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Abstract. First of all, we define an intuitionistic quotient mapping and obtain its some properties. Second, we define some types continuities, open and closed mappings. And we investigate relationships among them and give some examples. Finally, we introduce the notions of an intuitionistic subspace and the heredity, and obtain some properties of each concept.

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1. Introduction

In 1996, Coker [5] introduced the concept of an intuitionistic set (called an intuitionistic crisp set by Salama et al.[17]) as the generalzation of an ordinary set and the specialization of an intuitionistic fuzzy set introduced by Atanassove [1]. After that time, many researchers [2, 6, 7, 8, 16, 18] applied the notion to topology, and Selvanayaki and Ilango [19] studied homeomorphisms in intuitionistic topological spaces. In particular, Bayhan and Coker [3] investigated separtion axioms in intuitionistic topological spaces. And they [4] dealt with pairwise separation axioms in intuitionistic topological spaces and some relationships between categories **Dbl-Top** and **Bitop**. Furthermore, Lee and Chu [15] introduced the category **ITop** and investigated some relationships between **ITop** and **Top**. Recently, Kim et al. [10] investigate the category **ISet** composed of intuitionistic sets and morphisms between them in the sense of a topological universe. Also, they [11, 12] studied some additional properties and give some examples related to closures, interiors in and separation axioms in intuitionistic topological spaces. Moreover, Lee at al [13] investigated limit points and nets in intuitionistic topological spaces and also they [14] studied intuitionistic equivalence relation.

In this paper, first of all, we define an intuitionistic quotient mapping and obtain its some properties. Second, we define some types continuities, open and closed mappings. And we investigate relationships among them and give some examples. Finally, we introduce the notions of an intuitionistic subspace and the heredity, and obtain some properties of each concept.

2. Preliminaries

In this section, we list the concepts of an intuitionistic set, an intuitionistic point, an intuitionistic vanishing point and operations of intuitionistic sets and some results obtained by [5, 6, 7, 11].

Definition 2.1 ([5])**.** Let X be a non-empty set. Then A is called an intuitionistic set (in short, IS) of X, if it is an object having the form

$$A = (A_T, A_F),$$

such that $A_T \cap A_F = \phi$, where A_T [resp. A_F] is called the set of members [resp. nonmembers] of A.

In fact, A_T [resp. A_F] is a subset of X agreeing or approving [resp. refusing or opposing] for a certain opinion, view, suggestion or policy.

The intuitionistic empty set [resp. the intuitionistic whole set] of X, denoted by ϕ_I [resp. X_I], is defined by $\phi_I = (\phi, X)$ [resp. $X_I = (X, \phi)$].

In general, $A_T \cup A_F \neq X$.

We will denote the set of all ISs of X as $IS(X)$.

It is obvious that $A = (A, \phi) \in IS(X)$ for each ordinary subset A of X. Then we can consider an IS of X as the generalization of an ordinary subset of X. Furthermore, it is clear that $A = (A_T, A_T, A_F)$ is an neutrosophic crisp set in X, for each $A \in IS(X)$. Thus we can consider a neutrosophic crisp set in X as the generalization of an IS of X. Moreover, we can consider an intuitionistic set in X as an intuitionistic fuzzy set in X from Remark 2.2 in [11].

Definition 2.2 ([5])**.** Let $A, B \in IS(X)$ and let $(A_j)_{j\in J} \subset IS(X)$.

(i) We say that A is contained in B, denoted by $A \subset B$, if $A_T \subset B_T$ and $A_F \supset B_F$.

(ii) We say that A equals to B, denoted by $A = B$, if $A \subset B$ and $B \subset A$.

(iii) The complement of A denoted by A^c, is an IS of X defined as:

$$A^c = (A_F, A_T).$$

(iv) The union of A and B, denoted by $A \cup B$, is an IS of X defined as:

$$A \cup B = (A_T \cup B_T, A_F \cap B_F).$$

(v) The union of $(A_j)_{j\in J}$, denoted by $\bigcup_{j\in J} A_j$ (in short, $\bigcup A_j$), is an IS of X defined as:

$$\bigcup_{j\in J} A_j = (\bigcup_{j\in J} A_{j,T}, \bigcap_{j\in J} A_{j,F}).$$

(vi) The intersection of A and B, denoted by $A \cap B$, is an IS of X defined as:

$$A \cap B = (A_T \cap B_T, A_F \cup B_F).$$

(vii) The intersection of $(A_j)_{j\in J}$, denoted by $\bigcap_{j\in J} A_j$ (in short, $\bigcap A_j$), is an IS of X defined as:

$$\bigcap_{j\in J} A_j = (\bigcap_{j\in J} A_{j,T}, \bigcup_{j\in J} A_{j,F}).$$

(viii) $A - B = A \cap B^c$.

(ix) $[\ ]A = (A_T, {A_T}^c)$, $< > A = ({A_F}^c, A_F)$.

From Propositions 3.6 and 3.7 in [10], we can easily see that $(IS(X), \cup, \cap, ^c, \phi_I, X_I)$ is a Boolean algebra except the following conditions:

$$A \cup A^c \neq X_I,\ A \cap A^c \neq \phi_I.$$

However, by Remark 2.12 in [11], $(IS_*(X), \cup, \cap, ^c, \phi_I, X_I)$ is a Boolean algebra, where

$$IS_*(X) = \{A \in IS(X) : A_T \cup A_F = X\}.$$

Definition 2.3 ([5])**.** Let $f : X \to Y$ be a mapping, and let $A \in IS(X)$ and $B \in IS(Y)$. Then

(i) the image of A under f, denoted by $f(A)$, is an IS in Y defined as:

$$f(A) = (f(A)_T, f(A)_F),$$

where $f(A)_T = f(A_T)$ and $f(A)_F = (f(A^c_F))^c$.

(ii) the preimage of B, denoted by $f^{-1}(B)$, is an IS in X defined as:

$$f^{-1}(B) = (f^{-1}(B)_T, f^{-1}(B)_F),$$

where $f^{-1}(B)_T = f^{-1}(B_T)$ and $f^{-1}(B)_F = f^{-1}(B_F)$.

Result 2.4. (See [5], Corollary 2.11) *Let $f : X \to Y$ be a mapping and let $A, B, C \in IS(X)$, $(A_j)_{j\in J} \subset IS(X)$ and $D, E, F \in IS(Y)$, $(D_k)_{k\in K} \subset IS(Y)$. Then the followings hold:*

(1) *if $B \subset C$, then $f(B) \subset f(C)$ and if $E \subset F$, then $f^{-1}(E) \subset f^{-1}(F)$.*

(2) *$A \subset f^{-1}f(A))$ and if f is injective, then $A = f^{-1}f(A))$,*

(3) *$f(f^{-1}(D)) \subset D$ and if f is surjective, then $f(f^{-1}(D)) = D$,*

(4) *$f^{-1}(\bigcup D_k) = \bigcup f^{-1}(D_k)$, $f^{-1}(\bigcap D_k) = \bigcap f^{-1}(D_k)$,*

(5) *$f(\bigcup A_j) = \bigcup f(A_j)$, $f(\bigcap A_j) \subset \bigcap f(A_j)$,*

(6) *$f(A) = \phi_I$ if and only if $A = \phi_I$ and hence $f(\phi_I) = \phi_I$, in particular if f is surjective, then $f(X_I) = Y_I$,*

(7) *$f^{-1}(Y_I) = Y_I$, $f^{-1}(\phi_I) = \phi_I$.*

(8) *if f is surjective, then $f(A)^c \subset f(A^c)$ and furthermore, if f is injective, then $f(A)^c = f(A^c)$,*

(9) *$f^{-1}(D^c) = (f^{-1}(D))^c$.*

Definition 2.5 (See [5])**.** Let X be a non-empty set, $a \in X$ and let $A \in IS(X)$.

(i) The form $(\{a\}, \{a\}^c)$ [resp. $(\phi, \{a\}^c)$]is called an intuitionistic point [resp. vanishing point] of X and denoted by a_I [resp. a_{IV}].

(ii) We say that a_I [resp. a_{IV}] is contained in A, denoted by $a_I \in A$ [resp. $a_{IV} \in A$], if $a \in A_T$ [resp. $a \notin A_F$].

We will denote the set of all intuitionistic points or intuitionistic vanishing points in X as $IP(X)$.

Definition 2.6 ([6]). Let X be a non-empty set and let $\tau \subset IS(X)$. Then τ is called an intuitionistic topology (in short IT) on X, it satisfies the following axioms:
(i) $\phi_I, X_I \in \tau$,
(ii) $A \cap B \in \tau$, for any $A, B \in \tau$,
(iii) $\bigcup_{j \in J} A_j \in \tau$, for each $(A_j)_{j \in J} \subset \tau$.
In this case, the pair (X, τ) is called an intuitionistic topological space (in short, ITS) and each member O of τ is called an intuitionistic open set (in short, IOS) in X. An IS F of X is called an intuitionistic closed set (in short, ICS) in X, if $F^c \in \tau$.

It is obvious that $\{\phi_I, X_I\}$ is the smallest IT on X and will be called the intuitionistic indiscreet topology and denoted by $\tau_{I,0}$. Also $IS(X)$ is the greatest IT on X and will be called the intuitionistic discreet topology and denoted by $\tau_{I,1}$. The pair $(X, \tau_{I,0})$ [resp. $(X, \tau_{I,1})$] will be called the intuitionistic indiscreet [resp. discreet] space.

We will denote the set of all ITs on X as $IT(X)$. For an ITS X, we will denote the set of all IOSs [resp. ICSs] on X as $IO(X)$ [resp. $IC(X)$].

Result 2.7 ([6], Proposition 3.5). *Let (X, τ) be an ITS. Then the following two ITs on X can be defined by:*

$$\tau_{0,1} = \{[\ ]U : U \in \tau\}, \tau_{0,2} = \{<>U : U \in \tau\}.$$

Furthermore, the following two ordinary topologies on X can be defined by (See [3]):

$$\tau_1 = \{U_T : U \in \tau\},\ \tau_2 = \{U_F^c : U \in \tau\}.$$

We will denote two ITs $\tau_{0,1}$ and $\tau_{0,2}$ defined in Result 2.7 as

$$\tau_{0,1} = [\ ]\tau \text{ and } \tau_{0,2} = <> \tau.$$

Moreover, for an IT τ on a set X, we can see that (X, τ_1, τ_2) is a bitopological space by Kelly [9] (Also see Proposition 3.1 in [4]).

