Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 13, No. 6, (June 2017), pp. 759–768
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

New properties of $S_{\alpha,\beta}$ and $T_{\alpha,\beta}$ intuitionistic fuzzy modal operators

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Received 6 October 2016; Revised 15 November 201y; Accepted 27 November 2016

Abstract. K. T. Atanassov defined Intuitionistic Fuzzy Modal Operators firstly in 1999[2]. After then, new Intuitionistic Fuzzy Modal Operators called one type and second type modal operators on Intuitionistic Fuzzy Sets introduced by different authors[3, 4, 5, 7, 10, 11, 13, 16]. Several properties of these operators had been studied by researchers.
New properties of intuitionistic fuzzy modal operators called $S_{\alpha,\beta}$ and $T_{\alpha,\beta}$ are investigated.

2010 AMS Classification: 03E72, 47S40

Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy modal operators, OTMOs diagram

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1. Introduction

In 1965, Zadeh [17] introduced the Fuzzy Set Theory as an extension of crisp sets. The concept of Intuitionistic fuzzy sets was introduced by Atanassov in 1986 [1], that is an extension of fuzzy sets by expanding the truth value set to the lattice $[0, 1] \times [0, 1]$.

Intuitionistic fuzzy modal operators introduced by Atanassov[1, 2] which are called One Type Intuitionistic Fuzy Modal Operators and Second Type Intuitionistic Fuzy Modal Operators. Then severel extensions of these operators defined by different authors[3, 4, 5, 10, 11]. Some algebraic and characteristic properties of these operators were examined. Also, the effect of modal operators on algebraic structures were studied by several authors[8, 9, 15]. In 2014, the author introduced the concept of Uni-type Intuitionistic Fuzzy Modal Operators and examined some relationships of them[12].

In this study, we examine some properties of second type intuitionistic fuzzy modal operators $S_{\alpha,\beta}$ and $T_{\alpha,\beta}$. In addition, we obtaine some relationships between $S_{\alpha,\beta}$ and $T_{\alpha,\beta}$ with one type intuitionistic fuzzy modal operators.

2. Preliminaries

Definition 2.1 ([1]). An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the degree of membership of x in A , $\nu_A(x), (\nu_A : X \to [0,1])$is called the degree of non- membership of x in A ,and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \text{ for all } x \in X.$$

The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$.

The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$

Definition 2.2 ([1]). An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$), if for all $x \in X, \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 2.3 ([1]). Let $A \in IFS$ and let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$. Then the complement of A denoted by A^c, is defined by

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}.$$

The following operations of two IFSs A and B on X is defined by

$$A \sqcap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\},$$
$$A \sqcup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\},$$

$$A@B = \left\{\left\langle x, \frac{\mu_A(x) + \mu_B(x)}{2}, \frac{\nu_A(x) + \nu_B(x)}{2}\right\rangle : x \in X\right\}.$$

The notion of Second Type Intuitionistic Fuzzy Modal Operators was firstly introduced by Atanassov as following:

Definition 2.4 ([1]). Let X be universal and $A \in IFS(X)$. Then
(i) $\Box(A) = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$,
(ii) $\Diamond(A) = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle : x \in X\}$.

Definition 2.5 ([2]). Let X be universal and $A \in IFS(X)$, $\alpha \in [0,1]$. Then

$$D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1-\alpha)\pi_A(x)\rangle : x \in X\}.$$

Definition 2.6 ([2]). Let X be universal and $A \in IFS(X)$, $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$. Then

$$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + \beta\pi_A(x)\rangle : x \in X\}.$$

Definition 2.7 ([2]). Let X be universal and $A \in IFS(X)$, $\alpha, \beta \in [0,1]$. Then

$$G_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x)\rangle : x \in X\}.$$

Definition 2.8 ([2]). Let X be universal and $A \in IFS(X)$, $\alpha, \beta \in [0,1]$. Then
(i) $H_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta\pi_A(x)\rangle : x \in X\}$,
(ii) $H^*_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta(1 - \alpha\mu_A(x) - \nu_A(x))\rangle : x \in X\}$,
(iii) $J_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \beta\nu_A(x)\rangle : x \in X\}$,
(iv) $J^*_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha(1 - \mu_A(x) - \beta\nu_A(x)), \beta\nu_A(x)\rangle : x \in X\}$.

