Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 14, No. 5, (November 2017), pp. 433–443
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 14, No. 5, November 2017

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Received 4 January 2017; Revised 2 February 2017; Accepted 27 March 2017

Abstract. In this paper, we shall define k-pseudo similarity (right k-pseudo similar or left k-pseudo similar) for intuitionistic fuzzy matrices and prove that, for a pair of intuitionistic fuzzy matrices $A, B \in (IF)_n$, if A is said to be right (left) k-pseudo similar to B then A^s is said to be right (left) k-pseudo similar to B^s for any integer $s \geq 1$, but the converse is not true which is illustrated by an example. Also prove that, A is said to be right (left) k-pseudo similar to B if and only if B^T is said to be left (right) k-pseudo similar to A^T. We exhibit that the k-pseudo similarity on A and B preserve k-regularity of the intuitionistic fuzzy matrices A and B. As a special case, for $k = 1$ it reduces to pseudo similar intuitionistic fuzzy matrices [3].

2010 AMS Classification: $03E72, 15B15$

Keywords: Intuitionistic fuzzy matrix (IFM), k-pseudo similar, k-regular fuzzy matrix, k-g inverse.

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1. Introduction

We deal with the fuzzy matrices that is the matrices over the fuzzy algebra with support $[0, 1]$ under the max-min operations $\{+, .\}$ defined as $a + b = max\{a, b\}$ and $a.b = min\{a, b\}$ for all $a, b \in \{F : F = [0, 1]\}$. Let F_{mn} the set of all $m \times n$ fuzzy matrices over the fuzzy algebra F. A matrix $A \in F_{m\times n}$ is said to be regular if there exists X such that $AXA = A$; X is called a generalized (g^-) inverse of A and is denoted by A^-. A development of theory of fuzzy matrices analogous to that of Boolean matrices is made by Kim and Roush [5]. Atanassov has introduced and developed the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [1]. A study on regularity and various g-inverse of intuitionistic fuzzy matrices over intuitionistic fuzzy algebra are discussed in [10]. Basic properties of intuitionistic fuzzy matrices as a generalization of the results on fuzzy matrices have been derived by Pal and Khan [4]. Meenakshi and Gandhimathi have studied on regularity of intuitionistic fuzzy matrices [8]. In [11], some properties on both idempotent intuitionistic fuzzy matrices and idempotent intuitionistic fuzzy matrices of T-type are discussed. In [12], a problem of reducing intuitionistic fuzzy matrices is examined and some useful properties are obtained with respect to nilpotent intuitionistic fuzzy matrices. In [6], some properties of a transitive fuzzy matrix are examined and the canonical form of the transitive fuzzy matrix is given using the properties also obtained a canonical form of the transitive intuitionistic fuzzy matrix. In [13], szpilrajn's theorem on ordering is generalized to intuitionistic fuzzy orderings. In [14], Riyaz Ahmad Padder and Murugadas have introduced the max-max operations on intuitionistic fuzzy matrices to study the conditions for convergence of intuitionistic fuzzy matrices. In [2], Cho has discussed the consistency of fuzzy matrix equations. Recently, Meenakshi and Jenita have introduced the concept of k-regular fuzzy matrix as a generalization of regular fuzzy matrix [9]. Further to learn about fuzzy matrix theory and applications one may refer [7]. In this paper, we have introduced the concept of k-pseudo similar intuitionistic fuzzy matrices(IFM) as a generalization of pseudo similar intuitionistic fuzzy matrices [3].

2. Preliminaries

In this paper, we are concerned with fuzzy matrices, that is matrices over a fuzzy algebra FM(FN) with support $[0, 1]$, under maxmin(minmax) operations and the usual ordering of real numbers. Let $(IF)_{m\times n}$ be the set of all intuitionistic fuzzy matrices of order $m \times n$, $F^M_{m\times n}$ be the set of all fuzzy matrices of order $m \times n$, under the maxmin composition and $F^N_{m\times n}$ be the set of all fuzzy matrices of order $m \times n$, under the minmax composition.

If $A = (a_{ij}) \in (IF)_{m\times n}$, then $A = (\langle a_{ij\mu}, a_{ij\vartheta}\rangle)$, where $a_{ij\mu}$ and $a_{ij\vartheta}$ are the membership values and non membership values of a_{ij} in A respectively with respect to the fuzzy sets μ and ϑ, maintaining the condition $0 \le a_{ij\mu} + a_{ij\vartheta} \le 1$.

