Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 13, No. 6, (June 2017), pp. 729–742
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Decision making with distance and cosine similarity measures for intuitionistic hesitant fuzzy sets

Hong Wang, Dandan Li

Received 21 July 2016; Revised 2 September 2016; Accepted 9 November 2016

Abstract. The definition of intuitionistic hesitant fuzzy sets(IHFSs) is developed based on intuitionistic fuzzy sets(IFSs) and hesitant fuzzy sets(HFSs) and it allows the membership of an element to be a set of several possible intuitionistic fuzzy values. The intuitionistic hesitant fuzzy sets(IHFSs) is a new and flexible tool in representing hesitant information in decision making. Distance and cosine similarity measures have been applied widely in many research domains and practical fields. In this paper, we firstly proposed some distance and similarity measures for IHFSs based on Hamming distance, Euclidean distance and generalized distance, especially, a new cosine similarity measure for IHFSs is proposed and the corresponding cosine distance measures are given. It is shown that all three parameters (membership degree, non membership degree and degree of hesitation) describing intuitionistic hesitant fuzzy sets should be taken into account when calculating those distance and cosine similarity measures. In the end, a new cosine similarity measure and distance measures are applied to multiple attribute decision-making.

2010 AMS Classification: 03E72, 08A72

Keywords: Intuitionistic hesitant fuzzy sets, Distance measure, Similarity measure, Cosine similarity measure, Cosine distance measure.

Corresponding Author: Dandan Li (1607330917@qq.com)

1. Introduction

Fuzzy sets were first introduced by Zadeh [33], and some researches about distances and similarity measures of fuzzy sets have also been proposed. They were later extended to interval-valued fuzzy sets(IVFSs) [34] by Zadeh, and the subsets of the interval [0,1] were used for membership value rather than exact numberical values. Further, fuzzy sets were generalized to intuitionistic fuzzy sets(IFSs) by Atanassov [1], which are characterized by a membership function and a non-membership function. Then, Atanassov and Gargov [2] proposed the interval-valued intuitionistic fuzzy sets(IVIFSs), whose membership and non-membership are represented by interval numbers. Recently, Torra [23] introduced the concept of hesitant fuzzy sets(HFSs) as an extension fuzzy sets in which the membership degree of a given element to a set, is defined as a set of possible values. Xu and Xia [30] proposed a variety of distance measures for hesitant fuzzy sets, based on which the corresponding similarity measures can be obtained. Xu and Xia [31] defined the distance and correlation measures for hesitant fuzzy information and discussed their properties in detail. HFSs are highly useful in handling the situations where people have hesitancy in providing their preferences over objects in the decision-making process [12, 13, 14, 15]. Chen et al. [6] generalized the concept of HFSs to interval-valued hesitant fuzzy sets(IVHFSs) in which the membership degrees of an element to a given set are not exactly defined but denoted by several possible interval values. Furthermore, Zhu et al. [35] proposed dual hesitant fuzzy sets(DHFSs), which encompass fuzzy sets, IFSs, HFSs, and fuzzy multisets as special cases, whose membership degrees and non-membership degrees are represented by a set of possible values. If the idea of DHFSs is used from a new perspective, Chen et al. [7] introduced the notion of intuitionistic hesitant fuzzy sets(IHFSs), which extended the hesitant fuzzy sets to intuitionistic fuzzy environments and permitted the membership of an element to be a set of several possible intuitionistic fuzzy values. Thus, the IHFSs is a very useful tool to deal with the situations in which the experts hesitate between several possible intuitionistic fuzzy values to assess the degree to which an alternative satisfies an attribute.

A similarity measure is an important tool for determining the degree of similarity between two objects. Based on the similarity measures that are very useful in some ares, such as data analysis and classification, machine learning, pattern recognition, decision making and image processing. Some researches on similarity measures between fuzzy sets have been proposed and studied in recent years [5]. With the research of fuzzy sets, Li and Cheng [11] introduced several similarity measures between IFSs and applied the measures to pattern recognition. The similarity and distance of IFSs are counterparts, Szmidt and Kacprzyk [20, 21] introduced the Hamming distance and the Euclidean distance between IFSs and proposed a similarity measure between IFSs on the distance. Hung and Yang [9] proposed another method to calculate the distance between intuitionistic fuzzy sets based on the Hausdorff distance and used it to propose several similarity measures between intuitionistic fuzzy sets. Thereafter, other distance and similarity measures for IFSs and IVIFSs have been proposed [8, 24, 25, 28, 29]. Su and Xu [19] introduced a number of dual hesitant fuzzy distance measures and developed the similarity measures of DHFSs on distance. Liao and Xu [16] discussed distance and similarity measures for hesitant fuzzy linguistic term sets. Liao and Xu [17] also proposed novel cosine distance and similarity measures from a geometric point of view. In this paper, we propose a series of distance measures for intuitionistic hesitant fuzzy information, which include the membership degree and non-membership degree, hesitation degree.

The cosine similarity measures were defined as the inner product of two vectors divided by product of their lengths [4, 18]. Ye [32] introduced the cosine similarity measure and the weighted cosine similarity measure between IFSs, which are considered the membership degree and non-membership degree. However, Wan [10] introduced a new cosine similarity measure for IVIFSs. In his research, the method included membership, non-membership and hesitation degree are considered. Bai [3] proposed a cosine similarity measure for DHFSs and a weighted cosine similarity measure for DHFSs, and solved the problem of multi-criteria group decision-making. However, as we know, the cosine similarity measure for intuitionistic hesitant fuzzy sets has not been presented.

In this paper, we define the basic distances between the intuitionistic hesitant fuzzy sets: the normalized Hamming distance, the normalized Euclidean distance, generalized intuitionistic hesitant normalized distance. We propose a new cosine similarity measure for intuitionistic hesitant fuzzy sets, which not only involves the first two parameters(membership degree and non-membership degree), but also takes into account the third parameter(hesitation degree). To do so, the remainder of this paper is organized as follows: Section 2 reviews some basic concepts of fuzzy sets , intuitionistic fuzzy sets and intuitionistic hesitant fuzzy sets. Section 3 introduces the distance and similarity measures of IHFSs. Section 4 defines a new cosine similarity measure and a weighted cosine similarity measure between IHFSs. The corresponding cosine distance measure and weighted cosine distance measure between IHFSs are obtained. Some examples are presented to illustrate the developed approach in section 5. Finally, section 6 gives the conclusions.

2. Preliminaries

Definition 2.1 ([1])**.** An IFS A on the universe of discourse X, is defined as:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle |\ 0 \leq \mu_A(x) + \nu_A(x) \leq 1, x \in X\},$$

where the maps $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$, define the degree of membership and non-membership of the element $x \in X$, respectively.

