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Hermite-Hadamard type inequality for r_1-convex function and r_2-convex function using Sugeno integrals

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Abstract. In this paper we prove a fuzzy integral inequality for r_1-convex function and r_2-convex function. Some examples satisfying results are also given.

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1. Intoduction

In 1974 theory of fuzzy measures and fuzzy integrals was introduced by M. Sugeno in his Ph.D. thesis [11]. In [12, 13, 14], Hadamard inequalities for convex function is given. Since then many authors have worked on fuzzy integrals inequalities [2, 3, 4, 7, 8, 9, 10].

Motivated from the above results in this paper we present Hermite-Hadamard type inequality for r_1-convex function and r_2-convex function using Sugeno integrals.

2. Preliminaries

Now we give some basic definitions and properties of the fuzzy integral given in [11, 16]. Suppose that $\wp$ is a σ-algebra of subsets of X and $\mu : \wp \longrightarrow [0, \infty)$ be a non-negative, extended real valued set function. We say that μ is a fuzzy measure, if

(i) $\mu(\phi) = 0$,
(ii) $E, F \in \wp$ and $E \subset F$ imply $\mu(E) \leq \mu(F)$,
(iii) $\{E_n\} \subset \wp, E_1 \subset E_2 \subset ...$, imply $\lim_{n\longrightarrow\infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n)$,
(iv) $\{E_n\} \subset \wp$, $E_1 \supset E_2 \supset ...$, $\mu(E_1) < \infty$, imply $\lim_{n\longrightarrow\infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n)$.

If f is non-negative real-valued function defined on X, we denote the set $\{x \in X : f(x) \geq \alpha\} = \{x \in X : f \geq \alpha\}$ by F_α for $\alpha \geq 0$, where if $\alpha \leq \beta$, then $F_\beta \subset F_\alpha$.

Let $(X, \wp, \mu)$ be a fuzzy measure space, we denote M^+ the set of all non-negative measurable functions with respect to $\wp$.

Definition 2.1 ([11], Sugeno)**.** Let $(X, \wp, \mu)$ be a fuzzy measure space, $f \in M^+$ and $A \in \wp$, the Sugeno integral (or fuzzy integral) of f on A, with respect to the fuzzy measure μ, is defined as:

$$(s)\int_A f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(A \cap F_\alpha)], \tag{2.1}$$

when $A = X$,

$$(s)\int_X f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(F_\alpha)], \tag{2.2}$$

where $\bigvee$ and $\wedge$ denote the operations sup and inf on $[0, \infty)$, respectively.

Some of the properties of fuzzy integral are as follows.

Proposition 2.2 ([15])**.** *Let $(X, \wp, \mu)$ be fuzzy measure space, $A, B \in \wp$ and $f, g \in M^+$ then*

(1) $(s)\int_A f d\mu \leq \mu(A)$;
(2) $(s)\int_A k d\mu = k \wedge \mu(A)$, *$k$ for non-negative constant,*
(3) $(s)\int_A f d\mu \leq (s)\int_A g d\mu$, *for* $f \leq g$,
(4) $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \Longrightarrow (s)\int_A f d\mu \geq \alpha$,
(5) $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Longrightarrow (s)\int_A f d\mu \leq \alpha$,
(6) $(s)\int_A f d\mu > \alpha \Longleftrightarrow$ *there exists* $\gamma > \alpha$ *such that* $\mu(A \cap \{f \geq \gamma\}) > \alpha$,
(7) $(s)\int_A f d\mu < \alpha \Longleftrightarrow$ *there exists* $\gamma < \alpha$ *such that* $\mu(A \cap \{f \geq \gamma\}) < \alpha$.

Consider the distribution function F associated to f on A, that is, $F(\alpha) = \mu(A \cap \{f \geq \alpha\})$. Then from (4) *and* (5) *of Proposition 2.2, we have $F(\alpha) = \alpha \Longrightarrow (s)\int_A f d\mu = \alpha$. Fuzzy integral can be obtained by solving the equation $F(\alpha) = \alpha$.*

Definition 2.3 ([1])**.** a positive function f is called r-convex on $[a, b]$, if for each $x, y \in [a, b]$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq \begin{cases} [tf^r(x) + (1-t)f^r(y)]^{\frac{1}{r}}, & r \neq 0, \\ [f(x)]^t [f(y)]^{1-t}, & r = 0. \end{cases} \tag{2.3}$$

It is obvious 0-convex functions are simply log-convex functions and 1-convex functions are ordinary convex functions. Note that if f is r-convex $[a, b]$, then f^r is a convex function ($r > 0$).

