Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 14, No. 5, (November 2017), pp. 475–486
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 14, No. 5, November 2017

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A novel extension of cubic sets and its applications in BCK/BCI-algebras

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Received 23 April 2017 Revised 9 May 2017; Accepted 17 May 2017

Abstract. An extension of cubic sets is discussed, and its applications in BCK/BCI-algebras are considered. The notion of a cubic intuitionistic set is introduced, and related properties are investigated. Cubic intuitionistic subalgebras/ideals are introduced, and several properties are investigated.

2010 AMS Classification: 06F35, 03B60, 03B52.

Keywords: Cubic set, cubic intuitionistic set, internal (external) cubic intuitionistic set, cubic intuitionistic subalgebra, cubic intuitionistic ideal.

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1. Introduction

In 2012, Jun et al. [7] introduced cubic sets, and then this notion is applied to several algebraic structures (see [1], [4], [6], [8], [9], [10], [12], [13], [14]).

The aim of this paper is to introduce the notion of cubic intuitionistic set which is an extended concept of a cubic set. We introduce the notions of (left, right) internal cubic intuitionistic set, double left (right) internal cubic intuitionistic set, cross left (right) internal cubic intuitionistic set and (cross) external cubic intuitionistic set, and investigate related properties. We apply the cubic intuitionistic set to BCK/BCI-algebras. We introduce the concepts of cubic intuitionistic subalgebra and cubic intuitionistic ideal, and discuss several properties. We investigate relations between cubic intuitionistic subalgebra and cubic intuitionistic ideal.

2. Preliminaries

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the following axioms:

(I) $(\forall x,y,z \in X)$ $(((x*y)*(x*z))*(z*y)=0)$,
(II) $(\forall x,y \in X)$ $((x*(x*y))*y=0)$,
(III) $(\forall x \in X)$ $(x*x=0)$,

(IV) $(\forall x, y \in X)$ $(x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)$ $(0 * x = 0)$,

then X is called a *BCK-algebra*. We can define a partial ordering $\leq$ on X by $x \leq y$ if and only if $x * y = 0$. Any BCK/BCI-algebra X satisfies the following conditions:

$$(2.1) \qquad (\forall x \in X)\ (x * 0 = x),$$

$$(2.2) \qquad (\forall x, y, z \in X)\ (x \leq y \ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x),$$

$$(2.3) \qquad (\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y),$$

$$(2.4) \qquad (\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y).$$

Any BCI-algebra X satisfies the following condition:

$$(2.5) \qquad (\forall x \in X)\ (0 * (0 * x) = (0 * x) * (0 * x)),$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if $0 \in I$ and the following condition is valid.

$$(2.6) \qquad (\forall x, y \in X)\,(x * y \in I,\ y \in I \ \Rightarrow\ x \in I)\,.$$

We refer the reader to the books [5] and [11] for further information regarding BCK/BCI-algebras.

A *fuzzy set* in a set X is defined to be a function $\lambda : X \to [0, 1]$. For $I = [0, 1]$, denote by I^X the collection of all fuzzy sets in a set X. Define a relation $\leq$ on I^X as follows:

$$(2.7) \qquad (\forall \lambda, \mu \in I^X)\,(\lambda \leq \mu \iff (\forall x \in X)(\lambda(x) \leq \mu(x))).$$

The complement of $\lambda \in I^X$, denoted by λ^c, is defined by

$$(2.8) \qquad (\forall x \in X)\ (\lambda^c(x) = 1 - \lambda(x)).$$

An *intuitionistic fuzzy set* in a set X (see [2]) is defined to be an object of the form

$$\lambda = \{\langle x, \mu_\lambda(x), \nu_\lambda(x)\rangle \mid x \in X\}$$

where $\mu_\lambda(x) \in [0, 1]$ and $\nu_\lambda(x) \in [0, 1]$ with $\mu_\lambda(x) + \nu_\lambda(x) \leq 1$. The intuitionistic fuzzy set

$$\lambda = \{\langle x, \mu_\lambda(x), \nu_\lambda(x)\rangle \mid x \in X\}$$

is simply denoted by $\lambda(x) = (\mu_\lambda(x), \nu_\lambda(x))$ for $x \in X$ or $\lambda = (\mu_\lambda, \nu_\lambda)$.

Obviously every fuzzy set has the form

$$\{\langle x, \mu_\lambda(x), 1 - \mu_\lambda(x)\rangle \mid x \in X\}.$$

An *interval-valued intuitionistic fuzzy set* A over a set X (see [3]) is an object having the form

$$A = \{\langle x, \alpha_A(x), \beta_A(x)\rangle \mid x \in X\}$$

where $\alpha_A(x) \subseteq [0, 1]$ and $\beta_A(x) \subseteq [0, 1]$ are intervals and for every $x \in X$,

$$\sup \alpha_A(x) + \sup \beta_A(x) \leq 1$$

3. Cubic intuitionistic sets

Given two closed subintervals $D_1 = [D_1^-, D_1^+]$ and $D_2 = [D_2^-, D_2^+]$ of $[0, 1]$, we define the order "$\ll$" as follows:

$$D_1 \ll D_2 \;\Leftrightarrow\; D_1^- \leq D_2^- \text{ and } D_1^+ \leq D_2^+.$$

