\\ \geqslant& [0.5, 0.5].\end{aligned}$$

Since

$$\begin{aligned}&(x \wedge y) \to (x \otimes (x \to y))\\ \leqslant& ((x \otimes (x \to y)) \to z) \to ((x \wedge y) \to z)\\ \leqslant& ((x \to y) \otimes ((x \otimes (x \to y)) \to z)) \to ((x \to y) \otimes ((x \wedge y) \to z))\\ =& ((x \to y) \otimes ((x \to y) \to (x \to z))) \to ((x \to y) \otimes ((x \wedge y) \to z)),\end{aligned}$$

by (M12), (M17) and (M6), it follows from (IF4) and (IFD) that

$$\begin{aligned}&\widetilde{\mu_F}(((x \to y) \otimes ((x \to y) \to (x \to z))) \to ((x \to y) \otimes ((x \wedge y) \to z)))\\ \geqslant& \text{rmin}\{\widetilde{\mu_F}((x \wedge y) \to (x \otimes (x \to y))), [0.5, 0.5]\}\\ \geqslant& \text{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5].\end{aligned}$$

Thus, by (M6), (IF4) and (IF3), we deduce that

$$\begin{aligned}&\widetilde{\mu_F}((x \to (y \wedge z)) \to ((x \to y) \otimes ((x \wedge y) \to z)))\\ \geqslant& \text{rmin}\{\widetilde{\mu_F}(((x \to (y \wedge z)) \to ((x \to y) \otimes ((x \to y) \to (x \to z))))\\ &\otimes(((x \to y) \otimes ((x \otimes (x \to y)) \to z)) \to ((x \to y) \otimes ((x \wedge y) \to z)))), [0.5, 0.5]\}\\ \geqslant& \text{rmin}\{\widetilde{\mu_F}((x \to (y \wedge z)) \to ((x \to y) \otimes ((x \to y) \to (x \to z)))),\\ &\widetilde{\mu_F}(((x \to y) \otimes ((x \to y) \to (x \to z))) \to ((x \to y) \otimes ((x \wedge y) \to z))), [0.5, 0.5]\}\\ \geqslant& \text{rmin}\{[0.5, 0.5], [0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5].\end{aligned}$$

So, (2) holds.

(2)$\Longrightarrow$(1): Assume that (2) holds. By putting $x = 1$ in (2), for all $y, z \in L$, we have

$$\begin{aligned}&[0.5, 0.5]\\ \leqslant& \widetilde{\mu_F}((1 \to (y \wedge z)) \to ((1 \to y) \otimes ((1 \wedge y) \to z)))\\ =& \widetilde{\mu_F}((y \wedge z) \to (y \otimes (y \to z))).\end{aligned}$$

Then (IFD) holds. Thus $F \in \mathbf{IFDF}(L)$, by Definition 6.1.

(1)$\Longrightarrow$(3): Assume that $F \in \mathbf{IFDF}(L)$. For all $x, y \in L$, since $y \otimes (y \to x) \leqslant y \wedge x$, by using (M7), we have

$$(y \wedge x) \to (x \otimes (x \to y)) \leqslant (y \otimes (y \to x)) \to (x \otimes (x \to y)).$$

Then it follows from (IF4) and (IFD) that

$\widetilde{\mu_F}((y \otimes (y \to x)) \to (x \otimes (x \to y)))$
$\geqslant \mathrm{rmin}\{\widetilde{\mu_F}((y \wedge x) \to (x \otimes (x \to y))), [0.5, 0.5]\} \quad \geqslant \mathrm{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]$.
Thus, (3) holds.

(3)$\Longrightarrow$(1): Assume that (3) holds. Then, for all $x, y \in L$, by using (M10), we have
$\widetilde{\mu_F}((y \otimes (y \to (x \wedge y))) \to (x \otimes (x \to (x \wedge y))))$
$= \widetilde{\mu_F}((y \otimes (y \to x)) \to (x \otimes (x \to y))) \geqslant [0.5, 0.5]$.
Thus, by takeing $y = x \wedge z$ in the above inequality, we have
$[0.5, 0.5]$
$\leqslant \widetilde{\mu_F}(((x \wedge z) \otimes ((x \wedge z) \to (x \wedge (x \wedge z)))) \to (x \otimes (x \to (x \wedge (x \wedge z)))))$
$= \widetilde{\mu_F}((x \wedge z) \to (x \otimes (x \to (x \wedge z))))$
$= \widetilde{\mu_F}((x \wedge z) \to (x \otimes (x \to z)))$.
So, (IFD) holds. Hence $F \in \mathbf{IFDF}(L)$ by Definition 6.1. □