The simplest One Type Intuitionistic Fuzzy Modal Operators defined by Atanassov, in 1999.

Definition 2.9 ([2])**.** Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X), \alpha, \beta \in [0,1]$. Then
(i) $\boxplus A = \left\{\left\langle x, \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2}\right\rangle : x \in X\right\}$,
(ii) $\boxtimes A = \left\{\left\langle x, \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2}\right\rangle : x \in X\right\}$.

After this definition, in 2001, Atanassov, defined the extension of these operators as following:

Definition 2.10 ([3])**.** Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$, $\alpha, \beta \in [0,1]$. Then
(i) $\boxplus_\alpha A = \{\langle x, \alpha\mu_A(x), \alpha\nu_A(x) + 1 - \alpha\rangle : x \in X\}$,
(ii) $\boxtimes_\alpha A = \{\langle x, \alpha\mu_A(x) + 1 - \alpha, \alpha\nu_A(x)\rangle : x \in X\}$.

The operators $\boxplus_\alpha$and $\boxtimes_\alpha$ are the extensions of the operators $\boxplus$, $\boxtimes$, respectively.
In 2004, Dencheva introduced the second extension of $\boxplus_\alpha$and $\boxtimes_\alpha$.

Definition 2.11 ([13])**.** Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$, $\alpha, \beta \in [0,1]$. Then
(i) $\boxplus_{\alpha,\beta} A = \{\langle x, \alpha\mu_A(x), \alpha\nu_A(x) + \beta\rangle : x \in X\}$, where $\alpha + \beta \in [0,1]$,
(ii) $\boxtimes_{\alpha,\beta} A = \{\langle x, \alpha\mu_A(x) + \beta, \alpha\nu_A(x)\rangle : x \in X\}$,where $\alpha + \beta \in [0,1]$.

In 2006, the third extension of the above operators was studied by Atanassov. He defined the following operators;

Definition 2.12 ([4])**.** Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$. Then
(i) $\boxplus_{\alpha,\beta,\gamma}(A) = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x) + \gamma\rangle : x \in X\}$,
where $\alpha, \beta, \gamma \in [0,1], \max\{\alpha, \beta\} + \gamma \leqslant 1$.
(ii) $\boxtimes_{\alpha,\beta,\gamma}(A) = \{\langle x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x)\rangle : x \in X\}$,
where $\alpha, \beta, \gamma \in [0,1], \max\{\alpha, \beta\} + \gamma \leqslant 1$.

In 2007, author[10] defined a new operator named $E_{\alpha,\beta}$ and studied some of its properties. This operator is given below.

Definition 2.13 ([10])**.** Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$, $\alpha, \beta \in [0,1]$. We define the following operator:

$$E_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + 1 - \alpha), \alpha(\beta\nu_A(x) + 1 - \beta)\rangle : x \in X\}.$$

In the same year, Atanassov introduced the operator $\boxdot_{\alpha,\beta,\gamma,\delta}$ which is a natural extension of all these operators in [6].

Definition 2.14 ([6])**.** Let Xbe a set, $A \in IFS(X)$, $\alpha, \beta, \gamma, \delta \in [0,1]$ such that

$$\max(\alpha, \beta) + \gamma + \delta \leqslant 1.$$

Then the operator $\boxdot_{\alpha,\beta,\gamma,\delta}$ defined by

$$\boxdot_{\alpha,\beta,\gamma,\delta}(A) = \{\langle x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x) + \delta\rangle : x \in X\}.$$

In 2008, most general operator $\circledcirc_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$ was introduced.

Definition 2.15 ([5]). Let X be a set, $A \in IFS(X)$, $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0, 1]$ such that

$$\max(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \leqslant 1$$

and

$$\min(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \geq 0.$$

Then the operator $\circledcirc_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$ defined by

$$\circledcirc_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A) = \{\langle x, \alpha\mu_A(x) - \varepsilon\nu_A(x) + \gamma, \beta\nu_A(x) - \zeta\mu_A(x) + \delta\rangle : x \in X\}.$$

In 2010, Çuvalcıoğlu[11] defined a new operator which is a generalization of $E_{\alpha,\beta}$.