We also define the refined minimum (briefly, rmin) and refined maximum (briefly, rmax) as follows:

$$\begin{aligned}\text{rmin}\{D_1, D_2\} &= \left[\min\{D_1^-, D_2^-\}, \min\{D_1^+, D_2^+\}\right],\\ \text{rmax}\{D_1, D_2\} &= \left[\max\{D_1^-, D_2^-\}, \max\{D_1^+, D_2^+\}\right].\end{aligned}$$

Denote by $D[0, 1]$ the set of all closed subintervals of $[0, 1]$. In this paper we use the interval-valued intuitionistic fuzzy set

$$A = \{\langle x, \alpha_A(x), \beta_A(x)\rangle \mid x \in X\}$$

over X in which $\alpha_A(x)$ and $\beta_A(x)$ are closed subintervals of $[0, 1]$ for all $x \in X$. Also, we use the notations $\alpha_A^-(x)$ and $\alpha_A^+(x)$ to mean the left end point and the right end point of the interval $\alpha_A(x)$, respectively, and so we have $\alpha_A(x) = [\alpha_A^-(x), \alpha_A^+(x)]$. The interval-valued intuitionistic fuzzy set

$$A = \{\langle x, \alpha_A(x), \beta_A(x)\rangle \mid x \in X\}$$

over X is simply denoted by $A(x) = \langle \alpha_A(x), \beta_A(x)\rangle$ for $x \in X$ or $A = \langle \alpha_A, \beta_A\rangle$.

Definition 3.1. Let X be a nonempty set. By a *cubic intuitionistic set* in X we mean a structure

$$\mathcal{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in X\}$$

in which A is an interval-valued intuitionistic fuzzy set in X and λ is an intuitionistic fuzzy set in X.

A cubic intuitionistic set $\mathcal{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in X\}$ is simply denoted by $\mathcal{A} = \langle A, \lambda\rangle$.

Let $\mathcal{A} = \langle A, \lambda\rangle$ be a cubic intuitionistic set in a nonempty set X. Given $(\varepsilon, \delta) \in [0, 1] \times [0, 1]$ and $([s_\alpha, t_\alpha], [s_\beta, t_\beta]) \in D[0, 1] \times D[0, 1]$, we consider the sets

$$\begin{aligned}&\alpha_A[s_\alpha, t_\alpha] := \{x \in X \mid \alpha_A(x) \gg [s_\alpha, t_\alpha]\},\\ &\beta_A[s_\beta, t_\beta] := \{x \in X \mid \beta_A(x) \ll [s_\beta, t_\beta]\},\\ &\mu_\lambda[\varepsilon] := \{x \in X \mid \mu_\lambda(x) \leq \varepsilon\},\\ &\nu_\lambda[\delta] := \{x \in X \mid \nu_\lambda(x) \geq \delta\}.\end{aligned}$$

Definition 3.2. Let X be a nonempty set. A cubic intuitionistic set $\mathcal{A} = \langle A, \lambda\rangle$ is said to be

- *left internal* if $\mu_\lambda(x) \in \alpha_A(x)$ and $\nu_\lambda(x) \notin \beta_A(x)$ for all $x \in X$,
- *right internal* if $\mu_\lambda(x) \notin \alpha_A(x)$ and $\nu_\lambda(x) \in \beta_A(x)$ for all $x \in X$,
- *internal* if $\mu_\lambda(x) \in \alpha_A(x)$ and $\nu_\lambda(x) \in \beta_A(x)$ for all $x \in X$,
- *double left internal* if $\mu_\lambda(x) \in \alpha_A(x)$ and $\nu_\lambda(x) \in \alpha_A(x)$ for all $x \in X$,
- *double right internal* if $\mu_\lambda(x) \in \beta_A(x)$ and $\nu_\lambda(x) \in \beta_A(x)$ for all $x \in X$,
- *cross left internal* if $\nu_\lambda(x) \in \alpha_A(x)$ and $\mu_\lambda(x) \notin \beta_A(x)$ for all $x \in X$,

- *cross right internal* if $\nu_\lambda(x) \notin \alpha_A(x)$ and $\mu_\lambda(x) \in \beta_A(x)$ for all $x \in X$,
- *cross internal* if $\nu_\lambda(x) \in \alpha_A(x)$ and $\mu_\lambda(x) \in \beta_A(x)$ for all $x \in X$,
- *external* if $\mu_\lambda(x) \notin \alpha_A(x)$ and $\nu_\lambda(x) \notin \beta_A(x)$ for all $x \in X$.
- *cross external* if $\nu_\lambda(x) \notin \alpha_A(x)$ and $\mu_\lambda(x) \notin \beta_A(x)$ for all $x \in X$.