Theorem 6.3. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter. Then $F \in \mathbf{IFGF}(L)$ implies $F \in \mathbf{IFDF}(L)$. But the converse does not always hold.*

Proof. Assume that $F \in \mathbf{IFGF}(L)$. For all $x, y \in L$, since $x \otimes y \leqslant x \otimes (x \to y)$, we have $(x \wedge y) \to (x \otimes y) \leqslant (x \wedge y) \to (x \otimes (x \to y))$. It follows from (IF4) and Theorem 3.9(6) that
$\widetilde{\mu_F}((x \wedge y) \to (x \otimes (x \to y)))$
$\geqslant \min\{\widetilde{\mu_F}((x \wedge y) \to (x \otimes y)), [0.5, 0.5]\}$
$\geqslant \min\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]$.
Then, (IFD) holds. Thus $F \in \mathbf{IFDF}(L)$, by Definition 6.1.

Conversely, we consider the MTL-algebra L given in Example 4.2, Define an interval valued fuzzy set F on L by $\widetilde{\mu_F}(0) = \widetilde{\mu_F}(a) = \widetilde{\mu_F}(b) = [0.3, 0.4], \widetilde{\mu_F}(1) = [0.8, 0.9]$. Then $F \in \mathbf{IFDF}(L)$, but $F \notin \mathbf{IFGF}(L)$, because $\widetilde{\mu_F}(a \to (a \otimes a)) = \widetilde{\mu_F}(a) = [0.3, 0.4] < [0.5, .05]$. The proof is completed. □

Theorem 6.4. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter. Then $F \in \mathbf{IFDF}(L)$ implies $F \in \mathbf{IFSF}(L)$. But the converse does not always hold.*

Proof. Assume that $F \in \mathbf{IFDF}(L)$. For all $x \in L$, since
$(x'' \wedge x) \to (x'' \otimes (x'' \to x))$
$\leqslant (x'' \otimes (x'' \to x))' \to (x'' \wedge x)'$
$\leqslant (x'' \otimes (x'' \otimes (x'' \to x))') \to (x'' \otimes (x'' \wedge x)')$
$\leqslant (x'' \otimes (x'' \wedge x)')' \to (x'' \otimes (x'' \otimes (x'' \to x))')'$,
by using (IF4) and (IFD), we have
$\widetilde{\mu_F}((x'' \otimes (x'' \wedge x)')' \to (x'' \otimes (x'' \otimes (x'' \to x))')')$
$\geqslant \mathrm{rmin}\{\widetilde{\mu_F}((x'' \wedge x) \to (x'' \otimes (x'' \to x))), [0.5, 0.5]\}$
$\geqslant \mathrm{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]$.
Since $(x'' \otimes (x'' \wedge x)')' = (x'' \otimes x')' = x'' \to x'' = 1$, by $x \leqslant x''$ and (M19), we have $\widetilde{\mu_F}((x'' \otimes (x'' \wedge x)')') = \widetilde{\mu_F}(1) \geqslant [0.5, 0.5]$. Then
$\widetilde{\mu_F}((x'' \otimes (x'' \to (x'' \to x)'))')$
$= \widetilde{\mu_F}((x'' \otimes (x'' \otimes (x'' \to x))')')$

$\geqslant \text{rmin}\{\widetilde{\mu_F}((x'' \otimes (x'' \wedge x)')'), \widetilde{\mu_F}((x'' \otimes (x'' \wedge x)')' \rightarrow (x'' \otimes (x'' \otimes (x'' \rightarrow x))')'), [0.5, 0.5]\}$
$\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5].$

Since $x' \rightarrow (x'' \rightarrow x) = x'' \rightarrow (x' \rightarrow x) = (x' \rightarrow 0) \rightarrow (x' \rightarrow x) \geqslant 0 \rightarrow x = 1$, we have $x' \rightarrow (x'' \rightarrow x) = 1$. Thus $x' \leqslant x'' \rightarrow x$, i.e., $(x'' \rightarrow x)' \leqslant x''$. So, we get that

