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A novel generalization of uni-soft subalgebras and ideals in BCK/BCI-algebras

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Abstract. The notions of uni-soft subalgebra/ideal with thresholds are introduced, and related properties are investigated. Relations between uni-soft subalgebra/ideal and uni-soft subalgebra/ideal with thresholds are discussed. Characterizations of uni-soft subalgebra/ideal with thresholds are considered. Conditions for a soft set to be a uni-soft ideal with thresholds are provided. Conditions for a uni-soft ideal with thresholds to be a uni-soft subalgebra with thresholds are also provided.

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1. Introduction

Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [20]. The approach to uncertainty, which is an attribute of information, is outlined by Zadeh [22]. Molodtsov [18] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [16] described the application of soft set theory to a decision making problem. Maji et al. [15] also studied several operations on the theory of soft sets. Chen et al. [5] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. Many algebraic properties of soft sets are studied in [1, 2, 3, 6, 8, 10, 11, 12, 13, 14, 19, 23]. In order to lay a foundation for providing a soft algebraic tool in considering many problems that contain uncertainties, Jun [9] introduced the notion of uni-soft sets, and considered its application to BCK/BCI-algebras. He introduced the notions of uni-soft algebras, uni-soft (commutative) ideals and closed uni-soft ideals. He displayed several examples, and investigated related properties and relations. He provided conditions for a uni-soft ideal to be closed, and gave conditions for a uni-soft ideal to be a uni-soft commutative ideal, and discussed characterizations of (closed) uni-soft ideals and uni-soft commutative ideals. He established extension property for a union-soft commutative ideal.

The aim of this article is to consider a generalization of Jun's results in [9]. We introduce the notions of uni-soft subalgebra/ideal with thresholds, and investigate related properties. We discuss relations between uni-soft subalgebra/ideal and uni-soft subalgebra/ideal with thresholds, and consider characterizations of uni-soft subalgebra/ideal with thresholds. We provide conditions for a soft set to be a uni-soft ideal with thresholds. We also provide conditions for a uni-soft ideal with thresholds to be a uni-soft subalgebra with thresholds.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x,y,z\in X)\ (((x*y)*(x*z))*(z*y)=0)$,
(II) $(\forall x,y\in X)\ ((x*(x*y))*y=0)$,
(III) $(\forall x\in X)\ (x*x=0)$,
(IV) $(\forall x,y\in X)\ (x*y=0, y*x=0\ \Rightarrow\ x=y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x\in X)\ (0*x=0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following axioms:

(a1) $(\forall x\in X)\ (x*0=x)$,
(a2) $(\forall x,y,z\in X)\ (x\le y\ \Rightarrow\ x*z\le y*z,\ z*y\le z*x)$,
(a3) $(\forall x,y,z\in X)\ ((x*y)*z=(x*z)*y)$,
(a4) $(\forall x,y,z\in X)\ ((x*z)*(y*z)\le x*y)$,

where $x\le y$ if and only if $x*y=0$. In a BCI-algebra X, the following hold:

(b1) $(\forall x,y\in X)\,(x*(x*(x*y))=x*y)$,
(b2) $(\forall x,y\in X)\,(0*(x*y)=(0*x)*(0*y))$.

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x*y\in S$ for all $x,y\in S$. A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

(2.1) $0\in A$,

(2.2) $(\forall x\in X)\,(\forall y\in A)\,(x*y\in A\ \Rightarrow\ x\in A)$.

We refer the reader to the books [7, 17] for further information regarding BCK/BCI-algebras.

A soft set theory is introduced by Molodtsov [18], and Çağman et al. [4] provided new definitions and various results on soft set theory.

In what follows, let U be an initial universe set and E be a set of parameters. We say that the pair (U, E) is a *soft universe*. Let $\mathcal{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$.

Definition 2.1 ([4, 18])**.** A soft set $\mathscr{F}_A$ over U is defined to be the set of ordered pairs

$$\mathscr{F}_A := \{(x, f_A(x)) : x \in E,\ f_A(x) \in \mathcal{P}(U)\},$$

where $f_A : E \to \mathcal{P}(U)$ such that $f_A(x) = \varnothing$ if $x \notin A$.

The function f_A is called the approximate function of the soft set $\mathscr{F}_A$. The subscript A in the notation f_A indicates that f_A is the approximate function of $\mathscr{F}_A$.