We shall follow the matrix operations on intuitionistic fuzzy matrices as defined in [8].

For $A, B \in (IF)_{m\times n}$,

$A + B = (\langle \max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ij\vartheta}, b_{ij\vartheta}\}\rangle)$,

$AB = \left(\left\langle \max_k \min\{a_{ik\mu}, b_{kj\mu}\}, \min_k \max\{a_{ik\vartheta}, b_{kj\vartheta}\}\right\rangle\right)$.

Let us define the order relation on $(IF)_{m\times n}$ as:

$A \le B \Leftrightarrow a_{ij\mu} \le b_{ij\mu}$ and $a_{ij\vartheta} \ge b_{ij\vartheta}$, for all i and j.

In this work, we shall represent $A \in (IF)_{m\times n}$ as Cartesian product of fuzzy matrices.

For $A = (a_{ij}) \in (IF)_{m\times n}$. Let $A = (a_{ij}) = (\langle a_{ij\mu}, a_{ij\vartheta}\rangle) \in (IF)_{m\times n}$. We define $A_\mu = (a_{ij\mu}) \in F^M_{m\times n}$ as the membership part of A and $A_\vartheta = (a_{ij\vartheta}) \in F^N_{m\times n}$ as the non-membership part of A. Thus A is written as the Cartesian product A_μ and A_ϑ, $A =< A_\mu, A_\vartheta >$ with $A_\mu \in F^M_{m\times n}, A_\vartheta \in F^N_{m\times n}$.

Definition 2.1 ([8]). For $A, B \in (IF)_{m\times n}$, if $A =< A_\mu, A_\vartheta >$ and $B =< B_\mu, B_\vartheta >$, then $A + B =< A_\mu + B_\mu, A_\vartheta + B_\vartheta >$.

Definition 2.2 ([8]). For $A \in (IF)_{m\times p}, B \in (IF)_{p\times n}$ if $A =< A_\mu, A_\vartheta >$ and $B =< B_\mu, B_\vartheta >$, then

(i) $AB =< A_\mu B_\mu, A_\vartheta B_\vartheta >$, where $A_\mu B_\mu$ is the max min product in $F^M_{m\times n}$ and $A_\vartheta B_\vartheta$ is the min max product in $F^N_{m\times n}$,

(ii) $A^T =< A^T_\mu, A^T_\vartheta >$.

Definition 2.3 ([8]). A matrix $A \in (IF)_n$ is said to be invertible, if there exists $X \in (IF)_n$ such that $AX = XA = I_n =< I^M_n, I^N_n >$, where I_n is the identity matrix in $(IF)_n$.

Definition 2.4 ([8]). A square intuitionistic fuzzy matrix is called intuitionistic fuzzy permutation matrix, if every row and column contains exactly one $< 1, 0 >$ and all the other entries are $< 0, 1 >$. Let P_n be the set of all $n \times n$ permutation matrices in $(IF)_n$.

Definition 2.5 ([4]). An $A \in (IF)_{m\times n}$ is said to be regular, if there exists $X \in (IF)_{m\times n}$ satisfying $AXA = A$. In this case, X is called a generalized inverses (g-inverse) of A and is denoted by $\overline{A}$.

Let $A\{1\}$ be the set of all g-inverses of A.

Definition 2.6 ([3]). $A \in (IF)_m$ and $B \in (IF)_n$ are said to be pseudo similar, denoted by $A \cong B$, if there exist $X \in (IF)_{mn}$ and $Y \in (IF)_{nm}$ such that

$$A = XBY, B = YAX \text{ and } XYX = X.$$

Lemma 2.7 ([3]). *Let $A \in (IF)_m$ and $B \in (IF)_n$. Then the following are equivalent:*

(1) $A \cong B$,

(2) *there exist $X \in (IF)_{mn}, Y \in (IF)_{nm}$ such that $A = XBY, B = YAX, XYX = X$ and $YXY = Y$,*

(3) *There exist $X \in (IF)_{mn}, Y \in (IF)_{nm}$ such that $A = XBY, B = ZAX, XYX = X = XZX$.*