Example 3.3. Let $\mathcal{A} = \langle A, \lambda\rangle$ be a cubic intuitionistic set in $[0, 1]$ with $A(x) = \langle[0.2, 0.5], [0.1, 0.4]\rangle$ for all $x \in [0, 1]$. If $\lambda(x) = (0.3, 0.6)$ for all $x \in [0, 1]$, then $\mathcal{A} = \langle A, \lambda\rangle$ is left internal. If $\lambda(x) = (0.6, 0.3)$ for all $x \in [0, 1]$, then $\mathcal{A} = \langle A, \lambda\rangle$ is right internal. If $\lambda(x) = (0.2, 0.45)$ for all $x \in [0, 1]$, then $\mathcal{A} = \langle A, \lambda\rangle$ is double left internal. If $\lambda(x) = (0.15, 0.35)$ for all $x \in [0, 1]$, then $\mathcal{A} = \langle A, \lambda\rangle$ is double right internal. If $\lambda(x) = (0.25, 0.35)$ for all $x \in [0, 1]$, then $\mathcal{A} = \langle A, \lambda\rangle$ is internal. If $\lambda(x) = (0.1, 0.7)$ for all $x \in [0, 1]$, then $\mathcal{A} = \langle A, \lambda\rangle$ is external. If $\lambda(x) = (0.7, 0.1)$ for all $x \in [0, 1]$, then $\mathcal{A} = \langle A, \lambda\rangle$ is cross external.

Proposition 3.4. *Let $\mathcal{A} = \langle A, \lambda\rangle$ be a cubic intuitionistic set in a nonempty set X.*

(1) *If $\mathcal{A} = \langle A, \lambda\rangle$ is internal and $\alpha_A(x) \subseteq \beta_A(x)$ for all $x \in X$, then it is double right internal.*
(2) *If $\mathcal{A} = \langle A, \lambda\rangle$ is internal and $\alpha_A(x) \supseteq \beta_A(x)$ for all $x \in X$, then it is double left internal.*
(3) *If $\mathcal{A} = \langle A, \lambda\rangle$ is cross left internal and $\alpha_A(x) \subseteq \beta_A(x)$ for all $x \in X$, then it is right internal.*
(4) *If $\mathcal{A} = \langle A, \lambda\rangle$ is cross right internal and $\alpha_A(x) \supseteq \beta_A(x)$ for all $x \in X$, then it is left internal.*
(5) *If $\mathcal{A} = \langle A, \lambda\rangle$ is double left internal and $\alpha_A(x) \subseteq \beta_A(x)$ for all $x \in X$, then it is double right internal.*
(6) *If $\mathcal{A} = \langle A, \lambda\rangle$ is double right internal and $\alpha_A(x) \supseteq \beta_A(x)$ for all $x \in X$, then it is double left internal.*

Proof. Straightforward. □

Definition 3.5. The *complement* of a cubic intuitionistic set $\mathcal{A} = \langle A, \lambda\rangle$ in X is defined to be the cubic intuitionistic set

$$\mathcal{A}^c = \{\langle x, A^c(x), \lambda^c(x)\rangle \mid x \in X\}$$

in which

$$A^c = \{\langle x, \beta_A(x), \alpha_A(x)\rangle \mid x \in X\}$$

and

$$\lambda^c = \{\langle x, \nu_\lambda(x), \mu_\lambda(x)\rangle \mid x \in X\}.$$

We will use the simple notation $\mathcal{A}^c = \langle A^c, \lambda^c\rangle$ instead of

$$\mathcal{A}^c = \{\langle x, A^c(x), \lambda^c(x)\rangle \mid x \in X\}.$$

Theorem 3.6. *Let $\mathcal{A} = \langle A, \lambda\rangle$ be a cubic intuitionistic set in X.*

(1) *If $\mathcal{A} = \langle A, \lambda\rangle$ is left (right) internal, then $\mathcal{A}^c = \langle A^c, \lambda^c\rangle$ is right (left) internal.*
(2) *If $\mathcal{A} = \langle A, \lambda\rangle$ is internal, then so is $\mathcal{A}^c = \langle A^c, \lambda^c\rangle$.*
(3) *If $\mathcal{A} = \langle A, \lambda\rangle$ is left (right) external, then $\mathcal{A}^c = \langle A^c, \lambda^c\rangle$ is right (left) external.*

(4) *If $\mathcal{A} = \langle A, \lambda \rangle$ is external, then so is $\mathcal{A}^c = \langle A^c, \lambda^c \rangle$.*
(5) *If $\mathcal{A} = \langle A, \lambda \rangle$ is double left (right) internal, then $\mathcal{A}^c = \langle A^c, \lambda^c \rangle$ is double right (left) internal.*
(6) *If $\mathcal{A} = \langle A, \lambda \rangle$ is cross left (right) internal, then $\mathcal{A}^c = \langle A^c, \lambda^c \rangle$ is cross right (left) internal.*
(7) *If $\mathcal{A} = \langle A, \lambda \rangle$ is cross internal, then so is $\mathcal{A}^c = \langle A^c, \lambda^c \rangle$.*
(8) *If $\mathcal{A} = \langle A, \lambda \rangle$ is cross external, then so is $\mathcal{A}^c = \langle A^c, \lambda^c \rangle$.*

Proof. Straightforward. □

4. Cubic intuitionistic subalgebras of BCK/BCI-algebras

In what follows, let X be a BCK/BCI-algebra unless otherwise specified.