The pair $(\mu_A(x), \nu_A(x))$ is called an intuitionistic fuzzy value [27] and usually, it is denoted as $\alpha = (\mu_\alpha, \nu_\alpha)$, and the set of all IFVs on X denoted by V. $\pi_\alpha = 1-\mu_\alpha-\nu_\alpha$ is referred to as the degree of hesitation for $x \in X$ to A.

For any two IFVs $\alpha = (\mu_\alpha, \nu_\alpha)$, $\beta = (\mu_\beta, \nu_\beta)$. The following operations can be defined:

(i) $(\mu_\alpha, \nu_\alpha) \leq_{L^*} (\mu_\beta, \nu_\beta) \Leftrightarrow \mu_\alpha \leq \mu_\beta, \nu_\alpha \geq \nu_\beta$,

(ii) $(\mu_\alpha, \nu_\alpha) \bigvee (\mu_\beta, \nu_\beta) = (\max\{\mu_\alpha, \mu_\beta\}, \min\{\nu_\alpha, \nu_\beta\})$,

(iii) $(\mu_\alpha, \nu_\alpha) \bigwedge (\mu_\beta, \nu_\beta) = (\min\{\mu_\alpha, \mu_\beta\}, \max\{\nu_\alpha, \nu_\beta\})$,

(iv) the complement of an IFV (μ_α, ν_α): $(\mu_\alpha, \nu_\alpha)^c = (\nu_\alpha, \mu_\alpha)$,

where $L^* = \{(\mu, \nu) \in [0,1]^2 |\ \mu + \nu \leq 1\}$.

Definition 2.2 ([23])**.** Let X be a fixed set. Then a hesitant fuzzy set(HFS) A on X is defined in terms of a function $h_A(x)$ that when applied to X returns a finite subset of $[0,1]$.

To be easily understood, Xia and Xu [26] expressed the HFS by a mathematical symbol:

$$A = \{\langle x, h_A(x)\rangle | x \in X\},$$

where $h_A(x)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees of the element $x \in X$ to the set A. For convenience, Xia and Xu named $h_A(x)$ a hesitant fuzzy element(HFE).

Given three HFEs represented by h, h_1, h_2, Torra [23] defined some operations on them, which can be described as:

(i) $h^c = \{1 - \gamma | \gamma \in h\}$,

(ii) $h_1 \bigcup h_2 = \{\max(\gamma_1, \gamma_2) | \gamma_1 \in h_1, \gamma_2 \in h_2\}$,

(iii) $h_1 \bigcap h_2 = \{\min(\gamma_1, \gamma_2) | \gamma_1 \in h_1, \gamma_2 \in h_2\}$.

Definition 2.3 ([7]). Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite and nonempty universe of discourse, an intuitionistic hesitant fuzzy set (IHFS) on X is given in terms of a function that when applied to X returns a subset of V (the set of all IFVs on X).

To be easily understood, we express the IHFS by a mathematical symbol

$$A = \{\langle x, h_A(x)\rangle | x \in X\},$$

where $h_A(x)$ is a set of some intuitionistic fuzzy values (IFVs) in V, denoting the possible membership degrees and non-membership degrees of the element $x \in X$ to the set A. For convenience, we call $h = h_A(x)$ an intuitionistic hesitant fuzzy element (IHFE) and $l(h_A(x))$ is the number of IFVs in IHFE $h_A(x)$. The set of all IHFEs on X is denoted as H and the set of all the IHFSs on X are denoted as IHFSs(X), respectively. If $\alpha \in h$, then α is an IFV, and it can be denoted by $\alpha = (\mu_\alpha, \nu_\alpha)$.

For any $\alpha \in h$ if α is a real number in $[0, 1]$, then h reduces to a hesitant fuzzy element (HFE) and A reduces to a hesitant fuzzy set (HFS); for any $h \in A$, if h posses only one α, then h reduces to an intuitionistic fuzzy value (IFV) and A reduces to an intuitionistic fuzzy set.

Definition 2.4. Given three IHFEs represented by h, h_1, h_2, one defines some operations on them, which can be described as follows:

(i) $h^c = \{\alpha^c | \alpha \in h\} = \{(\nu_\alpha, \mu_\alpha) | \alpha \in h\}$.

(ii) $h_1 \bigcup h_2 = \{\alpha_1 \bigvee \alpha_2 | \alpha_1 \in h_1, \alpha_2 \in h_2\}$
$= \{(\mu_{\alpha_1} \bigvee \mu_{\alpha_2}, \nu_{\alpha_1} \bigwedge \nu_{\alpha_2}) | \alpha_1 \in h_1, \alpha_2 \in h_2\}$,

(iii) $h_1 \bigcap h_2 = \{\alpha_1 \bigwedge \alpha_2 | \alpha_1 \in h_1, \alpha_2 \in h_2\}$
$= \{(\mu_{\alpha_1} \bigwedge \mu_{\alpha_2}, \nu_{\alpha_1} \bigvee \nu_{\alpha_2}) | \alpha_1 \in h_1, \alpha_2 \in h_2\}$,

(iv) $\sup h = (\sup\{\mu_\alpha\}, \inf\{\nu_\alpha\})$, $\forall\ (\mu_\alpha, \nu_\alpha) \in h$,

(v) $\inf h = (\inf\{\mu_\alpha\}, \sup\{\nu_\alpha\})$, $\forall\ (\mu_\alpha, \nu_\alpha) \in h$.

Let $A = \langle x, h(x)\rangle$ be an IHFS on the reference set X, then empty IHFE, empty IHFS, full IHFE and full IHFS are defined as follows:

(vi) the empty IHFE : $h(x) = \{(0, 1)\}$ (for for short, $h(x) = [0]$), $x \in X$,

(vii) the empty IHFS : $h(x) = \{(0, 1)\}$ (for short, $A = \{[0]\}$), all $x \in X$,

(viii) the full IHFE : $h(x) = \{(1, 0)\}$ (for for short, $h(x) = [1]$), $x \in X$, $h(x) = [1]$,

(ix) the full IHFS : $h(x) = \{(1, 0)\}$ (for for short, $A = \{[1]\}$), for all $x \in X$.