Theorem 2.8 ([8]). *Let $A \in (IF)_{m\times n}$ be of the form $A = \langle A_\mu, A_\vartheta\rangle$. Then A is regular $\Leftrightarrow$ A_μ is regular in $F^M_{m\times n}$ under max-min composition and A_ϑ is regular in $F^N_{m\times n}$ under min-max composition. $A_\mu = (a_{ij\mu}) \in F^M_{m\times n}$ as the membership part of A and $A_\vartheta = (a_{ij\vartheta}) \in F^N_{m\times n}$ as the non-membership part of A.*

3. k-Pseudo Similar Intuitionistic Fuzzy Matrices

Definition 3.1. A matrix $A \in (IF)_n$, is said be right k-regular, if there exists a matrix $X \in (IF)_n$ such that $A^k XA = A^k$, for some positive integer k.

In this case, X is called a right k-g-inverse of A.

Let $A_r \left\{1^k\right\} = \left\{X/A^k XA = A^k\right\}$.

Definition 3.2. A matrix $A \in (IF)_n$, is said be left k-regular, if there exists a matrix $Y \in (IF)_n$ such that $AYA^k = A^k$, for some positive integer k.

In this case, Y is called a left k-g-inverse of A.

Let $A_\ell \left\{1^k\right\} = \left\{Y/AYA^k = A^k\right\}$.

In general, right k-regular is different from left k-regular. Then a right k-g-inverse need not be a left k-g-inverse (refer to Example 3.4). Thus forth we call a right k-regular (*or*) left k-regular IFM as a k-regular IFM.

Example 3.3. Let us consider $A = \begin{bmatrix} \langle 0.3, 0\rangle & \langle 0, 1\rangle \\ \langle 0.5, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix} \in (IF)_2$, where $A_\mu = \begin{bmatrix} 0.3 & 0 \\ 0.5 & 0.2 \end{bmatrix} \in F_2^M$ and $A_\vartheta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in F_2^N$. Since each row of A_μ cannot be expressed as linear combination of the other row, by Definition 2.5 of (5) , the rows are linearly independent. Then by Definition 2.6 of (2) ,they form a standard basis for the row space of A_μ.
For both permutation matrices $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_\mu P_1 A_\mu = \begin{bmatrix} 0.3 & 0 \\ 0.3 & 0.2 \end{bmatrix} \neq A_\mu$ and $A_\mu P_2 A_\mu = \begin{bmatrix} 0.3 & 0.2 \\ 0.5 & 0.2 \end{bmatrix} \neq A_\mu$. Thus A_μ is not regular by step 3 in Algorithm 1 of (2) . Namely, A_μ is regular iff $A_\mu P A_\mu = A_\mu$, for some permutation matrix P. Since A_ϑ is idempotent, A_ϑ itself is a g-inverse of A_ϑ, A_ϑ is regular under min max composition. So by Theorem 2.8, A is not regular.
For this A, $A^2 = \begin{bmatrix} \langle 0.3, 0\rangle & \langle 0, 1\rangle \\ \langle 0.3, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix}$. For $X = \begin{bmatrix} \langle 1, 0\rangle & \langle 0, 1\rangle \\ \langle 0, 0\rangle & \langle 0.2, 0\rangle \end{bmatrix}$, $A^2XA = A^2 = AXA^2$ holds. Hence A is 2-regular.

Example 3.4. Let $A = \begin{bmatrix} \langle 1, 0\rangle & \langle 0.5, 0.5\rangle & \langle 0, 0\rangle \\ \langle 0, 0\rangle & \langle 0, 1\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0, 0\rangle & \langle 0, 0\rangle \end{bmatrix}$. Then A_μ is not regular (See [9]). Thus by Theorem 2.8, A is not regular.
For this A, $A^2 = \begin{bmatrix} \langle 1, 0\rangle & \langle 0.5, 0\rangle & \langle 0.5, 0\rangle \\ \langle 0.5, 0\rangle & \langle 0, 0.5\rangle & \langle 0, 0\rangle \\ \langle 0.5, 0\rangle & \langle 0.5, 0\rangle & \langle 0, 0\rangle \end{bmatrix}$, $A^3 = \begin{bmatrix} \langle 1, 0\rangle & \langle 0.5, 0\rangle & \langle 0.5, 0\rangle \\ \langle 0.5, 0\rangle & \langle 0.5, 0\rangle & \langle 0, 0\rangle \\ \langle 0.5, 0\rangle & \langle 0.5, 0\rangle & \langle 0.5, 0\rangle \end{bmatrix}$.
For $X = \begin{bmatrix} \langle 1, 0\rangle & \langle 0, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0, 1\rangle & \langle 0.5, 0\rangle \\ \langle 0.5, 0\rangle & \langle 0.5, 0.5\rangle & \langle 0.5, 0\rangle \end{bmatrix}$, $A^3XA = A^3 \neq AXA^3$ holds. So X is a right 3-g inverse but X is not a 3-g inverse of A.

