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On linear operators in Felbin's fuzzy normed linear space

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Abstract. In this paper, we have obtained a few results on fuzzy boundedness of linear operators on Felbin's fuzzy normed linear spaces. The main result of this paper is an extension theorem for a strongly (weakly) fuzzy bounded linear operator. The existence of the inverse of a strongly (weakly) fuzzy bounded linear operator has also been established.

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1. Introduction

Based on the work of Kaleva and Seikkala in [6] on fuzzy metric space, Felbin [3] introduced fuzzy normed linear spaces. With this notion of fuzzy norm, fuzzy analogous of several classical concepts of normed linear spaces have been established [2, 4, 9, 10, 11, 14]. In [4], Felbin introduced the notion of fuzzy bounded linear operators over fuzzy normed linear spaces and defined the "fuzzy norm" for such an operator. Later these definitions were proved to be erroneous by Bag and Samanta [1]. They subsequently introduced a new definition of a fuzzy bounded linear operator and "fuzzy norm" for such an operator. Further, Xiao and Zhu [12, 13] and Ji, Qi and Wei [5] considered Felbin's type fuzzy norm in its general form for the study of linear operators.

In this paper, we present the notion of an extension and the inverse of a linear operator in Felbin's fuzzy normed linear space. Some results on fuzzy boundedness of linear operators have also been established. We have obtained generalized results using the more general right norm R; instead of the standard right and left norms.

2. Preliminaries

Throughout the paper, we denote the set of all real numbers by $\mathbb{R}$ and set of all positive real numbers by $\mathbb{R}^+$.

In this paper we have considered fuzzy real numbers (fuzzy intervals) in the sense of Xiao and Zhu [14] (slightly different from [3]), as follows:

Definition 2.1 ([14], Xiao and Zhu)**.** A mapping $\eta : \mathbb{R} \to [0,1]$ is called a fuzzy real number (fuzzy interval), whose α-level set is denoted by $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies the following:

(N1) There exists $t_\circ \in \mathbb{R}$ such that $\eta(t_\circ) = 1$,
(N2) for each $\alpha \in (0,1]$; $[\eta]_\alpha = [\eta^-_\alpha, \eta^+_\alpha]$, where $-\infty < \eta^-_\alpha \leq \eta^+_\alpha < +\infty$.

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $\mathcal{F}$. Since to each $r \in \mathbb{R}$, one can consider $\bar{r} \in \mathcal{F}$ defined by $\bar{r}(t) = 1$, if $t = r$ and $\bar{r}(t) = 0$ if $t \neq r$, $\mathbb{R}$ can be embedded in $\mathcal{F}$.

Further, η is called convex, if $\eta(t) \geq \min(\eta(s), \eta(r))$, where $s \leq t \leq r$.

If there exists a $t_\circ \in \mathbb{R}$ such that $\eta(t_\circ) = 1$, then η is called normal.

Definition 2.2 ([14], Xiao and Zhu)**.** Let $\eta \in \mathcal{F}$. Then η is called a positive fuzzy real number (fuzzy interval), if $\eta(t) = 0\ \forall t < 0$. The set of all positive fuzzy real numbers (fuzzy intervals) is denoted by $\mathcal{F}^+$.

Remark 2.3. As $\eta \in \mathcal{F}^+$ is upper semicontinuous, it follows that $\eta(t) = 0, \forall t \leq 0$.

A partial ordering " $\preceq$ " in $\mathcal{F}$ is defined by $\eta \preceq \delta$ if and only if $a^1_\alpha \leq a^2_\alpha$ and $b^1_\alpha \leq b^2_\alpha$, for all $\alpha \in (0,1]$, where $[\eta]_\alpha = [a^1_\alpha, b^1_\alpha]$ and $[\delta]_\alpha = [a^2_\alpha, b^2_\alpha]$ [3]. The strict inequality in $\mathcal{F}$ is defined by $\eta \prec \delta$ if and only if $a^1_\alpha < a^2_\alpha$ and $b^1_\alpha < b^2_\alpha$ for each $\alpha \in (0,1]$.