3. Main Results

The Hermite-Hadamard type inequality for r-convex function and s-convex function obtained in [5].

Theorem 3.1. *Let $f, g : [a, b] \longrightarrow (0, \infty)$ be r-convex and s-convex function respectively on $[a, b]$ and $r, s > 0$. Then the following inequality holds*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{2}\left(\frac{r}{r+2}\right)\frac{f^{r+2}(b)-f^{r+2}(a)}{f^r(b)-f^r(a)} + \frac{1}{2}\left(\frac{s}{s+2}\right)\frac{g^{s+2}(b)-g^{s+2}(a)}{g^s(b)-g^s(a)}, \tag{3.1}$$

$f(a) \neq f(b)$, $g(a) \neq g(b)$.

Now we give an example.

Example 3.2. Consider $X = [0, 1]$ and let μ be the Lebesgue measure on X. If we take $r = s = \frac{7}{8}$-convex functions, $f(x) = \frac{x^2}{2}$ and $g(x) = \frac{x^2}{2}$ on $[0, 1]$, we have f and g are r-convex and s-convex functions respectively, from simple calculation we get

$$(s)\int_0^1 \frac{x^4}{4} d\mu = 0.1380, \tag{3.2}$$

and the other hand

$$\frac{1}{2}\left(\frac{r}{r+2}\right)\frac{f^{r+2}(b)-f^{r+2}(a)}{f^r(b)-f^r(a)} + \frac{1}{2}\left(\frac{s}{s+2}\right)\frac{g^{s+2}(b)-g^{s+2}(a)}{g^s(b)-g^s(a)} = 0.076.$$

This proves that the inequality (3.1) is not satisfied for Sugeno integral.

Now we give Hermite-Hadamard type inequality for Sugeno integral using r_1-convex and r_2-convex functions.

Theorem 3.3. *Let $f, g : [0, 1] \longrightarrow [0, \infty)$ be the r_1-convex and r_2-convex functions respectively. Let $r_1, r_2 > 0$ and μ be the Lebesgue measure on $\boldsymbol{R}$ with $f(0) \neq f(1)$ and $g(0) \neq g(1)$.*

Case 1. *If $f(1) > f(0)$ and $g(1) > g(0)$, then*

$$(s)\int_0^1 fg d\mu \leq min\{\beta, 1\}, \tag{3.3}$$

where β is given by

$$1 - \left(\frac{\beta^{r_2} - g^{r_2}(0)}{g^{r_2}(1) - g^{r_2}(0)}\right) - \left(\frac{\beta^{r_1} - f^{r_1}(0)}{f^{r_1}(1) - f^{r_1}(0)}\right) + \left(\frac{\beta^{r_1} - f^{r_1}(0)}{f^{r_1}(1) - f^{r_1}(0)}\right).\left(\frac{\beta^{r_2} - g^{r_2}(0)}{g^{r_2}(1) - g^{r_2}(0)}\right) = \beta. \tag{3.4}$$

Case 2. *If $f(0) > f(1)$ and $g(0) > g(1)$, then*

$$(s)\int_0^1 fg d\mu \leq min\{\beta, 1\}, \tag{3.5}$$

where β is satisfying the following equation

$$\beta(f^{r_1}(1) - f^{r_1}(0))(g^{r_2}(1) - g^{r_2}(0)) - (\beta^{r_1} - f^{r_1}(0))(\beta^{r_2} - g^{r_2}(0)) = 0. \tag{3.6}$$

Proof. Since f and g are r_1-convex and r_2-convex function respectively, we have

$$\begin{aligned} f(x) =& f(x.1 + (1-x).0) \\ \leq& [x.f^{r_1}(1) + (1-x)f^{r_1}(0)]^{1/r_1} = h_1(x), \\ g(x) =& g(x.1 + (1-x).0) \\ \leq& [x.g^{r_2}(1) + (1-x)g^{r_2}(0)]^{1/r_2} = h_2(x). \end{aligned}$$