In what follows, denote by $S(U)$ the set of all soft sets over U by Çağman et al. [4].

Let U denote an initial universe set and assume that E, a set of parameters, has a binary operation.

Definition 2.2 ([9])**.** Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then $\mathscr{F}_A$ is called a *uni-soft subalgebra* over U if the approximate function f_A of $\mathscr{F}_A$ satisfies:

$$(\forall x, y \in A)\,(f_A(x * y) \subseteq f_A(x) \cup f_A(y)). \tag{2.3}$$

Definition 2.3 ([9])**.** Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A \in S(U)$. Then $\mathscr{F}_A$ is called a *uni-soft ideal* over U if the approximate function f_A of $\mathscr{F}_A$ satisfies:

$$(\forall x \in A)\,(f_A(0) \subseteq f_A(x)), \tag{2.4}$$

$$(\forall x, y \in A)\,(f_A(x) \subseteq f_A(x * y) \cup f_A(y)). \tag{2.5}$$

3. Uni-soft subalgebras/ideals with thresholds

For any $P, Q \in \mathcal{P}(U)$ with $P \subseteq Q$, we define an inclusion with thresholds (P, Q), denoted by "$\sqsubseteq_{(P,Q)}$" on $\mathcal{P}(U)$ as follows:

$$(\forall X, Y \in \mathcal{P}(U))\left(X \sqsubseteq_{(P,Q)} Y \iff X \cap Q \subseteq Y \cup P\right). \tag{3.1}$$

We define an equality with thresholds (P, Q), denoted by "$=_{(P,Q)}$", as follows:

$$(\forall X, Y \in \mathcal{P}(U))\left(X =_{(P,Q)} Y \iff X \sqsubseteq_{(P,Q)} Y \ \&\ Y \sqsubseteq_{(P,Q)} X\right). \tag{3.2}$$

Definition 3.1. Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra and let $f_E \in S(U)$. Then f_E is called a *uni-soft subalgebra* over U with thresholds (P, Q) if

$$(\forall x, y \in E)\left(f_E(x * y) \sqsubseteq_{(P,Q)} f_E(x) \cup f_E(y)\right). \tag{3.3}$$

Example 3.2. Consider a BCK-algebra $E = \{0, 1, 2, 3, 4\}$ with the Cayley table which is given in Table 1.
Take $U = E$ and define a soft set f_E over U as follows:

$$f_E : E \to \mathcal{P}(U),\ x \mapsto \{a \in E \mid a * x = 0\}.$$

Table 1. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Then $f_E(0) = \{0\}$, $f_E(1) = \{0, 1\}$, $f_E(2) = \{0, 1, 2\}$, $f_E(3) = \{0, 3\}$ and $f_E(4) = \{0, 4\}$. By routine calculations, we know that f_E is a uni-soft subalgebra over U with thresholds (P, Q) with $P = \{0, 2\}$ and $Q = \{0, 2, 4\}$.

Example 3.3. Consider a BCK-algebra $E = \{0, 1, 2, 3, 4\}$ with the Cayley table which is given in Table 2.

Table 2. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	3	2	0

Take $U = E$ and define a soft set g_E over U by $g_E(0) = \{0\}$, $g_E(1) = \{0, 1\}$, $g_E(2) = \{0, 4\}$, $g_E(3) = \{0, 1, 2, 3\}$ and $g_E(4) = \{0, 1\}$. Then g_E is not a uni-soft subalgebra with thresholds (P, Q) for $P = \{3\}$ and $Q = \{1, 2, 3\}$ since

$$g_E(4 * 2) \cap \{1, 2, 3\} = g_E(3) \cap \{1, 2, 3\} = \{1, 2, 3\} \nsubseteq \{0, 1, 3, 4\} = g_E(4) \cup g_E(2) \cup \{3\}.$$

Theorem 3.4. *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra and let $P, Q \in \mathcal{P}(U)$ with $P \subseteq Q$. Then every uni-soft subalgebra over U is a uni-soft subalgebra over U with thresholds (P, Q).*

Proof. If f_E is a uni-soft subalgebra over U, then

$$f_E(x * y) \cap Q \subseteq f_E(x * y) \subseteq f_E(x) \cup f_E(y) \subseteq f_E(x) \cup f_E(y) \cup P,$$

that is, $f_E(x * y) \sqsubseteq_{(P,Q)} f_E(x) \cup f_E(y)$ for all $x, y \in E$. Hence f_E is a uni-soft subalgebra over U with thresholds (P, Q). □

It is clear that every uni-soft subalgebra f_E over U with thresholds $(\varnothing, U)$ is a uni-soft subalgebra over U.

