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Abstract. In an attempt to develop multi-number system, in this paper, we introduce a concept of multi-integer system which forms a multi-integral domain. It is also shown that the multi-integer system is an extension of multi-natural number system.

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Keywords: Multiset, Multi-natural number, Multi-integer, Multi-ring, Multi-integral domain.

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1. Introduction

The term multiset (mset in short) as Knuth notes [22], was first suggested by N. G. de Bruijn [11] in a private correspondence to him. N. G. de Bruijn's interests in multisets grew out of his investigations into the combinatorial properties of the set of divisors of a number. A number or any of its divisors is expressible as a multiset of prime factors [2, 22]. The repeated prime factors of the number 72, although identical in all respects, are treated as multiplicity. So, it is convenient to accept a collection like $\{2, 2, 2, 3, 3\}$ of prime factors rather than a set like $\{2, 3\}$. In classical set theory, a set is a well-defined collection of distinct objects. If the repeated occurrences of any object are allowed in a collection, then that mathematical structure is called a multiset. Owing to aptness, multiset has replaced a variety of terms viz. list, bunch, heap, bag, sample, weighted set, occurrence set and fireset (finitely repeated element set) used in different contexts but conveying synonimity with mset. As an important generalisation of classical set theory, theory of multisets now have become an area of special interest in various subjects like mathematics, statistics, computer science, physics and philosophy [2, 9, 11, 13, 15, 26, 28, 30, 32]. Many authors like Yagar [32], Miyamoto [25], Hickman [17], Blizard [4], Girish and John [13, 14, 15, 29], D. Singh [30, 31], A. M. Ibrahim [18, 30, 31] etc. have studied the properties of multisets.

Some authors have also generalised the notion of multisets to form fuzzy multisets [23], Intuitionistic fuzzy multisets [3, 29], soft multisets [1, 14, 15, 24] etc.

In many situations, it is more convenient to consider a collection like multiset. e.g., the repeated eigen values of a matrix, prime factors of a positive integer, repeated observations in a statistical sample, data structure, information retrieval on the web, multicriteria decision making, knowledge presentation in data based system, biological systems and membrane computing [20, 21, 25, 26, 28, 30, 31, 32]. More studies on multisets can be found in [2, 4, 5, 6, 7, 10, 13, 16, 18, 19, 22]. Although the studies on multisets revolved around combinatorics in earlier times [2, 4], the modern research in this field about the structural development in multiset corpus is relatively new. Various research work on the multiset ordering [4, 12, 30], relations and functions in multiset context [5, 25], multiset topology [13, 14], multi group theory [27] etc. have been done recently by some researchers. In order to develop various structures on multisets we have started from the beginning. Our motif is to develop a multi-number system which a generalisation of the ordinary number system and also compatible with the multiset setting as number system plays an important role in mathematics. In a previous paper [8], we have introduced a concept of multi-natural number system from the axiomatic point of view and study its properties related to compositions and order relations. In this paper, we extend it to develop multi-integers and to study their properties. The organization of the paper is as follows:

Section 2 is the preliminary part where some definitions and results regarding multisets and multi-natural numbers have been introduced. In section 3, the notion of multi-difference system together with binary operations and order relation defined on it has been introduced. Several properties regarding multi-difference system have been studied and notions like multi-distributive property, general multiset, multi-integers, multi-ring, non-multi-zero divisor, multi-integral domain etc. have been also defined in this section. Finally, Multi-integer system has been introduced, its isomorphism with multi difference system and its existence and uniqueness have been established. The straightforward proofs of the propositions have been omitted.

2. Preliminaries

Definition 2.1 ([13, 18])**.** A multiset (or mset, in short) M drawn from a set X is represented by a function $Count_M$ or C_M defined as $C_M : X \to N \cup \{0\}$ where N represents the set of all natural numbers. Let M be an mset drawn from the set $X = \{x_1, x_2, ..., x_n\}$ with x_i appearing k_i times in M. It is denoted by $x_i \in^{k_i} M$. The mset M drawn from the set X is then denoted by $\{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$. Also $C_M(x)$ is the number of occurrences of the element x in the mset M. However, those elements which are not include in the mset M have zero count.

Example 2.2. Let $X = \{a, b, c, d, e\}$. Then $M = \{3/a, 2/b, 1/e\}$ is an mset drawn from X.

Definition 2.3 ([13])**.** Let M and P be two msets drawn from a set X. Then the followings are defined:

(i) $M = P$, if $C_M(x) = C_P(x)\ \forall x \in X$,

(ii) $M \subseteq P$, if $C_M(x) \leq C_p(x)$ $\forall x \in X$ (then we call P to be submset of M),
(iii) $P = M \cup N$, if $C_P(x) = max\{C_M(x), C_N(x)\}$ $\forall x \in X$,
(iv) $P = M \cap N$, if $C_P(x) = min\{C_M(x), C_N(x)\}$ $\forall x \in X$,
(v) $P = M \oplus N$, if $C_P(x) = C_M(x) + C_N(x)$ $\forall x \in X$,
(vi) $P = M \ominus N$, if $C_P(x) = max\{C_M(x) - C_N(x), 0\}$ $\forall x \in X$,
where $\oplus$ and $\ominus$ represents mset addition and mset subtraction respectively .

Let M be an mset drawn from a set X, then the support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$. i.e., M^* is an ordinary set and it is also called root set. The cardinality of an mset M drawn from a set X is denoted by $card(M)$ or $|M|$ and is given by $|M| = \sum_{x \in X} C_M(x)$.

Remark 2.4 ([13, 19])**.** A domain X is defined as a set of elements from which msets are constructed. The mset space $[X]^m$ is the set of all msets whose elements are in X such that no element in the mset occurs more than m times.

The mset space $[X]^\infty$ is the set of all msets over a domain X such that there is no limit on the number of occurrences of an element in an mset. If $X = \{x_1, x_2, ..., x_k\}$, then

$$[X]^m = \{\{m_1/x_1, m_2/x_2, ..., m_k/x_k\},$$

for $i = 1, 2, ..., m; m_i \in \{0, 1, 2, ..., m\}\}$.

Definition 2.5 ([13, 19])**.** Let X be a support set and $[X]^m$ be the mset space defined over X. Then the complement M^c of M in $[X]^m$ is an element of $[X]^m$ such that

$$C^c_M(x) = m - C_M(x), \forall x \in X.$$

Definition 2.6. (Different types of submsets)

(i) [13] Whole submset: A submset P of an mset M (i.e., $P \subseteq M$) is a whole submset of M with each element in P having full multiplicity as in M, i.e., $C_p(x) = C_M(x)$, $\forall x \in P^*$.

(ii) [13] Partial whole submset: A submset P of an mset M is a partial whole submset of M with at least one element in P having same multiplicity as in M, i.e., $C_p(x) = C_M(x)$, for some $x \in P^*$.

(iii) [13] Full submset: A submset P of an mset M is a full submset of M, if $M^* = P^*$ and $C_p(x) \leq C_M(x)$, $\forall x \in P^*$.

(iv) [8] Single whole submset single mset and single submset: A submset P of an mset M drawn from a set X is a single whole submset, if $C_P(x)$ is either $C_M(x)$ or 0, $\forall x \in P^*$ and $\{x \in P^* : C_P(x) = C_M(x)\}$ is a singleton set, say $\{a\}$, then let us denote it as $M_{\{a\}}(= P)$, i.e., a single whole submset is such a submset of a multiset for which exactly one element of the support set belongs to it with the same count as in the mset.

An mset is a single mset, if it has a singleton support set and a submset P of a mset M drawn from a set X is a single submset, if P is a single mset.
then immediately, each mset can be expressed as a union of all its single whole submsets. Thus $M = \underset{a \in M^*}{\cup} M_{\{a\}}$.

In this connection, we note that single whole submsets are pairwise disjoint.

Definition 2.7 ([8])**.** (Axiomatic definition of multi-natural numbers)

Let $(N, 1, \sigma)$ be the unique ordinary natural number system defined by Peano. Then

Axiom 1: For all $p, q \in N$, there exist a multi-natural number denoted by N_p^q,

Axiom 2: Two multi-natural numbers N_p^q and N_r^s are equal iff $p = r$ and $q = s$,

Axiom 3: For any multi-natural number N_p^q, $p, q \in N$, there exist a multi-natural number $N_{\sigma(P)}^q$ (defined to be the support successor of N_p^q) and another multi-natural number $N_p^{\sigma(q)}$ (defined to be multiplicity successor of N_p^q),

Axiom 4: N_1^q $\forall q \in N$ is not support successor of any multi-natural number. Also, N_p^1 $\forall p \in N$ is not multiplicity successor of any multi-natural number,

Axiom 5: Let $P(N_p^q)$ be any proposition involving a multi-natural number N_p^q. Suppose that $P(N_1^1)$ is true. Also suppose that whenever $P(N_p^q)$ is true. Then $P(N_{\sigma(p)}^q)$ and $P(N_p^{\sigma(q)})$ both are also true. Thus $P(N_p^q)$ is true, for every multi-natural number N_p^q.