Then from Proposition 2.2, we have

$$\begin{aligned} (s)\int_0^1 fgd\mu =& (s)\int_0^1 f(x.1+(1-x).0).g(x.1+(1-x).0)d\mu \\ \leq& (s)\int_0^1 [x.f^{r_1}(1)+(1-x)f^{r_1}(0)]^{1/r_1}.[x.g^{r_2}(1)+(1-x)g^{r_2}(0)]^{1/r_2}d\mu \\ =& (s)\int_0^1 h_1(x)h_2(x)d\mu. \end{aligned} \tag{3.7}$$

Now to obtain right hand side of (3.7), we consider the distribution function F given by:

$$\begin{aligned} F(\beta) =& \mu([0,1]\cap\{x|h_1(x)h_2(x)\geq\beta\}) \\ =& \mu([0,1]\cap\{x|h_1(x)\geq\beta\}).\mu([0,1]\cap\{x|h_2(x)\geq\beta\}) \\ =& (\mu([0,1]\cap\{x|[x.f^{r_1}(1)+(1-x)f^{r_1}(0)]^{1/r_1}\geq\beta\})) \\ & \times(\mu([0,1]\cap\{x|[x.g^{r_2}(1)+(1-x)g^{r_2}(0)]^{1/r_2}\geq\beta\})). \end{aligned} \tag{3.8}$$

Case 1. If $f(1) > f(0)$ and $g(1) > g(0)$, then from (3.8), we have

$$\begin{aligned} F(\beta) =& \mu\left([0,1]\cap\left\{x|x\geq\frac{\beta^{r_1}-f^{r_1}(0)}{f^{r_1}(1)-f^{r_1}(0)}\right\}\right).\mu\left([0,1]\cap\left\{x|x\geq\frac{\beta^{r_2}-g^{r_2}(0)}{g^{r_2}(1)-g^{r_2}(0)}\right\}\right) \\ =& \mu\left(\frac{\beta^{r_1}-f^{r_1}(0)}{f^{r_1}(1)-f^{r_1}(0)},1\right).\mu\left(\frac{\beta^{r_2}-g^{r_2}(0)}{g^{r_2}(1)-g^{r_2}(0)},1\right) \\ =& \left(1-\left(\frac{\beta^{r_1}-f^{r_1}(0)}{f^{r_1}(1)-f^{r_1}(0)}\right)\right).\left(1-\left(\frac{\beta^{r_2}-g^{r_2}(0)}{g^{r_2}(1)-g^{r_2}(0)}\right)\right) \end{aligned} \tag{3.9}$$

and solution of (3.9) is $F(\beta) = \beta$ given in (3.4). From Proposition 2.2, we have

$$(s)\int_0^1 fgd\mu \leq min\{\beta, 1\}.$$

Case 2. If $f(0) > f(1)$ and $g(0) > g(1)$, then from (3.8), we have

$$\begin{aligned} F(\beta) =& \mu\left([0,1]\cap\left\{x|x\leq\frac{\beta^{r_1}-f^{r_1}(0)}{f^{r_1}(1)-f^{r_1}(0)}\right\}\right).\mu\left([0,1]\cap\left\{x|x\leq\frac{\beta^{r_2}-g^{r_2}(0)}{g^{r_2}(1)-g^{r_2}(0)}\right\}\right) \\ =& \mu\left(0,\frac{\beta^{r_1}-f^{r_1}(0)}{f^{r_1}(1)-f^{r_1}(0)}\right).\mu\left(0,\frac{\beta^{r_2}-g^{r_2}(0)}{g^{r_2}(1)-g^{r_2}(0)}\right) \\ =& \left(\frac{\beta^{r_1}-f^{r_1}(0)}{f^{r_1}(1)-f^{r_1}(0)}\right).\left(\frac{\beta^{r_2}-g^{r_2}(0)}{g^{r_2}(1)-g^{r_2}(0)}\right) \end{aligned} \tag{3.10}$$

and solution of (3.10) is $F(\beta) = \beta$, given in (3.6). From Proposition 2.2, we have

$$(s)\int_0^1 fgd\mu \le min\{\beta, 1\}.$$

□

Remark 3.4. In case if we put $f(0) = f(1)$ and $g(0) = g(1)$ in Theorem 3.3 then

$$(s)\int_0^1 f(x)g(x)d\mu \le (s)\int_0^1 f(0)g(0)d\mu = f(0)g(0) \wedge 1.$$

Example 3.5. Consider $X = [0, 1]$ and let μ be the Lebesgue measure on X. If we take $f(x) = \frac{x^3}{3}$ and $g(x) = \frac{x^3}{3}$ then $f(x)$, $g(x)$ are $r_1 = r_2 = \frac{7}{8}$-convex function then from Theorem 3.3 we have

$$0.0712 = (s)\int_0^1 \frac{x^6}{9} d\mu \le min\{0.1781, 1\} = 0.1781. \tag{3.11}$$

Now we give the following theorem which is general case of Theorem 3.3.