Example 2.5. Let $X = \{x_1, x_2\}$ be the reference set and let $h(x_1) = \{(0.7, 0.2), (0.5, 0.3)\}$, $h(x_2) = \{(0.8, 0.1), (0.6, 0.2)\}$ be two IHFEs, respctively. Then A can be considered as an IHFS and is represented as follows:

$$A = \{\langle x_1, (0.7, 0.2), (0.5, 0.3)\rangle, \langle x_2, (0.8, 0.1), (0.6, 0.2)\rangle\}.$$

Definition 2.6. Let X be a reference set. Then an IHFS A on X is defined as follows:

$$A = \{\langle x, h_A(x)\rangle | x \in X\},$$

where the hesitancy degree $\pi_A(x)$ of an element x to an IHFS A is a set of some different values in $[0,1]$, and $\pi_A(x) = \bigcup_{\alpha\in h_A(x)}\{1 - \mu_\alpha(x) - \nu_\alpha(x)\}$.

Example 2.7. In Example 2.5, the hesitancy degrees of x_1 and x_2 can be caculated by above definition as the followings:

$$\pi_A(x_1) = \{1 - 0.7 - 0.2, 1 - 0.5 - 0.3\} = \{0.1, 0.2\},$$
$$\pi_A(x_2) = \{1 - 0.8 - 0.1, 1 - 0.6 - 0.2\} = \{0.1, 0.2\}.$$

3. Distance and similarity measures of intuitionistic hesitant fuzzy sets

Definition 3.1. Let A and B be two IHFSs on $X = \{x_1, x_2, \dots, x_n\}$. Then the distance measure between A and B, denoted by $d(A, B)$, is defined as the followings:
(i) $0 \le d(A, B) \le 1$,
(ii) $d(A, B) = 0$ if and only if $A = B$,
(iii) $d(A, B) = d(B, A)$.

Definition 3.2. Let A and B be two IHFSs on $X = \{x_1, x_2, \dots, x_n\}$. Then the similarity measure between A and B, denoted by $S(A, B)$, is defined as the followings:
(i) $0 \le S(A, B) \le 1$,
(ii) $S(A, B) = 1$ if and only if $A = B$,
(iii) $S(A, B) = S(B, A)$.

We define an intuitionistic hesitant normalized Hamming distance:

$$d_{ihnh}(A, B) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2l_i}\sum_{j=1}^{l_i} |\mu_A^{\sigma(j)}(x_i) - \mu_B^{\sigma(j)}(x_i)| + |\nu_A^{\sigma(j)}(x_i) - \nu_B^{\sigma(j)}(x_i)| + |\pi_A^{\sigma(j)}(x_i) - \pi_B^{\sigma(j)}(x_i)|\right)$$

and

an intuitionistic hesitant normalized Euclidean distance:

$$d_{ihne}(A, B) = \left\{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2l_i}\sum_{j=1}^{l_i} |\mu_A^{\sigma(j)}(x_i) - \mu_B^{\sigma(j)}(x_i)|^2 + |\nu_A^{\sigma(j)}(x_i) - \nu_B^{\sigma(j)}(x_i)|^2 + |\pi_A^{\sigma(j)}(x_i) - \pi_B^{\sigma(j)}(x_i)|^2\right)\right\}^{\frac{1}{2}}.$$

With the generalization of the two distances, a generalized intuitionistic hesitant normalized distance can be obtained:

$$d_{gihn}(A, B) = \left\{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2l_i}\sum_{j=1}^{l_i} |\mu_A^{\sigma(j)}(x_i) - \mu_B^{\sigma(j)}(x_i)|^\lambda + |\nu_A^{\sigma(j)}(x_i) - \nu_B^{\sigma(j)}(x_i)|^\lambda + |\pi_A^{\sigma(j)}(x_i) - \pi_B^{\sigma(j)}(x_i)|^\lambda\right)\right\}^{\frac{1}{\lambda}},$$

where $\lambda > 0$.

Let $l_i = \max\{l(h_A(x_i)), l(h_B(x_i))\}$, for each x_i in X, where $l(h_A(x_i))$ and $l(h_B(x_i))$ represent the number of IFVs in $h_A(x_i)$ and $h_B(x_i)$, respectively. All the elements in each $h_A(x_i)$ are arranged in increasing order, and then $h_A^{\sigma(j)}(x_i) = (\mu_A^{\sigma(j)}(x_i), \nu_A^{\sigma(j)}(x_i))$ is referred to as the jth largest value in $h_A(x_i)$. When $l(h_A(x_i)) \neq l(h_B(x_i))$, one can make them have the same number of elements through adding some elements to the IHFE which has less number of elements. According to the pessimistic principle, the smallest element will be added. Then, if $l(h_A(x_i)) < l(h_B(x_i))$, $l(h_A(x_i))$ should be extended by adding the minimum IFV in it until it has the same length as $h_B(x_i)$. In the same way, all the elements in each $\pi_A(x_i)$ are arranged in increasing order, and $\pi_A^{\sigma(j)}(x_i)$ is represented to as the jth largest value in $\pi_A(x_i)$.

In practical application, we should consider the weights of x_i $(i = 1, 2, \ldots, n)$ in X. Let $\omega = \{\omega_1, \omega_2, \ldots, \omega_n\}^T$ be the weight vector of x_i $(i = 1, 2, \ldots, n)$ with $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$. Then the generalized intuitionistic hesitant weighted normalized distance between IHFSs A and B is proposed as follows:

$$\begin{aligned} &d_{gihwn}(A,B)\\ &= \{\textstyle\sum_{i=1}^{n}(\frac{w_i}{2l_i}\sum_{j=1}^{l_i}|\mu_A^{\sigma(j)}(x_i)-\mu_B^{\sigma(j)}(x_i)|^\lambda+|\nu_A^{\sigma(j)}(x_i)-\nu_B^{\sigma(j)}(x_i)|^\lambda\\ &\qquad\qquad +|\pi_A^{\sigma(j)}(x_i)-\pi_B^{\sigma(j)}(x_i)|^\lambda)\}^{\frac{1}{\lambda}}, \end{aligned}$$

where $\lambda > 0$.

Next, we shall show that the proposed distance measures satisfy axiom definition of distance measure.

Proposition 3.3. *Let A and B be any IHFSs. Then the $d_{ihnh}(A, B)$ is the distance measure.*

Proof. (i) By the distance of intuitionistic fuzzy sets, we have

$$0 \leq \tfrac{1}{2l_i}\textstyle\sum_{j=1}^{l_i}|\mu_A^{\sigma(j)}(x_i)-\mu_B^{\sigma(j)}(x_i)|+|\nu_A^{\sigma(j)}(x_i)-\nu_B^{\sigma(j)}(x_i)|+|\pi_A^{\sigma(j)}(x_i)-\pi_B^{\sigma(j)}(x_i)| \leq 1.$$

Then $0 \leq d_{ihnh}(A, B) \leq 1$.