The set of all multi-natural numbers is denoted by $m(N)$. $p \in N$ and $q \in N$ are respectively the support and the multiplicity of a multi-natural number N_p^q.

Definition 2.8 ([8])**.** (Successor Functions) $S : m(N) \to m(N)$ defined by $S(N_p^q) = N_{\sigma(P)}^q$ is the support successor function. $M : m(N) \to m(N)$ defined by $S(N_p^q) = N_p^{\sigma(q)}$ is the multiplicity successor function. S and M both are one to one since σ is one to one.

Definition 2.9 ([8])**.** (Definition of addition)

There exists a unique function $A : m(N) \times m(N) \to m(N)$ with the following properties:

Axiom 1: $A(N_p^q, N_1^1) = S(N_p^q)$,

Axiom 2: $A(N_p^q, S(N_n^m)) = S(A(N_p^q, N_n^m))$,

Axiom 3: $A(N_p^q, M(N_n^m)) = M^{(q)}(A(N_p^q, N_n^m))$ which is called addition of two multi-natural numbers and it is given by $A(N_p^q, N_n^m) = N_{p+n}^{qm}, N_p^q, N_n^m \in m(N)$. $A(N_p^q, N_n^m)$ is also denoted by $N_p^q + N_n^m$.

Proposition 2.10 ([8])**.** Properties of addition:

(1) $S(N_p^q) = N_p^q + N_1^1$, $\forall N_p^q \in m(N)$.

(2) $N_p^q + (N_k^t + N_1^1) = (N_p^q + N_k^t) + N_1^1$, $\forall N_p^q, N_k^t \in m(N)$.

(3) $N_1^1 + N_p^q = N_p^q + N_1^1$, $\forall N_p^q \in m(N)$.

(4) $(N_p^q + N_1^1) + N_k^t = (N_p^q + N_k^t) + N_1^1$ $\forall N_p^q, N_k^t \in m(N)$.

(5) *The commutative law of addition:* $N_p^q + N_k^t = N_k^t + N_p^q$ $\forall N_p^q, N_k^t \in m(N)$.

(6) *The associative law of addition:* $(N_p^q + N_k^t) + N_m^n = N_p^q + (N_k^t + N_m^n)$, $\forall N_p^q, N_k^t, N_m^n \in m(N)$.

(7) *The cancellation law for addition:* $N_p^q + N_k^t = N_p^q + N_m^n \Rightarrow N_p^q = N_k^t$, $\forall N_p^q, N_k^t, N_m^n \in m(N)$.

Example 2.11. For two multi-natural number N_5^6 and N_3^4, $N_5^6 + N_3^4 = N_{5+3}^{6.4} = N_8^{24}$.

Definition 2.12 ([8])**.** (Definition of multiplication)

There exists a unique function $P : m(N) \times m(N) \to m(N)$ with the following properties:

Axiom 1: $P(N_p^q, N_1^1) = N_1^1$,

Axiom 2: $P(N_p^q, S(N_n^m)) = S^{(p)}(P(N_p^q, N_n^m))$,
Axiom 3: $P(N_p^q, M(N_n^m)) = M^{(p)}(P(N_p^q, N_n^m))$, $N_p^q, N_n^m \in m(N)$
which is called multiplication of two multi-natural numbers and it is given by $P(N_p^q, N_n^m) = N_{pn}^{qm}$, $N_p^q, N_n^m \in m(N)$. $P(N_p^q, N_n^m)$ is also denoted by $N_p^q \cdot N_n^m$.

Proposition 2.13 ([8])**.** Properties of multiplication:
(1) $P(N_1^1, N_p^q) = N_p^q = P(N_p^q, N_1^1)$ $\forall N_p^q \in m(N)$.
(2) *The commutative law of multiplication:* $P(N_p^q, N_n^m) = P(N_n^m, N_p^q)$, $\forall N_p^q, N_n^m \in m(N)$.
(3) *The associative law of multiplication:* $P(P(N_p^q, N_k^t), N_m^n) = P(N_p^q, P(N_k^t, N_m^n))$, $\forall N_p^q, N_k^t, N_m^n \in m(N)$.
(4) *P does not obey distributive property over A, i.e., in general,*

$$P(N_p^q, A(N_m^n, N_r^s)) \neq A(P(N_p^q, N_m^n), P(N_p^q, N_r^s)),$$

$N_p^q, N_r^s, N_m^n \in m(N)$.

Example 2.14. For two multi-natural numbers N_5^6 and N_3^4, $N_5^6 \cdot N_3^4 = N_{5\cdot 3}^{6\cdot 4} = N_{15}^{24}$.

Definition 2.15 ([8])**.** (Order on $m(N)$)
For $N_p^q, N_m^n \in m(N)$, $N_p^q = N_m^n$ iff ($p = m$ as well as $q = n$).
Also for $N_p^q, N_m^n \in m(N)$, N_p^q is greater than N_m^n, i.e., $N_p^q > N_m^n$, if $\exists N_r^s \in m(N)$ such that $N_p^q = N_m^n + N_r^s$ $(= N_{m+r}^{ns})$, i.e., if ($p > m$ as well as $n|q$).
Again, N_p^q is greater than or equal to N_m^n and we write $N_p^q \geq N_m^n$, if $N_p^q > N_m^n$ or $N_p^q = N_m^n$, i.e., if ($p > m$ as well as $n|q$) or if ($p = m$ as well as $n = q$).
The relation $\geq$ on $m(N)$ is a partial order relation which is not total.

Definition 2.16 ([8])**.** (Multi-number of elements in a multiset)
Let N be a single mset. Also, let x is the only element of N with $C_N(x) = n$. Then, we define N_1^n as the multi-number of elements in N.

Next, we consider an mset M whose support $N^* = \{x_1, x_2, ..., x_n\}$ is a finite set and multiplicity of each of its elements is finite and is given by the count function as $C_N(x_i) = t_i, i = 1, 2, ..., n$. Then we define the multi-number of elements in M as the sum of the multi-numbers of the elements in all its single whole submsets, i.e., $N_1^{t_1} + N_1^{t_2} + ... + N_1^{t_n} = N_n^{t_1 t_2 ... t_n}$.

Example 2.17. (1) The multi-number of elements in the multiset $\{a, a, a\}$ is N_1^3.
(2) The multi-number of elements in the multiset $\{b, b\}$ is N_1^2.
(3) The multi-number of elements in the multiset $\{a, a, a, b, b, c, c\}$ is $(N_1^3 + N_1^2) + N_1^2 = N_2^6 + N_1^2 = N_3^{12}$.
(4) The multi-number of elements in the multiset $\{a, a, a, a, a, a, a, a, a, a, a, a, b, c\}$ is $(N_1^{12} + N_1^1) + N_1^1 = N_2^{12} + N_1^1 = N_3^{12}$.
(5) The multi-number of the roots of the equation $(x-1)^2(x-2)^3 = 0$ is $N_1^2 + N_1^3 = N_2^6$.

3. The multi-integer system

Here we shall represent multi-integer system in terms of multi-natural numbers that we have already constructed in a previous paper [8].

First of all, we shall introduce the concept of Multi-Difference System together with some binary operations and order relation.

Let us now introduce the following binary relation on $m(N) \times m(N)$:

Definition 3.1. For $(N_a^b, N_c^d), (N_p^q, N_r^s) \in m(N) \times m(N)$, we say (N_a^b, N_c^d) is equivalent to (N_p^q, N_r^s) and we write $(N_a^b, N_c^d) \sim (N_p^q, N_r^s)$ iff $N_a^b + N_r^s = N_c^d + N_p^q$.

Theorem 3.2. *The relation $\sim$ is an equivalence relation defined on $m(N) \times m(N)$.*

Proof. Since $\forall (N_a^b, N_c^d) \in m(N) \times m(N)$, we have $N_a^b + N_c^d = N_c^d + N_a^b$ (by (5) of Proposition 2.10). Then $(N_a^b, N_c^d) \sim (N_a^b, N_c^d)$. Thus $\sim$ is a reflexive relation on $m(N) \times m(N)$.

Next, for $(N_a^b, N_c^d), (N_p^q, N_r^s) \in m(N) \times m(N)$, let $(N_a^b, N_c^d) \sim (N_p^q, N_r^s)$. Then

$$\begin{aligned} N_a^b + N_r^s = N_c^d + N_p^q &\Rightarrow N_r^s + N_a^b = N_p^q + N_c^d \text{ (by (5) of Proposition 2.10)} \\ &\Rightarrow N_p^q + N_c^d = N_r^s + N_a^b \\ &\Rightarrow (N_p^q, N_r^s) \sim (N_a^b, N_c^d). \end{aligned}$$

Thus $\sim$ is a symmetric relation on $m(N) \times m(N)$.