Definition 2.8 ([7]). Let X be an ITS, $p \in X$ and let $N \in IS(X)$. Then
(i) N is called a neighborhood of p_I, if there exists an IOS G in X such that

$$p_I \in G \subset N, \text{ i.e., } p \in G_T \subset N_T \text{ and } G_F \supset N_F,$$

(ii) N is called a neighborhood of p_{IV}, if there exists an IOS G in X such that

$$p_{IV} \in G \subset N, \text{ i.e., } G_T \subset N_T \text{ and } p \notin G_F \supset N_F.$$

We will denote the set of all neighborhoods of p_I [resp. p_{IV}] by $N(p_I)$ [resp. $N(p_{IV})$].

Result 2.9 ([11], Theorem 4.2). *Let (X, τ) be an ITS and let $A \in IS(X)$. Then*
(1) $A \in \tau$ if and only if $A \in N(a_I)$, for each $a_I \in A$,
(1) $A \in \tau$ if and only if $A \in N(a_{IV})$, for each $a_{IV} \in A$.

Result 2.10 ([7], Proposition 3.4). *Let (X, τ) be an ITS. We define the families*

$$\tau_I = \{G : G \in N(p_I), \text{ for each } p_I \in G\}$$

and

$$\tau_{IV} = \{G : G \in N(p_{IV}), \text{ for each } p_{IV} \in G\}.$$

Then $\tau_I, \tau_{IV} \in IT(X)$.

From the above Result, we can easily see that for an IT τ on a set X and each $U \in \tau$,

$$\tau_I = \tau \cup \{(U_T, S_U) : S_U \subset U_F\} \cup \{(\phi, S) : S \subset X\}$$

and

$$\tau_{IV} = \tau \cup \{(S_U, U_F) : S_U \supset U_T \text{ and } S_U \cap U_F = \phi\}.$$

Result 2.11 ([7], Proposition 3.5)**.** *Let* (X, τ) *be an ITS. Then* $\tau \subset \tau_I$ *and* $\tau \subset \tau_{IV}$.

Result 2.12 ([11], Corollary 4.6)**.** *Let* (X, τ) *be an ITS and let* IC_τ *[resp.* IC_{τ_I} *and* $IC_{\tau_{IV}}$*] be the set of all ICSs w.r.t.* τ *[resp.* τ_I *and* τ_{IV}*]. Then*

$$IC_\tau(X) \subset IC_{\tau_I}(X) \text{ and } IC_\tau(X) \subset IC_{\tau_{IV}}(X).$$

Definition 2.13 ([6])**.** Let (X, τ) be an ITS and let $A \in IS(X)$.

(i) The intuitionistic closure of A w.r.t. τ, denoted by $Icl(A)$, is an IS of X defined as:

$$Icl(A) = \bigcap \{K : K^c \in \tau \text{ and } A \subset K\}.$$

(ii) The intuitionistic interior of A w.r.t. τ, denoted by $Iint(A)$, is an IS of X defined as:

$$Iint(A) = \bigcup \{G : G \in \tau \text{ and } G \subset A\}.$$

3. Intuitionistic quotient spaces

In this section, we define an intuitionistic quotient mapping and obtain its some properties.

Definition 3.1 ([6])**.** Let X, Y be an ITSs. Then a mapping $f : X \to Y$ is said to be continuous, if $f^{-1}(V) \in IO(X)$, for each $V \in IO(Y)$.

The following is the immediate result of by the above definition.

Proposition 3.2. *Let* X, Y *be ITSs. Then*

(1) *the identity* $id : X \to X$ *is continuous,*
(2) *if* $f : X \to Y$ *and* $g : Y \to Z$ *are continuous, then* $g \circ f : X \to Z$ *is continuous,*
(3) *if* $f : X \to Y$ *is a constant mapping, then* f *is continuous,*
(4) *if* X *is an intuitionistic discrete space, then* f *is continuous,*
(5) *if* Y *is an intuitionistic indiscrete space, then* f *is continuous.*

Result 3.3 ([6], Proposition 4.4)**.** $f : X \to Y$ *is continuous if and only if* $f^{-1}(F) \in IC(X)$*, for each* $F \in IC(Y)$.

Result 3.4 ([6], Proposition 4.5)**.** *The followings are equivalent:*

(1) $f : X \to Y$ *is continuous,*
(2) $f^{-1}(Iint(B)) \subset Iint(f^{-1}(B))$*, for each* $B \in IS(Y)$,
(3) $Icl(f^{-1}(B)) \subset f^{-1}(Icl(B))$*, for each* $B \in IS(Y)$.

Result 3.5 ([15], Theorem 3.1)**.** *The followings are equivalent:*

(1) $f : X \to Y$ *is continuous,*
(2) $f(Icl(A)) \subset Icl(f(A))$*, for each* $a \in IS(X)$.

Definition 3.6. Let X, Y be ITSs. Then a mapping $f : X \to Y$ is said to be:
(i) open [6], if $f(A) \in IO(Y)$, for each $A \in IO(X)$,
(ii) closed [15], if $f(F) \in IC(Y)$, for each $F \in IC(X)$.

The following is the immediate result of the above definition.

Proposition 3.7. *Let X, Y be an ITSs.*
(1) *$f : X \to Y$ and $g : Y \to Z$ are open [resp. closed], then $g \circ f : X \to Z$ is open [resp. closed].*
(2) *If both X and Y are intuitionistic discrete spaces, then f is continuous and open.*

Result 3.8 ([15], Theorem 3.2)**.** *$f : X \to Y$ be continuous and injective. Then $Iintf(A) \subset f(Iint(A))$, for each $A \in IS(X)$.*

Result 3.9 ([15], Theorem 3.4)**.** *Let X, Y be ITSs. Then the followings are equivalent:*
(1) *$f : X \to Y$ is open,*
(2) *$f(Iint(A)) \subset Iint(f(A))$, for each $A \in IS(X)$,*
(3) *$Iint(f^{-1}(B)) \subset f^{-1}(Iint(B))$, for each $B \in IS(Y)$.*

The following is the immediate result of Results 3.8 and 3.9.

Corollary 3.10. *$f : X \to Y$ be continuous, open and injective. Then $Iintf(A) = f(Iint(A))$, for each $A \in IS(X)$.*

Result 3.11 ([15], Theorem 3.8)**.** *Let X, Y be ITSs and $f : X \to Y$ a mapping. Then f is closed if and only if $Iclf(A)) \subset f(Icl(A))$, for each $A \in IS(X)$.*

The following is the immediate result of Results 3.5 and 3.11.

Corollary 3.12. *Let X, Y be ITSs and $f : X \to Y$ a mapping. Then f is continuous and closed if and only if $Iclf(A)) = f(Icl(A))$, for each $A \in IS(X)$.*

Proposition 3.13. *Let (X, τ) be an ITS, let Y be a set and let $f : X \to Y$ be a mapping. We define a family $\tau_Y \subset IS(Y)$ as follows:*

$$\tau_Y = \{U \in IS(Y) : f^{-1}(U) \in \tau\}.$$

Then
(1) *$\tau_Y \in IT(Y)$,*
(2) *$f : (X, \tau) \to (Y, \tau_Y)$ is continuous,*
(3) *if σ is an IT on Y such that $f : (X, \tau) \to (Y, \sigma)$ is continuous, then τ_Y is finer than σ, i.e., $\sigma \subset \tau_Y$.*

Proof. (1) From Result 2.4 and the definition of an IT, we can easily show that $\tau_Y \in IT(Y)$.
(2) It is obvious that $f : (X, \tau) \to (Y, \tau_Y)$ is continuous, by the definition τ_Y.
(3) Let $U \in \sigma$. Since $f : (X, \tau) \to (Y, \sigma)$ is continuous, $f^{-1}(U) \in \tau$. Then by the definition τ_Y, $U \in \tau_Y$. Thus $\sigma \subset \tau_Y$. □

Definition 3.14. Let (X, τ) be an ITS, let Y be a set and let $f : X \to Y$ be a surjective mapping. Let $\tau_Y = \{U \in IS(Y) : f^{-1}(U) \in \tau\}$ be the IT on Y in Proposition 3.13. Then τ_Y is called the intuitionistic quotient topology on Y

induced by f. The pair (Y, τ_Y) is called an intuitionistic quotient space of X and f is called an intuitionistic quotient mapping.

From Proposition 3.13, the intuitionistic quotient mapping f is not only continuous but τ_Y is the finest topology on Y for which f is continuous. It is easy to prove that if (Y, σ) is an intuitionistic quotient space of (X, τ) with intuitionistic quotient mapping f, then F is closed in Y if and only if $f^{-1}(F)$ is closed in X.

Proposition 3.15. *Let (X, τ) and (Y, σ) be ITSs, let $f : X \to Y$ be a continuous surjective mapping and let τ_Y be the intuitionistic quotient topology on Y induced by f. If f is open or closed, then $\sigma = \tau_Y$.*

Proof. Suppose f is open. Since τ_Y is the finest topology on Y for which f is continuous, $\sigma \subset \tau_Y$. Let $U \in \tau_Y$. Then by the definition of τ_Y, $f^{-1}(U) \in \tau$. Since f is open and surjective, $U = f(f^{-1}(U)) \in \sigma$. Thus $U \in \sigma$. So $\tau_Y \subset \sigma$. Hence $\sigma = \tau_Y$.

Suppose f is closed. Then by the similar arguments, we can see that $\sigma = \tau_Y$. □

From Proposition 3.15, we can easily see that if $f : (X, \tau) \to (Y, \sigma)$ is open (or closed) continuous surjective, then f is an intuitionistic quotient mapping.

The following is the immediate result of Definition 3.14.

Proposition 3.16. *The composition of two intuitionistic quotient mappings is an intuitionistic quotient mapping.*

Theorem 3.17. *Let (X, τ) be an ITS, let Y be a set, let $f : X \to Y$ be a surjection, let τ_Y be the intuitionistic quotient topology on Y induced by f and let (Z, σ) be an ITS. Then a mapping $g : Y \to Z$ is continuous if and only if $g \circ f : X \to Z$ is continuous.*

Proof. Suppose $g : Y \to Z$ is continuous. Since $f : (X, \tau) \to (Y, \tau_Y)$ is continuous, by Proposition 3.2 (2), $g \circ f : (X, \tau) \to (Z, \sigma)$ is continuous.