Theorem 3.5. *Let $A = \langle A_\mu, A_\vartheta\rangle \in (IF)_n$. Then A is right k-regular IFM $\Leftrightarrow$ $A_\mu, A_\vartheta \in F_n$ are right k-regular.*

Proof. Let $A = \langle A_\mu, A_\vartheta\rangle \in (IF)_n$. Since A is right k-regular IFM, there exists $X \in (IF)_n$, such that $A^kXA = A^k$.
Let$X = \langle X_\mu, X_\vartheta\rangle \in (IF)_n$ with $X_\mu, X_\vartheta \in F_n$. Then by Definition 2.2, $A^kXA = A^k$. Thus $\langle A_\mu, A_\vartheta\rangle^k \langle X_\mu, X_\vartheta\rangle \langle A_\mu, A_\vartheta\rangle = \langle A_\mu, A_\vartheta\rangle^k$,

$$\left\langle A_\mu^k, A_\vartheta^k\right\rangle \langle X_\mu, X_\vartheta\rangle \langle A_\mu, A_\vartheta\rangle = \left\langle A_\mu^k, A_\vartheta^k\right\rangle,$$
$$\left\langle A_\mu^k X_\mu A_\mu, A_\vartheta^k X_\vartheta A_\vartheta\right\rangle = \left\langle A_\mu^k, A_\vartheta^k\right\rangle,$$
$$A_\mu^k X_\mu A_\mu = A_\mu^k \text{ and } A_\vartheta^k X_\vartheta A_\vartheta = A_\vartheta^k.$$

So $A_\mu, A_\vartheta \in F_n$ are right k-regular.
Conversely, suppose $A_\mu, A_\vartheta \in F_n$ are right k-regular. Then $A_\mu^k X_\mu A_\mu = A_\mu^k$ and $A_\vartheta^k X_\vartheta A_\vartheta = A_\vartheta^k$, for some $X_\mu, X_\vartheta \in F_n$. Thus X_μ is a right k-g inverse of A_μ and X_ϑ is a right k-g inverse of A_ϑ.
Now let us define the IFM $Z = \langle V, W\rangle$, where V is a right k-g inverse of A_μ and W is a right k-g inverse of A_ϑ. We claim that Z is a right k-g inverse of A. Then by

Definition 2.2,

$$A^k ZA = \langle A_\mu, A_\vartheta\rangle^k \langle V, W\rangle \langle A_\mu, A_\vartheta\rangle = \left\langle A_\mu^k V A_\mu, A_\vartheta^k W A_\vartheta\right\rangle = \left\langle A_\mu^k, A_\vartheta^k\right\rangle = A^k.$$

Thus A is right k-regular IFM. So the proof is done. □

Theorem 3.6. *Let $A = \langle A_\mu, A_\vartheta\rangle \in (IF)_n$. Then A is left k-regular IFM $\Leftrightarrow A_\mu, A_\vartheta \in F_n$ are left k-regular.*

Proof. This can be proved along the same lines as that of Theorem 3.5. □

Definition 3.7. $A \in (IF)_n$ is said to be right k-pseudo similar to $B \in (IF)_n$, denoted by $A \cong_r^k B$, if there exist $X, Y \in (IF)_n$ such that $A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$.

Definition 3.8. $A \in (IF)_n$ is said to be left k-pseudo similar to $B \in (IF)_n$, denoted by $A \cong_\ell^k B$, if there exist $X, Y \in (IF)_n$ such that $A = X^kBY, B = YAX, XYX^k = X^k$ and $YXY = Y$.