$\widetilde{\mu_F}((x'' \rightarrow x)'')$
$= \widetilde{\mu_F}((x'' \wedge (x'' \rightarrow x)')')$
$\geqslant \text{rmin}\{\widetilde{\mu_F}((x'' \otimes (x'' \rightarrow (x'' \rightarrow x)'))'), \widetilde{\mu_F}((x'' \otimes (x'' \rightarrow (x'' \rightarrow x)'))' \rightarrow (x'' \wedge (x'' \rightarrow x)')'), [0.5, 0.5]\}$
$\geqslant \text{rmin}\{[0.5, 0.5], \widetilde{\mu_F}((x'' \wedge (x'' \rightarrow x)') \rightarrow (x'' \otimes (x'' \rightarrow (x'' \rightarrow x)'))), [0.5, 0.5]\}$
$\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5].$

Hence, (IFS) holds. Therefore $F \in \textbf{IFSF}(L)$, by Definition 5.8.

The last part can be shown by Example 5.11. In that example, $F \in \textbf{IFSF}(L)$, but $F \notin \textbf{IFDF}(L)$, because $\widetilde{\mu_F}((c \wedge b) \rightarrow (c \otimes (c \rightarrow b))) = \widetilde{\mu_F}(b \rightarrow a) = \widetilde{\mu_F}(c) = [0.3, 0.4] < [0.5, 0.5]$. The proof is completed. □

Theorem 6.5. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter. Then $F \in \textbf{IFMVF}(L)$ implies $F \in \textbf{IFDF}(L)$. But the converse does not always hold.*

Proof. Assume that $F \in \textbf{IFMVF}(L)$. For all $x, y \in L$, since

$((x' \rightarrow y') \rightarrow y') \rightarrow (x' \vee y')$
$\leqslant ((x' \rightarrow y') \rightarrow y') \rightarrow (x \wedge y)'$
$= ((y \rightarrow x'') \otimes y)' \rightarrow (x \wedge y)' \leqslant (x \wedge y)'' \rightarrow ((y \rightarrow x'') \otimes y)''$
$\leqslant (x \wedge y) \rightarrow ((y \rightarrow x'') \otimes y)'',$

by using (IF4) and Theorem 3.7(3), we have

$\widetilde{\mu_F}((x \wedge y) \rightarrow ((y \rightarrow x'') \otimes y)'')$
$\geqslant \text{rmin}\{\widetilde{\mu_F}(((x' \rightarrow y') \rightarrow y') \rightarrow (x' \vee y')), [0.5, 0.5]\}$
$\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5].$

Since $F \in \textbf{IFMVF}(L)$ implies that $F \in \textbf{IFRF}(L)$, by Theorem 5.3, we get that $\widetilde{\mu_F}(((y \rightarrow x'') \otimes y)'' \rightarrow ((y \rightarrow x'') \otimes y)) \geqslant [0.5, 0.5]$ by (IFR). Then, we can obtain that

$\widetilde{\mu_F}((x \wedge y) \rightarrow ((y \rightarrow x'') \otimes y))$
$\geqslant \text{rmin}\{\widetilde{\mu_F}(((x \wedge y) \rightarrow ((y \rightarrow x'') \otimes y)'') \otimes (((y \rightarrow x'') \otimes y)'' \rightarrow ((y \rightarrow x'') \otimes y))), [0.5, 0.5]\}$
$\geqslant \text{rmin}\{((x \wedge y) \rightarrow ((y \rightarrow x'') \otimes y)''), \widetilde{\mu_F}(((y \rightarrow x'') \otimes y)'' \rightarrow ((y \rightarrow x'') \otimes y)), [0.5, 0.5]\}$
$\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5].$

Since $(y \otimes (y \rightarrow x'')) \rightarrow (y \otimes (y \rightarrow x)) \geqslant (y \rightarrow x'') \rightarrow (y \rightarrow x) \geqslant x'' \rightarrow x$, it follows from (IF4) and (IFR) that

$$\widetilde{\mu_F}((y \otimes (y \rightarrow x'')) \rightarrow (y \otimes (y \rightarrow x))) \geqslant \widetilde{\mu_F}(x'' \rightarrow x) \geqslant [0.5, 0.5].$$