Finally, for $(N_a^b, N_c^d), (N_p^q, N_r^s), (N_u^v, N_w^x) \in m(N) \times m(N)$, let $(N_a^b, N_c^d) \sim (N_p^q, N_r^s)$ and $(N_p^q, N_r^s) \sim (N_u^v, N_w^x)$. Then $N_a^b + N_r^s = N_c^d + N_p^q$ as well as $N_p^q + N_w^x = N_r^s + N_u^v$. Thus

$$\begin{aligned} & (N_a^b + N_r^s) + (N_p^q + N_w^x) = (N_c^d + N_p^q) + (N_r^s + N_u^v) \\ \Rightarrow\ & (N_a^b + N_w^x) + (N_r^s + N_a^b) = (N_c^d + N_u^v) + (N_r^s + N_a^b) \\ & \text{(by (5) and (6) of Proposition 2.10)} \\ \Rightarrow\ & N_a^b + N_w^x = N_c^d + N_u^v \text{ (by (7) of Proposition 2.10)} \\ \Rightarrow\ & (N_a^b, N_c^d) \sim (N_u^v, N_w^x). \end{aligned}$$

So $\sim$ is a transitive relation on $m(N) \times m(N)$. Hence $\sim$ is an equivalence relation on $m(N) \times m(N)$. □

Remark 3.3. Let us denote the set of all equivalence classes of $m(N) \times m(N)$ by $m_d(Z)$ and call it as multi-difference system. An element $[(N_a^b, N_c^d)]$ of $m_d(Z)$ will now be simply denoted by $[N_a^b, N_c^d]$ and accordingly $[N_a^b, N_c^d] = [N_p^q, N_r^s]$ iff $N_a^b + N_r^s = N_c^d + N_p^q$. Now we have only produced the elements of $m_d(Z)$. A bunch of elements can hardly be a system. We still need to define appropriate binary operations and order relations on it just as we did for $m(N)$ [8]. Before we do so, let us note the following fundamental relations between elements in $m_d(Z)$.

Remark 3.4. For $[N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z)$,

$$\begin{aligned} & [N_a^b, N_c^d] = [N_p^q, N_r^s] \\ \Leftrightarrow\ & N_a^b + N_r^s = N_c^d + N_p^q \\ \Leftrightarrow\ & N_{a+r}^{bs} = N_{c+p}^{dq} \text{ (by Definition 2.9)} \\ \Leftrightarrow\ & a + r = c + p \text{ and } bs = dq \text{ (by axiom 2 of Definition 2.7)} \\ \Leftrightarrow\ & a - c = p - r \text{ and } \tfrac{b}{d} = \tfrac{q}{s}. \end{aligned}$$

Lemma 3.5. $[N_a^b, N_c^d] = [N_a^b + N_k^t, N_c^d + N_k^t] = [N_k^t + N_a^b, N_k^t + N_c^d]$, $\forall N_a^b, N_c^d \in m_d(Z)$ *and* $\forall N_k^t \in m_d(Z)$.

Proof.

$$\begin{aligned} & [N_a^b, N_c^d] = [N_a^b + N_k^t, N_c^d + N_k^t] \\ \Leftrightarrow\ & N_a^b + (N_c^d + N_k^t) = N_c^d + (N_a^b + N_k^t) \\ \Leftrightarrow\ & (N_a^b + N_c^d) + N_k^t = (N_c^d + N_a^b) + N_k^t \text{ (by (6) of Proposition 2.10)} \end{aligned}$$

$\Leftrightarrow (N_a^b + N_c^d) + N_k^t = (N_a^b + N_c^d) + N_k^t$ (by (5) of Proposition 2.10)
which is a tautology. Also a similar tautology can be established for the second part. Then the result holds. □

Definition 3.6. (Addition on $m_d(Z)$)
∃ a well defined binary operation ⊕ on $m_d(Z)$ defined by:

$$[N_a^b, N_c^d] \oplus [N_p^q, N_r^s] = [N_a^b + N_p^q, N_c^d + N_r^s], \text{ for } [N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z).$$

To show that ⊕ is well-defined, we need to show that for any $[N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z)$, there is one and only one image under ⊕: Let $[N_a^b, N_c^d] = [N_p^q, N_r^s]$ and $[N_e^f, N_g^h] = [N_u^v, N_w^x]$. Then $[N_a^b, N_c^d] \oplus [N_e^f, N_g^h] = [N_a^b + N_e^f, N_c^d + N_g^h]$ and $[N_p^q, N_r^s] \oplus [N_u^v, N_w^x] = [N_p^q + N_u^v, N_r^s + N_w^x]$. On the other hand,
$[N_a^b, N_c^d] = [N_p^q, N_r^s] \Rightarrow N_a^b + N_r^s = N_c^d + N_p^q$
and
$[N_e^f, N_g^h] = [N_u^v, N_w^x] \Rightarrow N_e^f + N_w^x = N_g^h + N_u^v$.
Thus $(N_a^b + N_r^s) + (N_e^f + N_w^x) = (N_c^d + N_p^q) + (N_g^h + N_u^v)$
$\Rightarrow (N_a^b + N_e^f) + (N_r^s + N_w^x) = (N_c^d + N_g^h) + (N_p^q + N_u^v)$
(by (5) and (6) of Proposition 2.10)
$\Rightarrow [N_a^b + N_e^f, N_c^d + N_g^h] = [N_p^q + N_u^v, N_r^s + N_w^x]$
$\Rightarrow [N_a^b, N_c^d] \oplus [N_e^f, N_g^h] = [N_p^q, N_r^s] \oplus [N_u^v, N_w^x]$.
So ⊕ is well-defined.

Proposition 3.7. (Properties of addition on $m_d(Z)$) *Following properties of addition can be deduced:*
(1) *⊕ is commutative on $m_d(Z)$, since + is commutative on $m(N)$,*
(2) *⊕ is associative on $m_d(Z)$, since + is associative on $m(N)$,*
(3) *$[N_1^1, N_1^1]$ is the identity element in $m_d(Z)$ for ⊕,*
(4) *for each $[N_a^b, N_c^d] \in m_d(Z)$, its ⊕ - inverse exists and is given by $[N_c^d, N_a^b] \in m_d(Z)$ such that $[N_a^b, N_c^d] \oplus [N_c^d, N_a^b] = [N_1^1, N_1^1]$.*

Proof. The proofs of (1) and (2) are clear.
(3) $\forall [N_a^b, N_c^d] \in m_d(Z)$, by Lemma 3.5,

$$[N_a^b, N_c^d] \oplus [N_1^1, N_1^1] = [N_a^b + N_1^1, N_c^d + N_1^1] = [N_a^b, N_c^d].$$

Similarly, using Lemma 3.5, it can be shown that $[N_1^1, N_1^1] \oplus [N_a^b, N_c^d] = [N_a^b, N_c^d]$. Hence the result holds.
(4) By Lemma 3.5,

$$\begin{aligned}[N_a^b, N_c^d] \oplus [N_c^d, N_a^b] &= [N_a^b + N_c^d, N_c^d + N_a^b] \\ &= [N_a^b + N_c^d + N_1^1, N_c^d + N_a^b + N_1^1] \\ &= [N_1^1, N_1^1].\end{aligned}$$

Let us denote the ⊕ - inverse of $[N_a^b, N_c^d] \in m_d(Z)$ as $(-[N_a^b, N_c^d])$. □

Remark 3.8. $(m_d(Z), \oplus)$ is a commutative group.

Remark 3.9. From Definition 3.6 and Definition 2.9, we can write

$$[N_a^b, N_c^d] \oplus [N_p^q, N_r^s] = [N_{a+p}^{bq}, N_{c+r}^{ds}], \text{ for } [N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z).$$

Definition 3.10. (Multiplicationon $m_d(Z)$)
$\exists$ a well-defined binary operation $\odot$ on $m_d(Z)$ defined by:

$$[N_a^b, N_c^d] \odot [N_p^q, N_r^s] = [N_{ap+cr}^{bq}, N_{ar+cp}^{ds}], \text{ for } [N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z).$$

To show that $\odot$ is well-defined, we need to show that for any $[N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z)$, there is one and only one image under $\odot$: Let $[N_a^b, N_c^d] = [N_{a'}^{b'}, N_{c'}^{d'}]$ and $[N_p^q, N_r^s] = [N_{p'}^{q'}, N_{r'}^{s'}]$. Then

$$[N_a^b, N_c^d] \odot [N_p^q, N_r^s] = [N_{ap+cr}^{bq}, N_{ar+cp}^{ds}]$$

and

$$[N_{a'}^{b'}, N_{c'}^{d'}] \odot [N_{p'}^{q'}, N_{r'}^{s'}] = [N_{a'p'+c'r'}^{b'q'}, N_{a'r'+c'p'}^{d's'}].$$

On the other hand,

$[N_a^b, N_c^d] = [N_{a'}^{b'}, N_{c'}^{d'}]$
$\Rightarrow N_a^b + N_{c'}^{d'} = N_c^d + N_{a'}^{b'}$
$\Rightarrow N_{a+c'}^{bd'} = N_{c+a'}^{db'}$ (by Definition 2.9)
$\Rightarrow a + c' = c + a'$ and $bd' = db'$ (by Axiom 2 of Definition 2.7).