Definition 4.1. A cubic intuitionistic set $\mathcal{A} = \langle A, \lambda \rangle$ in X is called a *cubic intuitionistic subalgebra* of X if the following conditions are valid.

$$(\forall x, y \in X) \left(\begin{array}{l} \alpha_A(x * y) \gg \text{rmin}\{\alpha_A(x), \alpha_A(y)\} \\ \beta_A(x * y) \ll \text{rmax}\{\beta_A(x), \beta_A(y)\} \end{array} \right), \tag{4.1}$$

$$(\forall x, y \in X) \left(\begin{array}{l} \mu_\lambda(x * y) \leq \max\{\mu_\lambda(x), \mu_\lambda(y)\} \\ \nu_\lambda(x * y) \geq \min\{\nu_\lambda(x), \nu_\lambda(y)\} \end{array} \right). \tag{4.2}$$

Example 4.2. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Define a cubic intuitionistic set $\mathcal{A} = \langle A, \lambda \rangle$ in X as follows:

X	$A = \langle \alpha_A, \beta_A \rangle$	$\lambda = (\mu_\lambda, \nu_\lambda)$
0	$\langle [0.2, 0.6], [0.1, 0.4] \rangle$	$(0.2, 0.6)$
a	$\langle [0.1, 0.4], [0.3, 0.5] \rangle$	$(0.5, 0.4)$
b	$\langle [0.2, 0.6], [0.1, 0.4] \rangle$	$(0.3, 0.5)$
c	$\langle [0.1, 0.4], [0.3, 0.5] \rangle$	$(0.7, 0.2)$

It is routine to verify that $\mathcal{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic subalgebra of X.

Proposition 4.3. *If $\mathcal{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic subalgebra of X, then*

(1) *$\alpha_A(0)$ is an upper bound of $\{\alpha_A(x) \mid x \in X\}$.*
(2) *$\nu_\lambda(0)$ is an upper bound of $\{\nu_\lambda(x) \mid x \in X\}$.*
(3) *$\beta_A(0)$ is a lower bound of $\{\beta_A(x) \mid x \in X\}$.*
(4) *$\mu_\lambda(0)$ is a lower bound of $\{\mu_\lambda(x) \mid x \in X\}$.*

Proof. Let $x \in X$. Then $x * x = 0$, and so

$$\alpha_A(0) = \alpha_A(x * x) \gg \text{rmin}\{\alpha_A(x), \alpha_A(x)\} = \alpha_A(x),$$
$$\nu_\lambda(0) = \nu_\lambda(x * x) \geq \min\{\nu_\lambda(x), \nu_\lambda(x)\} = \nu_\lambda(x),$$
$$\beta_A(0) = \beta_A(x * x) \ll \text{rmax}\{\beta_A(x), \beta_A(x)\} = \beta_A(x),$$
$$\mu_\lambda(0) = \mu_\lambda(x * x) \leq \max\{\mu_\lambda(x), \mu_\lambda(x)\} = \mu_\lambda(x).$$

This completes the proof. □

Proposition 4.4. *If $\mathcal{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic subalgebra of a BCI-algebra X, then*

(1) $(\forall x \in X)$ *($\alpha_A(0 * x)$ is an upper bound of $\{\alpha_A(y) \mid y \in X\}$).*
(2) $(\forall x \in X)$ *($\beta_A(0 * x)$ is a lower bound of $\{\beta_A(y) \mid y \in X\}$).*
(3) $(\forall x \in X)$ *($\mu_\lambda(0 * x)$ is a lower bound of $\{\mu_\lambda(y) \mid y \in X\}$).*
(4) $(\forall x \in X)$ *($\nu_\lambda(0 * x)$ is an upper bound of $\{\nu_\lambda(y) \mid y \in X\}$).*

Proof. Using Proposition 4.3, we have

$$\alpha_A(0 * x) \gg \text{rmin}\{\alpha_A(0), \alpha_A(x)\} = \alpha_A(x),$$
$$\beta_A(0 * x) \ll \text{rmax}\{\beta_A(0), \beta_A(x)\} = \beta_A(x),$$
$$\mu_\lambda(0 * x) \leq \max\{\mu_\lambda(0), \mu_\lambda(x)\} = \mu_\lambda(x),$$
$$\nu_\lambda(0 * x) \geq \min\{\nu_\lambda(0), \nu_\lambda(x)\} = \nu_\lambda(x)$$

for all $x \in X$. This completes the proof. □

Proposition 4.5. *For a cubic intuitionistic subalgebra $\mathcal{A} = \langle A, \lambda\rangle$ of X, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} \alpha_A(x_n) = [1, 1]$, $\lim_{n\to\infty} \beta_A(x_n) = [0, 0]$, $\lim_{n\to\infty} \mu_\lambda(x_n) = 0$, and $\lim_{n\to\infty} \nu_\lambda(x_n) = 1$, then $\alpha_A(0) = [1, 1]$, $\beta_A(0) = [0, 0]$, $\mu_\lambda(0) = 0$ and $\nu_\lambda(0) = 1$.*

Proof. Using Proposition 4.3, we have $\alpha_A(0) \gg \alpha_A(x_n)$, $\beta_A(0) \ll \beta_A(x_n)$, $\mu_\lambda(0) \leq \mu_\lambda(x_n)$ and $\nu_\lambda(0) \geq \nu_\lambda(x_n)$. It follows from hypothesis that

$$[1, 1] \gg \alpha_A(0) \gg \lim_{n\to\infty} \alpha_A(x_n) = [1, 1],$$
$$[0, 0] \ll \beta_A(0) \ll \lim_{n\to\infty} \beta_A(x_n) = [0, 0],$$
$$0 \leq \mu_\lambda(0) \leq \lim_{n\to\infty} \mu_\lambda(x_n) = 0,$$
$$1 \geq \nu_\lambda(0) \geq \lim_{n\to\infty} \nu_\lambda(x_n) = 1.$$