Proposition 3.5. *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. If f_E is a uni-soft subalgebra over U with thresholds (P, Q), then*

$$(\forall x \in E) \left(f_E(0) \sqsubseteq_{(P,Q)} f_E(x) \right). \tag{3.4}$$

Proof. For any $x \in E$, we have

$$f_E(0) \cap Q = f_E(x * x) \cap Q \subseteq f_E(x) \cup f_E(x) \cup P = f_E(x) \cup P,$$

that is, $f_E(0) \sqsubseteq_{(P,Q)} f_E(x)$ for all $x \in E$. □

If we take $(P, Q) = (\varnothing, U)$ in Proposition 3.5, then we have the following corollary.

Corollary 3.6 ([9]). *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. If f_E is a uni-soft subalgebra over U, then*

$$(\forall x \in E)\,(f_E(0) \subseteq f_E(x))\,. \tag{3.5}$$

Proposition 3.7. *Let $(U, E) = (U, X)$ where X is a BCI-algebra. If f_E is a uni-soft subalgebra over U with thresholds (P, Q) satisfying the following condition*

$$(\forall x \in E)\,(f_E(x) \subseteq Q)\,, \tag{3.6}$$

then $f_E(x * (0 * y)) \sqsubseteq_{(P,Q)} f_E(x) \cup f_E(y)$ for all $x, y \in E$.

Proof. Using (3.3), (3.6) and Proposition 3.5, we have

$$\begin{aligned} f_E(x * (0 * y)) \cap Q &\subseteq f_E(x) \cup f_E(0 * y) \cup P \\ &= f_E(x) \cup (f_E(0 * y) \cap Q)) \cup P \\ &\subseteq f_E(x) \cup (f_E(0) \cup f_E(y) \cup P) \cup P \\ &= f_E(x) \cup (f_E(0) \cap Q) \cup f_E(y) \cup P \\ &\subseteq f_E(x) \cup f_E(y) \cup P, \end{aligned}$$

and so $f_E(x * (0 * y)) \sqsubseteq_{(P,Q)} f_E(x) \cup f_E(y)$ for all $x, y \in E$. □

Proposition 3.8. *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. Given a soft set f_E over U, we have*

$$(\forall x, y \in E)\,\big(f_E(x * y) \sqsubseteq_{(P,Q)} f_E(y) \Longrightarrow f_E(x) \sqsubseteq_{(P,Q)} f_E(0)\big)\,. \tag{3.7}$$

Proof. It is by taking $y = 0$ and using (a1). □

Proposition 3.9. *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. If f_E is a uni-soft subalgebra over U with thresholds (P, Q), then*

$$(\forall x \in E)\,\big(f_E(0) \cup P \sqsubseteq_{(P,Q)} f_E(x)\big)\,. \tag{3.8}$$

Proof. For any $x \in E$, we have

$$\begin{aligned} (f_E(0) \cup P) \cap Q &= (f_E(0) \cap Q) \cup (P \cap Q) \\ &= (f_E(0) \cap Q) \cup P \\ &= (f_E(x * x) \cap Q) \cup P \\ &\subseteq ((f_E(x) \cup f_E(x) \cup P)) \cup P \\ &= f_E(x) \cup P. \end{aligned}$$

Hence $f_E(0) \cup P \sqsubseteq_{(P,Q)} f_E(x)$ for all $x \in E$. □

Given a soft set f_E over U and a subset R of U with $R \subseteq Q$, we consider a set:

$$e(f_E; R) := \{x \in E \mid f_E(x) \subseteq P \cup R\}.$$

Theorem 3.10. *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. If a soft set f_E over U satisfies (3.6), then the following are equivalent.*

(1) *f_E is a uni-soft subalgebra over U with thresholds (P, Q).*
(2) *The set $e(f_E; R)$ is a subalgebra of E for all $R \subseteq Q$.*

Proof. Assume that f_E is a uni-soft subalgebra over U with thresholds (P, Q). Let $x, y \in e(f_E; R)$ for every $R \subseteq Q$. Then $f_E(x) \subseteq P \cup R$ and $f_E(y) \subseteq P \cup R$. It follows from (3.6) and (3.3) that

$$f_E(x * y) = f_E(x * y) \cap Q \subseteq f_E(x) \cup f_E(y) \cup P \subseteq P \cup R.$$

Hence $x * y \in e(f_E; R)$, and therefore $e(f_E; R)$ is a subalgebra of E for all $R \subseteq Q$.