Definition 2.16 ([11]). Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$, $\alpha, \beta, \omega \in [0, 1]$. Then

$$Z^{\omega}_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \omega - \omega.\beta)\rangle : x \in X\}.$$

Definition 2.17 ([11]). Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$, $\alpha, \beta, \omega, \theta \in [0, 1]$. Then

$$Z^{\omega,\theta}_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \theta - \theta.\beta)\rangle : x \in X\}.$$

The operator $Z^{\omega,\theta}_{\alpha,\beta}$ is a generalization of $Z^{\omega}_{\alpha,\beta}$,and also, $E_{\alpha,\beta}, \boxplus_{\alpha,\beta}, \boxtimes_{\alpha,\beta}$. The diagram of modal operators was created first in 2007. In following years, this diagram was expanded by defining new modal operators. The last diagram was given in [11], as in Figure 1.

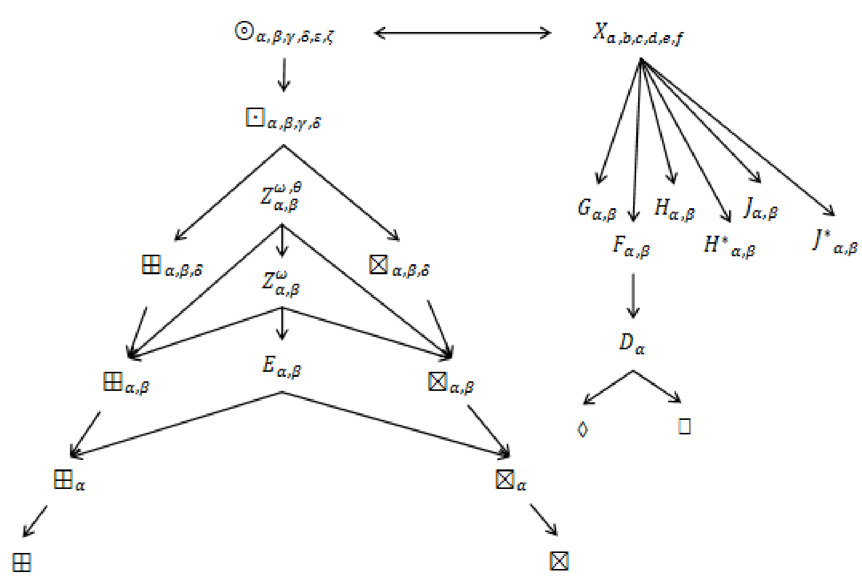

FIGURE 1.

The concept of Uni-type intuitionistic fuzzy modal operators introduced by the author in [12].

Definition 2.18 ([12]). Let X be a universal, $A \in IFS(X)$ and $\alpha, \beta, \omega \in [0,1]$. Then

(i) $\boxplus^{\omega}_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x)), \alpha(\beta\nu_A(x) + \omega - \omega\beta)\rangle : x \in X\}$,
(ii) $\boxtimes^{\omega}_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega\alpha), \alpha((1-\beta)\mu_A(x) + \nu_A(x))\rangle : x \in X\}$.

Definition 2.19 ([12]). Let X be a set and $A \in IFS(X)$, $\alpha, \beta, \omega, \theta \in [0,1]$. Then

$$E^{\omega,\theta}_{\alpha,\beta}(A) = \left\{ \left\langle \begin{array}{c} x, \beta((1-(1-\alpha)(1-\theta))\mu_A(x) + (1-\alpha)\theta\nu_A(x) + (1-\alpha)(1-\theta)\omega), \\ \alpha((1-\beta)\theta\mu_A(x) + (1-(1-\beta)(1-\theta))\nu_A(x) + (1-\beta)(1-\theta)\omega) \end{array} \right\rangle : x \in X \right\}.$$

Definition 2.20 ([12]). Let X be a set, $A \in IFS(X)$ and $\alpha, \beta \in [0,1]$. Then

$$B_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x)), \alpha((1-\beta)\mu_A(x) + \nu_A(x))\rangle : x \in X\}.$$

Definition 2.21 ([12]). Let X be a set, $A \in IFS(X)$ and $\alpha, \beta, \omega \in [0,1]$. Then

$$\boxminus_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\beta)\nu_A(x)), \alpha((1-\alpha)\mu_A(x) + \nu_A(x))\rangle : x \in X\}.$$

Uni-type intuitionistic fuzzy modal operators have relation to both types of operators and expanded the diagram. Then, in 2014 last one-type modal operators defined by authors as below.

Definition 2.22 ([16]). Let X be a set and $A \in IFS(X), \alpha, \beta, \omega \in [0,1]$ and $\alpha + \beta \leq 1$.