Suppose $g \circ f : (X, \tau) \to (Z, \sigma)$ is continuous and let $V \in \sigma$. Then $(g \circ f)^{-1}(V) \in \tau$ and $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$. Thus by the definition of τ_Y, $g^{-1}(V) \in \tau_Y$. So $g : (Y, \tau_Y) \to (Z, \sigma)$ is continuous. □

Theorem 3.18. *Let (X, τ) and (Y, σ) be ITSs and let $p : X \to Y$ be continuous. Then p is an intuitionistic quotient mapping if and only if for each ITS (Z, η) and each mapping $g : Y \to Z$, the continuity of $g \circ p$ implies that of g.*

Proof. The proof is similar to one of an ordinary topological space. □

Theorem 3.19. *Let $(X, \tau), (Y, \sigma)$ and (Z, η) be ITSs, let $p : (X, \tau) \to (Y, \sigma)$ be an intuitionistic quotient mapping and let $h : (X, \tau) \to (Z, \eta)$ be continuous. Suppose $h \circ p^{-1}$ is single-valued, i.e., for each $y \in Y$, h is constant on $p^{-1}(y_I)$. Then*

(1) $(h \circ p^{-1}) \circ p = h$ and $h \circ p^{-1}$ is continuous,

(2) $h \circ p^{-1}$ is open (closed) if and only if $h(U)$ is open (closed), whenever U is open (closed) in X such that $U = p^{-1}(p(U))$.

Proof. (1) Let $x \in X$. Then $x_I \in p^{-1}(p(x_I))$. Since h is constant on $p^{-1}(p(x_I))$, $h(x_I) = h(p^{-1}(p(x_I)))$. On the other hand, $h(p^{-1}(p(x_I))) = [(h \circ p^{-1}) \circ p](x_I)$. Thus $h = (h \circ p^{-1}) \circ p$. Since h is continuous and p is an intuitionistic quotient mapping, by Theorem 3.18, $h \circ p^{-1}$ is continuous.

(2) The proof is similar to one of an ordinary topological space. □

Theorem 3.20. *Let $(X,\tau),(Y,\sigma)$ and (Z,η) be ITSs, let $p:(X,\tau)\to(Y,\sigma)$ be an intuitionistic quotient mapping and let $g:Y\to Z$ be sujective. Then $g\circ p$ is an intuitionistic quotient mapping if and only if g is an intuitionistic quotient mapping.*

Proof. The proof is similar to one of an ordinary topological space. □

Definition 3.21 ([14])**.** Let X, Y be non-empty sets. Then R is called an intuitionistic relation (in short, IR) from X to Y, if it is an object having the form

$$R=(R_T,R_F)$$

such that $R_T, R_F\subset X\times Y$ and $R_T\cap R_F=\phi$, where R_T [resp. R_F] is called the set of members [resp. nonmembers] of R. In fact, $R\in IS(X\times Y)$. In general, $R_T\cup R_F\neq X\times Y$.

In particular, R is called an intuitionistic relation on X, if $R\in IS(X\times X)$.

The intuitionistic empty relation [resp. the intuitionistic whole relation] on X, denoted by $\phi_{R,I}$ [resp. $X_{R,I}$], is defined by $\phi_{R,I}=(\phi,X\times X)$ [resp. $X_{R,I}=(X\times X,\phi)$].

We will denote the set of all IRs on X [resp. from X to Y] as $IR(X\times X)$ [resp. $IR(X\times Y)$].

It is obvious that if $R\in IR(X\times Y)$, then R_T, R_F are ordinary relations from X to Y and conversely, if R_o is an ordinary relation from X to Y, then $(R_o,R_o^c)\in IR(X\times Y)$.

Definition 3.22 ([3])**.** Let X, Y be non-empty sets, let $R\in IR(X\times Y)$ and let $(p,q)\in X\times Y$.

(i) $(p,q)_I$ is said to belong to R, denoted by $(p,q)_I\in R$, if $(p,q)\in R_T$.

(ii) $(p,q)_{IV}$ is said to belong to R, denoted by $(p,q)_{IV}\in R$, if $(p,q)\notin R_F$.

Definition 3.23 ([14])**.** An IR R is called an intuitionistic equivalence relation (in short, IER) on X, if it satisfies the following conditins:

(i) intuitionistic reflexive, i.e., R_T is reflexive and R_F is irreflexive, i.e., $(x,x)\notin R_F$, for each $x\in X$,

(ii) intuitionistic symmetric, i.e., R_T and R_F are symmetric,

(iii) intuitionistic transitive, i.e., $R_T\circ R_T\subset R_T$ and $R_F\circ R_F\supset R_F$, where $S_T\circ R_T$ and denotes the ordinary composition and $S_F\circ R_F=(S_F^c\circ R_F^c)^c$.

We will denote the set of all IERs on X as $IE(X)$.

It is obvious that $R\in IE(X)$ if and only if R_T is an ordinary equivalence relation on X, R_F is irreflexive and $(R_F^c\circ R_F^c)^c\supset R_F$.

Definition 3.24 ([14])**.** Let $R\in IE(X)$ and let $x\in X$. Then the intuitionistic equivalence class (in short, IEC) of x_I modulo R, denoted by R_{x_I} or $[x_I]$, is an IS in X defined as:

$$R_{x_I}=\bigcup\{y_I\in X_I:(x,y)_I\in R\}.$$

In fact, $R_{x_I}=\bigcup\{y_I\in X_I:(x,y)\in R_T\}$.

We will denote the set of all IECs by R as X/R and $X/R=\{R_{x_I}:x\in X\}$ will be called an intuionistic quotient set (in short, IQS) of X by R.

Result 3.25 ([14], Proposition 4.23)**.** *Let $f:X\to Y$ be a mapping. Consider the IR R_f on X defined as: for each $(x,y)\in X\times X$, $(x,y)_I\in R_f$ if and only if*

$f(x_I) = f(y_I)$. *Then* $R_f \in IE(X)$.

In this case, R_f is called the intuitionistic equivalence relation determined by f.

Proposition 3.26. *Let* (X, τ) *and* (Y, σ) *be ITSs, let* $f : (X, \tau) \to (Y, \sigma)$ *be continuous and let* R_f *be the intuitionistic equivalence relation on* X *determined by* f. *Then*

(1) *the intuitionistic natural mapping* $p : (X, \tau) \to (X/R_f, \tau_{X/R_f})$ *is an intuitionistic quotient mapping, where* τ_{X/R_f} *denotes the intuitionistic quotient topology on* X/R_f,

(2) $f \circ p^{-1}$ *is continuous injective,*

(3) *if* f *is surjective, then bijective.*

Proof. (1) It is obvious.

(2) Suppose $x_I, y_I \in p^{-1}(z)$, for some $z = [a_I] \in X/R_f$. Then by the definition of R_f, $f(x_I) = f(y_I)$. Thus $f \circ p^{-1}$ is single-valued. So by Theorem 3.19 (1), $f \circ p^{-1}$ is continuous.

Now suppose $[a_I], [b_I] \in X/R_f$ and $f \circ p^{-1}([a_I]) = f \circ p^{-1}([b_I])$. Let $x_I \in p^{-1}([a_I])$ and $y_I \in p^{-1}([b_I])$. Then $f(x_I) = f(y_I)$. Thus $(x, y)_I \in R_f$. So $[a_I] = p(x_I) = p(y_I) = [b_I]$. Hence $f \circ p^{-1}$ is injective.

(3) Suppose f is surjective and let $y \in Y$. Then there is $x \in X$ such that $f(x) = y$. Since $X_I = \bigcup X/R_f$, $[x_I] \in X/R_f$ and $f \circ p^{-1}([x_I]) = y_I$. Thus $f \circ p^{-1}$ is surjective. So by (2), $f \circ p^{-1}$ is bijective □

Theorem 3.27. *Let* (X, τ) *and* (Y, σ) *be ITSs and let* $f : (X, \tau) \to (Y, \sigma)$ *be continuous surjective. Then* $f \circ p^{-1} : X/R_f \to Y$ *is a homeomorphism if and only if* f *is an intuitionistic quotient mapping.*

Proof. Suppose $f \circ p^{-1} : (X/R_f, \tau_{X/R_f} \to (Y, \sigma)$ be a homeomorphism and let σ_Y be the intuitionistic quotient topology on Y induced by $f \circ p^{-1}$. Then by Proposition 3.13, $\sigma = \sigma_Y$. Thus $f \circ p^{-1}$ is an intuitionistic quotient mapping. So by Theorem 3.20, $(f \circ p^{-1}) \circ p$ is an intuitionistic quotient mapping. On the other hand, $f = (f \circ p^{-1}) \circ p$. Hence f is an intuitionistic quotient mapping.

Suppose $f : (X, \tau) \to (Y, \sigma)$ is an intuitionistic quotient mapping. Since f is surjective, by Proposition 3.26 (3), $f \circ p^{-1}$ is bijective. Let U be any IOS in X/R_f such that $U = p^{-1}(p(U))$. Since $p^{-1}(p(U)) = f^{-1}(f(U))$, $f^{-1}(f(U))$ is open in X. Since f is an intuitionistic quotient mapping, $f(U) \in \tau$. Then by Theorem 3.19 (2), $f \circ p^{-1}$ is open. Thus $f \circ p^{-1}$ is a homeomorphism. □

Definition 3.28 ([14])**.** Let $(A_j)_{j \in J} \subset IS(X)$. Then $(A_j)_{j \in J}$ is called an intuitionistic partition of X, if it satisfies the following conditions:

(i) $A_j \neq \phi_I$, for each $j \in J$,

(ii) either $A_i \cap A_j = \phi_I$ or $A_i = A_j$, for any $i, j \in J$,

(iii) $\bigcup_{j \in J} A_j = X_I$.

Now we turn our attention toward another way of defining an intuitionistic quotient space.

Definition 3.29. Let (X, τ) be an ITS and let Σ be an intuitionistic partition of X. Let $p : X \to \Sigma$ be the mapping defined by: for each $x \in X$,

$$p(x_I) = D \text{ and } x_I \in D, \text{ for some } D \in \Sigma.$$

If τ_Σ is the intuitionistic quotient topology on Σ induced by p, then (Σ, τ_Σ) is called an intuitionistic quotient space and p is called the intuitionistic natural mapping of X onto Σ. The set Σ is called an intuitionistic decomposition of X and the intuitionistic quotient space (Σ, τ_Σ) is called an intuitionistic decomposition space or an intuitionistic identification of X.

Example 3.30. Let $X = \mathbb{N}$, let $A = (\{n \in \mathbb{N} : n \text{ is odd}\}, \{n \in \mathbb{N} : n \text{ is even}\}), B = (\{n \in \mathbb{N} : n \text{ is even}\}, \{n \in \mathbb{N} : n \text{ is odd}\})$ and let $\Sigma = \{A, B\}$. Consider the mapping $p : X \to \Sigma$ given by: for each $n \in X$,

$$p(n_I) = A, \text{ if } n \text{ is odd and } p(n_I) = B, \text{ if } n \text{ is even.}$$

Then clearly, Σ is an intuitionistic partition of X. Let τ be the usual intuitionistic topology on $\mathbb{N}$ and consider $\tau_\mathbb{N}$. Then clearly, $\tau_\mathbb{N}$ is the intuitionistic discrete topology on $\mathbb{N}$. Thus $p_{-1}(A), p_{-1}(B) \in \tau_\mathbb{N}$. So Σ is an intuitionistic decomposition of X.