Conversely, suppose that (2) is valid. Let $x, y \in E$ be such that $f_E(x) = R_1$ and $f_E(y) = R_2$. Then

$$f_E(x) = R_1 \subseteq P \cup R_1 \subseteq P \cup R,$$
$$f_E(y) = R_2 \subseteq P \cup R_2 \subseteq P \cup R,$$

where $R = R_1 \cup R_2 \subseteq Q$. Hence $x, y \in e(f_E; R)$, and so $x * y \in e(f_E; R)$ since $e(f_E; R)$ is a subalgebra of E. It follows from (3.6)

$$f_E(x * y) \cap Q = f_E(x * y) \subseteq P \cup R = R_1 \cup R_2 \cup P = f_E(x) \cup f_E(y) \cup P,$$

that is, $f_E(x * y) \sqsubseteq_{(P,Q)} f_E(x) \cup f_E(y)$ for all $x, y \in E$. Therefore f_E is a uni-soft subalgebra over U with thresholds (P, Q). □

Corollary 3.11 ([9]). *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra and let $f_E \in S(U)$. Then f_E is a uni-soft subalgebra over U if and only if the set*

$$e(f_E; \tau) = \{x \in E \mid f_E(x) \subseteq \tau\}$$

is a subalgebra of E for all $\tau \in \mathcal{P}(U)$.

Proof. It is by taking $(P, Q) = (\varnothing, U)$ and $R = \tau$ in Theorem 3.10. □

Definition 3.12. Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. A soft set f_E over U is called a *uni-soft ideal* over U with thresholds (P, Q) if

$$(\forall x \in E) \left(f_E(0) \sqsubseteq_{(P,Q)} f_E(x)\right), \tag{3.9}$$

$$(\forall x, y \in E) \left(f_E(x) \sqsubseteq_{(P,Q)} f_E(x * y) \cup f_E(y)\right). \tag{3.10}$$

It is clear that every uni-soft ideal over U with thresholds $(\varnothing, U)$ is a uni-soft ideal over U.

Example 3.13. Consider a BCI-algebra $E = \{0, 1, 2, a, b\}$ with the Cayley table which is given in Table 3.
For an initial universal set $U = \mathbb{N}$ consist of all natural numbers, define a soft set f_E over U by $f(0) = 12\mathbb{N}$, $f(1) = 3\mathbb{N}$, $f(2) = 4\mathbb{N}$, $f(a) = 2\mathbb{N}$ and $f(b) = \mathbb{N}$. By routine calculations, we know that f_E is a uni-soft ideal over U with thresholds (P, Q) for $\varnothing \neq P \subseteq Q \subseteq U$. If we define a soft set g_E over U by $g_E(0) = 4\mathbb{N}$,

$$g_E(1) = \{2n + 1 \mid n = \mathbb{N} \setminus \{1\}\},$$

Table 3. Cayley table for the binary operation "$*$"

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

$g_E(2) = 8\mathbb{N}$, $g_E(a) = 2\mathbb{N}$ and $g_E(b) = \{3n \mid n = 1, 2, 3, \cdots 33\}$, then g_E is not a uni-soft ideal over U with thresholds (P, Q) for $P = 6\mathbb{N}$ and $Q = 3\mathbb{N}$ since

$$g_E(b) \cap Q = \{3n \mid n = 1, 2, 3, \cdots 33\} \nsubseteq g_E(b * a) \cup g_E(a) \cup P.$$

Theorem 3.14. *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra and let $P, Q \in \mathcal{P}(U)$ be such that $P \subseteq Q$. Then every uni-soft ideal over U is a uni-soft ideal over U with thresholds (P, Q).*