Theorem 3.6. *Let $r_1, r_2 > 0$ and let μ be the Lebesgue measure on $\boldsymbol{R}$. Let $f, g : [a, b] \longrightarrow [0, \infty)$ be r_1-convex and r_2-convex function with $f(a) \ne f(b)$ and $g(a) \ne g(b)$.*

Case 1. *If $f(b) > f(a)$ and $g(b) > g(a)$, then*

$$(s)\int_a^b fgd\mu \le min\{\beta, b - a\},$$

where β is given by:

$$\begin{aligned}(b-a)^2 - (b-a)^2\left(\frac{\beta^{r_2} - g^{r_2}(a)}{g^{r_2}(b) - g^{r_2}(a)}\right) - (b-a)^2\left(\frac{\beta^{r_1} - f^{r_1}(a)}{f^{r_1}(b) - f^{r_1}(a)}\right)\\ + (b-a)^2\left(\frac{\beta^{r_1} - f^{r_1}(a)}{f^{r_1}(b) - f^{r_1}(a)}\right)\left(\frac{\beta^{r_2} - g^{r_2}(a)}{g^{r_2}(b) - g^{r_2}(a)}\right) = \beta.\end{aligned} \tag{3.12}$$

Case 2. *If $f(a) > f(b)$ and $g(a) > g(b)$, then*

$$(s)\int_a^b fgd\mu \le min\{\beta, b - a\},$$

where β satisfies the following equation

$$\beta(f^{r_1}(b) - f^{r_1}(a))(g^{r_2}(b) - g^{r_2}(a)) - (b-a)^2(\beta^{r_1} - f^{r_1}(a))(\beta^{r_2} - g^{r_2}(a)) = 0. \tag{3.13}$$

Proof. As f and g are r_1-convex and r_2-convex functions, $x \in [a,b]$, we have

$$\begin{aligned} f(x) =& f\left(\left(1-\frac{x-a}{b-a}\right).a+\frac{x-a}{b-a}.b\right)\right) \\ \leq& \left[\left(1-\frac{x-a}{b-a}\right).f^{r_1}(a)+\frac{x-a}{b-a}f^{r_1}(b)\right]^{\frac{1}{r_1}} = h_1(x), \\ g(x) =& g\left(\left(1-\frac{x-a}{b-a}\right).a+\frac{x-a}{b-a}.b\right) \\ \leq& \left[\left(1-\frac{x-a}{b-a}\right).g^{r_2}(a)+\frac{x-a}{b-a}g^{r_2}(b)\right]^{\frac{1}{r_2}} = h_2(x). \end{aligned}$$

Hence by (3) of Proposition 2.2, we have

$$\begin{aligned} (s)\int_a^b fgd\mu =& (s)\int_a^b f\left(\left(1-\frac{x-a}{b-a}\right).a+\frac{x-a}{b-a}.b\right).g\left(\left(1-\frac{x-a}{b-a}\right).a+\frac{x-a}{b-a}.b\right)d\mu \\ \leq& (s)\int_a^b \left[\left(1-\frac{x-a}{b-a}\right)f^{r_1}(a)+\frac{x-a}{b-a}f^{r_1}(b)\right]^{\frac{1}{r_1}}. \\ & \left[\left(1-\frac{x-a}{b-a}\right)g^{r_2}(a)+\frac{x-a}{b-a}g^{r_2}(b)\right]^{\frac{1}{r_2}}d\mu \\ =& (s)\int_a^b h_1(x)h_2(x)d\mu. \end{aligned}$$