(ii) When $A = B$, we can easily obtain $d_{ihnh}(A, B) = 0$. Now we need prove

$$d_{ihnh}(A,B) = 0 \Longrightarrow A = B.$$

Let $d_{ihnh}(A, B) = 0$. Then we get

$$\tfrac{1}{2l_i}\textstyle\sum_{j=1}^{l_i}|\mu_A^{\sigma(j)}(x_i)-\mu_B^{\sigma(j)}(x_i)|+|\nu_A^{\sigma(j)}(x_i)-\nu_B^{\sigma(j)}(x_i)|+|\pi_A^{\sigma(j)}(x_i)-\pi_B^{\sigma(j)}(x_i)| = 0.$$

Thus

$$\mu_A^{\sigma(j)}(x_i) = \mu_B^{\sigma(j)}(x_i), \nu_A^{\sigma(j)}(x_i) = \nu_B^{\sigma(j)}(x_i), \pi_A^{\sigma(j)}(x_i) = \pi_B^{\sigma(j)}(x_i).$$

So $h_A^{\sigma(j)}(x_i) = h_B^{\sigma(j)}(x_i)$. Hence $A = B$.

(iii) For any two intuitionistic hesitant fuzzy sets A and B,

$$\begin{aligned} &d_{ihnh}(A,B)\\ &= \tfrac{1}{n}\textstyle\sum_{i=1}^{n}(\frac{1}{2l_i}\sum_{j=1}^{l_i}|\mu_A^{\sigma(j)}(x_i)-\mu_B^{\sigma(j)}(x_i)|+|\nu_A^{\sigma(j)}(x_i)-\nu_B^{\sigma(j)}(x_i)|\\ &\qquad\qquad +|\pi_A^{\sigma(j)}(x_i)-\pi_B^{\sigma(j)}(x_i)|)\\ &= \tfrac{1}{n}\textstyle\sum_{i=1}^{n}(\frac{1}{2l_i}\sum_{j=1}^{l_i}|\mu_B^{\sigma(j)}(x_i)-\mu_A^{\sigma(j)}(x_i)|+|\nu_B^{\sigma(j)}(x_i)-\nu_A^{\sigma(j)}(x_i)|\\ &\qquad\qquad +|\pi_B^{\sigma(j)}(x_i)-\pi_A^{\sigma(j)}(x_i)|)\\ &= d_{ihnh}(B,A). \end{aligned}$$

Therefore, $d_{ihnh}(A, B)$ is a distance measure. □

Proposition 3.4. *Let A and B be any IHFSs. Then $d_{ihne}(A, B)$, $d_{gihn}(A, B)$ and $d_{gihwn}(A, B)$ are distance measures.*

Proof. Follows from Proposition 3.3. □

It is well known that distance measure and similarity measure are complementary concepts. Therefore, we may use the distance measure to define a similarity measure, and vice versa. Let Z be a monotone decreasing function. Since $0 \leq d(A,B) \leq 1$,

$$Z(1) \leq Z(d(A,B)) \leq Z(0).$$

This implies

$$0 \leq \frac{Z(d(A,B)) - Z(1)}{Z(0) - Z(1)} \leq 1.$$

Thus, we may define the similarity measure between IHFSs A and B as follows:

$$S(A,B) = \frac{Z(d(A,B)) - Z(1)}{Z(0) - Z(1)}.$$

From Definition 3.1 and the property of $Z(\cdot)$, it is evident that the similarity measure meets all the requirements listed in Definition 3.2.

We find that different formulas can be developed to calculate the similarity measure between IHFSs using different. The problem here is to select a useful and reasonable $Z(\cdot)$. For instance, (1) $Z(x) = 1 - x$ (2) $Z(x) = 1 - e^{-x}$ (3) $Z(x) = \frac{1}{1+x}$. So the similarity measures between A and B are defined as follows:

(i) $S_1(A,B) = 1 - d(A,B)$,
(ii) $S_2(A,B) = \frac{e^{-d(A,B)} - e^{-1}}{1 - e^{-1}}$,
(iii) $S_3(A,B) = \frac{1-d(A,B)}{1+d(A,B)}$.

Example 3.5. Let us consider following intuitionistic hesitant fuzzy sets A, B, $X = \{x_1, x_2\}$, where

$$A = \{\langle x_1, (0.5, 0.3), (0.7, 0.2)\rangle, \langle x_2, (0.8, 0.15), (0.9, 0.1)\rangle\},$$

and

$$B = \{\langle x_1, (0.6, 0.3), (0.8, 0.2)\rangle, \langle x_2, (0.5, 0.4), (0.7, 0.2)\rangle\}.$$

Then

$$\begin{aligned} & d_{ihnh}(A,B) \\ &= \tfrac{1}{2}[\tfrac{1}{4}(|0.5-0.6| + |0.3-0.3| + |0.2-0.1| + |0.7-0.8| + |0.2-0.2| + |0.1-0|) \\ &\quad + \tfrac{1}{4}(|0.8-0.5| + |0.15-0.4| + |0.1-0.05| + |0.9-0.7| + |0.1-0.2| + |0.1-0|)] \\ &= 0.17500, \\ & d_{ihne}(A,B) \\ &= \{\tfrac{1}{2}[\tfrac{1}{4}(|0.5-0.6|^2 + |0.3-0.3|^2 + |0.2-0.1|^2 + |0.7-0.8|^2 + |0.2-0.2|^2 + |0.1-0|^2) \\ &\quad + \tfrac{1}{4}(|0.8-0.5|^2 + |0.15-0.4|^2 + |0.1-0.05|^2 + |0.9-0.7|^2 + |0.1-0.2|^2 \\ &\quad + |0.1-0|^2)]\}^{\frac{1}{2}} \\ &= 0.17854. \end{aligned}$$

If we choose $d(A,B) = d_{ihnh}(A,B)$, then

$$S_1(A,B) = 1 - 0.175 = 0.82500,$$

$$S_2(A,B) = \frac{e^{-0.175} - e^{-1}}{1 - e^{-1}} = 0.74602,$$

$$S_3(A,B) = \frac{1 - 0.175}{1 + 0.175} = 0.70213.$$

4. Cosine similarity measure and cosine distance measure for IHFSs

Cosine similarity measures are defined as the inner product of two vectors divided by the product of their length [4, 18]. This is nothing but the cosine of the angle between the vector representations of the two fuzzy sets.

Definition 4.1 ([4, 18])**.** Assume that the $A = \{\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)\}$ and $B = \{\mu_B(x_1), \mu_B(x_2), \ldots, \mu_B(x_n)\}$ are two fuzzy sets in the universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, $x_i \in X$. A cosine similarity measure based on Battacharya's distance between two fuzzy sets A and B can be described as

$$C_F(A, B) = \frac{\sum_{i=1}^{n} \mu_A(x_i)\mu_B(x_i)}{\sqrt{\sum_{i=1}^{n} \mu_A^2(x_i)}\sqrt{\sum_{i=1}^{n} \mu_B^2(x_i)}}.$$

The cosine similarity measure takes values in the interval $[0, 1]$. It is undefined, if $\mu_A(x_i) = 0$ or $\mu_B(x_i) = 0$ $(i = 1, 2, \ldots, n)$.