Hence $\alpha_A(0) = [1, 1]$, $\beta_A(0) = [0, 0]$, $\mu_\lambda(0) = 0$ and $\nu_\lambda(0) = 1$. □

Theorem 4.6. *If $\mathcal{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic subalgebra of X, then the sets $\alpha_A[s, t]$, $\beta_A[s, t]$, $\mu_\lambda[\varepsilon]$ and $\nu_\lambda[\varepsilon]$ are subalgebras of X for all $[s, t] \in D[0, 1]$ and $\varepsilon \in [0, 1]$.*

Proof. For any $[s, t] \in D[0, 1]$ and $\varepsilon \in [0, 1]$, let $x, y \in X$ be such that

$$x, y \in \alpha_A[s, t] \cap \beta_A[s, t] \cap \mu_\lambda[\varepsilon] \cap \nu_\lambda[\varepsilon].$$

Then $\alpha_A(x) \gg [s,t]$, $\beta_A(x) \ll [s,t]$, $\mu_\lambda(x) \leq \varepsilon$, $\nu_\lambda(x) \geq \varepsilon$, $\alpha_A(y) \gg [s,t]$, $\beta_A(y) \ll [s,t]$, $\mu_\lambda(y) \leq \varepsilon$, and $\nu_\lambda(y) \geq \varepsilon$. It follows that

$$\begin{aligned}
&\alpha_A(x*y) \gg \text{rmin}\{\alpha_A(x), \alpha_A(y)\} \gg \text{rmin}\{[s,t],[s,t]\} = [s,t],\\
&\beta_A(x*y) \ll \text{rmax}\{\beta_A(x), \beta_A(y)\} \ll \text{rmax}\{[s,t],[s,t]\} = [s,t],\\
&\mu_\lambda(x*y) \leq \max\{\mu_\lambda(x), \mu_\lambda(y)\} \leq \max\{\varepsilon,\varepsilon\} = \varepsilon,\\
&\nu_\lambda(x*y) \geq \min\{\nu_\lambda(x), \nu_\lambda(y)\} \geq \min\{\varepsilon,\varepsilon\} = \varepsilon,
\end{aligned}$$

that is, $x*y \in \alpha_A[s,t]$, $x*y \in \beta_A[s,t]$, $x*y \in \mu_\lambda[\varepsilon]$ and $x*y \in \nu_\lambda[\varepsilon]$. Therefore $\alpha_A[s,t]$, $\beta_A[s,t]$, $\mu_\lambda[\varepsilon]$ and $\nu_\lambda[\varepsilon]$ are subalgebras of X for all $[s,t] \in D[0,1]$ and $\varepsilon \in [0,1]$. □

Corollary 4.7. *If $\mathcal{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic subalgebra of X, then*

$$\alpha_A[s,t] \cap \beta_A[s,t] \cap \mu_\lambda[\varepsilon] \cap \nu_\lambda[\varepsilon]$$

is a subalgebra of X for all $[s,t] \in D[0,1]$ and $\varepsilon \in [0,1]$.

Theorem 4.8. *Let $\mathcal{A} = \langle A, \lambda\rangle$ be a cubic intuitionistic set in X such that $\alpha_A[s_\alpha, t_\alpha]$, $\beta_A[s_\beta, t_\beta]$, $\mu_\lambda[\varepsilon]$ and $\nu_\lambda[\delta]$ are subalgebras of X for all $(\varepsilon,\delta) \in [0,1]\times[0,1]$ and $([s_\alpha,t_\alpha],[s_\beta,t_\beta]) \in D[0,1]\times D[0,1]$. Then $\mathcal{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic subalgebra of X.*

Proof. For any $x, y \in X$, putting

$$\begin{aligned}
&\text{rmin}\{\alpha_A(x), \alpha_A(y)\} := [s_\alpha, t_\alpha],\\
&\text{rmax}\{\beta_A(x), \beta_A(y)\} := [s_\beta, t_\beta],\\
&\max\{\mu_\lambda(x), \mu_\lambda(y)\} := \varepsilon, \text{ and}\\
&\min\{\nu_\lambda(x), \nu_\lambda(y)\} := \delta
\end{aligned}$$

imply that $\alpha_A(x) \gg [s_\alpha,t_\alpha]$, $\alpha_A(y) \gg [s_\alpha,t_\alpha]$, $\beta_A(x) \ll [s_\beta,t_\beta]$, $\beta_A(y) \ll [s_\beta,t_\beta]$, $\mu_\lambda(x) \leq \varepsilon$, $\mu_\lambda(y) \leq \varepsilon$, $\nu_\lambda(x) \geq \delta$, and $\nu_\lambda(y) \geq \delta$. Hence $x \in \alpha_A[s_\alpha,t_\alpha]$, $y \in \alpha_A[s_\alpha,t_\alpha]$, $x \in \beta_A[s_\beta,t_\beta]$, $y \in \beta_A[s_\beta,t_\beta]$, $x \in \mu_\lambda[\varepsilon]$, $y \in \mu_\lambda[\varepsilon]$, $x \in \nu_\lambda[\delta]$, and $y \in \nu_\lambda[\delta]$. It follows from hypothesis that $x*y \in \alpha_A[s_\alpha,t_\alpha]$, $x*y \in \beta_A[s_\beta,t_\beta]$, $x*y \in \mu_\lambda[\varepsilon]$, and $x*y \in \nu_\lambda[\delta]$ and so that