For $\eta \in \mathcal{F}$ and $\delta(> \bar{0}) \in \mathcal{F}^+$,

$$(\eta \oslash \delta)(t) = \sup_{s \in \mathbb{R}} \min\{\eta(st), \delta(s)\}, t \in \mathbb{R}.$$

From [3], we have the following result:

Proposition 2.4. *Let* $\eta, \delta \in \mathcal{F}$ *and* $[\eta]_\alpha = [a^1_\alpha, b^1_\alpha], [\delta]_\alpha = [a^2_\alpha, b^2_\alpha], \alpha \in (0,1]$. *Then*

(1) $[\eta \oplus \delta]_\alpha = [a^1_\alpha + a^2_\alpha, b^1_\alpha + b^2_\alpha]$,
(2) $[\eta \ominus \delta]_\alpha = [a^1_\alpha - b^2_\alpha, b^1_\alpha - a^2_\alpha]$,
(3) $[\eta \odot \delta]_\alpha = [a^1_\alpha a^2_\alpha, b^1_\alpha b^2_\alpha]$,
(4) $[\bar{1} \oslash \delta]_\alpha = [\frac{1}{b^2_\alpha}, \frac{1}{a^2_\alpha}], a^2_\alpha > 0, \forall \alpha \in (0,1]$.

The arithmetic operations $\oplus$, $\ominus$ and $\odot$ in $\mathcal{F}$ are defined as in [8].

We shall require the following result on fuzzy real numbers (fuzzy intervals).

Proposition 2.5. [1] *Let* $\{[a_\alpha, b_\alpha]; \alpha \in (0,1]\}$ *be a family of nested bounded closed intervals. Let the function* $\eta : \mathbb{R} \to [0,1]$ *be defined by* $\eta(t) = \bigvee\{\alpha \in (0,1] : t \in [a_\alpha, b_\alpha]\}$. *Then* η *is a fuzzy real number (fuzzy interval).* α*- level sets of* η *are denoted by* $[\eta]_\alpha = [\eta^-_\alpha, \eta^+_\alpha], \alpha \in (0,1]$.

Here η is the fuzzy real number generated by the family of nested bounded closed intervals $\{[a_\alpha, b_\alpha]; \alpha \in (0,1]\}$.

Following is the definition of fuzzy norm on a linear space as given by Xiao and Zhu [14]:

Definition 2.6. Let X be a vector space over $\mathbb{R}$ and the mappings L, R(respectively left norm and right norm): $[0,1]\times[0,1] \longrightarrow [0,1]$ be symmetric, non decreasing in both arguments and satisfying $L(0,0)=0$ and $R(1,1)=1$. Let $\parallel . \parallel$ a mapping from X into $\mathcal{F}^+$. Write

$$[\parallel x \parallel]_\alpha = [\parallel x \parallel_\alpha^1, \parallel x \parallel_\alpha^2], \text{ for } x \in X, 0 < \alpha \leq 1.$$

Then the quadruple $(X, \parallel . \parallel, L, R)$ is called a fuzzy normed linear space (briefly, FNS) and $\parallel . \parallel$ is a fuzzy norm, if the following axioms are satisfied:

(F1) $\parallel x \parallel = \bar{0}$ if and only if $x = \theta$, θ is the zero element of X,

(F2) $\parallel rx \parallel = \mid r \mid \parallel x \parallel, x \in X, r \in \mathbb{R}$,

(F3) for all $x, y \in X$,

(F3L) $\parallel x + y \parallel (s+t) \geq L(\parallel x \parallel (s), \parallel y \parallel (t))$ whenever $s \leq \parallel x \parallel_1^1, t \leq \parallel x \parallel_1^1$ and $s + t \leq \parallel x + y \parallel_1^1$,

(F3R) $\parallel x + y \parallel (s+t) \leq R(\parallel x \parallel (s), \parallel y \parallel (t))$ whenever $s \geq \parallel x \parallel_1^1, t \geq \parallel x \parallel_1^1$ and $s + t \geq \parallel x + y \parallel_1^1$.

Remark 2.7. Felbin [3] proved that if $L = \bigwedge$(Min) and $R = \bigvee$(Max), then the triangular inequality (F3) in Definition 2.6 is equivalent to

$$\parallel x + y \parallel \preceq \parallel x \parallel \oplus \parallel y \parallel .$$

In this case, $\parallel \parallel_\alpha^i; i = 1, 2$ are crisp norms on X for each $\alpha \in (0,1]$.

In our further discussion we will use the definitions of convergence of a sequence, Cauchy sequence and also of a complete fuzzy normed linear space as given in [3] and [14] maintaining their respective notations as far as possible.

Let $(X, \parallel \parallel)$ and $(Y, \parallel \parallel^\sim)$ be two fuzzy normed linear spaces and $T : X \to Y$ be a linear operator. Then we have the following definitions and results from [1].