(i) $L^{\omega}_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x) + \omega(1-\alpha), \alpha(1-\beta)\nu_A(x) + \alpha\beta(1-\omega)\rangle\, x \in X\}$.
(ii) $K^{\omega}_{\alpha,\beta}(A) = \{\langle x, \alpha(1-\beta)\mu_A(x) + \alpha\beta(1-\omega), \alpha\nu_A(x) + \omega(1-\alpha)\rangle\, x \in X\}$.

After this definition, we get the following diagram:

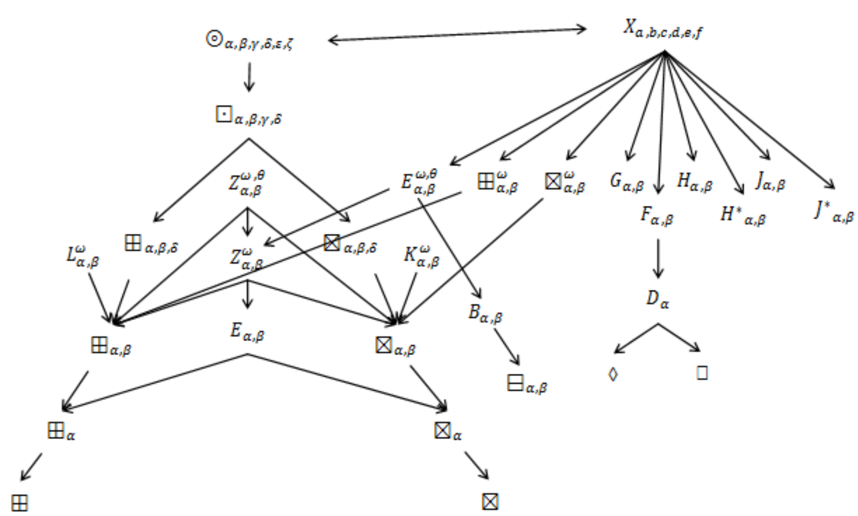

Figure 2.

Proposition 2.23 ([16]). *Let X be a universal, $A \in IFS(X)$ and $\alpha \in [0,1]$.*
(1) $L^1_{\alpha,0}(A) = \boxtimes_\alpha(A)$.
(2) $K^1_{\alpha,0}(A) = \boxplus_\alpha(A)$.

Proposition 2.24. [16]*Let X be a universal, $A \in IFS(X)$ and $\alpha, \beta, \alpha+\beta \in [0,1]$, $\alpha \neq 1$.*

(1) $L^{\frac{\beta}{1-\alpha}}_{\alpha,0}(A) = \boxtimes_{\alpha,\beta}(A)$.

(2) $K^{\frac{\beta}{1-\alpha}}_{\alpha,0}(A) = \boxplus_{\alpha,\beta}(A)$.

The second type intuitionistic fuzzy modal operator, represented by $\otimes_{\alpha,\beta,\gamma,\delta}$, was introduced in [7] and some properties were given.

Definition 2.25 ([7]). Let X be a set and $A \in IFS(X), \alpha, \beta, \gamma, \delta \in [0,1]$ and $\alpha+\beta \leq 1, \gamma+\delta \leq 1$. Then

$$\otimes_{\alpha,\beta,\gamma,\delta}(A) = \{\langle x, \alpha\mu_A(x) + \gamma\nu_A(x), \beta\mu_A(x) + \delta\nu_A(x)\rangle\}.$$

3. New properties of $S_{\alpha,\beta}$ and $T_{\alpha,\beta}$ intuitionistic fuzzy modal operators

In this section, we will give the definition of $S_{\alpha,\beta}$ and $T_{\alpha,\beta}$ modal operators which were introduced in [14] and we will give new results on these operators.

Definition 3.1 ([14]). Let X be a set and $A \in IFS(X), \alpha, \beta, \alpha+\beta \in [0,1]$.
(i) $T_{\alpha,\beta}(A)$
$= \{< x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x) + \alpha), \alpha(\nu_A(x) + (1-\beta)\mu_A(x)) >: x \in X\}$,
where $\alpha+\beta \in [0,1]$.
(ii) $S_{\alpha,\beta}(A)$
$= \{< x, \alpha(\mu_A(x) + (1-\beta)\nu_A(x)), \beta(\nu_A(x) + (1-\alpha)\mu_A(x) + \alpha) >: x \in X\}$,
where $\alpha+\beta \in [0,1]$.