Definition 4.2 ([32])**.** Assume that there are two IFSs A and B in a universe discourse $X = \{x_1, x_2, \ldots, x_n\}$. Based on the extension of the cosine measure between fuzzy sets, a cosine similarity measure between two IFSs A and B is defined as follows:

$$C_{IFS}(A, B) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mu_A(x_i)\mu_B(x_i) + \nu_A(x_i)\nu_B(x_i)}{\sqrt{\mu_A^2(x_i) + \nu_A^2(x_i)}\sqrt{\mu_B^2(x_i) + \nu_B^2(x_i)}}.$$

The cosine similarity measure of two IFSs A and B satisfies the following properties:

(1) $0 \leq C_{IFS}(A, B) \leq 1$;
(2) $C_{IFS}(A, B) = C_{IFS}(B, A)$;
(3) $C_{IFS}(A, B) = 1$ if $A = B$, i.e, $\mu_A(x_i) = \mu_B(x_i)$ and $\nu_A(x_i) = \nu_B(x_i)$.

Definition 4.3. Let A and B be two IHFSs on universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, denoted as $A = \{\langle x, h_A(x)\rangle | x \in X\}$ and $B = \{\langle x, h_B(x)\rangle | x \in X\}$, respectively. Then, a new cosine similarity measure between A and B is defined by

$$C_{IHFS}(A, B) = \frac{1}{n}\sum_{i=1}^{n} \frac{\sum_{j=1}^{l_i}[\mu_A^{\sigma(j)}(x_i)\mu_B^{\sigma(j)}(x_i) + \nu_A^{\sigma(j)}(x_i)\nu_B^{\sigma(j)}(x_i) + \pi_A^{\sigma(j)}(x_i)\pi_B^{\sigma(j)}(x_i)]}{\sqrt{\sum_{j=1}^{l_i}[(\mu_A^{\sigma(j)}(x_i))^2 + (\nu_A^{\sigma(j)}(x_i))^2 + (\pi_A^{\sigma(j)}(x_i))^2]}\sqrt{\sum_{j=1}^{l_i}[(\mu_B^{\sigma(j)}(x_i))^2 + (\nu_B^{\sigma(j)}(x_i))^2 + (\pi_B^{\sigma(j)}(x_i))^2]}}.$$

Theorem 4.4. *The cosine similarity measure between two IHFSs A and B satisfies the following properties:*

(1) $0 \leq C_{IHFS}(A, B) \leq 1$*,*
(2) $C_{IHFS}(A, B) = C_{IHFS}(B, A)$*,*
(3) $C_{IHFS}(A, B) = 1$*, if* $A = B$*.*

Proof. (1) The inequality $C_{IHFS}(A, B) \geq 0$ is obvious. Now let us prove $C_{IHFS}(A, B) \leq 1$.

According to the Cauchy - Schwarz inequality:

$$(x_1y_1 + x_2y_2 + \ldots + x_ny_n)^2 \leq (x_1^2 + x_2^2 + \ldots + x_n^2)(y_1^2 + y_2^2 + \ldots + y_n^2),$$

where $(x_1, x_2, \ldots, x_n) \in R^n$ and $(y_1, y_2, \ldots, y_n) \in R^n$, we can obtain:

$\sum_{j=1}^{l_i}[\mu_A^{\sigma(j)}(x_i)\mu_B^{\sigma(j)}(x_i)+\nu_A^{\sigma(j)}(x_i)\nu_B^{\sigma(j)}(x_i)+\pi_A^{\sigma(j)}(x_i)\pi_B^{\sigma(j)}(x_i)]$
$\leq \sqrt{\sum_{j=1}^{l_i}[(\mu_A^{\sigma(j)}(x_i))^2+(\nu_A^{\sigma(j)}(x_i))^2+(\pi_A^{\sigma(j)}(x_i))^2]}\sqrt{\sum_{j=1}^{l_i}[(\mu_B^{\sigma(j)}(x_i))^2+(\nu_B^{\sigma(j)}(x_i))^2+(\pi_B^{\sigma(j)}(x_i))^2]}.$

Thus

$$0\leq \frac{\sum_{j=1}^{l_i}[\mu_A^{\sigma(j)}(x_i)\mu_B^{\sigma(j)}(x_i)+\nu_A^{\sigma(j)}(x_i)\nu_B^{\sigma(j)}(x_i)+\pi_A^{\sigma(j)}(x_i)\pi_B^{\sigma(j)}(x_i)]}{\sqrt{\sum_{j=1}^{l_i}[(\mu_A^{\sigma(j)}(x_i))^2+(\nu_A^{\sigma(j)}(x_i))^2+(\pi_A^{\sigma(j)}(x_i))^2]}\sqrt{\sum_{j=1}^{l_i}[(\mu_B^{\sigma(j)}(x_i))^2+(\nu_B^{\sigma(j)}(x_i))^2+(\pi_B^{\sigma(j)}(x_i))^2]}}\leq 1.$$

So

$$0\leq \frac{1}{n}\sum_{i=1}^{n}\frac{\sum_{j=1}^{l_i}[\mu_A^{\sigma(j)}(x_i)\mu_B^{\sigma(j)}(x_i)+\nu_A^{\sigma(j)}(x_i)\nu_B^{\sigma(j)}(x_i)+\pi_A^{\sigma(j)}(x_i)\pi_B^{\sigma(j)}(x_i)]}{\sqrt{\sum_{j=1}^{l_i}[(\mu_A^{\sigma(j)}(x_i))^2+(\nu_A^{\sigma(j)}(x_i))^2+(\pi_A^{\sigma(j)}(x_i))^2]}\sqrt{\sum_{j=1}^{l_i}[(\mu_B^{\sigma(j)}(x_i))^2+(\nu_B^{\sigma(j)}(x_i))^2+(\pi_B^{\sigma(j)}(x_i))^2]}}$$
$\leq 1.$

Hence, $0\leq C_{IHFS}(A,B)\leq 1$.

(2) $C_{IHFS}(A,B)=C_{IHFS}(B,A)$ is obvious.

(3) $A=B\Rightarrow h_A(x_i)=h_B(x_i)$, $x_i\in X\Rightarrow C_{IHFS}(A,B)=1$. □

Based on the relationship between distance and similarity measures, the cosine distance measures for IHFSs can be introduced.