Definition 2.8. $T : X \to Y$ is said to be strongly fuzzy bounded, if there exists a real number $k > 0$ such that

$$\parallel Tx \parallel^\sim \oslash \parallel x \parallel \preceq \bar{k}, \forall x (\neq \theta) \in X.$$

Definition 2.9. $T : X \to Y$ is said to be weakly fuzzy bounded, if there exists a fuzzy interval $\eta \in \mathcal{F}^+, \eta > \bar{0}$ such that

$$\parallel Tx \parallel^\sim \oslash \parallel x \parallel \preceq \eta, \forall x (\neq \theta) \in X.$$

Remark 2.10. Bag and Samanta [1] proved with a suitable example that a strongly fuzzy bounded linear operator is weakly fuzzy bounded but not conversely (Example 4.2 of [1]).

In [1], Bag and Samanta defined the fuzzy norm $\parallel T \parallel^*$ of a linear operator T from a fuzzy normed linear space X to fuzzy normed linear space Y.

The fuzzy norm $\parallel T \parallel^*: \mathbb{R} \to [0,1]$ is a function defined as:

$$\parallel T \parallel^* (t) = \bigvee \{\alpha \in (0,1] : t \in [\parallel T \parallel_\alpha^{*1}, \parallel T \parallel_\alpha^{*2}]\},$$

where

$$\parallel T \parallel_\alpha^{*1} = \sup_{\substack{x \in X \\ x \neq \theta}} \frac{\parallel Tx \parallel_\alpha^{\sim 1}}{\parallel x \parallel_\alpha^2} \text{ and } \parallel T \parallel_\alpha^{*2} = \sup_{\substack{x \in X \\ x \neq \theta}} \frac{\parallel Tx \parallel_\alpha^{\sim 2}}{\parallel x \parallel_\alpha^1}.$$

Then the fuzzy norm $\parallel T \parallel^*$ is generated by the family of nested bounded closed intervals $\{[\parallel T \parallel_\alpha^{*1}, \parallel T \parallel_\alpha^{*2}] : \alpha \in (0,1]\}$.

Xiao and Zhu [14] gave the following definitions considering a right norm R which is more general.

Definition 2.11 ([14], Xiao and Zhu)**.** Let $(X, \|\|, L, R)$ be an fuzzy normed linear space, $\lim_{a\to 0^+} R(a,a) = 0$, $A \subseteq X, x_\circ \in X$. $x_\circ$ is called a point of closure of A if $\{x_\circ + N(\alpha,\alpha)\} \cap A \neq \phi$ for every $\alpha \in (0,1]$; $\bar{A}$ denotes the set of all points of closure of A. A is called fuzzy closed, if $\bar{A} = A$. A is called a fuzzy bounded set if for each $\alpha \in (0,1]$ there exists $M = M(\alpha) > 0$ such that $A \subseteq N(M,\alpha)$.

Lemma 2.12 ([14], Xiao and Zhu)**.** *Let $(X, \| \, . \, \|, L, R)$ be an fuzzy normed linear space, $\lim_{a\to 0^+} R(a,a) = 0$. Then*

(1) $A \subseteq X$, $x \in \bar{A}$ if and only if there exists $\{x_n\}_{n=1}^{\infty} \subseteq A$ such that $\lim_{n\to\infty} x_n = x$,

(2) $\{x_n\} \subseteq X$ is a Cauchy sequence if and only if for each $\alpha \in (0,1]$ there is $K \in \mathbb{Z}^+$ such that $x_m - x_n \in N(\alpha,\alpha)$, for all $m, n \geq K$,

(3) $A \subseteq X$ is fuzzy bounded if and only if $\lim_{n\to\infty} x_n/n = \theta$, for arbitrary $\{x_n\}_{n=1}^{\infty} \subseteq A$.

Lemma 2.13 ([14], Xiao and Zhu)**.** *Let $(X, \| \, . \, \|, L, R)$ be a fuzzy normed linear space, $\lim_{a\to 0^+} R(a,a) = 0$. Then (F3R) $\Rightarrow$ for each $\alpha \in (0,1]$, there is $\alpha_\circ \in (0,\alpha]$ such that $\| x + y \|_\alpha^2 \leq \| x \|_{\alpha_\circ}^2 + \| y \|_{\alpha_\circ}^2$, for each $x, y \in X$.*

3. Main results

3.1. **Extension of a linear operator.** In this section we introduce an extension of a linear operator in fuzzy normed linear spaces and prove a boundedness result for the extension operator. The definitions of range space and null space of a linear operator are considered as in Kreiszig [7].

Definition 3.1. Let $T : (X, \|\|, L, R) \to (Y, \|\|^\sim, L, R)$ be a linear operator. The restriction of the operator T to a subset A of X is denoted by $T \mid_A$ and is defined by

$$T \mid_A : A \to Y,\ T \mid_A (x) = Tx, \forall x \in A.$$

Definition 3.2. An extension of T from $M \subset X$ to X is an operator $\widetilde{T} : X \to Y$ such that $\widetilde{T} \mid_M = T, i.e., \widetilde{T}(x) = T(x)$, $\forall x \in M$.