Thus

$\widetilde{\mu_F}((x \wedge y) \rightarrow (y \otimes (y \rightarrow x)))$
$\geqslant \text{rmin}\{(((x \wedge y) \rightarrow ((y \rightarrow x'') \otimes y)) \otimes ((y \otimes (y \rightarrow x'')) \rightarrow (y \otimes (y \rightarrow x)))), [0.5, 0.5]\}$
$\geqslant \text{rmin}\{\widetilde{\mu_F}((x \wedge y) \rightarrow ((y \rightarrow x'') \otimes y)), \widetilde{\mu_F}((y \otimes (y \rightarrow x'')) \rightarrow (y \otimes (y \rightarrow x))), [0.5, 0.5]\}$

$\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]$.
So, (IFD) holds. Hence, $F \in \mathbf{IFDF}(L)$, by Definition 6.1.

Conversely, we consider the MTL-algebra L given in Example 4.2. Define an interval valued fuzzy set F on L, by $\widetilde{\mu_F}(0) = \widetilde{\mu_F}(a) = \widetilde{\mu_F}(b) = [0.3, 0.4], \widetilde{\mu_F}(1) = [0.8, 0.9]$. Then $F \in \mathbf{IFDF}(L)$, but $F \notin \mathbf{IFMVF}(L)$, because $\widetilde{\mu_F}(((b \to a) \to a) \to (b \vee a)) = \widetilde{\mu_F}(b) = [0.3, 0.4] < [0.5, .05]$. The proof is completed. □

Theorem 6.6. *Let L be an MTL-algebra and F an interval valued $(\in, \in\vee q)$-fuzzy filter. Then $F \in \mathbf{IFMVF}(L)$ if and only if $F \in \mathbf{IFRF}(L)$ and $F \in \mathbf{IFDF}(L)$.*

Proof. $\Longrightarrow$: Assume that $F \in \mathbf{IFMVF}(L)$. Then, by Theorem 5.3 and Theorem 6.5, we can see that $F \in \mathbf{IFRF}(L)$ and $F \in \mathbf{IFDF}(L)$.

$\Longleftarrow$: Assume that $F \in \mathbf{IFRF}(L)$ and $F \in \mathbf{IFDF}(L)$. For all $x, y \in L$, putting $u = x \vee y$. Then, we have

$$\begin{aligned} &\widetilde{\mu_F}(u' \to (x' \otimes (y \to x''))) \\ &= \widetilde{\mu_F}(u' \to (x' \otimes (x' \to y'))) \\ &= \widetilde{\mu_F}((x \vee y)' \to (x' \otimes (x' \to y'))) \\ &= \widetilde{\mu_F}((x' \wedge y') \to (x' \otimes (x' \to y'))) \geqslant [0.5, 0.5], \end{aligned}$$

by (M23), (M22) and (IFD).

On the other hand, since

$$\begin{aligned} &(u' \to (x' \otimes (y \to x''))) \to (((y \to x) \to x'') \to u'') \\ &= (u' \to (x' \otimes (y \to x''))) \to ((x' \otimes (y \to x))' \to u'') \\ &\geqslant (u' \to (x' \otimes (y \to x''))) \to (u''' \to (x' \otimes (y \to x))'') \\ &\geqslant (u' \to (x' \otimes (y \to x''))) \to (u' \to (x' \otimes (y \to x))) \\ &\geqslant (x' \otimes (y \to x'')) \to (x' \otimes (y \to x)) \\ &\geqslant (y \to x'') \to (y \to x) \\ &\geqslant x'' \to x, \end{aligned}$$

by (M19), (M17) and (M12), it follows from (IF4) and (IFR) that

$$\begin{aligned} &\widetilde{\mu_F}((u' \to (x' \otimes (y \to x''))) \to (((y \to x) \to x'') \to u'')) \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}(x'' \to x), [0.5, 0.5]\} \\ &\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]. \end{aligned}$$