It is clear that:

$$\begin{aligned} &\beta(\mu_A(x) + (1-\alpha)\nu_A(x) + \alpha) + \alpha(\nu_A(x) + (1-\beta)\mu_A(x)) \\ &= (\mu_A(x) + \nu_A(x))(\alpha+\beta-\alpha\beta) + \alpha\beta \\ &\leq \alpha+\beta-\alpha\beta+\alpha\beta \leq 1. \end{aligned}$$

These new operators are given in the diagram as Figure 3.

Theorem 3.2. *Let X be a set and $A \in IFS(X)$. If $\alpha, \beta, \alpha+\beta \in [0,1]$, then $T_{\alpha,\beta}(A)^c = S_{\alpha,\beta}(A^c)$.*

Proof. It is clear from definition. □

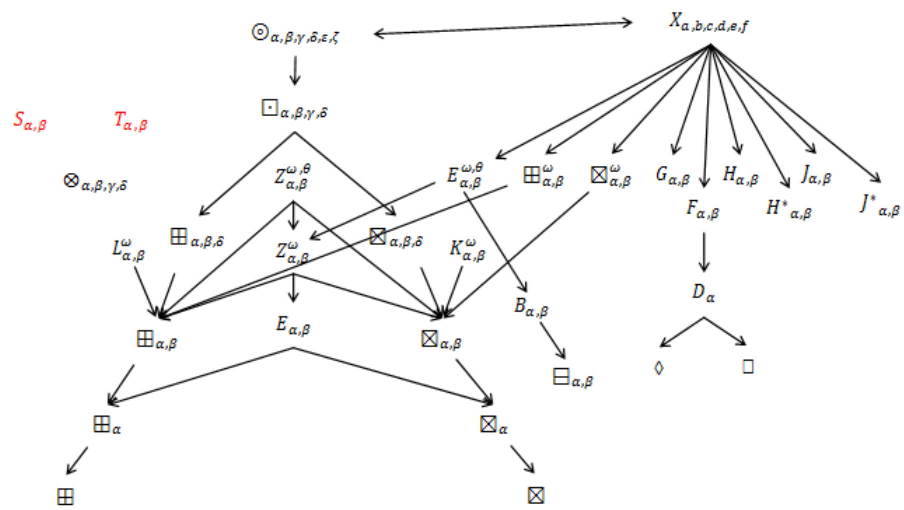

FIGURE 3.

Proposition 3.3. *Let X be a set and $A \in IFS(X)$. If $\alpha, \beta, \alpha + \beta \in [0, 1]$, then*

(1) $T_{\beta,\alpha}(A)^c \sqsubseteq T_{\alpha,\beta}(A^c)$,
(2) $S_{\alpha,\beta}(A^c) \sqsubseteq S_{\beta,\alpha}(A)^c$.

Proof. (1) From definition of this operators and complement of an intuitionistic fuzzy set, we get that,

$$\beta(\nu_A(x) + (1 - \alpha)\mu_A(x)) \leq \beta(\nu_A(x) + (1 - \alpha)\mu_A(x) + \alpha)$$

and

$$\alpha(\mu_A(x) + (1 - \beta)\nu_A(x) + \beta) \geq \alpha(\mu_A(x) + (1 - \beta)\nu_A(x)).$$

Then, we can say $T_{\beta,\alpha}(A)^c \sqsubseteq T_{\alpha,\beta}(A^c)$.

(2) We can show this inclusion as the same way. □

Theorem 3.4. *Let X be a set and $A \in IFS(X)$. If $\alpha, \beta, \alpha + \beta \in [0, 1]$ and $\beta \leq \alpha$, then*

(1) $T_{\alpha,\beta}(A) \sqsubseteq T_{\beta,\alpha}(A)$,
(2) $S_{\beta,\alpha}(A) \sqsubseteq S_{\alpha,\beta}(A)$.

Proof. It is clear. □

Theorem 3.5. *Let X be a set and $A, B \in IFS(X)$. If $\alpha, \beta, \alpha + \beta \in [0, 1]$ then*

(1) $T_{\alpha,\beta}(A) \sqcap T_{\alpha,\beta}(B) \sqsubseteq T_{\alpha,\beta}(A \sqcap B)$,
(2) $T_{\alpha,\beta}(A \sqcup B) \sqsubseteq T_{\alpha,\beta}(A) \sqcup T_{\alpha,\beta}(B)$.