Theorem 3.3. *Let $T : \mathcal{D} \to Y$ be a strongly fuzzy bounded linear operator, where $\mathcal{D}$ denotes the domain of the linear operator T that lies in the fuzzy normed linear space $(X, \|\|, L, R)$ and $(Y, \|\|^\sim, L, R)$ a complete fuzzy normed linear space with $\lim_{a\to 0^+} R(a,a) = 0$. Then T has an extension $\widetilde{T} : \overline{\mathcal{D}} \to Y$ such that $\widetilde{T}$ is strongly fuzzy bounded linear operator of fuzzy norm $\| \widetilde{T} \|^* = \| T \|^*$.*

Proof. Consider any $x \in \overline{\mathcal{D}}$. By Lemma 2.12(1), there exists a sequence (x_n) in $\mathcal{D}$ such that $\lim_{n\to\infty} x_n = x$. Since T is a strongly fuzzy bounded linear operator,

$\exists k \in \mathbb{R}^+$ such that $\| Tx \|^\sim \oslash \| x \| \preceq \bar{k}$, for all $x \in X$.

In particular, we have $\| Tx_m - Tx_n \|^\sim \oslash \| x_m - x_n \| \preceq \bar{k}$. This gives

$\frac{\|Tx_m - Tx_n\|_\alpha^{\sim 1}}{\|x_m - x_n\|_\alpha^2} \leq k$ and $\frac{\|Tx_m - Tx_n\|_\alpha^{\sim 2}}{\|x_m - x_n\|_\alpha^1} \leq k$
Since (x_n) converges, (x_n) is Cauchy. and then,

$$\lim_{n,m\to\infty} \| x_m - x_n \|_\alpha^2 = \lim_{n,m\to\infty} \| x_m - x_n \|_\alpha^1 = 0.$$

Thus we get $\lim_{n,m\to\infty} \| Tx_m - Tx_n \|_\alpha^{\sim 2} = 0$. So (Tx_n) is a Cauchy sequence in Y. As Y is complete, (Tx_n) converges.

Let $Tx_n \to y$ with $y \in Y$. Define $\widetilde{T} : \overline{\mathcal{D}} \to Y$ as $\widetilde{T}x = y$ for $x \in \overline{\mathcal{D}}$. Also since (x_n) is a sequence in $\mathcal{D} \subseteq X$, for all n,

$$\| Tx_n \|_\alpha^{\sim 1} \leq k \| x_n \|_\alpha^2 .$$

Now,

$$\begin{aligned} \| \widetilde{T}x \|_\alpha^{\sim 1} = \| y \|_\alpha^{\sim 1} = \| \lim_{n\to\infty} Tx_n \|_\alpha^{\sim 1} &= \lim_{n\to\infty} \| Tx_n \|_\alpha^{\sim 1} \\ &\leq k \lim_{n\to\infty} \| x_n \|_\alpha^2 = k \| \lim_{n\to\infty} x_n \|_\alpha^2 \\ &= k \| x \|_\alpha^2, \end{aligned}$$

i.e., $\frac{\|\widetilde{T}x\|_\alpha^{\sim 1}}{\|x\|_\alpha^2} \leq k$. In the same way, it can be proved that $\frac{\|\widetilde{T}x\|_\alpha^{\sim 2}}{\|x\|_\alpha^1} \leq k$.
Then $\| \widetilde{T}x \|^\sim \oslash \| x \| \preceq \bar{k}$. Thus $\widetilde{T}$ is strongly fuzzy bounded.

Next we shall show $\| T \|^* = \| \widetilde{T} \|^*$.

For $x \in \mathcal{D}$, $Tx = \widetilde{T}x$ gives $\| T \|^* = \| \widetilde{T} \|^*$.

For $x \in \overline{\mathcal{D}} - \mathcal{D}$, there exists a sequence (x_n) in $\mathcal{D}$ such that $\lim_{n\to\infty} x_n = x$. For all such $x \in \overline{\mathcal{D}} - \mathcal{D}$,

$$\begin{aligned} \| \widetilde{T} \|_\alpha^{*1} = \sup_x \frac{\| \widetilde{T}x \|_\alpha^{\sim 1}}{\| x \|_\alpha^2} &= \sup_x \frac{\| \lim_{n\to\infty} Tx_n \|_\alpha^{\sim 1}}{\| x \|_\alpha^2} \\ = \sup_x \frac{\| T(\lim_{n\to\infty} x_n) \|_\alpha^{\sim 1}}{\| x \|_\alpha^2} &= \sup_x \frac{\| Tx \|_\alpha^{\sim 1}}{\| x \|_\alpha^2}. \end{aligned}$$