Also,

$[N_p^q, N_r^s] = [N_{p'}^{q'}, N_{r'}^{s'}]$
$\Rightarrow N_p^q + N_{r'}^{s'} = N_r^s + N_{p'}^{q'}$
$\Rightarrow N_{p+r'}^{qs'} = N_{r+p'}^{sq'}$ (by Definition 2.9)
$\Rightarrow p + r' = r + p'$ and $qs' = sq'$ (by Axiom 2 of Definition 2.7).

Thus,

$(a - c)(p - r) = (a' - c')(p' - r')$ and $bqd's' = dsb'q'$
$\Rightarrow ap + cr + a'r' + c'p' = ar + cp + a'p' + c'r'$ and $bqd's' = dsb'q'$
$\Rightarrow N_{ap+cr+a'r'+c'p'}^{bqd's'} = N_{ar+cp+a'p'+c'r'}^{dsb'q'}$ (by Axiom 2 of Definition 2.7)
$\Rightarrow N_{ap+cr}^{bq} + N_{a'r'+c'p'}^{d's'} = N_{ar+cp}^{ds} + N_{a'p'+c'r'}^{b'q'}$ (by Definition 2.9)
$\Rightarrow [N_{ap+cr}^{bq}, N_{ar+cp}^{ds}] = [N_{a'p'+c'r'}^{b'q'}, N_{a'r'+c'p'}^{d's'}]$
$\Rightarrow [N_a^b, N_c^d] \odot [N_p^q, N_r^s] = [N_{a'}^{b'}, N_{c'}^{d'}] \odot [N_{p'}^{q'}, N_{r'}^{s'}]$.

So $\odot$ is well-defined.

Proposition 3.11. Properties of multiplication on $m_d(Z)$

(1) *$\odot$ is commutative on $m_d(Z)$.*
(2) *$\odot$ is associative on $m_d(Z)$.*
(3) *The identity element exist for $\odot$ in $m_d(Z)$ and is $[N_2^1, N_1^1]$.*
(4) $[N_a^1, N_b^1] \odot ([N_p^q, N_r^s] \oplus [N_x^y, N_z^t])$
$= ([N_a^1, N_b^1] \odot [N_p^q, N_r^s]) \oplus ([N_a^1, N_b^1] \odot [N_x^y, N_z^t])$.
(5) (Remark on distributive property)
$[N_a^b, N_c^d] \odot ([N_p^q, N_r^s] \oplus [N_x^y, N_z^t])$
$\neq ([N_a^b, N_c^d] \odot [N_p^q, N_r^s]) \oplus ([N_a^b, N_c^d] \odot [N_x^y, N_z^t])$, *in general.*

Actually, $[N_a^b, N_c^d] \odot ([N_p^q, N_r^s] \oplus [N_x^y, N_z^t]) = [N_a^b, N_c^d] \odot [N_{p+x}^{qy}, N_{r+z}^{st}]$
$= [N_{ap+ax+cr+cz}^{bqy}, N_{ar+az+cp+cx}^{dst}]$.

But, $([N_a^b, N_c^d] \odot [N_p^q, N_r^s]) \oplus ([N_a^b, N_c^d] \odot [N_x^y, N_z^t])$
$= [N_{ap+cr}^{bq}, N_{ar+cp}^{ds}] \oplus [N_{ax+cz}^{by}, N_{az+cx}^{dt}]$

$= [N^{b^2qy}_{ap+cr+ax+cz}, N^{d^2st}_{ar+cp+az+cx}]$
$= [N^b_2, N^d_1] \odot [N^{bqy}_{ap+cr+ax+cz}, N_{ar+cp+az+cx}]$, (by Lemma 3.5).

(5) (Multi-distributive property)
$\forall [N^b_a, N^d_c], [N^q_p, N^s_r], [N^y_x, N^t_z] \in m_d(Z)$,
$[N^b_2, N^d_1] \odot ([N^b_a, N^d_c] \odot ([N^q_p, N^s_r] \oplus [N^y_x, N^t_z]))$
$= ([N^b_a, N^d_c] \odot [N^q_p, N^s_r]) \oplus ([N^b_a, N^d_c] \odot [N^y_x, N^t_z])$.

Let us define the above property to be the multi-distributive property of $\odot$ over $\oplus$ on $m_d(Z)$.

Proof. The proofs of (1) and (2) are obvious.

(3) $\forall [N^b_a, N^d_c] \in m_d(Z)$, by Lemma 3.5,

$$[N^b_a, N^d_c] \odot [N^1_2, N^1_1] = [N^b_{2a+c}, N^d_{a+2c}] = [N^b_a + N^1_{a+c}, N^d_c + N^1_{a+c}] = [N^b_a, N^d_c].$$

(5) The proof is omitted. □

Remark 3.12. (Order on $m_d(Z)$): After defining two binary operations on $m_d(Z)$, the next natural thing is to order the elements of $m_d(Z)$. Our aim is to define an order that will make $m_d(Z)$ a partially ordered multi-integral domain. In this connection, we shall first define a subset of $m_d(Z)$ that serves as the set of multi-natural numbers. Intuitively, this set should turn out eventually to resemble $m(N)$. So, we are representing the following notation:

Definition 3.13. We define the subset $m_d(N_Z)$ of $m_d(Z)$ by:

$$m_d(N_Z) = \{[N^m_n + N^1_1, N^1_1] \in m_d(Z) : N^m_n \in m(N)\}.$$

The following theorem tells us that $m_d(N_Z)$ appears to be indeed a very good model of $m(N)$.

Proposition 3.14. $[N^v_u, N^x_w] \in m_d(N_Z) \Leftrightarrow u - v \in N$ *and* $x|v$.

Proof. Suppose $[N^v_u, N^x_w] \in m_d(N_Z)$. Then $\exists N^\beta_\alpha \in m(N)$ such that $[N^v_u, N^x_w] = [N^\beta_\alpha + N^1_1, N^1_1]$. Thus $N^v_u + N^1_1 = N^x_w + (N^\beta_\alpha + N^1_1)$. By (6) and (7) of Proposition 2.10, $N^v_u = N^x_w + N^\beta_\alpha$. So $N^v_u = N^{x\beta}_{w+\alpha}$. Hence $u = w + \alpha$ and $v = x\beta; u, v, w, x, \alpha, \beta \in N$. Therefore $u - w \in N$ and $x|v$.

The converse is immediate. □

Theorem 3.15. *For the set $m_d(N_Z)$ the following hold:*

(1) *$(m_d(N_Z), \oplus)$ is a sub semigroup of $(m_d(Z), \oplus)$,*
(2) *$(m_d(N_Z), \odot)$ is a sub semigroup of $(m_d(Z), \odot)$,*
(3) *$(m_d(N_Z), \oplus)$ is isomorphic to $(m(N), +)$ and $(m_d(N_Z), \odot)$ is isomorphic to $(m(N), \cdot)$ as semi group under the same isomorphism,*
(4) *for every $x \in m_d(Z)$, $\exists y, z \in m_d(N_Z)$ such that $x = y \oplus (-z)$.*

Proof. Clearly, $m_d(N_Z)$ is a subset of $m_d(Z)$.

(1) Let $[N^m_n + N^1_1, N^1_1], [N^q_p + N^1_1, N^1_1] \in m_d(N_Z)$. Then
$[N^m_n + N^1_1, N^1_1] \oplus [N^q_p + N^1_1, N^1_1]$
$= [N^m_n + N^q_p + N^1_1 + N^1_1, N^1_1 + N^1_1]$ (by Definition 3.6)

$= [(N_n^m + N_p^q) + N_1^1, N_1^1] \in m_d(N_Z)$ (by Lemma 3.5).
Thus, $m_d(N_Z)$ is closed under $\oplus$. So $(m_d(N_Z), \oplus)$ is a sub semigroup of $(m_d(Z), \oplus)$.

(2) Let $[N_n^m + N_1^1, N_1^1], [N_p^q + N_1^1, N_1^1] \in m_d(N_Z)$. Then

$[N_n^m + N_1^1, N_1^1] \odot [N_p^q + N_1^1, N_1^1]$
$= [N_{n+1}^m, N_1^1] \odot [N_{p+1}^q, N_1^1]$
$= [N_{(n+1)(p+1)+1}^{mq}, N_{(n+1)+(p+1)}^1] = [N_{np+(n+p+1+1)}^{mq}, N_{n+p+1+1}^1]$
$= [(N_{np}^{mq} + N_1^1) + N_{n+p+1}^1, N_1^1 + N_{n+p+1}^1]$
$= [N_{np}^{mq} + N_1^1, N_1^1] \in m_d(N_Z)$ (by Lemma 3.5).