Remark 3.9. In particular for k=1, Definitions 3.7 and 3.8 are identical. Then k-pseudo similar is reduced to Lemma 2.7. However, both right and left k-pseudo similarity of intuitionistic fuzzy matrices are not symmetric as in the case of pseudo similarity of intuitionistic fuzzy matrices.

Lemma 3.10. *Let $A, B \in (IF)_n$. If $A \cong_r^k B$, then we have the following:*
(1) $A^s = XB^sY$, *for any integer* $s \geq 1$,
(2) $BYX = YXB = B$,
(3) $AXY = XYA = A$,
(4) $B^s = YA^sX$, *for any integer* $s \geq 1$.

Proof. Since $A \cong_r^k B, A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$.
(1) Since $A = XBY$, $A^2 = (XBY)(XBY) = X(BYX)BY$. Thus

$$BYX = \left(YAX^k\right)YX = YA\left(X^kYX\right) = YAX^k = B.$$

So, $A^2 = (XBY)(XBY) = X(BYX)BY = XBBY = XB^2Y$. Hence in general, $A^s = XB^sY$, for any integer $s \geq 1$.
(2) $YXB = YX\left(YAX^k\right) = (YXY)AX^k = YAX^k = B$
and $BYX = \left(YAX^k\right)YX = YA\left(X^kYX\right) = YAX^k = B$.
(3) $AXY = (XBY)XY = XB(YXY) = XBY = A$
and $XYA = XY(XBY) = X(YXB)Y = XBY = A$.
(4) Clearly, $B = YXB$. Then $B^s = YXB^s$. Thus $B^s = YX(B^sYX) = Y(XB^sY)X = YA^sX$. □

Lemma 3.11. *Let $A, B \in (IF)_n$. If $A \cong_\ell^k B$. Then we have the following:*
(1) $B^s = YA^sX$, *for any integer* $s \geq 1$,
(2) $AXY = XYA = A$,
(3)) $BYX = YXB = B$,
(4) $A^s = XB^sY$, *for any integer* $s \geq 1$.

Proof. This can be proved as that of Lemma 3.10 and then omitted. □

Theorem 3.12. *Let $A, B \in (IF)_n$ such that $A \cong_r^k B$. A is right$(left)$ k-regular $\Leftrightarrow$ B is right$(left)$ k-regular.*

Proof. Since $A \cong_r^k B, A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$. When the positive integers k and s are same in Lemma 3.10, we have

$$A^k = XB^kY, BYX = YXB = B, AXY = XYA = A$$

and

$$B^k = YA^kX, \text{ for any integer } k \geq 1.$$

Let A be right k-regular, i.e., $A^kGA = A^k$. Then G is a right k-g-inverse of A. Choose $U = YGX$. We claim that U is a right k-g-inverse of B. Then

$$\begin{aligned} B^kUB &= \left(YA^kX\right)(YGX)B = Y\left(A^kXY\right)G(XB) \\ &= YA^kG(XBYX) = YA^kG(AX) \\ &= Y\left(A^kGA\right)X = YA^kX = B^k. \end{aligned}$$

Conversely, assume that B is right k-regular, i.e., $B^kUB = B^k$. Then U is a right k-g-inverse of B. Choose $G = XUY$ We prove that , G is a right k-g-inverse of A. Then

$$\begin{aligned} A^kGA &= \left(XB^kY\right)(XUY)(XBY) \\ &= X\left(B^kYX\right)U(YBX)Y \\ &= XB^kUBY = XB^kY \\ &= A^k. \end{aligned}$$

On the other hand, A is left k-regular $\Leftrightarrow$ B is left k-regular can be proved in the same manner. Thus the proof is done. □

Theorem 3.13. *Let $A, B \in (IF)_n$ such that $A \cong_\ell^k B$. Then A is right$(left)$ k-regular $\Leftrightarrow$ B is right$(left)$ k-regular.*

Proof. This can be proved as that of Theorem 3.12 and then omitted. □

Remark 3.14. For $k = 1$, Theorems 3.12 and 3.13 reduces to the following.