Proof. Let f_E be a uni-soft ideal over U. Then

$$f_E(0) \cap Q \subseteq f_E(0) \subseteq f_E(x) \subseteq f_E(x) \cup P$$

and

$$f_E(x) \cap Q \subseteq f_E(x) \subseteq f_E(x * y) \cup f_E(y) \subseteq f_E(x * y) \cup f_E(y) \cup P,$$

that is, $f_E(0) \sqsubseteq_{(P,Q)} f_E(x)$ and $f_E(x) \sqsubseteq_{(P,Q)} f_E(x * y) \cup f_E(y)$ for all $x, y \in E$. Therefore f_E is a uni-soft ideal over U with thresholds (P, Q). □

Proposition 3.15. *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. If a soft set f_E over U is a uni-soft ideal over U with thresholds (P, Q) and satisfies the condition (3.6), then for any $x, y, z \in E$,*

(1) $x \leq y \;\Rightarrow\; f_E(x) \sqsubseteq_{(P,Q)} f_E(y)$,
(2) $f_E(x * y) = f_E(0) \;\Rightarrow\; f_E(y) \sqsubseteq_{(P,Q)} f_E(x)$,
(3) $f_E((x * y) * (x * z)) \sqsubseteq_{(P,Q)} f_E(z * y)$,
(4) $f_E((x * z) * (y * z)) \sqsubseteq_{(P,Q)} f_E(x * y)$.

Proof. (1) Let $x, y \in E$ be such that $x \leq y$. Then $x * y = 0$, and so

$$\begin{aligned} f_E(x) \cap Q &\subseteq f_E(x * y) \cup f_E(y) \cup P \\ &= f_E(0) \cup f_E(y) \cup P \\ &= (f_E(0) \cap Q) \cup f_E(y) \cup P \\ &\subseteq (f_E(y) \cup P) \cup f_E(y) \cup P \\ &= f_E(y) \cup P, \end{aligned}$$

that is, $f_E(x) \sqsubseteq_{(P,Q)} f_E(y)$.

(2) It is similar to the proof of (1).
(3) It is straightforward from (I) and (1).
(4) It is straightforward from (a4) and (1). □

Corollary 3.16 ([9]). *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. If a soft set f_E over U is a uni-soft ideal over U, then for any $x, y, z \in E$,*

(1) $x \leq y \;\Rightarrow\; f_E(x) \subseteq f_E(y)$,
(2) $f_E(x * y) = f_E(0) \;\Rightarrow\; f_E(y) \subseteq f_E(x)$,
(3) $f_E((x * y) * (x * z)) \subseteq f_E(z * y)$,
(4) $f_E((x * z) * (y * z)) \subseteq f_E(x * y)$.

Proof. It is by taking $(P, Q) = (\varnothing, U)$ in Proposition 3.15. □

Proposition 3.17. *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra and let f_E be a uni-soft ideal over over U with thresholds (P, Q) satisfying (3.6). Then*

$$(\forall x, y, z \in E)\left(x * y \leq z \;\Rightarrow\; f_E(x) \sqsubseteq_{(P,Q)} f_E(y) \cup f_E(z)\right). \tag{3.11}$$

Proof. Let $x, y, z \in E$ be such that $x * y \leq z$. Then $(x * y) * z = 0$, and so

$$\begin{aligned} f_E(x) \cap Q &\subseteq f_E(x * y) \cup f_E(y) \cup P \\ &= (f_E(x * y) \cap Q) \cup f_E(y) \cup P \\ &\subseteq (f_E((x * y) * z) \cup f_E(z) \cup P) \cup f_E(y) \cup P \\ &= f_E(0) \cup f_E(z) \cup f_E(y) \cup P \\ &= (f_E(0) \cap Q) \cup f_E(z) \cup f_E(y) \cup P \\ &\subseteq f_E(y) \cup f_E(z) \cup P \end{aligned}$$

by (3.6), (3.9) and (3.10). Therefore $f_E(x) \sqsubseteq_{(P,Q)} f_E(y) \cup f_E(z)$. □

Corollary 3.18 ([9]). *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. If f_E is a uni-soft ideal over over U with thresholds $(\varnothing, U)$, then the following assertion is valid.*

$$(\forall x, y, z \in E)\left(x * y \leq z \;\Rightarrow\; f_E(x) \subseteq f_E(y) \cup f_E(z)\right). \tag{3.12}$$

We provide conditions for a soft set to be a uni-soft ideal with thresholds.