Thus $m_d(N_Z)$ is closed under $\odot$. So $(m_d(N_Z), \odot)$ is a sub semigroup of $(m_d(Z), \odot)$.

(3) Define $\phi : m_d(N_Z) \to m(N)$ by:

$$\phi([N_n^m + N_1^1, N_1^1]) = N_n^m, N_n^m \in m(N).$$

We first show that ϕ is a well-defined: Let, $[N_p^q + N_1^1, N_1^1] = [N_n^m + N_1^1, N_1^1]$. Then

$(N_p^q + N_1^1) + N_1^1 = N_1^1 + (N_n^m + N_1^1)$
$\Leftrightarrow N_p^q = N_n^m$, i.e., $[N_p^q + N_1^1, N_1^1] = [N_n^m + N_1^1, N_1^1]$
$\Leftrightarrow \phi([N_p^q + N_1^1, N_1^1]) = \phi([N_n^m + N_1^1, N_1^1])$.

Thus ϕ is well-defined.

Suppose $\phi([N_p^q + N_1^1, N_1^1]) = \phi([N_n^m + N_1^1, N_1^1])$. Then $N_p^q = N_n^m$. Thus $[N_p^q + N_1^1, N_1^1] = [N_n^m + N_1^1, N_1^1]$. So ϕ is one to one.

On the other hand, for any $N_n^m \in m(N)$, consider $[N_n^m + N_1^1, N_1^1] \in m_d(N_Z)$ so that $\phi([N_n^m + N_1^1, N_1^1]) = N_n^m$. Then ϕ is onto.

Now take any $[N_p^q + N_1^1, N_1^1] \in m_d(N_Z)$. Then

$\phi([N_p^q + N_1^1, N_1^1] \oplus [N_n^m + N_1^1, N_1^1])$
$= \phi([N_p^q + N_1^1 + N_n^m + N_1^1, N_1^1 + N_1^1])$
$= \phi([N_p^q + N_n^m + N_1^1, N_1^1])$
$= \phi([N_{p+n}^{qm} + N_1^1, N_1^1])$
$= N_{p+n}^{qm} = N_p^q + N_n^m$
$= \phi([N_p^q + N_1^1, N_1^1]) + \phi([N_n^m + N_1^1, N_1^1])$.

Thus $(m_d(N_Z), \oplus)$ is isomorphic to $(m(N), +)$.

Similarly, we can show that

$$\phi([N_p^q + N_1^1, N_1^1] \odot [N_n^m + N_1^1, N_1^1]) = \phi([N_p^q + N_1^1, N_1^1]) \cdot \phi([N_n^m + N_1^1, N_1^1]).$$

So $(m_d(N_Z), \odot)$ is isomorphic to $(m(N), \cdot)$.

(4) For any $x = [N_a^b, N_c^d] \in m_d(Z)$, take $y = [N_a^b + N_1^1, N_1^1]$ and $z = [N_c^d + N_1^1, N_1^1] \in m_d(N_Z)$ such that $y + (-z) = [N_a^b + N_1^1, N_1^1] \oplus (-[N_c^d + N_1^1, N_1^1]) = [N_a^b + N_1^1, N_1^1] \oplus [N_1^1, N_c^d + N_1^1] = [N_a^b + N_1^1 + N_1^1, N_1^1 + N_c^d + N_1^1] = [N_a^b, N_c^d] = x$ (by Proposition 3.6 and Definition 3.5). □

Definition 3.16. Let us define each member of $m_d(Z)$ as a multi-integer. Let us also define each member of $m_d(N_Z)$ as a positive multi-integer.

Remark 3.17. From Theorem 3.15, it is clear that $m_d(N_Z)$ is embedded in $m(N)$ as a structure. So, one can call each member of $m(N)$ as the positive multi-integer.

Definition 3.18. We now define the set of negative multi-integers as follows:
Let us define the subset $(-m_d(N_Z))$ of $m_d(Z)$ by:

$$(-m_d(N_Z)) = \{[N_c^d, N_a^b] : [N_a^b, N_c^d] \in m_d(N_Z)\}.$$

Let us also define every member of $(-m_d(N_Z))$ as negative multi-integer.

Definition 3.19. (Positive multi-integer, Negative multi-integer, Zero, Special multi-integer, and Multi-zero)
Define $m_d(Z_S) = m_d(Z) - (m_d(N_Z) \cup (-m_d(N_Z)) \cup \{[N_1^1, N_1^1]\})$.
We have defined every member of $m_d(N_Z)$ as a positive multi-integer, every member of $(-m_d(N_Z))$ as a negative multi-integer, let us now define $[N_1^1, N_1^1]$ as zero and every member of $m_d(Z_S)$ as special multi-integer. Also we define any multi-integer of the form $[N_a^p, N_a^q]$ as multi-zero which is obviously either a special multi-integer or zero.

Theorem 3.20. *If the product of two multi-integers be zero, then at least one of them must be a multi-zero.*

Proof. For $[N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z)$,
$[N_a^b, N_c^d] \odot [N_p^q, N_r^s] = [N_1^1, N_1^1]$(by Remark 3.4)
$\Rightarrow [N_{ap+cr}^{bq}, N_{ar+cp}^{ds}] = [N_1^1, N_1^1]$
$\Rightarrow N_{ap+cr}^{bq} + N_1^1 = N_{ar+cp}^{ds} + N_1^1$
$\Rightarrow N_{ap+cr}^{bq} = N_{ar+cp}^{ds} \Rightarrow ap + cr = ar + cp$ and $bq = ds$ (by Remark 3.4)
$\Rightarrow (a - c)(p - r) = 0$ and $bq = ds$
$\Rightarrow$ (either $a = c$ or $p = r$) and $bq = ds$
$\Rightarrow$ atleast one of $[N_a^b, N_c^d]$ or $[N_p^q, N_r^s]$ is a multi-zero. □

Definition 3.21. (Order on $m_d(Z)$) Let $[N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z)$. We define $[N_a^b, N_c^d] > [N_p^q, N_r^s]$, if $[N_a^b, N_c^d] \oplus (-[N_p^q, N_r^s]) \in m_d(N_Z)$, i.e., $\exists [N_n^m + N_1^1, N_1^1] \in m_d(N_Z)$ such that

$[N_a^b, N_c^d] \oplus (-[N_p^q, N_r^s]) = [N_n^m + N_1^1, N_1^1]$ or$[N_a^b, N_c^d] = [N_p^q, N_r^s] \oplus [N_n^m + N_1^1, N_1^1]$.

Also, we define $[N_a^b, N_c^d] \geq [N_p^q, N_r^s]$ if $[N_a^b, N_c^d] > [N_p^q, N_r^s]$ or $[N_a^b, N_c^d] = [N_p^q, N_r^s]$.

Remark 3.22. Let us denote $[N_a^b, N_c^d] \oplus (-[N_p^q, N_r^s])$ as $[N_a^b, N_c^d] - [N_p^q, N_r^s]$.

Theorem 3.23. *(Partial order relation) $\geq$ defined on $m_d(Z)$ is a partial order relation.*

Proof. Immediately, $\geq$ is a reflexive relation on $m_d(Z)$.

For $x = [N_a^b, N_c^d]$, $y = [N_p^q, N_r^s] \in m_d(Z)$, let $[N_a^b, N_c^d] \geq [N_p^q, N_r^s]$ as well as $[N_p^q, N_r^s] \geq [N_a^b, N_c^d]$. If possible, let $[N_a^b, N_c^d] \neq [N_p^q, N_r^s]$. Then $x > y$ also $y > x$. Thus $(x - y)$ and $(y - x) = -(x - y)$ both $\in m_d(Z)$ which is impossible, since $(m_d(N_Z), \oplus)$ is isomorphic to $(m(N), +)$ and $(m(N), +)$ is a monoid but not a group. So our assumption is wrong. Hence $x = y$. Therefore $\geq$ is an antisymmetric relation on $m_d(Z)$.