4. Some types of intuitionistic continuities

In this section, we define some types continuities, open and closed mappings. And we investigate relationships among them and give some examples.

Definition 4.1. Let $(X, \tau), (Y, \sigma)$ be an ITSs. Then a mapping $f : X \to Y$ is said to:

(i) σ-τ-continuous, if it is continuous in the sense of Definition 3.1,
(ii) σ-τ_I-continuous, if for each $V \in \sigma$, $f^{-1}(V) \in \tau_I$,
(iii) σ-τ_{IV}-continuous, if for each $V \in \sigma$, $f^{-1}(V) \in \tau_{IV}$,
(iv) σ_I-τ-continuous, if for each $V \in \sigma_I$, $f^{-1}(V) \in \tau$,
(v) σ_I-τ_I-continuous, if for each $V \in \sigma_I$, $f^{-1}(V) \in \tau_I$
(vi) σ_I-τ_{IV}-continuous, if for each $V \in \sigma_I$, $f^{-1}(V) \in \tau_{IV}$,
(vii) σ_{IV}-τ-continuous, if for each $V \in \sigma_{IV}$, $f^{-1}(V) \in \tau$,
(viii) σ_{IV}-τ_I-continuous, if for each $V \in \sigma_{IV}$, $f^{-1}(V) \in \tau_I$,
(ix) σ_{IV}-τ_{IV}-continuous, if for each $V \in \sigma_{IV}$, $f^{-1}(V) \in \tau_{IV}$

The followings are the immediate results of Definition 4.1 and Result 2.11.

Proposition 4.2. *Let $(X, \tau), (Y, \sigma)$ be an ITSs, $f : X \to Y$ be a mapping and let $p \in X$.*

(1) If f is continuous, then it is both σ-τ_I-continuous and σ-τ_{IV}-continuous.
(2) If σ_I-τ-continuous, then both σ_I-τ_I-continuous and σ_I-τ_{IV}-continuous.
(3) σ_{IV}-τ-continuous, then both σ_{IV}-τ_I-continuous and σ_{IV}-τ_{IV}-continuous.

The followings explain relationships among types of intutionistic continuities.

Example 4.3. (See Example 3.6 in [7]) (1) Let $X = \{a, b, c, d\}$ and consider ITs τ on X given by:

$$\tau = \{\phi_I, X_I, A_1, A_2, A_3, A_4\},$$

where

$$A_1 = (\{a, b\}, \{d\}),\ A_2 = (\{c\}, \{b, d\}),\ A_3 = (\phi, \{b, d\}),\ A_4 = (\{a, b, c\}, \{d\}).$$

Moreover,

$$\tau_I = \tau \bigcup \{A_i : i = 5, 6, \cdots, 23\},\ \tau_{IV} = \tau \cup \{A_{24}, A_{25}\},$$

where
$A_5 = (\{c\},\{b\}),\ A_6 = (\{c\},\{d\}),\ A_7 = (\{a,b\},\phi),\ A_8 = (\{a,b,c\},\phi),$
$A_9 = (\{c\},\phi),\ A_{10} = (\phi,\{a\}),\ A_{11} = (\phi,\{b\}),\ A_{12} = (\phi,\{c\}),$
$A_{13} = (\phi,\{d\}),\ A_{14} = (\phi,\{a,b\}),\ A_{15} = (\phi,\{a,c\}),\ A_{16} = (\phi,\{a,d\}),$
$A_{17} = (\phi,\{b,c\}),\ A_{18} = (\phi,\{c,d\}),\ A_{19} = (\phi,\{a,b,c\}),\ A_{20} = (\phi,\{a,b,d\}),$
$A_{21} = (\phi,\{a,c,d\}),\ A_{22} = (\phi,\{b,c,d\}),\ A_{23} = (\phi,\phi),$
$A_{24} = (\{a,c\},\{b,d\}),\ A_{25} = (\{a\},\{b,d\}).$

Let $Y = \{1,2,3,4,5\}$ and let us consider ITS (Y,σ) given by:

$$\sigma = \{\phi_I, X_I, B_1, B_2\},$$

where $B_1 = (\{1,2,3\},\{5\})$, $B_2 = (\{3\},\{4,5\})$. Then we can easily find τ_I and τ_{IV}:

$$\sigma_I = \sigma \cup \{B_3, B_4, B_5, B_6\} \cup \Im,$$

where $B_3 = (\{1,2,3\},\phi)$, $B_4 = (\{3\},\{4\})$,$B_5 = (\{3\},\{5\})$, $B_6 = (\{3\},\phi)$,
$\Im = \{(\phi, S) : S \subset Y\}$
and

$$\sigma_{IV} = \sigma \cup \{B_7, B_8, B_9, B_{10}, B_{11}, B_{12}, B_{13}, B_{14}, B_{15}, B_{16}, B_{17}, B_{18}\},$$

where $B_7 = (\{1,2,3,4\},\{5\})$, $B_8 = (\{1,3\},\{4,5\})$, $B_9 = (\{2,3\},\{4,5\})$,
$B_{10} = (\{1,2,3\},\{4,5\})$, $B_{11} = (\{1,3\},\{4\})$, $B_{12} = (\{2,3\},\{4\})$,
$B_{13} = (\{1,2,3\},\{4\})$, $B_{14} = (\{1,3\},\{5\})$, $B_{15} = (\{2,3\},\{5\})$,
$B_{16} = (\{1,2,3\},\{5\})$, $B_{17} = (\{1,2,3\},\phi)$, $B_{18} = (\{1,2,3,4\},\phi)$.

Now let $f : X \to Y$ be the mapping defined by:

$$f(a) = f(b) = 1, f(c) = 4, f(d) = 5.$$

(i) $f^{-1}(B_1) = A_1 \in \tau, f^{-1}(B_2) = A_{18} \in \tau_I$. Then f is not continuous but σ-τ_I-continuous.

(ii) We can easily see that $f^{-1}(U) \in \tau_I$, for each $U \in \sigma_I$. Then f is σ_I-τ_I-continuous.

(iii) $f^{-1}(B_1),\ f^{-1}(B_7) = (\{a,b,c\},\{d\} \notin \tau_{IV}$. Then f is neither σ-τ_{IV}-continuous nor σ_{IV}-τ_{IV}-continuous.

(iv)$f^{-1}(B_8) = (\{a\},\{c,d\}) \notin \tau_I$. Then f is not σ_{IV}-τ_I-continuous.

(2) Let $X = \{a,b,c,d\}, Y = \{1,2,3,4,5\}$ and consider ITs τ and σ on X and Y, respectively given by:
$\tau = \{\phi_I, X_I, A_1, A_2, A_3, A_4\}$
and
$\sigma = \{\phi_I, Y_I, B_1\},$
where $A_1 = (\{a,b\},\{d\}), A_2 = (\{b,d\},\{a,c\}), A_3 = (\{b\},\{a,c,d\}), A_4 = (\{a,b,d\},\phi)$ and $B_1 = (\{1,2\},\{3,4\})$.
Then
$\tau_I = \tau \cup \{A_i : i = 5, \cdots, 15\} \cup \Im_X$ and $\tau_{IV} = \tau \cup \{A_{17}\}$,
where $A_5 = (\{a,b\},\phi), A_6 = (\{b,d\},\phi), A_7 = (\{b,d\},\{a\}), A_8 = (\{b,d\},\{c\}),$
$A_9 = (\{b\},\phi), A_{10} = (\{b\},\{a\}), A_{11} = (\{b\},\{c\}), A_{12} = (\{b\},\{d\}),$
$A_{13} = (\{b\},\{a,c\}), A_{14} = (\{b\},\{a,d\}), A_{15} = (\{b\},\{c,d\}),$
$\Im_X = \{(\phi,S) : S \subset X\}, A_{17} = (\{a,b,c\},\{d\})$
and
$\sigma_I = \sigma \cup \{B_2, B_3, B_4\} \cup \Im_Y,$

$\sigma_{IV} = \sigma \cup \{B_5\}$,
where $B_2 = (\{1, 2\}, \phi), B_3 = (\{1, 2\}, \{3\}), B_4 = (\{1, 2\}, \{4\})$,
$\Im_Y = \{(\phi, S) : S \subset Y\}, B_5 = (\{1, 2, 5\}, \{3, 4\})$.

Let $g : X \to Y$ be the mapping defined by:

$$g(a) = 3, g(b) = 1, g(c) = 4, g(d) = 2.$$

(i) $g^{-1}(B_1) = A_2 \in \tau$. Then g is continuous.

(ii) $g^{-1}(B_2) = A_6,\ g^{-1}(B_3) = A_7,\ g^{-1}(B_4) = A_8 \in \tau_I$ but $g^{-1}(B_2) \notin \tau_{IV}$. Then g is σ_I-τ_I-continuous but not σ_I-τ_{IV}-continuous.

(iii) $g^{-1}(B_5) = A_2 \in \tau$ but $g^{-1}(B_5) \notin \tau_I$ and $g^{-1}(B_5) \notin \tau_{IV}$. Then g is σ_{IV}-τ-continuous but neither σ_{IV}-τ_I-continuous nor σ-τ_{IV}-continuous.

Theorem 4.4. *Let $(X, \tau), (Y, \sigma)$ be the ITSs. Then*

(1) *$f : (X, \tau) \to (Y, \sigma)$ is continuous if and only if $f : (X, [\ ]\tau) \to (Y, [\ ]\sigma)$ is continuous,*

(2) *$f : (X, \tau) \to (Y, \sigma)$ is continuous if and only if $f : (X, <>\tau) \to (Y, <>\sigma)$ is continuous.*

Proof. (1) Suppose $f : (X, \tau) \to (Y, \sigma)$ is continuous and let $(V_T, V_T^c) \in [\ ]\sigma$. Then by the definition of $[\ ]\sigma$, there is $V \in \sigma$ such that $[\ ]V = (V_T, V_T^c)$. Thus by the hypothesis, $f^{-1}(V) \in \tau$. So $[\ ]f^{-1}(V) = f^{-1}([\ ]V) \in [\ ]\tau$. Hence $f : (X, [\ ]\tau) \to (Y, [\ ]\sigma)$ is continuous.

Conversely, suppose $f : (X, [\ ]\tau) \to (Y, [\ ]\sigma)$ is continuous and let $V \in \sigma$. Then clearly, $[\ ]V \in [\ ]\sigma$. Thus by the hypothesis, $f^{-1}([\ ]V) = [\ ]f^{-1}(V) \in [\ ]\tau$. So $f^{-1}(V) \in \tau$. Hence $f : (X, \tau) \to (Y, \sigma)$.