Which finally gives $\| \widetilde{T} \|_\alpha^{*1} = \| T \|_\alpha^{*1}$. In the similar way, it can be shown that

$$\| \widetilde{T} \|_\alpha^{*2} = \| T \|_\alpha^{*2} .$$

Then the two families

$\{[\| T \|_\alpha^{*1}, \| T \|_\alpha^{*2}] : 0 < \alpha \leq 1\}$ and $\{[\| \widetilde{T} \|_\alpha^{*1}, \| \widetilde{T} \|_\alpha^{*2}] : 0 < \alpha \leq 1\}$

are the same. Thus the fuzzy numbers generated by these two families are also the same (using Proposition 2.5).
So $\| T \|^* = \| \widetilde{T} \|^*$. □

Example 3.4. Consider $X = Y = \mathbb{R}$, the linear space of all real numbers and $L = \min$ and $R = \max$. Define the fuzzy norms $\| . \|$ and $\| . \|^\sim$ on $\mathbb{R}$ as follows:

$$\| x \| (t) = \begin{cases} \frac{|x|}{t} & \text{if } | x | < t, \\ 1 & \text{if } | x | = t = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\| x \|^{\sim}(t) = \begin{cases} 1 & \text{if } | x |= t, \\ 0 & \text{otherwise.} \end{cases}$$

Then the α-level sets of $\| x \|$ and $\| x \|^{\sim}$ are given by

$$[\| x \|]_\alpha = [| x |, \frac{| x |}{\alpha}]$$

and

$$[\| x \|]^{\sim}_\alpha = [| x |, | x |].$$

Define a mapping $T : \mathcal{D} \to X$, $\mathcal{D} = (-1, 1)$ as $Tx = x, \forall x \in \mathcal{D}$. Clearly T is linear and strongly fuzzy bounded. For

$$\frac{\| Tx \|^{\sim 1}_\alpha}{\| x \|^2_\alpha} = \frac{| x |}{\frac{|x|}{\alpha}} = \alpha \leq 1, \forall x \neq \theta \in \mathcal{D} \tag{3.4.1}$$

and

$$\frac{\| Tx \|^{\sim 2}_\alpha}{\| x \|^1_\alpha} = \frac{| x |}{| x |} = 1, \forall x(\neq \theta) \in \mathcal{D}. \tag{3.4.2}$$

From (3.4.1) and (3.4.2), $\| Tx \|^{\sim} \oslash \| x \| \preceq \bar{1}$, $\forall x(\neq \theta) \in \mathcal{D}$. Then T is strongly fuzzy bounded.

As $R = \max$ satisfies $\lim\limits_{a \to 0^+} R(a, a) = 0$, using Lemma 2.12, we get $\overline{\mathcal{D}} = [-1, 1]$.

Now define a function $\widetilde{T} : \overline{\mathcal{D}} \to Y$ as follows:

$$\widetilde{T}x = \begin{cases} -1 & \text{if } x = -1, \\ 1 & \text{if } x = 1, \\ Tx & \text{if } x(\neq -1, 1) \in \overline{\mathcal{D}}. \end{cases}$$

Then $\widetilde{T} |_{\mathcal{D}}= T$ and such $\widetilde{T}$ is an extension of T.

Also,

$$\| T \|^{*1}_\alpha = \sup_{x(\neq\theta)\in\mathcal{D}} \frac{\| Tx \|^{\sim 1}_\alpha}{\| x \|^2_\alpha} = \sup_{x(\neq\theta)\in\mathcal{D}} \frac{| x |}{\frac{|x|}{\alpha}} = \alpha \leq 1,$$

$$\| T \|^{*2}_\alpha = \sup_{x(\neq\theta)\in\mathcal{D}} \frac{\| Tx \|^{\sim 2}_\alpha}{\| x \|^1_\alpha} = \sup_{x(\neq\theta)\in\mathcal{D}} \frac{| x |}{| x |} = 1,$$

$$\| \widetilde{T} \|^{*1}_\alpha = \sup_{x(\neq\theta)\in\overline{\mathcal{D}}} \frac{\| \widetilde{T}x \|^{\sim 1}_\alpha}{\| x \|^2_\alpha} = \max\{\frac{\| \widetilde{T}(-1) \|^{\sim 1}_\alpha}{\| (-1) \|^2_\alpha}, \frac{\| \widetilde{T}(1) \|^{\sim 1}_\alpha}{\| (1) \|^2_\alpha}, \sup_{x\in\overline{\mathcal{D}}-\{-1,1\}} \frac{\| \widetilde{T}x \|^{\sim 1}_\alpha}{\| x \|^2_\alpha}\}$$