Theorem 3.15 ([3])**.** *Let $A \in (IF)_m$ and $B \in (IF)_n$ such that $A \cong B$. Then A is a regular matrix $\Leftrightarrow$ B is a regular matrix.*

Lemma 3.16. *Let $A, B \in (IF)_n$ and suppose $A \cong_r^k B$. Then there exist $X, Y \in (IF)_n$ such that $A = XBY, B = YAX^k$ and XY is k-potent.*

Proof. Since $A \cong_r^k B, A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$. Then

$$\begin{aligned} (XY)^k &= (XY)^{k-1}XY \\ &= (XY)^{k-2}XYXY \\ &= (XY)^{k-2}X(YXY) \\ &= (XY)^{k-2}XY \\ &= \ldots\ldots. \\ &= XY. \end{aligned}$$

Thus the proof is done. □

Remark 3.17. The converse of the above Lemma need not be true. This is illustrated in the following.

Example 3.18. Let us consider $X = \begin{bmatrix} \langle 0.3, 0.3\rangle & \langle 0, 1\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.2, 0.2\rangle \end{bmatrix}$ and $Y = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.5, 0\rangle & \langle 0.5, 0\rangle \end{bmatrix}$. For $A = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.2, 0.3\rangle & \langle 0.2, 0.3\rangle \end{bmatrix}$ and $B = \begin{bmatrix} \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.2, 0.3\rangle & \langle 0.2, 0.3\rangle \end{bmatrix}$, $A = XBY, B = YAX^2$ and $(XY)^2 = XY$. Then XY is 2-potent, but $X^2YX \neq X^2$ and $YXY \neq Y$. Here A is not right 2-pseudo similar to B.

Lemma 3.19. *Let $A, B \in (IF)_n$. If $A \cong_\ell^k B$, then there exist $X, Y \in (IF)_n$ such that $A = X^kBY, B = YAX$ and YX is k-potent.*

Proof. Since $A \cong_\ell^k B$, $A = X^kBY, B = YAX, XYX^k = X^k$ and $YXY = Y$. Thus

$$\begin{aligned}(YX)^k &= (YX)^{k-1}YX \\ &= (YX)^{k-2}YXYX \\ &= (YX)^{k-2}(YXY)X \\ &= (YX)^{k-2}YX \\ &= \ldots\ldots. \\ &= YX.\end{aligned}$$

So the proof is done. ∎

Theorem 3.20. *Let $A, B \in (IF)_n$. Then the following are equivalent:*
(1) $A \cong_r^k B$,
(2) $B^T \cong_\ell^k A^T$,
(3) $PAP^T \cong_r^k PBP^T$, *for some permutation matrix $P \in (IF)_n$.*

Proof. (1) ⇔ (2) : This is direct by taking transpose on both sides and by using $(A^T)^T = A$ and $(AX)^T = X^TA^T$.

(2) ⇔ (3) : Suppose $A \cong_r^k B$. Then $A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$. Thus

Since $A = XBY$,

$$PAP^T = PXBYP^T = (PXP^T)(PBP^T)(PYP^T). \tag{3.1}$$

Since $B = YAX^k$,

$$\begin{aligned}PBP^T &= PYAX^kP^T = (PYP^T)(PAP^T)(PX^kP^T) \\ &= (PYP^T)(PAP^T)(PXP^T)^k.\end{aligned} \tag{3.2}$$

Since $X^kYX = X^k$, $PX^kP^T = PX^kYXP^T$. Thus

$$PX^kP^T = (PX^kP^T)(PYP^T)(PXP^T). \tag{3.3}$$

On the other hand, $(PXP^T)^k = (PXP^T)^k(PYP^T)(PXP^T)$. Since $Y = YXY$, $PYP^T = PYXYP^T$. Thus

$$PYP^T = (PYP^T)(PXP^T)(PYP^T). \tag{3.4}$$

So $PAP^T \cong_r^k PBP^T$.

Conversely, suppose $PAP^T \cong_r^k PBP^T$. Pre multiply by P^T and post multiply by P in Equations (3.1) to (3.4), we get $A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$. Then $A \cong_r^k B$. Thus the proof is done. ∎

Example 3.21. The above Theorem 3.20 is illustrated in this example.
Let us consider $A = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}$, $B = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.1, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}$, $X = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.3, 0.5\rangle \end{bmatrix}$ and $Y = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.1, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}$. Here $X \neq XYX$.
For this X, $X^2 = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}$.