(2) The proof is similar to (1). □

Proposition 4.5. *Let (X, τ) be the ITS such that $\tau \subset IS_*(X)$. Then $\tau = \tau_{IV}$ and $\tau = [\ ]\tau = <>\tau$.*

Proof. By Result 2.11, it is clear that $\tau \subset \tau_{IV}$. Let $G \in \tau_{IV}$. By Result 2.10, $G \in N(p_{IV})$, for each $p_{IV} \in G$. Then there exists $U_{p_{IV}} \in \tau$ such that $p_{IV} \in U_{p_{IV}} \subset G$. Since $\tau \subset IS_*(X)$, $p \in (U_{p_{IV}})_T$ and $p \notin (U_{p_{IV}})_F$. Thus

$$(U_{p_{IV}})_T = \bigcup_{p_{IV} \in G, p \in U_{p_{IV}})_T} \{p\} \text{ and } (U_{p_{IV}})_F = \bigcap_{p_{IV} \in G, p \notin U_{p_{IV}})_F} \{p\}^c.$$

So $G = \bigcup_{p_{IV} \in G} U_{p_{IV}} \in \tau$, i.e., $\tau_{IV} \subset \tau$. Hence $\tau = \tau_{IV}$.

The proof of second part is clear. □

The followings are the immediate results of Propositions 4.2 and 4.5.

Corollary 4.6. *Let (X, τ) be the ITS such that $\tau \subset IS_*(X)$, (Y, σ) be an ITS and let $f : X \to Y$ be a mapping. Then*

(1) *f is continuous if and only if it is σ-τ_{IV}-continuous,*

(2) *f is σ_I-τ-continuous if and only if it is σ_I-τ_{IV}-continuous,*

(3) *f is σ_{IV}-τ-continuous if and only if it is σ_{IV}-τ_{IV}-continuous.*

The followings are the immediate results of Propositions 4.2, 4.5 and Corollary 4.6.

Corollary 4.7. *Let $(X, \tau), (Y, \sigma)$ be the ITSs such that $\tau \subset IS_*(X)$, $\sigma \subset IS_*(Y)$ and let $f : X \to Y$ be a mapping. Then the followings are equivalent:*

(1) *f is continuous,*
(2) *f is σ-τ_{IV}-continuous,*
(3) *f is σ_{IV}-τ_{IV}-continuous.*

Definition 4.8. Let $(X, \tau), (Y, \sigma)$ be an ITSs and let $p \in Y$. Then a mapping $f : X \to Y$ is said to be:
(i) τ-σ-open, if it is open in the sense of Definition 3.6,
(ii)$'$ τ-σ-closed, if it is closed in the sense of Definition 3.6,
(ii) τ-σ_I-open, if $f(U) \in \sigma_I$, for each $U \in \tau$,
(ii)$'$ τ-σ_I-closed, if $f(F) \in IC_{\sigma_I}(Y)$, for each $F \in IC_\tau(X)$,
(iii) τ-σ_{IV}-open, if $f(U) \in \sigma_{IV}$, for each $U \in \tau$,
(iii)$'$ τ-σ_{IV}-closed, if $f(F) \in IC_{\sigma_{IV}}(Y)$, for each $F \in IC_\tau(X)$,
(iv) τ_I-σ-open, if $f(U) \in \sigma$, for each $U \in \tau_I$,
(iv)$'$ τ_I-σ-closed, if $f(F) \in IC_\sigma(Y)$, for each $F \in IC_{\tau_I}(X)$,
(v) τ_I-σ_I-open, if $f(U) \in \sigma_I$, for each $U \in \tau_I$,
(v)$'$ τ_I-σ_I-closed, if $f(F) \in IC_{\sigma_I}(Y)$, for each $F \in IC_{\tau_I}(X)$,
(vi) τ_I-σ_{IV}-open, if $f(U) \in \sigma_{IV}$, for each $U \in \tau_I$,
(vi)$'$ τ_I-σ_{IV}-closed, if $f(F) \in IC_{\sigma_{IV}}(Y)$, for each $F \in IC_{\tau_I}(X)$,
(vii) τ_{IV}-σ-open, if $f(U) \in \sigma$, for each $U \in \tau_{IV}$,
(vii)$'$ τ_V-closed, if $f(F) \in IC_\sigma(Y)$, for each $F \in IC_{\tau_{IV}}(X)$,
(viii) τ_{IV}-σ_I-open, if $f(U) \in \sigma_I$, for each $U \in \tau_{IV}$,
(viii)$'$ τ_{IV}-σ_I-closed, if $f(F) \in IC_{\sigma_I}(Y)$, for each $F \in IC_{\tau_{IV}}(X)$,
(ix) τ_{IV}-σ-open, if $f(U) \in \sigma$, for each $U \in \tau_{IV}$,
(ix)$'$ τ_{IV}-σ-closed, if $f(F) \in IC_\sigma(Y)$, for each $F \in IC_{\tau_{IV}}(X)$,
(x) τ_{IV}-σ_I-open, if $f(U) \in \sigma_I$, for each $U \in \tau_{IV}$,
(x)$'$ τ_{IV}-σ_I-closed, if $f(F) \in IC_{\sigma_I}(Y)$, for each $F \in IC_{\tau_{IV}}(X)$,
(xi) τ_{IV}-σ_{IV}-open, if $f(U) \in \sigma_{IV}$, for each $U \in \tau_{IV}$,
(xi)$'$ τ_{IV}-σ_{IV}-closed, if $f(F) \in IC_{\sigma_{IV}}(Y)$, for each $F \in IC_{\tau_{IV}}(X)$.

The followings are the immediate results of Definition 4.8, and Results 2.11 and 2.12.

Proposition 4.9. *Let $(X, \tau), (Y, \sigma)$ be an ITSs, $p \in Y$ and let $f : X \to Y$ be a mapping.*
(1) *If f is open, then it is both τ-σ_I-open and τ-σ_{IV}-open.*
(2) *If f is closed, then it is both τ-σ_I-closed and τ-σ_{IV}-closed.*
(3) *If f is τ_I-σ-open, then it is both τ_I-σ_I-open and τ_I-σ_{IV}-open.*
(4) *If f is τ_I-σ-closed, then it is both τ_I-σ_I-closed and τ_I-σ_{IV}-closed.*
(5) *If f is τ_{IV}-σ-open, then it is both τ_{IV}-σ_I-open and τ_{IV}-σ_{IV}-open.*
(6) *If f is τ_{IV}-σ-closed, then it is both τ_{IV}-σ_I-closed and τ_{IV}-σ_{IV}-closed.*

The followings explain relationships among types of intutionistic openness and closedness.

Example 4.10. Let $X = \{1, 2, 3, 4, 5\}$, $Y = \{a, b, c, d\}$ and consider ITs (X, τ) and σ on X and Y, respectively given by:

$$\tau = \{\phi_I, X_I, A_1, A_2, A_3, A_4\},\ \sigma = \{\phi_I, Y_I, B_1, B_2, B_3, B_4\},$$

where

$$A_1=(\{1,2,3\},\{5\}),\ A_2=(\{3\},\{4\}),\ A_3=(\{3\},\{4,5\}),\ A_4=(\{1,2,3\},\phi),$$

$$B_1=(\{a,b\},\{d\}),\ B_2=(\{b\},\{c\}),\ B_3=(\{b\},\{c,d\}),\ B_4=(\{a,b\},\phi).$$

Then clearly,

$$F_1=(\{5\},\{1,2,3\}),F_2=(\{4\},\{3\}),F_3=(\{4,5\},\{3\}),F_4=(\phi,\{1,2,3\})\in IC(X)$$

and

$$E_1=(\{d\},\{a,b\}),\ E_2=(\{c\},\{b\}),\ E_3=(\{c,d\},\{b\}),\ E_4=(\phi,\{a,b\})\in IC(Y).$$

Furthermore, $\tau_I=\tau\cup\{A_5,A_6\}\cup\Im_X$, $\tau_{IV}=\tau\cup\{A_7,\cdots,A_{18}\}$
and

$$\sigma_I=\sigma\cup\{B_5,B_6\}\cup\Im_Y,\ \sigma_{IV}=\sigma\cup\{B_7,\cdots,B_{13}\},$$

where $A_5=(\{3\},\phi),A_6=(\{3\},\{5\}),\Im_X=\{(\phi,S):S\subset X\}$,
$A_7=(\{1,2,3,4\},\{5\}),\ A_8=(\{1,3\},\{4\}),\ A_9=(\{2,3\},\{4\})$,
$A_{10}=(\{3,5\},\{4\}),\ A_{11}=(\{1,2,3\},\{4\}),\ A_{12}=(\{2,3,5\},\{4\})$,
$A_{13}=(\{1,2,3,5\},\{4\}),\ A_{14}=(\{1,3\},\{4,5\}),\ A_{15}=(\{2,3\},\{4,5\})$,
$A_{16}=(\{1,2,3\},\{4,5\}),\ A_{17}=(\{1,2,3,4\},\phi),\ A_{18}=(\{1,2,3,5\},\phi)$
and
$B_5=(\{b\},\phi),\ B_6=(\{b\},\{d\}),\ \Im_Y=\{(\phi,S):S\subset Y\}$,
$B_7=(\{a,b,c\},\{d\}),\ B_8=(\{a,b\},\{c\}),\ B_9=(\{b,d\},\{c\})$,
$B_{10}=(\{a,b,d\},\{c\}),\ B_{11}=(\{a,b\},\{c,d\}),\ B_{12}=(\{a,b,c\},\phi)$
$B_{13}=(\{a,b,d\},\phi)$.
Thus $IC_{\tau_I}(X)=IC(X)\cup\{F_5,F_6\}\cup\Im^c_X$, $IC_{\tau_{IV}}(X)=IC(X)\cup\{F_7,\cdots,F_{18}\}$
and

$$IC_{\sigma_I}(Y)=IC_Y\cup\{E_5,E_6\}\cup\Im^c_Y,\ IC_{\sigma_{IV}}(Y)=IC_Y\cup\{E_7,\cdots,E_{13}\},$$

where $F_5=(\phi,\{3\}),F_6=(\{5\},\{3\}),\Im^c_X=\{(S,\phi):S\subset X\}$,
$F_7=(\{5\},\{1,2,3,4\}),\ F_8=(\{4\},\{1,3\}),\ F_9=(\{4\},\{2,3\})$,
$F_{10}=(\{4\},\{3,5\},\ F_{11}=(\{4\},\{1,2,3\}),\ F_{12}=(\{4\},\{2,3,5\})$,
$F_{13}=(\{4\},\{1,2,3,5\}),\ F_{14}=(\{4,5\},\{1,3\}),\ F_{15}=(\{4,5\},\{2,3\})$,
$F_{16}=(\{4,5\},\{1,2,3\}),\ F_{17}=(\phi,\{1,2,3,4\}),\ F_{18}=(\phi,\{1,2,3,5\})$
and
$E_5=(\phi,\{b\}),\ E_6=(\{d\},\{b\}),\ \Im^c_Y=\{(S,\phi):S\subset Y\}$,
$E_7=(\{d\},\{a,b,c\}),\ E_8=(\{c\},\{a,b\}),\ E_9=(\{c\},\{b,d\})$,
$E_{10}=(\{c\},\{a,b,d\}),\ E_{11}=(\{c,d\},\{a,b\}),\ E_{12}=(\phi,\{a,b,c\})$,
$E_{13}=(\phi,\{a,b,d\})$.