$$= \max\{\frac{| (-1) |}{\frac{(-1)}{\alpha}}, \frac{| 1 |}{\frac{|1|}{\alpha}}, \| T \|^{*1}_\alpha\} = \max\{\alpha, \alpha, \alpha\} = \alpha \leq 1$$

and

$$\| \widetilde{T} \|^{*2}_\alpha = \sup_{x(\neq\theta)\in\overline{X}} \frac{\| \widetilde{T}x \|^{\sim 2}_\alpha}{\| x \|^1_\alpha} = \max\{\frac{\| \widetilde{T}(-1) \|^{\sim 2}_\alpha}{\| (-1) \|^1_\alpha}, \frac{\| \widetilde{T}(1) \|^{\sim 2}_\alpha}{\| (1) \|^1_\alpha}, \sup_{x\in\overline{\mathcal{D}}-\{-1,1\}} \frac{\| \widetilde{T}x \|^{\sim 2}_\alpha}{\| x \|^1_\alpha}\}$$

$$= \max\{\frac{| (-1) |}{| (-1) |}, \frac{| 1 |}{| 1 |}, \| T \|^{*2}_\alpha\} = \max\{1, 1, 1\} = 1.$$

Thus we have $\| T \|^{*1}_\alpha = \| \widetilde{T} \|^{*1}_\alpha$ and $\| T \|^{*2}_\alpha = \| \widetilde{T} \|^{*2}_\alpha$. So, the fuzzy real numbers $\| T \|^*$ and $\| \widetilde{T} \|^*$ generated by the families of closed intervals

$$\{[\| T \|_\alpha^{*1}, \| T \|_\alpha^{*2}], \alpha \in (0,1]\} \text{ and } \{[\| \widetilde{T} \|_\alpha^{*1}, \| \widetilde{T} \|_\alpha^{*2}], \alpha \in (0,1]\},$$

respectively are the same, i.e., $\| T \|^* = \| \widetilde{T} \|^*$.

3.2. **Inverse of a linear operator.** In this section we define the inverse of a linear operator on fuzzy normed linear space. An existence theorem is established.

Definition 3.5. Let $(X, \|\|, L, R)$ and $(Y, \|\|^\sim, L, R)$ be fuzzy normed linear space and T be a linear operator from X to Y. Let $\mathcal{R}(T)$ and $\mathcal{N}(T)$ be the range space and null space of T, respectively. If T is bijective, then T^{-1} exists and for $A \subseteq Y$,

$$T^{-1}[A \cap \mathcal{R}(T)] = \{x \in X : Tx \in A\}.$$

First we prove the following lemma.

Lemma 3.6. *Let T be a strongly fuzzy bounded linear operator from a fuzzy normed linear space $(X, \|\|, L, R)$ to a fuzzy normed linear space $(Y, \|\|^\sim, L, R)$.*
(1) If $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} Tx_n = Tx$.
In this case T is called continuous [14].
(2) If $\lim_{a\to 0^+} R(a,a) = 0$, Then the null space $\mathcal{N}(T)$ of T is fuzzy closed.

Proof. (1) Suppose $\lim_{n\to\infty} x_n = x$. Then $\lim_{n\to\infty} \| x_n - x \|_\alpha^1 = \lim_{n\to\infty} \| x_n - x \|_\alpha^2 = 0$. Here $\| T \|_\alpha^{*2} = \sup_{x_n - x \neq 0} \dfrac{\| Tx_n - Tx \|_\alpha^{\sim 2}}{\| x_n - x \|_\alpha^1}$. Thus $\| Tx_n - Tx \|_\alpha^{\sim 2} \leq \| T \|_\alpha^{*2} \| x_n - x \|_\alpha^1$. So we have $\lim_{n\to\infty} \| Tx_n - Tx \|_\alpha^{\sim 2} = 0$. Hence $\lim_{n\to\infty} Tx_n = Tx$.