Finally, for $x = [N_a^b, N_c^d]$, $y = [N_p^q, N_r^s]$, $z = [N_m^n, N_u^v] \in m_d(Z)$, let $x \geq y$ as well as $y \geq z$. If either $x = y$ or $y = z$, then immediately, $x \geq y$. Consider the case when $x > y$ and $y > z$. Then $(x-y), (y-z) \in m_d(N_Z)$. Thus $(x-y) \oplus (y-z) \in m_d(N_Z)$. Again, $(x - z) = (x - y) \oplus (y - z)$. So $(x - z) \in m_d(N_Z)$. Accordingly, $x \geq z$. Hence $\geq$ is a transitive relation on $m_d(Z)$. Therefore $\geq$ is a partial order relation on $m_d(Z)$. □

Proposition 3.24. *For* $[N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z)$*,* $[N_a^b, N_c^d] > [N_p^q, N_r^s]$ *if and only if* $a - c > p - r$ *and* $dq|bs$*.*

Proof. For $[N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z)$, let $[N_a^b, N_c^d] > [N_p^q, N_r^s]$. Then
$\exists [N_n^m + N_1^1, N_1^1] \in m_d(N_Z)$ such that

$$\begin{aligned}[N_a^b, N_c^d] &= [N_p^q, N_r^s] \oplus [N_n^m + N_1^1, N_1^1]\\ &= [N_p^q + N_n^m + N_1^1, N_r^s + N_1^1]\\ &= [N_p^q + N_n^m, N_r^s] \text{ (by Remark 3.5)}\\ &= [N_{p+n}^{qm}, N_r^s].\end{aligned}$$

Thus $a - c = p + n - r$ and $\frac{b}{d} = \frac{qm}{s}$ (by Remark 3.4). So $a - c > p - r$ and $dq|bs$.

The converse can be immediately be obtained by reversing the arguement. □

Remark 3.25. $(m_d(Z), \geq)$ is a poset but not a chain. Immediately, $(m_d(Z), \geq)$ do not obey the Law of Trichotomy, e.g., $[N_2^3 + N_1^1, N_1^1]$ and $[N_2^2 + N_1^1, N_1^1]$ are two incomparable elements of $(m_d(Z), \geq)$.

Proposition 3.26. $\forall [N_a^b, N_c^d] \in m_d(Z)$*,* $[N_a^b, N_c^d] \not> [N_a^b, N_c^d]$*.*

Proof. Since $a - c \not> a - c$, $\forall a, c \in N$ with $a \neq c$, from Proposition 3.24, the above proposition immediately follows. □

Proposition 3.27. *For* $[N_a^b, N_c^d], [N_e^f, N_g^h] \in m_d(Z)$*,* $[N_a^b, N_c^d] > [N_e^f, N_g^h]$ *if and only if* $[N_a^b, N_c^d] \oplus [N_u^v, N_w^x] > [N_e^f, N_g^h] \oplus [N_u^v, N_w^x]$*,* $\forall [N_u^v, N_w^x] \in m_d(Z)$*.*

Proof. For $[N_a^b, N_c^d], [N_e^f, N_g^h] \in m_d(Z)$, let $[N_a^b, N_c^d] > [N_e^f, N_g^h]$. Then from Proposition 3.24, $a - c > e - g$ and $df|bh$. Thus

$$(a + u) - (c + w) > (e + u) - (g + w), \forall u, w \in N$$

and

$$(dx)(fv)|(bv)(hx), \forall v, x \in N.$$

So $[N_{a+u}^{bv}, N_{c+w}^{dx}] > [N_{e+u}^{fv}, N_{g+w}^{hx}]$, i.e., $[N_a^b, N_c^d] \oplus [N_u^v, N_w^x] > [N_e^f, N_g^h] \oplus [N_u^v, N_w^x]$, $\forall [N_u^v, N_w^x] \in m_d(Z)$.

The converse can be immediately be obtain by reversing the arguement. □

Proposition 3.28. *For* $[N_a^b, N_c^d], [N_e^f, N_g^h], [N_u^v, N_w^x], [N_p^q, N_r^s] \in m_d(Z)$*, if* $[N_a^b, N_c^d] > [N_e^f, N_g^h]$ *and* $[N_u^v, N_w^x] > [N_p^q, N_r^s]$*, then* $[N_a^b, N_c^d] \oplus [N_u^v, N_w^x] > [N_e^f, N_g^h] \oplus [N_p^q, N_r^s]$*.*

Proof. Since $\forall a, c, e, g, u, w, p, r \in N$, $a - c > e - g$ and $u - w > p - r \Leftrightarrow (a + u) - (c + w) > (e + p) - (g + r)$. Also since $\forall b, d, f, h, v, s, q, x \in N$, $df|bh$ and $xq|vs \Leftrightarrow (dx)(fq)|(bv)(hs)$. Then the result immediately follows from Proposition 3.24. □

Proposition 3.29. *For* $[N_a^b, N_c^d], [N_e^f, N_g^h] \in m_d(Z)$*, if* $[N_a^b, N_c^d] \geq [N_e^f, N_g^h]$*, then* $[N_a^b, N_c^d] \oplus [N_2^1, N_1^1] > [N_e^f, N_g^h]$*.*

Proposition 3.30. $\forall [N_a^b, N_c^d] \in m_d(Z)$*,* $[N_a^b, N_c^d] \oplus [N_e^f, N_g^h] > [N_a^b, N_c^d]$*,* $\forall [N_e^f, N_g^h] \in m_d(N_Z)$*.*

Proposition 3.31. *For* $[N_a^b, N_c^d], [N_e^f, N_g^h], [N_u^v, N_w^x], [N_p^q, N_r^s] \in m_d(Z)$*,* $[N_a^b, N_c^d] \oplus [N_u^v, N_w^x] = [N_e^f, N_g^h] \oplus [N_p^q, N_r^s]$ *if and only if* $[N_a^b, N_c^d] = [N_e^f, N_g^h]$*.*

Proposition 3.32. *For* $[N_a^b, N_c^d], [N_e^f, N_g^h] \in m_d(Z)$, $[N_a^b, N_c^d] > [N_e^f, N_g^h]$ *if and only if* $[N_a^b, N_c^d] \odot [N_u^v, N_w^x] > [N_e^f, N_g^h] \odot [N_u^v, N_w^x]$, $\forall [N_u^v, N_w^x] \in m_d(N_Z)$.

Proof. Since $[N_u^v, N_w^x] \in m_d(N_Z)$, from Proposition 3.14, $u - w \in N$ and $x|v$. Then $a - c > e - g \Leftrightarrow (a-c)(u-w) > (e-g)(u-w)$, i.e., $(au + cw) - (aw + cu) > (eu + gw) - (ew + gu)$. Also, $df|bh \Leftrightarrow (dx)(fv)|(bv)(hx)$. Thus

$$[N_a^b, N_c^d] > [N_e^f, N_g^h]$$
$$\Leftrightarrow [N_a^b, N_c^d] \odot [N_u^v, N_w^x] > [N_e^f, N_g^h] \odot [N_u^v, N_w^x] \text{ (by Proposition 3.24).}$$ □

Definition 3.33. (General multiset, Real multiset and Natural multiset)

(i) Let X be a nonempty set. A general mset M drawn from X is characterized by a relation ρ_M between X and R (R being the set of all real numbers).

If $(x, r) \in \rho_M$, for some $x \in X$ and $r \in R - \{0\}$, then we represent it by writing $X_x^r \in M$.

(ii) Let X be a nonempty set. A real mset M drawn from X is characterized by a function $Count_M$ or $C_M : X \to R$.

If $C_M(x) = r$, for some $x \in X$ and $r \in R - \{0\}$, then we represent it by writing $X_x^r \in M$. Also, we shall denote a real mset M drawn from X as $\{X_{x_1}^{k_1}, X_{x_2}^{k_2}, ..., X_{x_n}^{k_n}, ...\}$, where $C_M(x_i) = k_i$, $x_i \in X$ and $k_i \in R - \{0\}$.

(iii) Let X be a nonempty set. A natural mset M drawn from X is characterized by a function $Count_M$ or $C_M : X \to N \cup \{0\}$.

If $C_M(x) = r$, for some $x \in X$ and $r \in N - \{0\}$, then we represent it by writing $X_x^r \in M$. Also, we shall denote a simple mset M drawn from X as $\{X_{x_1}^{k_1}, X_{x_2}^{k_2}, ..., X_{x_n}^{k_n}, ...\}$, where $C_M(x_i) = k_i$, $x_i \in X$ and $k_i \in R - \{0\}$. $k_i \in R - \{0\}$ is called the multiplicity of the element $x_i \in X$ in M.

Example 3.34. Consider the set $X = \{a, b, c\}$. Consider the relation ρ_M between X and R where $\rho_M = \{(a, \frac{1}{4}), (b, 3), (b, \sqrt{2})\}$. Then ρ_M represents a general mset M drawn from X which is given by $M = \{X_a^{\frac{1}{4}}, X_b^3, X_b^{\sqrt{2}}\}$.

Next, consider the function $C_M : X \to R$ defined by $C_M(a) = \frac{1}{4}$, $C_M(b) = 3$ and $C_M(c) = 0$. Then C_M represents a real mset M drawn from X which is given by $M = \{X_a^{\frac{1}{4}}, X_b^3\}$.