According to Definition 4.3, the cosine distance measure between two IHFSs A and B is obtained immediately:

$$d_1(A,B)=1-\frac{1}{n}\sum_{i=1}^{n}\frac{\sum_{j=1}^{l_i}[\mu_A^{\sigma(j)}(x_i)\mu_B^{\sigma(j)}(x_i)+\nu_A^{\sigma(j)}(x_i)\nu_B^{\sigma(j)}(x_i)+\pi_A^{\sigma(j)}(x_i)\pi_B^{\sigma(j)}(x_i)]}{\sqrt{\sum_{j=1}^{l_i}[(\mu_A^{\sigma(j)}(x_i))^2+(\nu_A^{\sigma(j)}(x_i))^2+(\pi_A^{\sigma(j)}(x_i))^2]}\sqrt{\sum_{j=1}^{l_i}[(\mu_B^{\sigma(j)}(x_i))^2+(\nu_B^{\sigma(j)}(x_i))^2+(\pi_B^{\sigma(j)}(x_i))^2]}}.$$

As $C_{IHFS}(A,B)$ satisfies the properties of cosine similarity measure between two IHFSs, its corresponding cosine distance measure satisfies the properties of distance measure.

In practical application, we should consider the weights of x_i $(i=1,2,\ldots,n)$ in X. Let $\omega=\{\omega_1,\omega_2,\ldots,\omega_n\}^T$ be the weight vector of x_i $(i=1,2,\ldots,n)$ with $\omega_i\geq 0$ and $\sum_{i=1}^{n}\omega_i=1$. The weighted cosine similarity measure between IHFSs A and B is proposed as follows

$$C_{WIHFS}(A,B)=\sum_{i=1}^{n}\omega_i\frac{\sum_{j=1}^{l_i}[\mu_A^{\sigma(j)}(x_i)\mu_B^{\sigma(j)}(x_i)+\nu_A^{\sigma(j)}(x_i)\nu_B^{\sigma(j)}(x_i)+\pi_A^{\sigma(j)}(x_i)\pi_B^{\sigma(j)}(x_i)]}{\sqrt{\sum_{j=1}^{l_i}[(\mu_A^{\sigma(j)}(x_i))^2+(\nu_A^{\sigma(j)}(x_i))^2+(\pi_A^{\sigma(j)}(x_i))^2]}\sqrt{\sum_{j=1}^{l_i}[(\mu_B^{\sigma(j)}(x_i))^2+(\nu_B^{\sigma(j)}(x_i))^2+(\pi_B^{\sigma(j)}(x_i))^2]}}.$$

If we take $\omega_i=\frac{1}{n}(i=1,2,\ldots,n)$, then there is $C_{WIHFS}(A,B)=C_{IHFS}(A,B)$.

Obviously, the weighted cosine similarity measure of two IHFSs A and B also satisfies the following properties:

(1) $0\leq C_{WIHFS}(A,B)\leq 1$,

(2) $C_{WIHFS}(A,B)=C_{WIHFS}(B,A)$,

(3) $C_{WIHFS}(A,B)=1$ if $A=B$.

Similarity to the previous proof method in Theorem 4.4, we can prove that properties(1)-(3)(omitted).

The corresponding weighted cosine distance measure is given as:

$$d_2(A,B)=1-\sum_{i=1}^{n}\omega_i\frac{\sum_{j=1}^{l_i}[\mu_A^{\sigma(j)}(x_i)\mu_B^{\sigma(j)}(x_i)+\nu_A^{\sigma(j)}(x_i)\nu_B^{\sigma(j)}(x_i)+\pi_A^{\sigma(j)}(x_i)\pi_B^{\sigma(j)}(x_i)]}{\sqrt{\sum_{j=1}^{l_i}[(\mu_A^{\sigma(j)}(x_i))^2+(\nu_A^{\sigma(j)}(x_i))^2+(\pi_A^{\sigma(j)}(x_i))^2]}\sqrt{\sum_{j=1}^{l_i}[(\mu_B^{\sigma(j)}(x_i))^2+(\nu_B^{\sigma(j)}(x_i))^2+(\pi_B^{\sigma(j)}(x_i))^2]}}.$$

5. Illustrative example

In this section, we present a handling for multiple attribute decision making with intuitionistic hesitant fuzzy information.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ be a set of attributes. If the decision makers provide several possible intuitionistic fuzzy values for the alternative A_i $(i = 1, 2, \ldots, m)$ under the attribute C_j $(j = 1, 2, \ldots, n)$ with the attribute weigh vector $\omega = \{\omega_1, \omega_2, \ldots, \omega_n\}^T$, such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. These values can be considered as intuitionistic hesitant fuzzy element $d_{ij} = h_{A_i}(C_j)$ $(j = 1, 2, \ldots, n; i = 1, 2, \ldots, m)$, therefore, we can derive an intuitionistic hesitant fuzzy decision matrix $D = (d_{ij})_{m\times n}$. We define the ideal alternative $A^* = \{\langle C_j, d_j^*\rangle | C_j \in C\}$ $(j = 1, 2, \ldots, n)$, where $d_j^* = \{(1, 0)\}$ $(j = 1, 2, \ldots, n)$ is defined as the ideal IHFE. The larger the value of weighted cosine similarity measure $C_{WIHFS}(A^*, A_i)$, the better the alternative A_i, as the alternative A_i is closer to the ideal alternative A^*. Therefore, all the alternatives can be ranked according to the weighted cosine similarity measures so that the best alternative can be selected.

Example 5.1. It is supposed that a person wants to buy a house. There is a panel with four possible alternatives A_i(i=1, 2, 3, 4) to buy the house. He must take a decision according to the following three attributes: (1) C_1 is the environmental impact analysis (2) C_2 is the price analysis (3) C_3 is the cosiness analysis. The attribute weight vector is given as $\omega = (0.3, 0.5, 0.2)^T$. The four possible alternatives A_i(i=1, 2, 3, 4) are to be evaluated using the intuitionistic hesitant fuzzy information by three decision makers under the three attributes C_j(j=1, 2, 3) as listed in the following intuitionistic hesitant fuzzy decision matrix D:

$$D = \begin{pmatrix} \{(0.8, 0.1), (0.6, 0.2), (0.5, 0.4)\} & \{(0.6, 0.4), (0.4, 0.5)\} & \{(0.9, 0.1), (0.6, 0.3), (0.5, 0.4)\} \\ \{(0.7, 0.3), (0.5, 0.4), (0.4, 0.4)\} & \{(0.6, 0.3), (0.4, 0.3), (0.2, 0.5)\} & \{(0.8, 0.2), (0.4, 0.3)\} \\ \{(0.7, 0.2), (0.5, 0.2), (0.3, 0.6)\} & \{(0.9, 0.1), (0.7, 0.2), (0.5, 0.4)\} & \{(0.5, 0.4), (0.4, 0.6), (0.3, 0.6)\} \\ \{(0.5, 0.3), (0.4, 0.5), (0.3, 0.6)\} & \{(0.7, 0.2), (0.5, 0.3)\} & \{(0.8, 0.1), (0.7, 0.3)\} \end{pmatrix}.$$

We use the generalized intuitionistic hesitant weighted normalized distance measure to calculate the distance between each alternative and the ideal alternative. The derived results are shown in Table 1 with the different values of the parameter.