Theorem 3.19. *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. If a soft set f_E over U satisfies (3.9) and (3.11), then f_E is a uni-soft ideal over U with thresholds (P, Q).*

Proof. Since $x * (x * y) \leq y$ for all $x, y \in E$, it follows from (3.11) that

$$f_E(x) \sqsubseteq_{(P,Q)} f_E(x * y) \cup f_E(y).$$

Hence f_E is a uni-soft ideal over U with thresholds (P, Q). □

Proposition 3.20. *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra and let f_E be a uni-soft ideal over U with thresholds (P, Q) and satisfies the condition (3.6). Then the following assertion is valid.*

$$(\forall x, y, z \in E)\left(f_E(x * y) \sqsubseteq_{(P,Q)} f_E(x * z) \cup f_E(z * y)\right), \tag{3.13}$$

Proof. Since $(x * y) * (x * z) \leq z * y$ for all $x, y, z \in E$, we have

$$f_E((x * y) * (x * z)) \sqsubseteq_{(P,Q)} f_E(z * y)$$

by Proposition 3.15(1). It follows from (3.10) and (3.6) that

$$\begin{aligned} f_E(x*y)\cap Q &\subseteq f_E((x*y)*(x*z))\cup f_E(x*z)\cup P \\ &= (f_E((x*y)*(x*z))\cap Q)\cup f_E(x*z)\cup P \\ &\subseteq (f_E(z*y)\cup P)\cup f_E(x*z)\cup P \\ &= f_E(x*z)\cup f_E(z*y)\cup P. \end{aligned}$$

Therefore $f_E(x*y) \sqsubseteq_{(P,Q)} f_E(x*z)\cup f_E(z*y)$ for all $x,y,z\in E$. □

Corollary 3.21 ([9]). *Let $(U,E)=(U,X)$ where X is a BCK/BCI-algebra If f_E is a uni-soft ideal over U with thresholds $(\varnothing,U)$, then the following assertion is valid.*

$$(\forall x,y,z\in E)\,(f_E(x*y)\subseteq f_E(x*z)\cup f_E(z*y))\,.$$

Lemma 3.22. *Let $(U,E)=(U,X)$ where X is a BCK/BCI-algebra If f_E is a soft set over U satisfying the condition (3.6), then the inclusion relation "$\sqsubseteq_{(P,Q)}$" is a transitive relation.*

Proof. Assume that $f_E(x)\sqsubseteq_{(P,Q)} f_E(y)$ and $f_E(y)\sqsubseteq_{(P,Q)} f_E(z)$ for $x,y,z\in E$. Then $f_E(x)\cap Q\subseteq f_E(y)\cup P$ and $f_E(y)\cap Q\subseteq f_E(z)\cup P$. It follows that

$$\begin{aligned} f_E(x)\cap Q\subseteq f_E(y)\cup P &= (f_E(y)\cap Q)\cup P \\ &\subseteq (f_E(z)\cup P)\cup P = f_E(z)\cup P. \end{aligned}$$

Hence $f_E(x)\sqsubseteq_{(P,Q)} f_E(z)$. □

Proposition 3.23. *Let $(U,E)=(U,X)$ where X is a BCK/BCI-algebra and let f_E be a uni-soft ideal over U with thresholds (P,Q) and satisfies the condition (3.6). Then the following assertions are equivalent.*

$$(\forall x,y\in E)\left(f_E(x*y)\sqsubseteq_{(P,Q)} f_E((x*y)*y)\right), \tag{3.14}$$

$$(\forall x,y,z\in E)\left(f_E((x*z)*(y*z))\sqsubseteq_{(P,Q)} f_E((x*y)*z)\right). \tag{3.15}$$

Proof. Assume that (3.14) is valid and let $x,y,z\in E$. Then

$$f_E((x*z)*(y*z)) = f_E((x*(y*z))*z)\sqsubseteq_{(P,Q)} f_E(((x*(y*z))*z)*z).$$

Since $((x*(y*z))*z)*z=((x*z)*(y*z))*z\leq (x*y)*z$, we have

$$f_E(((x*(y*z))*z)*z)\sqsubseteq_{(P,Q)} f_E((x*y)*z)$$

by Proposition 3.15(1). It follows from Lemma 3.22 that

$$f_E((x*z)*(y*z))\sqsubseteq_{(P,Q)} f_E((x*y)*z).$$

Conversely, suppose that (3.15) is valid. If we take $y=z$ in (3.15), then

$$f_E(x*z)=f_E((x*z)*0)=f_E((x*z)*(z*z))\sqsubseteq_{(P,Q)} f_E((x*z)*z)$$

and thus (3.14) is valid. □

If we take $(P,Q)=(\varnothing,U)$ in Proposition 3.23, then we have the following corollary.