$$\begin{aligned}
&\alpha_A(x*y) \gg [s_\alpha,t_\alpha] = \text{rmin}\{\alpha_A(x), \alpha_A(y)\},\\
&\beta_A(x*y) \ll [s_\beta,t_\beta] = \text{rmax}\{\beta_A(x), \beta_A(y)\},\\
&\mu_\lambda(x*y) \leq \varepsilon = \max\{\mu_\lambda(x), \mu_\lambda(y)\},\\
&\nu_\lambda(x*y) \geq \delta = \min\{\nu_\lambda(x), \nu_\lambda(y)\}
\end{aligned}$$

Therefore $\mathcal{A} = \langle A, \lambda\rangle$ is a cubic intuitionistic subalgebra of X. □

The following theorem gives us a way to establish a new cubic intuitionistic subalgebra from old one in BCI-algebras.

Theorem 4.9. *Given a cubic intuitionistic subalgebra $\mathcal{A} = \langle A, \lambda\rangle$ of a BCI-algebra X, let $\mathcal{A}^* = \langle A^*, \lambda^*\rangle$ be a cubic intuitionistic set in X defined by $\alpha^*_A(x) = \alpha_A(0*x)$, $\beta^*_A(x) = \beta_A(0*x)$, $\mu^*_\lambda(x) = \mu_\lambda(0*x)$ and $\nu^*_\lambda(x) = \nu_\lambda(0*x)$ for all $x \in X$. Then $\mathcal{A}^* = \langle A^*, \lambda^*\rangle$ is a cubic intuitionistic subalgebra of X.*

Proof. Note that $0 * (x * y) = (0 * x) * (0 * y)$ for all $x, y \in X$. Hence

$$\begin{aligned}\alpha_A^*(x * y) &= \alpha_A(0 * (x * y)) = \alpha_A((0 * x) * (0 * y)) \\ &\gg \text{rmin}\{\alpha_A(0 * x), \alpha_A(0 * y)\} \\ &= \text{rmin}\{\alpha_A^*(x), \alpha_A^*(y)\},\end{aligned}$$

$$\begin{aligned}\beta_A^*(x * y) &= \beta_A(0 * (x * y)) = \beta_A((0 * x) * (0 * y)) \\ &\ll \text{rmax}\{\beta_A(0 * x), \beta_A(0 * y)\} \\ &= \text{rmax}\{\beta_A^*(x), \beta_A^*(y)\},\end{aligned}$$

$$\begin{aligned}\mu_\lambda^*(x * y) &= \mu_\lambda(0 * (x * y)) = \mu_\lambda((0 * x) * (0 * y)) \\ &\leq \max\{\mu_\lambda(0 * x), \mu_\lambda(0 * y)\} \\ &= \max\{\mu_\lambda^*(x), \mu_\lambda^*(y)\},\end{aligned}$$

and

$$\begin{aligned}\nu_\lambda^*(x * y) &= \nu_\lambda(0 * (x * y)) = \nu_\lambda((0 * x) * (0 * y)) \\ &\geq \min\{\nu_\lambda(0 * x), \nu_\lambda(0 * y)\} \\ &= \min\{\nu_\lambda^*(x), \nu_\lambda^*(y)\}\end{aligned}$$

for all $x, y \in X$. Therefore $\mathcal{A}^* = \langle A^*, \lambda^* \rangle$ is a cubic intuitionistic subalgebra of X. □

5. Cubic intuitionistic ideals of BCK/BCI-algebras

Definition 5.1. A cubic intuitionistic set $\mathcal{A} = \langle A, \lambda \rangle$ is called a *cubic intuitionistic ideal* of X if the following conditions are valid.

$$\left\{\begin{array}{l}\alpha_A(0) \text{ is an upper bound of } \{\alpha_A(x) \mid x \in X\} \\ \beta_A(0) \text{ is a lower bound of } \{\beta_A(x) \mid x \in X\}\end{array}\right. \tag{5.1}$$

$$\left\{\begin{array}{l}\mu_\lambda(0) \text{ is a lower bound of } \{\mu_\lambda(x) \mid x \in X\} \\ \nu_\lambda(0) \text{ is an upper bound of } \{\nu_\lambda(x) \mid x \in X\}\end{array}\right. \tag{5.2}$$

$$(\forall x, y \in X)\left(\begin{array}{l}\alpha_A(x) \gg \text{rmin}\{\alpha_A(x * y), \alpha_A(y)\} \\ \beta_A(x) \ll \text{rmax}\{\beta_A(x * y), \beta_A(y)\}\end{array}\right) \tag{5.3}$$

$$(\forall x, y \in X)\left(\begin{array}{l}\mu_\lambda(x) \leq \max\{\mu_\lambda(x * y), \mu_\lambda(y)\} \\ \nu_\lambda(x) \geq \min\{\nu_\lambda(x * y), \nu_\lambda(y)\}\end{array}\right). \tag{5.4}$$

Example 5.2. Let $X = \{0, a, 1, 2, 3\}$ be a BCI-algebra with the following Cayley table.