Let $f,\ g,\ h:X\to Y$ be the mappings defined by:

$$f(1)=a,f(2)=f(3)=b,f(4)=c,f(5)=d,$$

$$g(1)=a,g(2)=g(5)=d,g(3)=b,g(4)=c,$$

$$h(1)=h(2)=a,h(3)=b,h(4)=c,h(5)=d.$$

Then we can easily check the followings:

(i) f is both open and τ_I-σ-closed but not closed; f is both τ_I-σ_I-open and τ_I-σ_I-open; f is τ_{IV}-σ_{IV}-open but not τ_{IV}-σ_{IV}-closed.

(ii) g is τ-σ_{IV}-open but neither open nor τ-σ_I-open; g is τ_I-σ_{IV}-open but neither τ_I-σ-open nor τ_I-σ_I-open; g is τ_{IV}-σ_{IV}-open but neither τ_{IV}-σ-open nor τ_{IV}-σ_I-open; g is both closed and τ_I-σ_I-closed but neither τ_I-σ-closed nor τ_I-σ_{IV}-closed; g is τ_{IV}-σ_{IV}-closed but neither τ_{IV}-σ-closed nor τ_{IV}-σ_I-closed.

(iii) h is both open and closed; h is both τ_I-σ_I-open and τ_I-σ_I-closed; h is both τ_{IV}-σ_{IV}-open and τ_{IV}-σ_{IV}-closed.

Example 4.11. Let $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c\}$ and consider ITs (X, τ) and σ on X and Y, respectively given by:

$$\tau = \{\phi_I, X_I, A_1, A_2, A_3, A_4\},\ \sigma = \{\phi_I, Y_I, B_1, B_2, B_3, B_4\},$$

where

$$A_1 = (\{1, 2\}, \{3\}),\ A_2 = (\{1, 4\}, \{3\}),\ A_3 = (\{1\}, \{2, 3\}),\ A_4 = (\{1, 2, 4\}, \{3\}),$$

$$B_1 = (\{a, b\}, \{c\}),\ B_2 = (\{b\}, \{a\}),\ B_3 = (\{b\}, \{a, c\}),\ B_4 = (\{a, b\}, \phi).$$

Then clearly,

$$F_1 = (\{3\}, \{1, 2\}), F_2 = (\{3\}, \{1, 4\}), F_3 = (\{2, 3\}, \{1\}), F_4 = (\{3\}, \{1, 2, 4\}) \in IC(X)$$

and

$$E_1 = (\{c\}, \{a, b\}),\ E_2 = (\{a\}, \{b\}),\ E_3 = (\{a, c\}, \{b\}),\ E_4 = (\phi, \{a, b\}) \in IC(Y).$$

Furthermore, $\tau_I = \tau \cup \{A_5, \cdots, A_{12}\} \cup \Im_X$, $\tau_{IV} = \tau \cup \{A_{13}\}$
and

$$\sigma_I = \sigma \cup \{B_5, B_6\} \cup \Im_Y,\ \sigma_{IV} = \sigma \cup \{B_7\},$$

where $A_5 = (\{1, 2\}, \phi), A_6 = (\{1, 4\}, \{2\}), A_7 = (\{1, 4\}, \{3\}),$
$A_8 = (\{1, 4\}, \phi),\ A_9 = (\{1\}, \{2\}), A_{10} = (\{1\}, \{3\}), A_{11} = (\{1\}, \phi),$
$A_{12} = (\{1, 2, 4\}, \phi), \Im_X = \{(\phi, S) : S \subset X\}, A_{13} = (\{1, 2, 4\}, \{3\})$
and
$B_5 = (\{b\}, \phi),\ B_6 = (\{b\}, \{c\}),\ \Im_Y = \{(\phi, S) : S \subset Y\}, B_7 = (\{a, b, c\}, \{d\}).$

Thus $IC_{\tau_I}(X) = IC(X) \cup \{F_5, \cdots, F_{12}\} \cup \Im^c_X$, $IC_{\tau_{IV}}(X) = IC(X) \cup \{F_{13}\}$
and

$$IC_{\sigma_I}(Y) = IC_Y \cup \{E_5, E_6\} \cup \Im^c_Y,\ IC_{\sigma_{IV}}(Y) = IC_Y \cup \{E_7\},$$

where $F_5 = (\phi, \{1, 2\}), F_6 = (\{2\}, \{1, 4\}), F_7 = (\{3\}, \{1, 4\})\ F_8 = (\phi, \{1, 4\}),$
$F_9 = (\{2\}, \{1\}), F_{10} = (\{3\}, \{1\},\ F_{11} = (\phi, \{1\}),\ F_{12} = (\phi, \{1, 2, 4\}),$
$\Im^c_X = \{(S, \phi) : S \subset X\}, F_{13} = (\{4\}, \{1, 2, 3, 5\})$
and
$E_5 = (\phi, \{b\}),\ E_6 = (\{c\}, \{b\}),\ \Im^c_Y = \{(S, \phi) : S \subset Y\},\ E_7 = (\{d\}, \{a, b, c\}).$

Let $f : X \to Y$ be the mappings defined by:

$$f(1) = f(2) = b, f(3) = f(4) = a.$$

Then we can easily check that:

f is τ-σ_I-open but neither τ-σ_I-closed nor open. In fact, f is neither the remainder's type open nor the remainder's type closed.

5. Intuitionistic subspaces

In this section, we introduce the notions of an intuitionistic subspace and the heredity, and obtain some properties of each concept.

Definition 5.1 ([6]). Let (X, τ) be an ITS.

(i) A subfamily β of τ is called an intutionistic base (in short, IB) for τ, if for each $A \in \tau$, $A = \phi_I$ or there exists $\beta^{'} \subset \beta$ such that $A = \bigcup \beta^{'}$.

(ii) A subfamily σ of τ is called an intutionistic subbase (in short, ISB) for τ, if the family $\beta = \{\bigcap \sigma^{'} : \sigma^{'}$ is a finite subset of $\sigma\}$ is a base for τ.

In this case, the IT τ is said to be generated by σ. In fact, $\tau = \{\phi_I\} \cup \{\bigcup \beta^{'} : \beta^{'} \subset \beta\}$.

Example 5.2. (1) ([6], Example 3.10) Let $\sigma = \{((a, b), (-\infty, a]) : a, b \in \mathbb{R}\}$ be the family of ISs in $\mathbb{R}$. Then σ generates an IT τ on $\mathbb{R}$, which is called the "usual left intuitionistic topology" on $\mathbb{R}$. In fact, the IB β for τ can be written in the form

$\beta = \{\mathbb{R}_I\} \cup \sigma$ and τ consists of the following ISs in $\mathbb{R}$:

$\phi_I, \mathbb{R}_I$;

$(\cup(a_j, b_j), (-\infty, c])$,

where $a_j, b_j, c \in \mathbb{R}$, $\{a_j : j \in J\}$ is bounded from below, $c < inf\{a_j : j \in J\}$;

$(\cup(a_j, b_j), \phi)$,

where $a_j, b_j \in \mathbb{R}$, $\{a_j : j \in J\}$ is not bounded from below.

Similarly, one can define the "usual right intuitionistic topology" on $\mathbb{R}$ using an analogue construction.

(2) ([6], Example 3.11) Consider the family σ of ISs in $\mathbb{R}$

$$\sigma = \{((a, b), (-\infty, a_1] \cup [b_1, \infty)) : a, b, a_1, b_1 \in \mathbb{R}, a_1 \leq a, b_1 \leq b\}.$$

Then σ generates an IT τ on $\mathbb{R}$, which is called the "usual intuitionistic topology" on $\mathbb{R}$. In fact, the IB β for τ can be written in the form $\beta = \{\mathbb{R}_I\} \cup \sigma$ and the elements of τ can be easily written down as in the above example.

(3) ([11], Example 3.10 (3)) Consider the family $\sigma_{[0,1]}$ of ISs in $\mathbb{R}$

$$\sigma_{[0,1]} = \{([a, b], (-\infty, a) \cup (b, \infty)) : a, b \in \mathbb{R} \text{ and } 0 \leq a \leq b \leq 1\}.$$

Then $\sigma_{[0,1]}$ generates an IT $\tau_{[0,1]}$ on $\mathbb{R}$, which is called the "usual unit closed interval intuitionistic topology" on $\mathbb{R}$. In fact, the IB $\beta_{[0,1]}$ for $\tau_{[0,1]}$ can be written in the form $\beta_{[0,1]} = \{\mathbb{R}\} \cup \sigma_{[0,1]}$ and the elements of τ can be easily written down as in the above example.

In this case, $([0, 1], \tau_{[0,1]})$ is called the "intuitionistic usual unit closed interval" and will be denoted by $[0, 1]_I$, where $[0, 1]_I = ([0, 1], (-\infty, 0) \cup (1, \infty))$.

Definition 5.3 ([11]). Let $a, b \in \mathbb{R}$ such that $a \leq b$. Then

(i) (the closed interval) $[a, b]_I = ([a, b], (-\infty, a) \cup (b, \infty))$,

(ii) (the open interval) $(a, b)_I = ((a, b), (-\infty, a] \cup [b, \infty))$,

(iii) (the half open interval or the half closed interval)

$(a, b]_I = ((a, b], (-\infty, a] \cup (b, \infty))$, $[a, b)_I = ([a, b), (-\infty, a) \cup [b, \infty))$,

(iv) (the half intuitionistic real line)

$(-\infty, a]_I = ((-\infty, a], (a, \infty))$, $(-\infty, a)_I = ((-\infty, a), [a, \infty))$,

$[a, \infty)_I = ([a, \infty), (-\infty, a))$, $(a, \infty)_I = ((a, \infty), (-\infty, a])$,

(v) (the intuitionistic real line) $(-\infty, \infty)_I = ((-\infty, \infty), \phi) = \mathbb{R}_I$.

Definition 5.4. Let (X, τ) be a ITS and let $A \in IS(X)$. Then the collection

$$\tau_A = \{U \cap A : U \in \tau\}$$

is called the subspace topology or relative topology on A.