Together with the above two results and (IF6), we get

$$\begin{aligned} &\widetilde{\mu_F}(((y \to x) \to x'') \to u'') \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}(u' \to (x' \otimes (y \to x''))), \widetilde{\mu_F}((u' \to (x' \otimes (y \to x''))) \\ &\qquad \to (((y \to x) \to x'') \to u'')), [0.5, 0.5]\} \\ &\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]. \end{aligned}$$

Thus, by (M6), (IF3) and (IFR), we have

$$\begin{aligned} &\widetilde{\mu_F}(((y \to x) \to x'') \to u) \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}((((y \to x) \to x'') \to u'') \otimes (u'' \to u)), [0.5, 0.5]\} \\ &\geqslant \text{rmin}\{\widetilde{\mu_F}(((y \to x) \to x'') \to u''), \widetilde{\mu_F}(u'' \to u), [0.5, 0.5]\} \\ &\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]. \end{aligned}$$

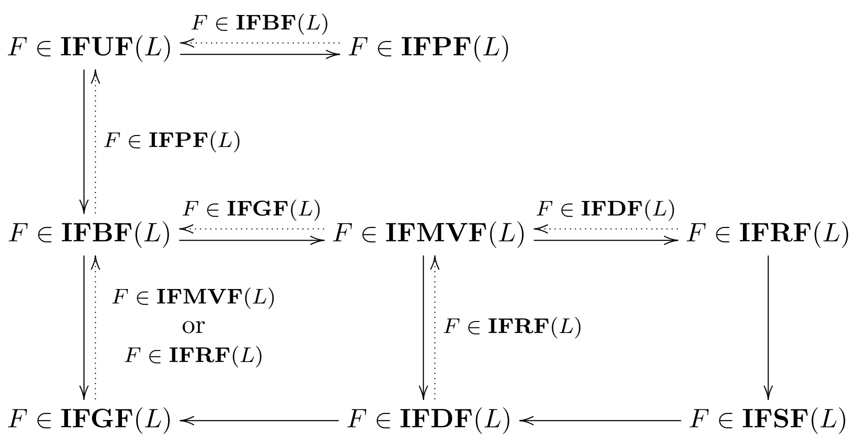
Since $x \leqslant x''$, we have $(y \to x) \to x \leqslant (y \to x) \to x''$. This implies $((y \to x) \to x'') \to u \leqslant ((y \to x) \to x) \to u$. So, by using (IF4), we obtain

$$\begin{aligned} &\widetilde{\mu_F}(((y \to x) \to x) \to (x \vee y)) \\ &= \widetilde{\mu_F}(((y \to x) \to x) \to u) \geqslant \text{rmin}\{\widetilde{\mu_F}(((y \to x) \to x'') \to u), [0.5, 0.5]\} \\ &\geqslant \text{rmin}\{[0.5, 0.5], [0.5, 0.5]\} = [0.5, 0.5]. \end{aligned}$$

Hence $F \in \mathbf{IFMVF}(L)$, by Theorem 3.7(3). □

7. Conclusions

As well known, the filters play an important role in investigating the structure of a logical system. In this paper, the interval valued $(\in, \in\vee q)$-fuzzy filter theory in MTL-algebras is further studied. Some new properties and equivalent characterizations of interval valued $(\in, \in\vee q)$-fuzzy Boolean, MV- and G-filters are given. We proposed the concepts of interval valued $(\in, \in\vee q)$-fuzzy prime, ultra, regular, strong and divisible filters in MTL-algebras and characterized several of their properties. At the same time, we also established the relationships between various interval valued $(\in, \in\vee q)$-fuzzy filters. It is proved that an interval valued $(\in, \in\vee q)$-fuzzy filter is an interval valued $(\in, \in\vee q)$-fuzzy MV-filter if and only if it is both an interval valued $(\in, \in\vee q)$-fuzzy regular filter and an interval valued $(\in, \in\vee q)$-fuzzy divisible filter. Results obtained in this paper not only enrich the content of interval valued $(\in, \in\vee q)$-fuzzy filter theory in MTL-algebras, but also show interactions of algebraic technique and interval valued fuzzifying method in the studying logic problems. We hope that more links of interval valued fuzzy sets and logics emerge by the stipulating of this work. Finally, we show relationships among interval valued $(\in, \in\vee q)$-fuzzy filters in MTL-algebras by the following figure.



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