Now $A = XBY = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}$,

$$B = YAX^2 = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.1, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix},$$

$$X^2 = X^2YX = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}$$

and

$$Y = YXY = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.1, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}.$$

Then $A \cong_r^k B$.

For a given A and B,

$$A^T = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}, B^T = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix},$$

$$X^T = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.3, 0.5\rangle \end{bmatrix}, Y^T = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}$$

and

$$\left(X^T\right)^2 = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}. \text{ For this } X^T, Y^T \in (IF)_2\,,$$

$$A^T = Y^TB^TX^T = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix},$$

$$B^T = \left(X^T\right)^2 A^TY^T = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix},$$

$$\left(X^T\right)^2 = X^TY^T\left(X^T\right)^2 = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix},$$

$$Y^T = Y^TX^TY^T = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}.$$

Thus $B^T \cong_\ell^k A^T$.

Consider a intuitionistic fuzzy permutation matrix $P = \langle P_\mu, P\vartheta\rangle = \begin{bmatrix} \langle 1, 0\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 1, 0\rangle \end{bmatrix}$.
For this P, $P^T = \begin{bmatrix} \langle 1, 0\rangle & \langle 0, 1\rangle \\ \langle 0, 1\rangle & \langle 1, 0\rangle \end{bmatrix}$. On the other hand,

$$PAP^T = X\left(PBP^T\right)Y = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}$$

and

$$PBP^T = Y\left(PAP^T\right)X^2 = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.1, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}.$$

So $PAP^T \cong_r^k PBP^T$ for some permutation matrix $P \in (IF)_n$.

Theorem 3.22. *Let $A, B \in (IF)_n$. Then the following are equivalent:*

(1) $A \cong_{\ell}^{k} B$,
(2) $B^T \cong_{r}^{k} A^T$,
(3)) $PAP^T \cong_{\ell}^{k} PBP^T$, *for some permutation matrix* $P \in (IF)_n$.

Proof. Proof of the theorem is similar to Theorem 3.20 and hence omitted. □

Theorem 3.23. *Let* $A, B \in (IF)_n$. *If* $A \cong_{r}^{k} B$, *then* $A^s \cong_{r}^{k} B^s$, *for any integer* $s \geq 1$.

Proof. Suppose $A \cong_{r}^{k} B$. Then $A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$.
Prove that, $A^s \cong_{r}^{k} B^s$. By Lemma 3.10(1), $A^s = XB^sY$, for any integer $s \geq 1$.
Next prove that, $B^s = YA^sX^k$.
By Lemma 3.10(2), $BYX = YXB = B$. Then

$$\begin{aligned} B^s &= YXB^s = YXB^{s-1}B = YXB^{s-1}\left(YAX^k\right) \\ &= Y\left(XB^{s-1}Y\right)AX^k = Y\left(A^{s-1}\right)AX^k \\ &= YA^sX^k. \end{aligned}$$

Thus $A^s \cong_{r}^{k} B^s$, for any integer $s \geq 1$. So the proof is done. □

Remark 3.24. The converse of the above theorem need not be true. This is illustrated in the following.

Example 3.25. Let us consider $X = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.3, 0.5\rangle \end{bmatrix}$ and $Y = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.1, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}$. For $X^2 = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}$,
$XYX \neq X, X^2YX = X^2$ and $YXY = Y$.
For $A = \begin{bmatrix} \langle 0.3, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.5, 0.5\rangle \end{bmatrix}$ and $B = \begin{bmatrix} \langle 0.5, 0.5\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.1, 0.5\rangle & \langle 0.1, 0.5\rangle \end{bmatrix}$,
$A^2 = XB^2Y$ and $B^2 = YA^2X^2$.
Then A^2 is right 2-pseudo similar to B^2. But $A \neq XBY$ and $B = YAX^2$. Here A is not right 2-pseudo similar to B.

Theorem 3.26. *Let* $A, B \in (IF)_n$. *If* $A \cong_{\ell}^{k} B$, *then* $A^s \cong_{\ell}^{k} B^s$, *for any integer* $s \geq 1$.