Example 5.5. (1) Let $\tau = \{U \subset \mathbb{R} : 0_I \in U \text{ or } U = \phi_I\}$ and let

$$A = ([1, 2], ((-\infty, 1), (2, \infty)) \in IS(\mathbb{R}).$$

Then we can easily show that τ is an IT on $\mathbb{R}$ and τ_A is the subspace topology on A.

(2) Let $X = \{a, b, c, d\}$ be a set and consider the IT τ given by:

$$\tau = \{\phi_I, X_I, A_1, A_2, A_3, A_4\},$$

where $A_1 = (\{a, b\}, \{c\}), A_2 = (\{a, c\}, \{b, d\}), A_3 = (\{a\}, \{b, c, d\}), A_4 = (\{a, b, c\}, \phi)$.

Let $A = (\{a, d\}, \{b, c\})$. Then

$$\begin{aligned}\tau_A &= \{\phi_I \cap A, X_I \cap A, A_1 \cap A, A_2 \cap A, A_3 \cap A, A_4 \cap A\} \\ &= \{\phi_I, A, (\{a\}, \{b, c\}), (\{a\}, \{b, c, d\}), (\{a\}, \{d\})\}.\end{aligned}$$

(3) Let $(\mathbb{R}, \tau)$ be the usual intuitionistic topological space. Consider

$$A = ([0, 1], (-\infty, 0) \cup (1, \infty)) \in IS(\mathbb{R}).$$

Then $\tau_A = \tau_{[0,1]}$.

(4) Let τ be the usual intuitionistic topology on $\mathbb{R}$ and let $U \subset [0, 1]_I$ such that $0_I, 1_I \notin U$. Then $U \in \tau_{[0,1]}$ if and only if $U \in \tau$. Suppose $0 < b < 1$, for $b \in \mathbb{R}$. Consider $(-1, b)_I = ((-1, b), (-\infty, b] \cup [b, \infty))$ and $(b, 2)_I = ((b, 2), (-\infty, b] \cup [2, \infty))$. Then $(-1, b)_I \cap [0, 1]_I = [0, b)_I \in \tau_{[0,1]}$ and $(b, 2)_I \cap [0, 1]_I = (b, 1]_I \in \tau_{[0,1]}$. Thus

$$\beta = \{(a, b)_I : 0 < a < b < 1\} \cup \{[0, b)_I : 0 < b < 1\} \cup \{(b, 1]_I : 0 < b < 1\}$$

is a base for $\tau_{[0,1]}$.

(5) Let $\tau = \{U \subset IS(\mathbb{R}) : 0_I \in U \text{ or } U = \phi_I\}$. Then we can easily prove that τ is an IT on $\mathbb{R}$. Let $A = [1, 2]_I \in IS(\mathbb{R})$ and let $x_I, x_{IV} \in A$. Then clearly, $\{0_I, x_I, x_{IV}\} \in \tau$ and $\{0_I, x_I, x_{IV}\} \cap A = \{x_I, x_{IV}\} \in \tau_A$. Thus τ_A is the intuitionistic discrete topology.

The following is the immediate result of Definition 5.4.

Proposition 5.6. *Let (X, τ) be an ITS and let $A \in IS(X)$. Then τ_A is an IT on A.*

Definition 5.7. Let (X, τ) be a ITS, let $A \in IS(X)$ and let τ_A be the subspace topology on A. Then the pair (A, τ_A) is called a subspace of (X, τ) and each member of τ_A is called a relatively open set (in short, an open set in A).

Example 5.8. (1) Let $(\mathbb{R}, \tau)$ be the usual intuitionistic topological space. Then $tau_{\mathbb{Z}}$ is the intuitionistic discrete topology on $\mathbb{Z}$.

(2) If τ is the intuitionistic discrete topology on a set X and $A \in IS(X)$, then τ_A is the intuitionistic discrete topology on A.

(3) If τ is the intuitionistic indiscrete topology on a set X and $A \in IS(X)$, then τ_A is the intuitionistic indiscrete topology on A.

The followings are the immediate results of Definition 5.4.

Proposition 5.9. *Let (X, τ) be an ITS and let $A, B \in IS(X)$ such that $A \subset B$. Then $\tau_A = (\tau_B)_A$ where $(\tau_B)_A$ denotes the subspace topology on A by τ_B.*

Proposition 5.10. *Let (X, τ) be an ITS, let $A \in IS(X)$ and let β be a base for τ. Then $\beta_A = \{B \cap A : B \in \beta\}$ is a base for τ_A.*

Proposition 5.11. *Let (X, τ) be an ITS and let $A \in \tau$. If $U \in \tau_A$, then $U \in \tau$.*

Theorem 5.12. *Let (X, τ) be an ITS, let $A, B \in IS_*(X)$ such that $B \subset A$. Then B is closed in (A, τ_A) if and only if there exists $F \in IC(X)$ such that $B = A \cap F$.*

Proof. Suppose B is closed in (A, τ_A). Then $A - B \in \tau_A$. Thus there exists $U \in \tau$ such that $A - B = A \cap B^c = A \cap U$, i.e., $A_T \cap B_F = A_T \cap U_T$ and $A_F \cup B_T = A_F \cup U_F$. Since $B \subset A$ and $A, B \in IS_*(X)$, we have $B_T = A_T \cap U_F$ and $B_F = A_F \cup U_T$, i.e., $B = A \cap U^c$. Since $U \in \tau$, $U^c \in IC(X)$. So B is closed in A.

Conversely, suppose there exists $F \in IC(X)$ such that $B = A \cap F$. Then $F^c \in \tau$. Since $A, B \in IS_*(X)$, it is clear that $A - B = A \cap F^c$. Thus $A - B \in \tau_A$. So B is closed in A. □

The following is the immediate result of Theorem 5.12.

Corollary 5.13. *Let (X, τ) be an ITS such that $\tau \subset IS_*(X)$, let $A \in IC(X)$ and let $B \in IS_*(X)$. If B is closed in A, then $B \in IC(X)$.*

Proposition 5.14. *Let (X, τ) be an ITS such that $\tau \subset IS_*(X)$, let $A, B \in IS_*(X)$ such that $B \subset A$. Then $cl_{\tau_A}(B) = A \cap Icl(B)$, where $cl_{\tau_A}(B)$ denotes the closure of B in (A, τ_A).*

Proof. Since $Icl(B) \in IC(X)$, $A \cap Icl(B)$ is closed in (A, τ_A). Since $B \subset A \cap Icl(B)$ and $cl_{\tau_A}(B) = \bigcap\{F : F \text{ is closed in } A \text{ and } B \subset F\}$, $cl_{\tau_A}(B) \subset A \cap Icl(B)$.

On the other hand, $cl_{\tau_A}(B)$ is closed in A. Then by Theorem 5.12, there exists $F \in IC(X)$ such that $cl_{\tau_A}(B) = A \cap F$. Since $B \subset cl_{\tau_A}(B)$, $B \subset F$. Thus $Icl(B) \subset F$. So $A \cap Icl(B) \subset A \cap F$. Hence $A \cap Icl(B) \subset cl_{\tau_A}(B)$. Therefore $cl_{\tau_A}(B) = A \cap Icl(B)$. □

Theorem 5.15. *Let (X, τ) be an ITS, let $A, U \in IS(X)$ such that $A \subset U$ and let $a \in X$.*

(1) *If $a_I \in A$, then $U \in N_{\tau_A}(a_I)$ if and only if there exists $V \in N(a_I)$ such that $U = A \cap V$, where $N_{\tau_A}(a_I)$ denotes the set of all neighborhoods of a_I in (A, τ_A).*

(2) *If $a_{IV} \in A$, then $U \in N_{\tau_A}(a_{IV})$ if and only if there exists $V \in N(a_{IV})$ such that $U = A \cap V$, where $N_{\tau_A}(a_V)$ denotes the set of all neighborhoods of a_{IV} in (A, τ_A).*

Proof. Suppose $U \in N_{\tau_A}(a_I)$. Then there exists $G \in \tau_A$ such that $a_I \in G \subset U$. Since $G \in \tau_A$, there exists $H \in \tau$ such that $G = A \cap H$. Let $V = U \cup H$. Then clearly, $a_I \in H \subset V$. Thus $V \in N(a_I)$. Since $G = A \cap H$, $U = A \cap V$. So the necessary condition holds.

The proof of the converse is easy.

(2) The proof is similar. □

Proposition 5.16. *Let $(X, \tau), (Y, \sigma)$ be ITSs and let $A \in IS(X), B \in IS(Y)$.*

(1) *The inclusion mapping $i : A \to X$ is continuous.*

(2) *If $f : X \to Y$ is continuous, then $f|_A : A \to Y$ is continuous.*
(3) *If $f : X \to B$ is continuous, then the mapping $g : X \to Y$ defined by $g(x) = f(x)$, for each $x \in X$ is continuous.*
(4) *If $f : X \to Y$ is continuous and $f(X_I) \subset B$, then the mapping $g : X \to B$ defined by $g(x) = f(x)$, for each $x \in X$ is continuous.*

Proof. (1) Let $U \in \tau$. Then clearly, $A \cap U \in \tau_A$ and $i^{-1}(U) = A \cap U$. Thus i is continuous.
(2) Let $U \in \sigma$. Then clearly, $f^{-1}(U) \in \tau$. Thus $A \cap f^{-1}(U) \in \tau_A$ and $(f|_A)^{-1}(U) = A \cap f^{-1}(U)$. Thus $(f|_A)^{-1}(U) \in \tau_A$. So $f|_A$ is continuous.
(3) Let $U \in \sigma$. Then clearly, $B \cap U \in \sigma_B$. Since $f : X \to B$ is continuous, $f^{-1}(B \cap U) = f^{-1}(U) \in \tau$. Since $g(x) = f(x)$, for each $x \in X$, $g^{-1}(U) = f^{-1}(U)$. Thus $g^{-1}(U) \in \tau$. So g is continuous.
(4) Let $U \in \sigma_B$. Then there is $V \in \sigma$ such that $U = B \cap V$. Since $f : X \to Y$ is continuous, $f^{-1}(V) \in \tau$. On the other hand,

$$g^{-1}(U) = g^{-1}(B) \cap g^{-1}(V) = X \cap f^{-1}(V) = f^{-1}(V).$$

Thus $g^{-1}(U) \in \tau$. So g is continuous. □

Proposition 5.17. *Let X, Y be ITSs, let $f : X \to Y$ be a mapping, let $\{U_j : j \in J\} \subset IO(X)$ such that $X_I = \bigcup_{j \in J} U_j$ and let $f|_{U_j} : U_j \to Y$ is continuous, for each $j \in J$. Then so is f.*

Proof. Let $V \in IO(Y)$ and let $j \in J$. Then by the hypothesis, $(f|_{U_j})^{-1}(V) \in IO(U_j)$. Since $U_j \in IO(X)$, by Proposition 5.16 (2), $(f|_{U_j})^{-1}(V) \in IO(X)$. Thus $f^{-1}(V) = \bigcup_{j \in J} (f|_{U_j})^{-1}(V) \in IO(X)$. So f is continuous. □

Proposition 5.18. *Let (X, τ) be an ITS such that $\tau \subset IS_*(X)$, let (Y, σ) be an ITS, let $A, B \in IC(X)$ such that $X_I = A \cup B$ and let $f : A \to Y$, $g : B \to Y$ be continuous such that $f(x) = g(x)$, for each $x \in A_T \cap B_T$. Define $h : X \to Y$ as follows:*

$$h(x) = f(x), \forall x \in A_T \text{ and } h(x) = g(x), \forall x \in B_T.$$

Then h is continuous.