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Define a cubic intuitionistic set $\mathcal{A} = \langle A, \lambda \rangle$ in X as follows:

X	$A = \langle \alpha_A, \beta_A \rangle$	$\lambda = (\mu_\lambda, \nu_\lambda)$
0	$\langle [0.3, 0.6], [0.1, 0.4] \rangle$	$(0.2, 0.7)$
a	$\langle [0.2, 0.4], [0.3, 0.6] \rangle$	$(0.2, 0.7)$
1	$\langle [0.1, 0.3], [0.5, 0.7] \rangle$	$(0.6, 0.2)$
2	$\langle [0.1, 0.3], [0.5, 0.7] \rangle$	$(0.4, 0.5)$
3	$\langle [0.1, 0.3], [0.5, 0.7] \rangle$	$(0.6, 0.2)$

Then $\mathcal{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic ideal of X.

Proposition 5.3. *Let $\mathcal{A} = \langle A, \lambda \rangle$ be a cubic intuitionistic ideal of X. If the inequality $x * y \leq z$ holds in X, then*

$$\begin{aligned} &\alpha_A(x) \gg \mathrm{rmin}\{\alpha_A(y), \alpha_A(z)\}, \\ &\beta_A(x) \ll \mathrm{rmax}\{\beta_A(y), \beta_A(z)\}, \\ &\mu_\lambda(x) \leq \max\{\mu_\lambda(y), \mu_\lambda(z)\}, \\ &\nu_\lambda(x) \geq \min\{\nu_\lambda(y), \nu_\lambda(z)\}. \end{aligned} \tag{5.5}$$

Proof. Let $x, y, z \in X$ be such that $x * y \leq z$. Then $(x * y) * z = 0$, and so

$$\begin{aligned} &\alpha_A(x * y) \gg \mathrm{rmin}\{\alpha_A((x * y) * z), \alpha_A(z)\} = \mathrm{rmin}\{\alpha_A(0), \alpha_A(z)\} = \alpha_A(z), \\ &\beta_A(x * y) \ll \mathrm{rmax}\{\beta_A((x * y) * z), \beta_A(z)\} = \mathrm{rmax}\{\beta_A(0), \beta_A(z)\} = \beta_A(z), \\ &\mu_\lambda(x * y) \leq \max\{\mu_\lambda((x * y) * z), \mu_\lambda(z)\} = \max\{\mu_\lambda(0), \mu_\lambda(z)\} = \mu_\lambda(z), \\ &\nu_\lambda(x * y) \geq \min\{\nu_\lambda((x * y) * z), \nu_\lambda(z)\} = \min\{\nu_\lambda(0), \nu_\lambda(z)\} = \nu_\lambda(z). \end{aligned}$$

This completes the proof. □

We provide conditions for a cubic intuitionistic set to be a cubic intuitionistic ideal.

Theorem 5.4. *Let $\mathcal{A} = \langle A, \lambda \rangle$ be a cubic intuitionistic set in X in which conditions (5.1) and (5.2) are valid. If $\mathcal{A} = \langle A, \lambda \rangle$ satisfies the condition (5.5) for all $x, y, z \in X$ with $x * y \leq z$. Then $\mathcal{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic ideal of X.*

Proof. Since $x * (x * y) \leq y$ for all $x, y \in X$, it follows from (5.5) that

$$\begin{aligned} &\alpha_A(x) \gg \mathrm{rmin}\{\alpha_A(x * y), \alpha_A(y)\}, \\ &\beta_A(x) \ll \mathrm{rmax}\{\beta_A(x * y), \beta_A(y)\}, \\ &\mu_\lambda(x) \leq \max\{\mu_\lambda(x * y), \mu_\lambda(y)\}, \\ &\nu_\lambda(x) \geq \min\{\nu_\lambda(x * y), \nu_\lambda(y)\}. \end{aligned}$$

Therefore $\mathcal{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic ideal of X. □

Lemma 5.5. *Every cubic intuitionistic ideal $\mathcal{A} = \langle A, \lambda \rangle$ in X satisfies the following condition.*

$$(5.6) \qquad (\forall x, y \in X) \left(x \leq y \ \Rightarrow \left\{ \begin{array}{l} \alpha_A(x) \gg \alpha_A(y), \ \beta_A(x) \ll \beta_A(y) \\ \mu_\lambda(x) \leq \mu_\lambda(y), \ \nu_\lambda(x) \geq \nu_\lambda(y) \end{array} \right. \right).$$

Proof. Assume that $x \leq y$ for all $x, y \in X$. Then $x * y = 0$, and so

$$\begin{aligned}
&\alpha_A(x) \gg \text{rmin}\{\alpha_A(x*y), \alpha_A(y)\} = \text{rmin}\{\alpha_A(0), \alpha_A(y)\} = \alpha_A(y),\\
&\beta_A(x) \ll \text{rmax}\{\beta_A(x*y), \beta_A(y)\} = \text{rmax}\{\beta_A(0), \beta_A(y)\} = \beta_A(y),\\
&\mu_\lambda(x) \leq \max\{\mu_\lambda(x*y), \mu_\lambda(y)\} = \max\{\mu_\lambda(0), \mu_\lambda(y)\} = \mu_\lambda(y),\\
&\nu_\lambda(x) \geq \min\{\nu_\lambda(x*y), \nu_\lambda(y)\} = \min\{\nu_\lambda(0), \nu_\lambda(y)\} = \nu_\lambda(y)
\end{aligned}$$

by (5.1), (5.2), (5.3) and (5.4). □

Theorem 5.6. *In a BCK-algebra X, every cubic intuitionistic ideal is a cubic intuitionistic subalgebra.*