To obtain right hand side of above inequality, we consider the distribution function F as

$$\begin{aligned} F(\beta) =& \mu\big([a,b]\cap\{x|h_1(x)h_2(x)\geq\beta\}\big) \\ =& \mu\big([a,b]\cap\{x|h_1(x)\geq\beta\}\big).\mu\big([a,b]\cap\{x|h_2(x)\geq\beta\}\big) \\ =& \mu\left([a,b]\cap\left\{x|\left[\left(1-\frac{x-a}{b-a}\right)f^{r_1}(a)+\frac{x-a}{b-a}f^{r_1}(b)\right]^{\frac{1}{r_1}}\geq\beta\right\}\right). \\ & \mu\left([a,b]\cap\left\{x|\left[\left(1-\frac{x-a}{b-a}\right)g^{r_2}(a)+\frac{x-a}{b-a}g^{r_2}(b)\right]^{\frac{1}{r_2}}\geq\beta\right\}\right). \end{aligned} \tag{3.14}$$

Case 1. If $f(b) > f(a)$ and $g(b) > g(a)$, then from (3.14), we have

$$\begin{aligned} F(\beta) =& \mu\left([a,b]\cap\left\{x|x\geq(b-a)\left(\frac{\beta^{r_1}-f^{r_1}(a)}{f^{r_1}(b)-f^{r_1}(a)}\right)+a\right\}\right). \\ & \mu\left([a,b]\cap\left\{x|x\geq(b-a)\left(\frac{\beta^{r_2}-g^{r_2}(a)}{g^{r_2}(b)-g^{r_2}(a)}\right)+a\right\}\right) \\ =& \mu\left((b-a)\left(\frac{\beta^{r_1}-f^{r_1}(a)}{f^{r_1}(b)-f^{r_1}(a)}\right)+a,b\right).\mu\left((b-a)\left(\frac{\beta^{r_2}-g^{r_2}(a)}{g^{r_2}(b)-g^{r_2}(a)}\right)+a,b\right) \\ =& \left((b-a)-(b-a)\left(\frac{\beta^{r_1}-f^{r_1}(a)}{f^{r_1}(b)-f^{r_1}(a)}\right)\right).\left((b-a)-(b-a)\left(\frac{\beta^{r_2}-g^{r_2}(a)}{g^{r_2}(b)-g^{r_2}(a)}\right)\right) \end{aligned} \tag{3.15}$$

and the solution of (3.15) is $F(\beta) = \beta$, given in (3.12). From (1) of Proposition 2.2, we have

$$(s)\int_0^1 fgd\mu \leq min\{\beta, b-a\}.$$

Case 2. If $f(a) > f(b)$ and $g(a) > g(b)$, then from (3.14), we have

$$\begin{aligned}
F(\beta) =& \mu\left([a,b] \cap \left\{x | x \leq (b-a)\left(\frac{\beta^{r_1} - f^{r_1}(a)}{f^{r_1}(b) - f^{r_1}(a)}\right) + a\right\}\right). \\
& \mu\left([a,b] \cap \left\{x | x \leq (b-a)\left(\frac{\beta^{r_2} - g^{r_2}(a)}{g^{r_2}(b) - g^{r_2}(a)}\right) + a\right\}\right) \\
=& \mu\left(a, (b-a)\left(\frac{\beta^{r_1} - f^{r_1}(a)}{f^{r_1}(b) - f^{r_1}(a)}\right) + a\right).\mu\left(a, (b-a)\left(\frac{\beta^{r_2} - g^{r_2}(a)}{g^{r_2}(b) - g^{r_2}(a)}\right) + a\right) \\
=& \left((b-a)\left(\frac{\beta^{r_1} - f^{r_1}(a)}{f^{r_1}(b) - f^{r_1}(a)}\right)\right).\left((b-a)\left(\frac{\beta^{r_2} - g^{r_2}(a)}{g^{r_2}(b) - g^{r_2}(a)}\right)\right)
\end{aligned} \tag{3.16}$$

and the solution of (3.16) is $F(\beta) = \beta$, given in (3.13). By (1) of Proposition 2.2, we have

$$(s)\int_0^1 fgd\mu \leq min\{\beta, b-a\}.$$

□

Remark 3.7. If we put $f(a) = f(b)$ and $g(a) = g(b)$, then from Theorem 3.6, we have

$$(s)\int_a^b f(x)g(x)d\mu \leq (s)\int_a^b f(a)g(a)d\mu = f(a)g(a) \wedge b-a.$$

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