Corollary 3.24 ([9]). *Let $(U,E)=(U,X)$ where X is a BCK/BCI-algebra. If f_E is a uni-soft ideal over U, then the following assertions are equivalent.*

$$(\forall x, y \in E)\,(f_E(x*y) \subseteq f_E((x*y)*y))\,, \tag{3.16}$$

$$(\forall x, y, z \in E)\,(f_E((x*z)*(y*z)) \subseteq f_E((x*y)*z))\,. \tag{3.17}$$

Theorem 3.25. *Let $(U,E)=(U,X)$ where X is a BCK-algebra and let f_E be a soft set over U satisfying the condition (3.6). If f_E is a uni-soft ideal over U with thresholds (P,Q), then it is a uni-soft subalgebra over U with the same thresholds (P,Q).*

Proof. Let f_E be a uni-soft ideal over U with thresholds (P,Q). Since $x*y \leq x$ for all $x, y \in E$, it follows from Proposition 3.15(1) that

$$f_E(x*y) \sqsubseteq_{(P,Q)} f_E(x). \tag{3.18}$$

Hence

$$\begin{aligned} f_E(x*y)\cap Q &\subseteq f_E(x)\cup P = (f_E(x)\cap Q)\cup P \\ &\subseteq f_E(x*y)\cup f_E(y)\cup P \\ &= (f_E(x*y)\cap Q)\cup f_E(y)\cup P \\ &\subseteq (f_E(x)\cup P)\cup f_E(y)\cup P \\ &= f_E(x)\cup f_E(y)\cup P \end{aligned}$$

by (3.18), (3.6) and (3.10), and so $f_E(x*y) \sqsubseteq_{(P,Q)} f_E(x)\cup f_E(y)$. Therefore f_E is a uni-soft subalgebra over U with thresholds (P,Q). □

If we take $(P,Q)=(\varnothing,U)$ in Theorem 3.25, then, we have the following corollary.

Corollary 3.26 ([9]). *Let $(U,E)=(U,X)$ where X is a BCK-algebra. Then every uni-soft ideal is a uni-soft subalgebra.*

Lemma 3.27. *Let $(U,E)=(U,X)$ where X is a BCK/BCI-algebra. Then every soft set f_E over U satisfies the following assertion.*

$$(\forall x,y,z)\left(f_E(x) \sqsubseteq_{(P,Q)} f_E(y) \;\Rightarrow\; f_E(x)\cup f_E(z) \sqsubseteq_{(P,Q)} f_E(y)\cup f_E(z)\right).$$

Proof. Straightforward. □

We provide conditions for a uni-soft ideal with thresholds (P,Q) to be a uni-soft subalgebra with the same thresholds (P,Q).

Theorem 3.28. *Let $(U,E)=(U,X)$ where X is a BCI-algebra. Let f_E be a uni-soft ideal over U with thresholds (P,Q) and satisfies (3.6) and the condition*

$$(\forall x \in E)\left(f_E(0*x) \sqsubseteq_{(P,Q)} f_E(x)\right). \tag{3.19}$$

Then f_E is a uni-soft subalgebra over U with thresholds (P,Q).

Proof. For any $x, y \in E$, we have

$$f_E(x*y) \sqsubseteq_{(P,Q)} f_E((x*y)*x)\cup f_E(x) = f_E(0*y)\cup f_E(x) \sqsubseteq_{(P,Q)} f_E(x)\cup f_E(y).$$

by (3.19) and Lemma 3.27. It follows from Lemma 3.22 that $f_E(x*y) \sqsubseteq_{(P,Q)} f_E(x)\cup f_E(y)$. Therefore f_E is a uni-soft subalgebra over U with thresholds (P,Q). □

Theorem 3.29. *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. If f_E is a soft set over U satisfying (3.6), then the following are equivalent.*