Table 1 . The generalized intuitionistic hesitant weighted normalized distances among A_i and A^*

	A_1	A_2	A_3	A_4	order
$\lambda = 1$	0.426666	0.520000	0.420000	0.430000	$A_3 < A_1 < A_4 < A_2$
$\lambda = 2$	0.420516	0.487167	0.428951	0.412311	$A_4 < A_1 < A_3 < A_2$
$\lambda = 3$	0.432649	0.501863	0.458239	0.432322	$A_4 < A_1 < A_3 < A_2$
$\lambda = 5$	0.526128	0.532191	0.475468	0.475448	$A_4 < A_3 < A_1 < A_2$

We consider the weights the criteria $C = \{C_1, C_2, C_3\}$ to obtain a more reasonable result. The weighted cosine similarity measures between $A_i(i = 1, 2, 3, 4)$ and A^* can be obtained in Table 2.

Table 2 . Weighted cosine similarity measures between A_i and A^*

A_i	$C_{WIHFS}(A_i, A^*)$
A_1	0.829809
A_2	0.707870
A_3	0.795515
A_4	0.817990

The result show that the alternative A_1 is the best choice according to the maximum value among four weighted cosine similarity measures.

In next, we use a medical diagnosis example to illustrate the distance and cosine similarity measures formulas.

Example 5.2 ([22]). To make a proper diagnosis D={Viral fever, Malaria, Typhoid, Stomach problem, Chest problem} for a patient with given values of symptoms S={Temperature, Headache, Cough, Stomach pain, Chest pain}, a medical knowledge base is necessary that involves elements described in terms of intuitionistic hesitant fuzzy sets. The data are given in Table 3, and each symptom is described by an IHFE. The set of patients is P={Al, Bob, Joe, Ted} and the symptoms are given in Table 4.

Table 3 . Symptom characteristics for the considered diagnoses in terms of IHFSs.

	Temperature	Headache	Cough	Stomach pain	Chest pain
Viral fever	{ (0.8,0.1),(0.6,0.1), (0.4,0.0)}	{(0.6,0.3), (0.4,0.5)}	{(0.5,0.3), (0.4,0.3)}	{(0.1,0.7), (0.0,0.8)}	{(0.1,0.8), (0.1,0.9)}
Malaria	{(0.8,0.0),(0.7,0.0), (0.6,0.1)}	{(0.2,0.6), (0.1,0.7)}	{(0.8,0.1), (0.7,0.3)}	{(0.1,0.8), (0.0,0.9)}	{(0.1,0.8), (0.05,0.8)}
Typhoid	{(0.4,0.3), (0.3,0.3)}	{(0.8,0.1),(0.7,0.1), (0.6,0.2)}	{(0.2,0.6), (0.1,0.9)}	{(0.2,0.7), (0.1,0.8)}	{(0.1,0.9), (0.0,0.9)}
Stomach problem	{(0.2,0.8), (0.1,0.8)}	{(0.3,0.4),(0.2,0.5)}	{(0.3,0.6), (0.2,0.8)}	{(0.8,0.0)}	{(0.1,0.9)}
Chest problem	{(0.2,0.7), (0.1,0.8)}	{(0.0,0.8)}	{(0.3,0.7), (0.2,0.8)}	{(0.2,0.8)}	{(0.9,0.1), (0.8,0.1)}

Table 4 . Symptom characteristics for the considered patients in terms of IHFSs.

	Temperature	Headache	Cough	Stomach pain	Chest pain
Al	{(0.9,0.1),(0.8,0.1), (0.7,0.2)}	{(0.7,0.2), (0.6,0.2)}	{(0.8,0.2), (0.6,0.2)}	{(0.3,0.7), (0.2,0.8)}	{(0.1,0.6), (0.1,0.5)}
Bob	{(0.1,0.8),(0.0,0.9)}	{(0.6,0.4), (0.5,0.4)}	{(0.2,0.7), (0.1,0.8)}	{(0.7,0.2), (0.6,0.4)}	{(0.2,0.7), (0.1,0.8)}
Joe	{(0.8,0.1), (0.6,0.2)}	{(0.8,0.2), (0.7,0.2)}	{(0.3,0.7), (0.1,0.8)}	{(0.1,0.8), (0.0,0.9)}	{(0.0,0.6)}
Ted	{(0.6,0.1), (0.5,0.3)}	{(0.7,0.3), (0.5,0.4)}	{(0.8,0.1), (0.7,0.2)}	{(0.5,0.4), (0.4,0.3)}	{(0.4,0.2), (0.3,0.4)}

To derive a diagnosis for each patient, we utilize the normalized Hamming distance and cosine similarity measure formulas between the symptoms characteristic of each diagnosis and that of each patient. All the results for the considered patients are listed in Table 5 and Table 6.

Table 5 . The normalized Hamming distances of symptoms for each patient to the considered set of possible diagnoses.

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Al	**0.23833**	0.25000	0.30833	0.54000	0.52833
Bob	0.40500	0.50167	0.31333	**0.19000**	0.38500
Joe	0.26167	0.31500	**0.19667**	0.49000	0.47500
Ted	**0.32167**	0.38750	0.41333	0.50000	0.47500

Table 6. The cosine similarity measures of symptoms for each patient to the considered set of possible diagnoses.

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Al	**0.90337**	0.86845	0.80204	0.62637	0.50260
Bob	0.66549	0.55733	0.81057	**0.93024**	0.69879
Joe	0.87848	0.77137	**0.99552**	0.58578	0.55589
Ted	**0.81195**	0.78152	0.71408	0.63679	0.62149

From Table 5 and Table 6, it is obvious that Al and Ted suffer from Viral fever, Bob suffers from Stomach problem, Joe suffers from Typhoid.