Proof. Let $F \in IC(Y)$. Since $f : A \to Y$ and $g : B \to Y$ are continuous, by Result 3.3, $f^{-1}(F)$ is closed in A and $g^{-1}(F)$ is closed in B. Since $A, B \in IC(X)$, by Corollary 5.13, $f^{-1}(F), g^{-1}(F) \in IC(X)$. On the other hand, $h^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$. Then $h^{-1}(F) \in IC(X)$. Thus by Result 3.3, h is continuous. □

Definition 5.19. An intuitionistic topological property P is said to be hereditary if every subspace of an ITS with P also has P.

For separation axioms in intuitionistic topological spaces, see [3, 12].

Proposition 5.20. (1) *$T_0(i)$ is hereditary, i.e., every subspace of a $T_0(i)$-space is $T_0(i)$.*
(2) *$T_1(i)$ is hereditary, i.e., every subspace of a $T_1(i)$-space is $T_1(i)$.*
(3) *$T_2(i)$ is hereditary, i.e., every subspace of a $T_2(i)$-space is $T_2(i)$.*

Proof. Let (X, τ) be an ITS and let $A \in IS(X)$.

(1) Suppose (X, τ) is $\mathrm{T}_0(i)$ and let $x_I \neq y_I \in A$. Then clearly, $x \neq y \in X$. Thus by the hypothesis, there exists $U \in \tau$ such that $x_I \in U, y_I \notin U$ or $x_I \notin U, y_I \in U$. Let $V = A \cap U$. Then clearly, $V \in \tau_A$. Moreover, $x_I \in V, y_I \notin V$ or $x_I \notin V, y_I \in V$. Thus (A, τ_A) is $\mathrm{T}_0(i)$.

(2) Suppose (X, τ) is $\mathrm{T}_1(i)$ and let $x_I \neq y_I \in A$. Then clearly, $x \neq y \in X$. Thus by the hypothesis, there exists $G, H \in \tau$ such that $x_I \in G, y_I \notin G$ and $x_I \notin H, y_I \in H$. Let $U = A \cap G$ and let $V = A \cap H$. Then clearly, $U, V \in \tau_A$. Moreover, $x_I \in U, y_I \notin U$ or $x_I \notin V, y_I \in V$. Thus (A, τ_A) is $\mathrm{T}_1(i)$.

(3) Suppose (X, τ) is $\mathrm{T}_2(i)$ and let $x_I \neq y_I \in A$. Then clearly, $x \neq y \in X$. Thus by the hypothesis, there exists $G, H \in \tau$ such that $x_I \in G, y_I \in H$ and $G \cap H = \phi_I$. Let $U = A \cap G$ and let $V = A \cap H$. Then clearly, $U, V \in \tau_A$. Since $G \cap H = \phi_I$, $U \cap V = \phi_I$. Moreover, $x_I \in U$ and $y_I \in V$. So (A, τ_A) is $\mathrm{T}_2(i)$. □

Proposition 5.21. *Let (X, τ) be an ITS such that $\tau \subset IS_*(X)$.*

(1) *$T_3(i)$ is hereditary, i.e., every subspace of a $T_3(i)$-space is $T_3(i)$.*

(2) *An intuitionistic complete regularity is hereditary, i.e., every subspace of intuitionistic complete regular space is intuitionistic complete regular.*

Proof. (1) Suppose (X, τ) be $\mathrm{T}_3(i)$ and let $A \in IS_*(X)$. Since (X, τ) is $\mathrm{T}_1(i)$, by Proposition 5.20 (2), (A, τ_A) is $\mathrm{T}_1(i)$. Let B be closed in (A, τ_A) such that $x_I \in B^c$. Then by Theorem 5.12, there exists $F \in IC(X)$ such that $B = A \cap F$. Since $x_I \in B^c$, $x_I \in F^c$. Thus by hypothesis, there exist $U, V \in \tau$ such that $F \subset U$, $x_I \in V$ and $U \cap V = \phi_I$. So $A \cap U, A \cap V \in \tau_A$ and $(A \cap U) \cap (A \cap V) == \phi_I$. Moreover, $F \subset A \cap U$ and $x_I \in A \cap V$. Hence (A, τ_A) is $\mathrm{T}_3(i)$.

(2) Suppose (X, τ) be an intuitionistic complete regular space and let $A \in IS_*(X)$. Since (X, τ) is $\mathrm{T}_1(i)$, by Proposition 5.20 (2), (A, τ_A) is $\mathrm{T}_1(i)$. Let B be closed in A such that $x_I \in B^c$. Then by Theorem 5.12, there exists $F \in IC(X)$ such that $B = A \cap F$. Since $x_I \in B^c$, $x_I \in F^c$. Thus by the hypothesis, there exists a continuous mapping $f : X \to [0, 1]_I$ such that $f(x_I) = 1_I$ and $f(y_I) = 0_I$, for each $y_I \in F$. Since $f : X \to [0, 1]_I$ is continuous, by Proposition 5.16 (2), $f|_A : A \to [0, 1]_I$ is continuous. Let $y_I \in B$. Since $B = A \cap F$, $y_I \in F$. So $f|_A(y_I) = f(y_I) = 0_I$. Moreover, $f|_A(x_I) = f(x_I) = 1_I$. Hence (A, τ_A) is intuitionistic complete regular. □

Proposition 5.22. *Let (X, τ) be an ITS such that $\tau \subset IS_*(X)$ and let $A \in IC(X)$. If (X, τ) is $T_4(i)$, then (A, τ_A) is $T_4(i)$.*

Proof. Suppose (X, τ) is $\mathrm{T}_4(i)$ and let $A \in IC(X)$. Since (X, τ) is $\mathrm{T}_1(i)$, by Proposition 5.20 (2), (A, τ_A) is $\mathrm{T}_1(i)$. Let B and C be closed in A such that $B \cap C = \phi_I$. Then by Theorem 5.12, there exists $F_1, F_2 \in IC(X)$ such that $B = A \cap F_1$ and $C = A \cap F_2$. Since $A \in IC(X)$, $B, C \in IC(X)$. Thus by the hypothesis, $U, V \in \tau$ such that $B \subset U$, $C \subset V$ and $U \cap V = \phi_I$. So $A \cap U, A \cap V \in \tau_A$ and $(A \cap U) \cap (A \cap V) = \phi_I$. Moreover, $B \subset A \cap U$ and $C \subset A \cap V$. Hence (A, τ_A) is $\mathrm{T}_4(i)$. □

6. Conclusions

In this paper, we mainly dealt with some properties of quotient mappings, various types of continuities, open and closed mappings in intuitionistic topological spaces.

In particular, we defined continuities, open and closed mappings under the global sense but did not define them under the local sense.

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References

[1] K. Atanassov, Intuitionistic fuzzy sets, VII ITKR's Session, Sofia (September, 1983) (in Bugaria).
[2] C. Bavithra, M. K. Uma and E. Roja, Feeble compactness of intuitionistic fell topological space, Ann. Fuzzy Math. Inform. 11 (3) (1016) 485–494.
[3] Sadik Bayhan and D. Coker, On separation axioms in intuitionistic topological spaces, IJMMS 27 (10) (2001) 621–630.
[4] Sadik Bayhan and D. Coker, Pairwise separation axioms in intuitionistic topological spaces, Hacettepe Journal of Mathematics and Statistics 34 S (2005) 101–114.
[5] D. Coker A note on intuitionistic sets and intuitionistic points, Tr. J. of Mathematics 20 (1996) 343–351.
[6] D. Coker An introduction to intuitionistic topological spaces, VUSEFAL 81 (2000) 51–56.
[7] E. Coskun and D. Coker On neighborhood structures in intuitionistic topological spaces, Math. Balkanica (N. S.) 12 (3–4) (1998) 283–909.
[8] Taha H. Jassim Completely normal and weak completely normal in intuitionistic topological spaces, International Journal of Scientific and Engineering Research 4 (10) (2013) 438–442.
[9] J. C. Kelly, Bitopological spaces, Proc. London Math. Soc. 13 (1963) 71-0-89.
[10] J. H. Kim, P. K. Lim, J. G. Lee, K. Hur, The category of intutionistic sets, To be submitted.
[11] J. H. Kim, P. K. Lim, J. G. Lee, K. Hur, Intuitionistic topological spaces, To be submitted.
[12] J. H. Kim, P. K. Lim, J. G. Lee, K. Hur, Separation axioms in intuitionistic topological spaces, To be submitted.
[13] J. G. Lee, P. K. Lim, J. H. Kim, K. Hur, Intuitionistic limit points and nets, To be submitted.
[14] J. G. Lee, P. K. Lim, J. H. Kim, K. Hur, Intuitionistic relations, To be submitted.
[15] S. J. Lee and J. M. Chu, Categorical properties of intuitinistic topological spaces, Commun. Korean Math. Soc. 24 (4) (2009) 595–603.
[16] Ahmet Z. Ozcelik and Serkan Narli, On submaximality in intuitionistic topological spaces, International Scholarly and Scientific Research and Innovation 1 (1) (2007) 64–66.
[17] A. A. Salama, Mohamed Abdelfattah and S. A. Alblowi, Some Intuitionistic Topological Notions of Intuitionistic Region, Possible Application to GIS Topological Rules, International Journal of Enhanced Research in Management and Computer Applications 3 (5) (2014) 4–9.
[18] S. Selvanayaki and Gnanambal Ilango, IGPR-continuity and compactness intuitionistic topological spaces, British Journal of Mathematics and Computer Science 11 (2) (2015) 1–8.
[19] S. Selvanayaki and Gnanambal Ilango, Homeomorphism on intuitionistic topological spaces, Ann. Fuzzy Math. Inform. 11 (6) (2016) 957–966.

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