Finally, consider the function $C_M : X \to N \cup \{0\}$ defined by $C_M(a) = 1$, $C_M(b) = 3$ and $C_M(c) = 0$. Then C_M represents a natural mset M drawn from X which is given by $M = \{X_a^1, X_b^3\}$. Also, $m(N)$ is a general mset drawn from N.

Remark 3.35. (1) Clearly, general mset is a generalization of real mset. Also, real mset is a generalization of natural mset.

(2) Let A' and B' be two general msets drawn from the sets A and B respectively. If for $a \in A \cap B$ and $r \in R - \{0\}$, $A_a^r \in A'$ and $B_a^r \in B'$, then we shall consider $A_a^r = B_a^r$.

(3) We note that for all $i, j \in N$, Z_i^j and N_i^j both are immediately identical, i.e., $Z_i^j = N_i^j, \forall i, j \in N$.

Theorem 3.36. (Isomorphism theorem) *Let us consider the general mset* $m(\widehat{Z})$ *which is the universal relation between* Z *and* Q^+ *(*Q^+ *is the set of all positive rational numbers). i.e,* $Z_p^q \in m(\widehat{Z})$ *iff* $p \in Z$ *and* $q \in Q^+$.

Then $(m_d(Z), \oplus, \odot, \geq)$ *and* $(m(\widehat{Z}), \widehat{\oplus}, \widehat{\odot}, \widehat{\geq})$ *are isomorphic.*

Proof. Let us define two binary operations $\widehat{\oplus}$ and $\widehat{\odot}$ on $m(\widehat{Z})$ as follows:

For $Z_p^q, Z_r^s \in m(\widehat{Z})$, $Z_p^q \widehat{\oplus} Z_r^s = Z_{p+r}^{qs}$ and $Z_p^q \widehat{\odot} Z_r^s = Z_{pr}^{qs}$.

Also, define $\widehat{>}$ on $m(\widehat{Z})$ as follows: For $Z_p^q, Z_r^s \in m(\widehat{Z})$, $Z_p^q \widehat{>} Z_r^s$ iff $\exists Z_a^b \in m(\widehat{Z})$ with $a, b \in N$ such that $Z_p^q = Z_r^s \widehat{\oplus} Z_a^b$.

For $Z_p^q, Z_r^s \in m(\widehat{Z})$, we define $Z_p^q = Z_r^s$ iff $p = r$ and $q = s$.

Also, for $Z_p^q, Z_r^s \in m(\widehat{Z})$, we define $Z_p^q \widehat{\geq} Z_r^s$ iff $Z_p^q \widehat{>} Z_r^s$ or $Z_p^q = Z_r^s$.

Let us now define a function $\tau : m_d(Z) \to m(\widehat{Z})$ as follows:

$$\tau([N_a^b, N_c^d]) = Z_{a-c}^{\frac{b}{d}},\ [N_a^b, N_c^d] \in m_d(Z).$$

Then for $[N_a^b, N_c^d], [N_{a'}^{b'}, N_{c'}^{d'}] \in m_d(Z)$,

$[N_a^b, N_c^d] = [N_{a'}^{b'}, N_{c'}^{d'}]$

$\Leftrightarrow a - c = a' - c'$ and $\frac{b}{d} = \frac{b'}{d'}$ (by Remark 3.4.)

$\Leftrightarrow Z_{a-c}^{\frac{b}{d}} = Z_{a'-c'}^{\frac{b'}{d'}}$

$\Leftrightarrow \tau([N_a^b, N_c^d]) = \tau([N_{a'}^{b'}, N_{c'}^{d'}])$.

Thus τ is well-defined and one-one. Next let $Z_p^q \in m(\widehat{Z})$. Then $p \in Z$ and $q \in Q^+$. Thus $\exists a, c; b, d \in N$ such that $p = a - c$ and $q = \frac{b}{d}$. So $[N_a^b, N_c^d] \in m_d(Z)$. Also, $\tau([N_a^b, N_c^d]) = Z_{a-c}^{\frac{b}{d}} = Z_p^q$. Hence τ is onto. Therefore, τ is a bijection.

Now let, $[N_a^b, N_c^d], [N_{a'}^{b'}, N_{c'}^{d'}] \in m_d(Z)$. Then

$\tau([N_a^b, N_c^d] \oplus [N_{a'}^{b'}, N_{c'}^{d'}])$

$= \tau([N_a^b + N_{a'}^{b'}, N_c^d + N_{c'}^{d'}])$

$= \tau([N_{a+a'}^{bb'}, N_{c+c'}^{dd'}])$

$= Z_{(a+a')-(c+c')}^{\frac{bb'}{dd'}}$

$= Z_{a-c}^{\frac{b}{d}} \widehat{\oplus} Z_{a'-c'}^{\frac{b'}{d'}}$

$= \tau([N_a^b, N_c^d]) \widehat{\oplus} \tau([N_{a'}^{b'}, N_{c'}^{d'}])$.

Also, $\tau([N_a^b, N_c^d] \odot [N_{a'}^{b'}, N_{c'}^{d'}]) = \tau([N_{aa'+cc'}^{bb'}, N_{ac'+ca'}^{dd'}]) = Z_{(aa'+cc')-(ac'+ca')}^{\frac{bb'}{dd'}}$.

Furthermore,

$\tau([N_a^b, N_c^d]) \widehat{\odot} \tau([N_{a'}^{b'}, N_{c'}^{d'}])$

$= Z_{a-c}^{\frac{b}{d}} \widehat{\odot} Z_{a'-c'}^{\frac{b'}{d'}}$

$= Z_{(a-c)(a'-c')}^{\frac{bb'}{dd'}}$

$= Z_{(aa'+cc')-(ac'+ca')}^{\frac{bb'}{dd'}}$.

Thus $\tau([N_a^b, N_c^d] \odot [N_{a'}^{b'}, N_{c'}^{d'}]) = \tau([N_a^b, N_c^d]) \widehat{\odot} \tau([N_{a'}^{b'}, N_{c'}^{d'}])$.

Next, for $[N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z)$, let $[N_a^b, N_c^d] > [N_p^q, N_r^s]$. Then $\exists [N_m^n + N_1^1, N_1^1] \in m_d(N_Z)$ such that $[N_a^b, N_c^d] = [N_p^q, N_r^s] \oplus [N_m^n + N_1^1, N_1^1]$ or $[N_a^b, N_c^d] = [N_p^q + N_m^n + N_1^1, N_r^s + N_1^1] = [N_p^q + N_m^n, N_r^s] = [N_{p+m}^{qn}, N_r^s]$. Thus

$a - c = p + m - r$ and $\frac{b}{d} = \frac{qn}{s}$

$\Rightarrow Z_{a-c}^{\frac{b}{d}} = Z_{p-r+m}^{\frac{q}{s}n}$

$\Rightarrow Z_{a-c}^{\frac{b}{d}} = Z_{p-r}^{\frac{q}{s}} \widehat{\oplus} Z_m^n$

$\Rightarrow Z_{a-c}^{\frac{b}{d}} \widehat{>} Z_{p-r}^{\frac{q}{s}}$ (since $m, n \in N$)
$\Rightarrow \tau([N_a^b, N_c^d]) \widehat{>} \tau([N_p^q, N_r^s]$.
So $(m_d(Z), \oplus, \odot, \geq)$ and $(m(\widehat{Z}), \widehat{\oplus}, \widehat{\odot}, \widehat{\geq})$ are isomorphic. □

Remark 3.37. (Properties of $(m(\widehat{Z}), \widehat{\oplus}, \widehat{\odot}, \widehat{\geq})$
Since $(m_d(Z), \oplus, \odot, \geq)$ and $(m(\widehat{Z}), \widehat{\oplus}, \widehat{\odot}, \widehat{\geq})$ are isomorphic, $(m(\widehat{Z}), \widehat{\oplus})$ is a commutative group, $(m(\widehat{Z}), \widehat{\odot})$ is a commutative monoid and $\widehat{\odot}$ obey multi-distributive property over $\widehat{\oplus}$. Also, $(m(\widehat{Z}), \widehat{\geq})$ is a poset. Moreover, $\widehat{\geq}$ defined on $m(\widehat{Z})$ is an extension of $\geq$ defined on $m(N)$.

Remark 3.38. $(m(\widehat{Z}), \widehat{\oplus})$ is a commutative group and $(m(\widehat{Z}), \widehat{\odot})$ is a commutative monoid but $(m(\widehat{Z}), \widehat{\oplus}, \widehat{\odot})$ is not a ring, since $\widehat{\odot}$ can not be distributed over $\widehat{\oplus}$. But $\widehat{\odot}$ obeys multi-distributive property over $\widehat{\oplus}$. Let us now introduce a new concept of multi-ring replacing distributive property by multi-distributive property and $(m(\widehat{Z}), \widehat{\oplus}, \widehat{\odot})$ to be such a multi-ring.