Proof. (1) Let $\alpha, \beta \in [0,1]$. Then

$$\beta(1-\alpha)\min(\nu_A(x), \nu_B(x)) \quad \leq \quad \beta(1-\alpha)\max(\nu_A(x), \nu_B(x)).$$

Thus

$$\beta\left(\min\left(\mu_A(x), \mu_B(x)\right) + (1-\alpha)\min\left(\nu_A(x), \nu_B(x)\right) + \alpha\right)$$
$$\leq \beta\left(\min\left(\mu_A(x), \mu_B(x)\right) + (1-\alpha)\max\left(\nu_A(x), \nu_B(x)\right) + \alpha\right).$$

On the other hand

$$\alpha(1-\beta)\max(\mu_A(x), \mu_B(x)) \quad \geq \quad \alpha(1-\beta)\min(\mu_A(x), \mu_B(x)).$$

So

$$\alpha\left(\max\left(\nu_A(x), \nu_B(x)\right) + (1-\beta)\max\left(\mu_A(x), \mu_B(x)\right)\right)$$
$$\geq \alpha\left(\max(\nu_A(x), \nu_B(x)) + (1-\beta)\min(\mu_A(x), \mu_B(x))\right).$$

Hence we see that $T_{\alpha,\beta}(A) \sqcap T_{\alpha,\beta}(B) \sqsubseteq T_{\alpha,\beta}(A \sqcap B)$.
(2) It can be shown easily. □

Theorem 3.6. *Let X be a set and $A, B \in IFS(X)$. If $\alpha, \beta, \alpha+\beta \in [0,1]$, then*
(1) $S_{\alpha,\beta}(A \sqcup B) \sqsubseteq S_{\alpha,\beta}(A) \sqcup S_{\alpha,\beta}(B)$,
(2) $S_{\alpha,\beta}(A) \sqcap S_{\alpha,\beta}(B) \sqsubseteq S_{\alpha,\beta}(A \sqcap B)$.

Proof. (1) Let $\alpha, \beta \in [0,1]$. Then

$$\alpha(1-\beta)\min(\nu_A(x), \nu_B(x)) \quad \leq \quad \alpha(1-\beta)\max(\nu_A(x), \nu_B(x)).$$

Thus

$$\alpha\left(\max\left(\mu_A(x), \mu_B(x)\right) + (1-\beta)\min\left(\nu_A(x), \nu_B(x)\right)\right)$$
$$\leq \alpha\left(\max\left(\mu_A(x), \mu_B(x)\right) + (1-\beta)\max\left(\nu_A(x), \nu_B(x)\right)\right).$$

On the other hand

$$\beta(1-\alpha)\max(\mu_A(x), \mu_B(x)) \quad \geq \quad \beta(1-\alpha)\min(\mu_A(x), \mu_B(x)).$$

So

$$\beta\left(\min\left(\nu_A(x), \nu_B(x)\right) + (1-\alpha)\max\left(\mu_A(x), \mu_B(x)\right) + \alpha\right)$$
$$\geq \beta\left(\min\left(\nu_A(x), \nu_B(x)\right) + (1-\alpha)\min\left(\mu_A(x), \mu_B(x)\right) + \alpha\right).$$

Hence, $S_{\alpha,\beta}(A \sqcup B) \sqsubseteq S_{\alpha,\beta}(A) \sqcup S_{\alpha,\beta}(B)$.
(2) Can be proved similarly. □

Theorem 3.7. *Let X be a set and $A, B \in IFS(X)$. If $\alpha, \beta, \alpha+\beta \in [0,1]$, then*
(1) $T_{\alpha,\beta}(A@B) = T_{\alpha,\beta}(A)@T_{\alpha,\beta}(B)$,
(2) $S_{\alpha,\beta}(A@B) = S_{\alpha,\beta}(A)@S_{\alpha,\beta}(B)$.