(1) *f_E is a uni-soft ideal over U with thresholds (P, Q).*
(2) *The set $e(f_E; R)$ is an ideal of E for all $R \subseteq Q$.*

Proof. Assume that f_E is a uni-soft ideal over U with thresholds (P, Q). Let R be any subset of U such that $R \subseteq Q$. Using (3.6) and (3.9) induces

$$f_E(0) = f_E(0) \cap Q \subseteq f_E(x) \cup P \subseteq P \cup R$$

for $x \in e(f_E; R)$. Hence $0 \in e(f_E; R)$. Let $x, y \in E$ be such that $x * y \in e(f_E; R)$ and $y \in e(f_E; R)$. Then $f_E(x * y) \subseteq P \cup R$ and $f_E(y) \subseteq P \cup R$. It follows from (3.6) and (3.10) that

$$f_E(x) = f_E(x) \cap Q \subseteq f_E(x * y) \cup f_E(y) \cup P \subseteq P \cup R.$$

Thus $x \in e(f_E; R)$, and therefore $e(f_E; R)$ is an ideal of E.

Conversely, suppose that $e(f_E; R)$ is an ideal of E for all $R \subseteq Q$. If there exists $a \in E$ such that "$f_E(0) \sqsubseteq_{(P,Q)} f_E(a)$" does not hold, then

$$f_E(0) = f_E(0) \cap Q \nsubseteq f_E(a) \cup P \subseteq P \cup R_a$$

for some subset R_a of U with $f_E(a) \subseteq R_a$. Hence $0 \notin e(f_E; R_a)$ which is a contradiction. Therefore $f_E(0) \sqsubseteq_{(P,Q)} f_E(x)$ for all $x \in E$. For any $x, y \in E$, let $f_E(x * y) = R_1$ and $f_E(y) = R_2$. Then

$$f_E(x * y) = R_1 \subseteq P \cup R_1 \subseteq P \cup R,$$
$$f_E(y) = R_2 \subseteq P \cup R_2 \subseteq P \cup R,$$

where $R = R_1 \cup R_2 \subseteq Q$. Thus $x * y \in e(f_E; R)$ and $y \in e(f_E; R)$. Since $e(f_E; R)$ is an ideal of E, it follows that $x \in e(f_E; R)$. Hence

$$f_E(x) \cap Q = f_E(x) \subseteq P \cup R = P \cup R_1 \cup R_2 = f_E(x * y) \cup f_E(y) \cup P,$$

and so $f_E(x) \sqsubseteq_{(P,Q)} f_E(x * y) \cup f_E(y)$. Therefore f_E is a uni-soft ideal over U with thresholds (P, Q). □

Corollary 3.30 ([9])**.** *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra and let $f_E \in S(U)$. Then f_E is a uni-soft ideal over U if and only if the set*

$$e(f_E; \tau) = \{x \in E \mid f_E(x) \subseteq \tau\}$$

is an ideal of E for all $\tau \in \mathcal{P}(U)$.

Proof. It is by taking $(P, Q) = (\varnothing, U)$ and $R = \tau$ in Theorem 3.29. □

Conclusion

We have considered a generalization of Jun's results in [9]. We have introduced the notions of uni-soft subalgebra/ideal with thresholds, and investigated related properties. We have discussed relations between uni-soft subalgebra/ideal and uni-soft subalgebra/ideal with thresholds, and considered characterizations of uni-soft subalgebra/ideal with thresholds. We have provided conditions for a soft set to be a uni-soft ideal with thresholds. We also have provided conditions for a uni-soft ideal with thresholds to be a uni-soft subalgebra with thresholds.

In consecutive research, we will find examples to show that if $(U, E) = (U, X)$ in which X is a BCI-algebra, then Theorem 3.25 is not true in general. We will also consider the following questions.

Question 1. Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra and let $P, Q \in \mathcal{P}(U)$ with $P \subseteq Q$. Then is every uni-soft subalgebra over U with thresholds (P, Q) a uni-soft subalgebra over U?

Question 2. Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra and let $P, Q \in \mathcal{P}(U)$ with $P \subseteq Q$. Then is every uni-soft ideal over U with thresholds (P, Q) a uni-soft ideal over U?

We will apply the idea/result in this paper to other type of ideals and related algebraic structures.

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