(2) Let $x \in \overline{\mathcal{N}(T)}$. Then by Lemma 2.12 (1), there exists a sequence (x_n) in $\mathcal{N}(T)$ such that $\lim_{n\to\infty} x_n = x$. Using (1), we get $\lim_{n\to\infty} Tx_n = Tx$. Also for each n, $Tx_n = 0$. Thus$Tx = 0$. So $x \in \mathcal{N}(T)$. Hence $\mathcal{N}(T)$ is fuzzy closed. □

Theorem 3.7. *Let T be a strongly fuzzy bounded linear operator from a complete fuzzy normed linear space $(X, \|\|, L, R)$ to a fuzzy normed linear space $(Y, \|\|^\sim, L, R)$ with $\lim_{a\to 0^+} R(a,a) = 0$. Suppose that for some $c > 0$, $\| Tx \|^\sim \oslash \| x \| \succeq \bar{c}$, $\forall x \in X$. Then the range $\mathcal{R}(T)$ of T is fuzzy closed. Further if $\mathcal{R}(T) = Y$, then T is invertible and $\| T^{-1} \|_\alpha^{*2} \leq \frac{1}{c}$.*

Proof. Let (y_n) be a sequence in $\mathcal{R}(T)$ and $\lim_{n\to\infty} y_n = y$. Then corresponding to each y_n, there is a x_n in X such that $Tx_n = y_n$. Thus $\lim_{n\to\infty} Tx_n = y$. Using the given hypothesis, we get

$$\| Tx_n \|^\sim \oslash \| x_n \| \succeq \bar{c},$$

i.e.,

$$\| Tx_n \|_\alpha^{\sim 2} \geq c \| x_n \|_\alpha^1 \text{ and } \| Tx_n \|_\alpha^{\sim 1} \geq c \| x_n \|_\alpha^2.$$

As (Tx_n) converges, it is a Cauchy sequence. Using the above inequality, we get

$$c \| x_m - x_n \|_\alpha^2 \leq \| Tx_m - Tx_n \|_\alpha^{\sim 1} \leq \| Tx_m - Tx_n \|_\alpha^{\sim 2},$$

which implies $\lim_{m,n\to\infty} \| x_m - x_n \|_\alpha^2 = 0$. So (x_n) is Cauchy. As X is complete, (x_n) converges in X.

Let $\lim_{n\to\infty} x_n = x$, $x \in X$. As T is strongly fuzzy bounded, using Lemma 3.6 (1), we have $\lim_{n\to\infty} Tx_n = Tx$. Then we finally obtain $Tx = y$. Thus $y \in \mathcal{R}(T)$. So $\mathcal{R}(T)$ is fuzzy closed.

Let us show the second part.

If $\mathcal{R}(T) = Y$, then obviously T is onto.

Let us assume that $x_1 \neq x_2$, $x_1, x_2 \in X$. Then it follows from given hypothesis that

$$\| Tx_1 \|_\alpha^{\sim 1} \neq \| Tx_2 \|_\alpha^{\sim 1} \text{ and } \| Tx_1 \|_\alpha^{\sim 2} \neq \| Tx_2 \|_\alpha^{\sim 2}.$$

Thus $Tx_1 \neq Tx_2$. So T is one-one. Hence T is invertible.

Let T^{-1} be the inverse of T. Then We have

$$\| T^{-1} \|_\alpha^{2*} = \sup_{x\in Y, x\neq\theta} \frac{\| T^{-1}x \|_\alpha^2}{\| x \|_\alpha^{\sim 1}}.$$

For $x \in Y, x \neq \theta$, let $T^{-1}x = y \in X$, i.e., $Ty = x$. Applying $\| Ty \|^\sim \oslash \| y \| \succeq \bar{c}$, it can be obtained that

$$\frac{\| Ty \|_\alpha^{\sim 1}}{\| y \|_\alpha^2} \geq c \Rightarrow \| y \|_\alpha^2 \leq c^{-1} \| Ty \|_\alpha^{\sim 1} \Rightarrow \| T^{-1}x \|_\alpha^2 \leq c^{-1} \| x \|_\alpha^{\sim 1}.$$

Since x is arbitrary, $\sup_{x\in Y, x\neq\theta} \frac{\| T^{-1}x \|_\alpha^2}{\| x \|_\alpha^{\sim 1}} \leq c^{-1}$. Thus $\| T^{-1} \|_\alpha^{2*} \leq c^{-1}$. □

Corollary 3.8. *Let $T : \mathcal{D} \to Y$ be a weakly fuzzy bounded linear operator from the domain $\mathcal{D}$ that lies in a fuzzy normed linear space $(X, \|\|, L, R)$ to the complete fuzzy normed linear space $(Y, \|\|^\sim, L, R)$ where $\lim_{a\to 0^+} R(a, a) = 0$. Then T has an extension $\widetilde{T} : \overline{\mathcal{D}} \to Y$ such that $\widetilde{T}$ is weakly fuzzy bounded linear operator of fuzzy norm $\| \widetilde{T} \|^* = \| T \|^*$.*

Corollary 3.9. *Let T be a weakly fuzzy bounded linear operator from a fuzzy normed linear space $(X, \|\|, L, R)$ to a fuzzy normed linear space $(Y, \|\|^\sim, L, R)$.*