6. Conclusions

In this paper, we developed some operational rules for intuitionistic hesitant fuzzy elements. Based on the traditional Hamming distance, Euclidean distance, and generalized distance, we proposed a series of distance measures and cosine similarity measures between two IHFSs by considering the degrees of membership, nonmembership and hesitancy in IHFSs. The cosine distance and weighted cosine distance measures for IHFSs have been introduced as well. Then, the generalized weighted distance measures and the weighted cosine similarity measures were applied to decision-making problem and medical diagnosis. Through the distance measures and cosine similarity measures between the ideal alternative and each alternative, we can choose the ranking order of alternative and the best one.

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.

[2] K. T. Atanassov and G. Gargov, Interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 31 (1989) 343–349.

[3] Z. Y. Bai, Cosine similarity measures for dual hesitant fuzzy sets and their application in multicriteria decision-making problem, Information Technology Journal 12 (15) (2013) 3363–3368.

[4] A. Bhattacharya, On a measure of divergence of two multinomial populations, Sankhya 7 (4) (1945) 401–406.

[5] S. M. Chen, Similarity measures between vague sets and between elements, IEEE Transactions on Systems Man and Cybernetics 27 (1) (1997) 153–158.

[6] N. Chen, Z. S. Xu, and M. M. Xia, Interval-valued hesitant preference relations and their applications to group decision making, Knowledge-Based Systems 37 (2) (2013) 528–540.

[7] X. M. Chen, J. M. Li, L. Qian and X. D. Hu, Distance and similarity measures for intuitionistic hesitant fuzzy sets, Advances in Intelligent Systems Research (2016).

[8] P. Grzegorzewski, Distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the Hausdorff metric, Fuzzy Sets and Systems 148 (2004) 319–328.

[9] W. L. Hung and M. S. Yang, Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance, Pattern Recognition Letters 25 (14) (2004) 1603–1611.

[10] W. K. I. Wan and L. Abdullah, A new cosine similarity measure for interval-valued intuitionistic fuzzy sets and their applications, AIP Conference Proceedings 1522 (2013) 292–298.

[11] D. F. Li, and C. T. Cheng, New similarity measures of intuitionistic fuzzy sets and applications to pattern recognitions, Pattern Recognition Letters 23 (1) (2002) 221–225.

[12] H. Liao, Subtraction and division operations over hesitant fuzzy sets, Journal of Intelligent and Fuzzy Systems 27 (1) (2014) 65-72.

[13] H. Liao and Z. S. Xu, Some new hybrid weighted aggregation operators under hesitant fuzzy multi-criteria decision making environment, Journal of Intelligent and Fuzzy Systems Applications in Engineering and Technology 26 (4) (2014) 1601–1617.

[14] H. Liao and Z. S. Xu, Extended hesitant fuzzy hybrid weighted aggregation operators and their application in decision making, Soft Computing 19 (9) (2015) 2551–2564.

[15] H. Liao, Z. S. Xu and X. J. Zeng, Novel correlation coefficients between hesitant fuzzy sets and their application in decision making, Knowledge-Based Systems 82 (2015) 115–127.
[16] H. Liao, Z. S. Xu and X. J. Zeng, Distance and similarity measures for hesitant fuzzy linguistic term sets and their application in multi-criteria decision making, Inform. Sci. 271 (3) (2014) 125–142.
[17] H. Liao and Z. S. Xu. Approaches to manage hesitant fuzzy linguistic information based on the cosine distance and similarity measures for HFLTSs and their application in qualitative decision making, Expert Systems with Applications 42 (12) (2015) 5328–5336.
[18] G. Salton and M. J. Mcgill, Introduction to modern information retrieval, McGraw-Hill, N. Y. These (1983).
[19] Z. Su, Z. Xu and H. Liu, etal, Distance and similarity measures for dual hesitant fuzzy sets and their applications in pattern recognition, Journal of Intelligent and Fuzzy Systems 29 (2) (2015) 731–745.
[20] E. Szmidt and J. Kacprzyk, Distances between intuitionistic fuzzy sets, Fuzzy Sets and Systems 114 (3) (2000) 505–518.
[21] E. Szmidt and J. Kacprzyk, A new concept of a similarity measure for intuitionistic fuzzy sets and its use in group decision making, Modeling Decisions for Artificial Intelligence 3558 (2005) 272–282.
[22] E. Szmidt and J. Kacprzyk, A similarity measure for intuitionistic fuzzy sets and its application in supporting medical diagnostic reasoning, Artificial Intelligence and Soft Computing - ICAISC 2004. Springer Berlin Heidelberg (2004) 388-393.
[23] V. Torra, Hesitant fuzzy sets, International Journal of Intelligent Systems 25 (6) (2010) 529–539.
[24] W. Q. Wang and X. L. Xin, Distance measure between intuitionistic fuzzy sets, Pattern Recognition Letters 26 (13) (2005) 2063-2069.
[25] M. M. Xia and Z. S. Xu, Some new similarity measures for intuitionistic fuzzy value and their application in group decision making, Journal of Systems Science and Systems Engineering 19 (4) (2010) 430–452.
[26] M. M. Xia and Z. S. Xu, Hesitant fuzzy information aggregation in decision making, Internat. J. Approx. Reason. 52 (3) (2011) 395–407.
[27] Z. S. Xu, Intuitionistic fuzzy aggregation operators, IEEE Transactions on Fuzzy Systems 14 (6) (2008) 1179–1187.
[28] Z. S. Xu, Some similarity measures of intuitionistic fuzzy sets and their applications to multiple attribute decision making, Fuzzy Optim. Decis. Mak. 6 (2) (2007) 109–121.
[29] Z. S. Xu and J. Chen, An overview of distance and similarity measures of intuitionistic fuzzy sets, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems 16 (4) (2008) 529–555.
[30] Z. S. Xu and M. M. Xia, Distance and similarity measures for hesitant fuzzy sets, Inform. Sci. 181 (11) (2011) 2128–2138.
[31] Z. S. Xu and M. M. Xia, On distance and correlation measures of hesitant fuzzy information, International Journal of Intelligent Systems 26 (5) (2011) 410–425.
[32] J. Ye, Cosine similarity measures for intuitionistic fuzzy sets and their application, Mathematical and Computer Modelling An International Journal 53 (1-2) (2011) 91–97.
[33] L. A. Zadeh, Fuzzy sets, Information and Control 8 (3) (1965) 338–353.
[34] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, Inform. Sci. 8 (3) (1975) 199–249.
[35] B. Zhu, Z. S. Xu and M. M. Xia, Dual hesitant fuzzy sets, J. Appl. Math. 2012 (11) (2012) 2607–2645.

Hong Wang (whdw218@163.com)
College of mathematics and Computer Science, Shan'xi Normal University, Linfen, Shan'xi 041004, China

Dandan Li (1607330917@qq.com)
College of mathematics and Computer Science, Shan'xi Normal University, Linfen, Shan'xi 041004, China