Proof. (1)

$$\begin{aligned} T_{\alpha,\beta}(A@B) &= \{\left\langle \begin{array}{c} x, \beta(\frac{\mu_A(x)+\mu_B(x)}{2} + (1-\alpha)\frac{\nu_A(x)+\nu_B(x)}{2} + \alpha), \\ \alpha(\frac{\nu_A(x)+\nu_B(x)}{2} + (1-\beta)\frac{\mu_A(x)+\mu_B(x)}{2}) \end{array} \right\rangle <: x \in X\} \\ &= \{\left\langle \begin{array}{c} x, \beta(\frac{\mu_A(x)+(1-\alpha)\nu_A(x)+\mu_B(x)+(1-\alpha)\nu_B(x)}{2} + \alpha), \\ \alpha(\frac{\nu_A(x)+(1-\beta)\mu_A(x)+\nu_B(x)+(1-\beta)\mu_B(x)}{2}) \end{array} \right\rangle : x \in X\} \\ &= T_{\alpha,\beta}(A)@T_{\alpha,\beta}(B) \end{aligned}$$

(2)The proof is clear. □

Theorem 3.8. *Let X be a set and $A \in IFS(X)$. If $\alpha, \beta, \alpha+\beta \in [0,1]$, then*

(1) $\boxplus_\alpha(T_{\alpha,\beta}(A)) \sqsubseteq T_{\alpha,\beta}(\boxplus_\alpha(A)$,
(2) $\boxplus_{\alpha,\beta}(T_{\alpha,\beta}(A)) \sqsubseteq T_{\alpha,\beta}(\boxplus_{\alpha,\beta}(A)$.

Proof. (1) If we use $1-\alpha \geq 0$, then $\beta(1-\alpha)^2 + \alpha\beta \geq \alpha^2\beta$. Thus

$$\begin{aligned} &\alpha\beta\mu_A(x) + \alpha\beta(1-\alpha)\nu_A(x) + \beta(1-\alpha)^2 + \alpha\beta \\ \geq\ &\alpha\beta\mu_A(x) + \alpha\beta(1-\alpha)\nu_A(x) + \alpha^2\beta \end{aligned}$$

and

$$\begin{aligned} &\alpha^2\nu_A(x) + \alpha^2(1-\beta)\mu_A(x) + \alpha(1-\alpha) \\ =\ &\alpha^2\nu_A(x) + \alpha^2(1-\beta)\mu_A(x) + \alpha(1-\alpha). \end{aligned}$$

So, $\boxplus_\alpha(T_{\alpha,\beta}(A)) \sqsubseteq T_{\alpha,\beta}(\boxplus_\alpha(A)$.

(2) Proof of this inclusion is similar. □

Theorem 3.9. *Let X be a set and $A \in IFS(X)$. If $\alpha, \beta, \alpha+\beta \in [0,1]$, then*

(1) $S_{\alpha,\beta}(\boxtimes_\alpha(A)) \sqsubseteq \boxtimes_\alpha(S_{\alpha,\beta}(A))$,
(2) $S_{\alpha,\beta}(\boxtimes_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\alpha,\beta}(S_{\alpha,\beta}(A))$.

Proof. (2) If we use $\alpha\beta \leq \alpha$, then

$$\alpha^2\mu_A(x) + \alpha^2(1-\beta)\nu_A(x) + \alpha\beta \leq \alpha^2\mu_A(x) + \alpha^2(1-\beta)\nu_A(x) + \beta$$

and

$$\alpha\beta + \beta^2(1-\alpha) \geq \alpha^2\beta.$$

Thus

$$\begin{aligned} &\alpha\beta\nu_A(x) + \alpha\beta(1-\alpha)\mu_A(x) + \alpha\beta + \beta^2(1-\alpha) \\ \geq\ &\alpha\beta\nu_A(x) + \alpha\beta(1-\alpha)\mu_A(x) + \alpha^2\beta. \end{aligned}$$

So, $S_{\alpha,\beta}(\boxtimes_{\alpha,\beta}(A)) \sqsubseteq \boxtimes_{\alpha,\beta}(S_{\alpha,\beta}(A))$. □

4. Conclusions

In this paper, some properties of intuitionistic fuzzy modal operators $S_{\alpha,\beta}$ and $T_{\alpha,\beta}$ are given. We obtained some relationships of these operators with some intuitionistic fuzzy operations and one type modal operators, $\boxtimes_\alpha$,$\boxplus_\alpha$, $\boxtimes_{\alpha,\beta}$,$\boxplus_{\alpha,\beta}$. In subsequent studies, relationships with second type intuitionistic fuzzy modal operators and uni-type intuitionistic fuzzy modal operators can be examined.

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