(1) *If $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} Tx_n = Tx$.*

(2) *If $\lim_{a\to 0^+} R(a, a) = 0$, Then the null space $\mathcal{N}(T)$ of T is fuzzy closed.*

Corollary 3.10. *Let T be a weakly fuzzy bounded linear operator from a complete fuzzy normed linear space $(X, \|\|, L, R)$ to a fuzzy normed linear space $(Y, \|\|^\sim, L, R)$ with $\lim_{a\to 0^+} R(a, a) = 0$. Suppose that for some $\mathcal{C} \succ \bar{0}$ in $\mathcal{F}$, $\| Tx \|^\sim \oslash \| x \| \succeq \mathcal{C}$. Then the range $\mathcal{R}(T)$ of T is fuzzy closed. Furthermore, if $\mathcal{R}(T) = Y$, then T is invertible and $\| T^{-1} \|_\alpha^{*2} \leq \frac{1}{\mathcal{C}_\alpha^2}$.*

The proofs of the above corollaries are same as in Theorem 3.3, Lemma 3.6 and Theorem 3.7 using Remark 2.10.

Remark 3.11. In connection to our Lemma 3.6 and Corollary 3.9, we show that the converse is not true in general, i.e., a continuous linear operator need not always be weakly fuzzy bounded and hence not strongly fuzzy bounded.

Example 3.12. Let X be a vector space over $\mathbb{R}$ and $B = \{e_i\}_{i=1}^\infty$ a basis for $X(dimX = \infty)$. With $L = \min$ and $R = \max$, let us define the fuzzy norms on X as follows:

$$\| x \|_\circ (t) = \begin{cases} 1, & \text{if } t = \sum_{i=1}^{n} | a_i | \\ 0, & \text{otherwise} \end{cases}$$

and

$$\| x \| (t) = \begin{cases} \sum_{i=1}^{n} \frac{| a_i |}{t^i}, & \text{if } \sum_{i=1}^{n} | a_i | \leq t, \\ 1, & \text{if } t = \sum_{i=1}^{n} | a_i | = 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $x = \sum_{i=1}^{n} a_i e_i$. Then the α-level sets of $\| \, . \, \|_\circ$ and $\| \, . \, \|$ are

$$[\| x \|_\circ]_\alpha = [\sum_{i=1}^{n} | a_i |, \sum_{i=1}^{n} | a_i |] \text{ and } [\| x \|]_\alpha = [\sum_{i=1}^{n} | a_i |, \sum_{i=1}^{n} \frac{| a_i |}{\alpha^i}].$$

Let $T : (X, \| \, . \, \|) \to (X, \| \, . \, \|_\circ)$ be a linear operator defined by $Te_n = ne_n$ for each $e_n \in B$.

(i) T is fuzzy norm continuous: Let $\{x_n\}$ be a sequence in $(X, \| \, . \, \|)$ such that $x_n \to 0$ and let $x_n = \sum_{i=1}^{k_n} a_{n_i} e_i$. Then $Tx_n = \sum_{i=1}^{k_n} i a_{n_i} e_i$. Thus $\| Tx_n \|_{\circ\alpha}^2 = \sum_{i=1}^{k_n} | a_{n_i} i |$, for all $\alpha \in (0, 1]$.

Let $\beta < 1$. Then we have

$$\| Tx_n \|_{\circ\alpha}^2 = \sum_{i=1}^{k_n} | a_{n_i} i | = \sum_{i=1}^{k_n} i \, | a_{n_i} | \leq \sum_{i=1}^{k_n} \frac{1}{\beta^i} | a_{n_i} | = \| x_n \|_\beta^2 .$$

As $x_n \to 0$, $\| x_n \|_\beta^2 \to 0$, for all $\beta \in (0, 1]$. Thus it gives $\| Tx_n \|_{\circ\alpha}^2 \to 0$, for all $\alpha \in (0, 1]$. So T is fuzzy norm continuous.

(ii) T is not weakly fuzzy bounded: Let T be weakly fuzzy bounded. Then there exists a fuzzy real number $\eta \succ \bar{0}$ such that $\| Tx \|_\circ \oslash \| x \| \preceq \eta$, for all $x \in X$. Thus $\| Tx \|_{\circ 1}^2 = \| ne_n \|_{\circ 1}^2 = | n | = n \leq \eta_1^2 \| e_n \|_1^2 = \eta_1^2$. So $n \leq \eta_1^2$, for all $n \in \mathbb{N}$. It gives $\eta_1^2 = \infty$ which is a contradiction. Hence T is not weakly fuzzy bounded.

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