Proof. Let $\mathcal{A} = \langle A, \lambda \rangle$ be a cubic intuitionistic ideal of a BCK-algebra X. Since $x * y \leq x$ for all $x, y \in X$, we have $\alpha_A(x*y) \gg \alpha_A(x)$, $\beta_A(x*y) \ll \beta_A(x)$, $\mu_\lambda(x*y) \leq \mu_\lambda(x)$ and $\nu_\lambda(x*y) \geq \nu_\lambda(x)$ by Lemma 5.5. It follows from (5.3) and (5.4) that

$$\begin{aligned}
&\alpha_A(x*y) \gg \alpha_A(x) \gg \text{rmin}\{\alpha_A(x*y), \alpha_A(y)\} \gg \text{rmin}\{\alpha_A(x), \alpha_A(y)\},\\
&\beta_A(x*y) \ll \beta_A(x) \ll \text{rmax}\{\beta_A(x*y), \beta_A(y)\} \ll \text{rmax}\{\beta_A(x), \beta_A(y)\},\\
&\mu_\lambda(x*y) \leq \mu_\lambda(x) \leq \max\{\mu_\lambda(x*y), \mu_\lambda(y)\} \leq \max\{\mu_\lambda(x), \mu_\lambda(y)\},\\
&\nu_\lambda(x*y) \geq \nu_\lambda(x) \geq \min\{\nu_\lambda(x*y), \nu_\lambda(y)\} \geq \min\{\nu_\lambda(x), \nu_\lambda(y)\}.
\end{aligned}$$

Therefore $\mathcal{A} = \langle A, \lambda \rangle$ is a cubic intuitionistic subalgebra of X. □

The converse of Theorem 5.6 is not true in general as seen in the following example.

Example 5.7. The cubic intuitionistic subalgebra $\mathcal{A} = \langle A, \lambda \rangle$ in Example 4.2 is not a cubic intuitionistic ideal of X Since

$$\begin{aligned}
\alpha_A(a) = [0.1, 0.4] &\not\gg \text{rmin}\{\alpha_A(a*b), \alpha_A(b)\}\\
&= \text{rmin}\{\alpha_A(0), \alpha_A(b)\} = \alpha_A(b) = [0,2, 0.6],
\end{aligned}$$

$$\begin{aligned}
\beta_A(a) = [0.3, 0.5] &\not\ll \text{rmax}\{\beta_A(a*b), \beta_A(b)\}\\
&= \text{rmax}\{\alpha_A(0), \alpha_A(b)\} = \alpha_A(b) = [0,1, 0.4],
\end{aligned}$$

$$\begin{aligned}
\mu_\lambda(a) = 0.5 &\not\leq \max\{\mu_\lambda(a*b), \mu_\lambda(b)\}\\
&= \max\{\mu_\lambda(0), \mu_\lambda(b)\} = \mu_\lambda(b) = 0.3,
\end{aligned}$$

and/or

$$\begin{aligned}
\nu_\lambda(a) = 0.4 &\not\geq \min\{\nu_\lambda(a*b), \nu_\lambda(b)\}\\
&= \min\{\nu_\lambda(0), \nu_\lambda(b)\} = \nu_\lambda(b) = 0.5.
\end{aligned}$$

We establish a characterization of a cubic intuitionistic ideal in a BCK-algebra.

Theorem 5.8. *For a cubic intuitionistic set* $\mathcal{A} = \langle A, \lambda \rangle$ *in a* BCK*-algebra* X*, the following assertions are equivalent.*

(1) $\mathcal{A} = \langle A, \lambda \rangle$ *is a cubic intuitionistic ideal of* X.
(2) $\mathcal{A} = \langle A, \lambda \rangle$ *is a cubic intuitionistic subalgebra of* X *satisfying* (5.5) *for all* $x, y, z \in X$ *with* $x * y \leq z$.

Proof. (1) ⇒ (2). It follows from Theorem 5.6 and Proposition 5.3.
(2) ⇒ (1). It is by Proposition 4.3 and Theorem 5.4. □

Conclusions

In this paper, we have introduced the notion of cubic intuitionistic set which is a generalization of a cubic set. We have introduce the notions of (left, right) internal cubic intuitionistic set, double left (right) internal cubic intuitionistic set, cross left (right) internal cubic intuitionistic set and (cross) external cubic intuitionistic set, and have investigated related properties. We have applied the cubic intuitionistic set to BCK/BCI-algebras, and have introduced the concepts of cubic intuitionistic subalgebra and cubic intuitionistic ideal. We have discussed several properties, and have investigated relations between cubic intuitionistic subalgebra and cubic intuitionistic ideal.

6. Acknowledgements

The author wishes to thanks the anonymous reviewers for their valuable suggestions.

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