Proof. This is similar to Theorem 3.23 and then omitted. □

Theorem 3.27. *Let* $A, B, C \in (IF)_n$. *If* $A \cong_{r}^{k} B$ *and* $B \cong_{r}^{k} C$, *then* $A \cong_{r}^{k} C$ *and if there exist matrices* X, Y, Z *and* L *with* $Y \in X\left\{1_r^k\right\}, Z \in L\left\{1_r^k\right\}, X \in Y\left\{1\right\}, L \in Z\left\{1\right\}$, *and* $XL = LX$ *satisfying any one of the following:*
(1) $LZY = Y$,
(2) $ZYX = Z$,
(3) $XLZ = X$,
(4) $YXL = L$.

Proof. Since $A \cong_{r}^{k} B, A = XBY, B = YAX^k, X^kYX = X^k$ and $YXY = Y$. Since $B \cong_{r}^{k} C, B = LCZ, C = ZBL^k, L^kZL = L^k$ and $ZLZ = Z$. Thern,

$$A = XBY = X\left(LCZ\right)Y = \left(XL\right)C\left(ZY\right).$$

and

$$C = ZBL^k = Z\left(YAX^k\right)L^k = (ZY)\,A\left(X^kL^k\right) = (ZY)\,A\,(XL)^k.$$

To prove, $A \cong_r^k C$, it is enough to prove that $ZY \in (XL)\left\{1_r^k\right\}$ and $XL \in (ZY)\{1\}$.

Suppose (1) holds. Then

$$\begin{aligned}(XL)^k(ZY)(XL) &= X^kL^k(ZY)(XL) = X^kL^{k-1}(LZY)(XL)\\ &= X^kL^{k-1}(Y)(XL) = L^{k-1}X^k(Y)(XL)\\ &= L^{k-1}\left(X^kYX\right)L = L^{k-1}X^kL\\ &= (XL)^k\end{aligned}$$

and

$$(ZY)(XL)(ZY) = ZYX(LZY) = ZYXY = Z(YXY) = ZY.$$

Suppose (2) holds. Then

$$\begin{aligned}(XL)^k(ZY)(XL) &= (XL)^k(ZYX)L\\ &= (XL)^kZY = X^kL^kZL\\ &= X^kL^k = (XL)^k\end{aligned}$$

and

$$(ZY)(XL)(ZY) = (ZYX)LZY = ZLZY = (ZLZ)Y = ZY.$$

Suppose (3) holds. Then

$$\begin{aligned}(XL)^k(ZY)(XL) &= (XL)^{k-1}(XL)(ZY)(XL)\\ &= (XL)^{k-1}(XLZ)(YXL)\\ &= (XL)^{k-1}XYXL = L^{k-1}X^{k-1}XYXL\\ &= L^{k-1}X^kYXL = L^{k-1}X^kL\\ &= (XL)^k\end{aligned}$$

and

$$(ZY)(XL)(ZY) = ZY(XLZ)Y = ZYXY = Z(YXY) = ZY.$$

Suppose (4) holds. Then

$$\begin{aligned}(XL)^k(ZY)(XL) &= (XL)^kZ(YXL)\\ &= (XL)^kZL = X^kL^kZL = X^kL^k\\ &= (XL)^k\end{aligned}$$

and $\quad (ZY)(XL)(ZY) = Z(YXL)ZY = ZLZY = (ZLZ)Y = ZY.$

Thus the proof is done. □

Theorem 3.28. *Let $A, B, C \in (IF)_n$. If $A \cong_\ell^k B$ and $B \cong_\ell^k C$, then $A \cong_\ell^k C$ and if there exist matrices X, Y, Z and L with $Y \in X\left\{1_\ell^k\right\}, Z \in L\left\{1_\ell^k\right\}, X \in Y\{1\}, L \in Z\{1\}$, and $XL = LX$ satisfying any one of the following:*

(1) $LZY = Y$,

(2) $ZYX = Z$,

(3) $XLZ = X$,

(4) $YXL = L$.

Proof. This is similar to that of Theorem 3.27 and then omitted. □

4. Conclusion

In this paper, the concept of k-regular intuitionistic fuzzy matrix as a generalization of regular intuitionistic fuzzy matrix is introduced. k-pseudo similar intuitionistic fuzzy matrix is defined and the properties are discussed.

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