Definition 3.39. (General mset drawn from a ring) Let $(X, +, \cdot)$ be ring. Let M be a general mset drawn from X. Consider two functions $\oplus : M \times M \to X \times R$ and $\odot : M \times M \to X \times R$ defined as follows: For $X_a^r, X_b^s \in M$,

$$X_a^r \oplus X_b^s = X_{a+b}^{rs} \text{ and } X_a^r \odot X_b^s = X_{a \cdot b}^{rs}.$$

Let us call $\oplus$ and $\odot$ respectively as m-addition and m-multiplication defined on M induced by the ring $(X, +, \cdot)$. Also let M be closed under both the operations $\oplus$ and $\odot$. Then immediately $\oplus$ obey commutative and associative property on M. So, $(M, \oplus)$ is then commutative semi groups. Also, immediately $\odot$ obey associative property on M. So, $(M, \odot)$ is a semi group. We define M to be a general mset drawn from the ring $(X, +, \cdot)$.

Theorem 3.40. *Let M be a general mset drawn from a ring $(X, +, \cdot)$. Then $\odot$ obey multi-distributive property over $\oplus$.*

Definition 3.41. (Multi-ring) Let M be a general mset drawn from a ring $(X, +, \cdot)$. $\oplus$ and $\odot$ are m-addition and m-multiplication defined on M induced by the ring $(X, +, \cdot)$. If the structure $(M, \oplus, \odot)$ satisfies the following:
(i) $(M, \oplus)$ is an abelian group,
(ii) $(M, \cdot)$ is a semigroup,
(ii) $\odot$ is multi-distributive over $\oplus$,
then we define $(M, \oplus, \odot)$ to be a multi-ring induced by the ring $(X, +, \cdot)$ on M, e.g., $(m(\widehat{Z}), \widehat{\oplus}, \widehat{\odot})$ is a multi-ring induced by the ring $(Z, +, \cdot)$.

Remark 3.42. Let $(M, \oplus, \odot)$ be the multi-ring induced by the ring $(X, +, \cdot)$ on the general mset M drawn from X. Let θ be the zero element in $(X, +, \cdot)$. Then X_θ^1 must be the zero element in $(M, \oplus, \odot)$. Let us also define any element in M of the form X_θ^r for some $r \in R - \{0\}$ to be the multi-zero elements of M such that the product of any element of the multi-ring with a multi-zero element of the same is again a multi-zero of the multi-ring. Clearly, the zero element in a multi-ring is a multi-zero element.

Remark 3.43. In the multi-ring $(m(\widehat{Z}), \widehat{\oplus}, \widehat{\odot})$, the non-zero multi-zeros are the only divisors of zero.

Theorem 3.44. *In a multi-ring, the non-zero multi-zero elements are divisors of zero.*

Definition 3.45. A multi-ring is said to have no non-multi-zero divisors of zero if its non-zero multi-zero elements are the only divisors of zero.

Remark 3.46. The multi-ring $(m(\widehat{Z}), \widehat{\oplus}, \widehat{\odot})$ induced by the ring $(Z, +, \cdot)$ has no non-multi-zero divisors of zero.

Definition 3.47. (Multi-integral domain) Let M be a general mset drawn from an integral domain $(X, +, \cdot)$. If the structure $(M, \oplus, \odot)$ satisfies the following:
(i) $(M, \oplus)$ is an commutative group,
(ii) $(M, \odot)$ is a commutative monoid,
(iii) $\odot$ is multi-distributive over $\oplus$,
(iv) M has no non-multi-zero divisors of zero,
then we define it to be a multi-integral domain induced by the integral domain $(X, +, \cdot)$ on M, e.g., $(m(\widehat{Z}), \widehat{\oplus}, \widehat{\odot})$ is a multi-integral domain induced by the integral domain $(Z, +, \cdot)$.

It is worth noting that if M be a general mset drawn from an integral domain $(X, +, \cdot)$, then immediately $(M, \oplus, \odot)$ has no non-multi-zero divisors of zero.

Theorem 3.48. $(m(\widehat{Z}), \widehat{\oplus}, \widehat{\odot}, \widehat{\geq})$ *is a partially ordered multi-integral domain drawn from the integral domain* $(Z, +, \cdot)$.

Definition 3.49. (Definition of Multi-integer system) A partially ordered multi-integral domain $(M, \oplus, \odot, \geq)$ is called a multi-integer system, if $\exists$ a subset N_M of M such that

(i) both $(N_M, \oplus)$ and $(N_M, \odot)$ are semigroups and under the same isomorphism $\phi : N_M \to N$, we have $(N_M, \oplus) \cong (m(N), +)$ and $(N_M, \odot) \cong (m(N), \cdot)$ as semigroup. Furthermore, for every $x, y \in N_M$, we have $x > y \Rightarrow \phi(x) > \phi(y)$.

(ii) for every $x \in M$, $\exists y, z \in N_M$ such that $x = y \oplus (-z)$.

Theorem 3.50. *(Existence and uniqueness of multi-integer system) Multi-integer system exists and any two multi-integer systems are isomorphic.*

Proof. We have previously shown that the system $(m(\widehat{Z}), \widehat{\oplus}, \widehat{\odot}, \widehat{\geq})$ is a partially ordered multi-integral domain drawn from the integral domain $(Z, +, \cdot)$.

Consider the subset $m(N_{\widehat{Z}}) = \{Z_a^b : a, b \in N\}$ of $m(\widehat{Z})$. Again, $a, b \in N$ implies $Z_a^b = N_a^b$. Then $m(N_{\widehat{Z}}) = m(N)$.

Also consider the restrictions of $\widehat{\oplus}$ and $\widehat{\odot}$ defined on $m(N_{\widehat{Z}})$. Then immediately, they are $+$ and $\cdot$ defined on $m(N)$.
Thus both $(m(N_{\widehat{Z}}), \widehat{\oplus})$ and $(m(N_{\widehat{Z}}), \widehat{\odot})$ are sub semigroups of $(m(\widehat{Z}), \widehat{\oplus})$ and $(m(\widehat{Z}), \widehat{\odot})$, respectively and they are isomorphic to $(m(N), +)$ and $(m(N), \cdot)$, respectively under the same isomorphism $\phi : m(N_{\widehat{Z}}) \to m(N)$ defined by $\phi(Z_p^q) = N_p^q, Z_p^q \in m(N_{\widehat{Z}})$.

Now let $Z_p^q, Z_m^n \in m(N_{\widehat{Z}})$ such that $Z_p^q \widehat{>} Z_m^n$. Since $p, m; q, n \in N$, $Z_p^q = N_p^q$ and $Z_m^n = N_m^n$.

Now $Z_p^q \widehat{>} Z_m^n \Rightarrow \exists Z_a^b \in m(\widehat{Z})$ with $a, b \in N$ such that $Z_p^q = Z_m^n \widehat{\oplus} Z_a^b$. Again $a, b \in N$ implies $Z_a^b = N_a^b$. Then $N_p^q = N_m^n \widehat{\oplus} N_a^b$. Thus $N_p^q = N_m^n + N_a^b$, i.e., $N_p^q > N_m^n$, i.e., $\phi(Z_p^q) > \phi(Z_m^n)$. So $\forall Z_p^q, Z_m^n \in m(N_{\widehat{Z}})$, $Z_p^q \widehat{>} Z_m^n \Rightarrow \phi(Z_p^q) > \phi(Z_m^n)$.

Finally let, $x = Z_a^r \in m(\widehat{Z})$. Then $a \in Z$ and $b \in Q^+$. Thus $\exists b, c; p, q \in N$ such that $a = b - c$ and $r = \frac{p}{q}$. So $x = Z_a^r = Z_{b-c}^{\frac{p}{q}} = Z_b^p \oplus Z_{-c}^{\frac{1}{q}} = Z_b^p \oplus (-Z_c^q) = y \oplus (-z)$, say, where $y = Z_b^p, z = Z_c^q \in m(N_{\widehat{Z}})$, since $b, c; p, q \in N$. Hence, $(m(\widehat{Z}), \widehat{\oplus}, \widehat{\odot}, \widehat{\geq})$ is a multi-integer system and so multi-integer system exists.

Next let $(m(Z), \oplus, \odot, \geq)$ and $(m(Z'), \oplus', \odot', \geq')$ be any two multi-integer systems ($m(Z)$ and $m(Z')$ being two general msets). Then by transitivity of isomorphism $\phi : m(N_Z) \rightarrow m(N_{Z'})$ such that

$\forall y, z \in m(N_Z)$, $\phi(y \oplus z) = \phi(y) \oplus' \phi(z)$ and $\phi(y \odot z) = \phi(y) \odot' \phi(z)$,
$y > z \Rightarrow \phi(y) >' \phi(z)$.

Also, for any $x \in m(Z)$, $\exists y_x, z_x \in m(N_Z)$ such that $x = y_x \oplus (-z_x)$.

Define $\psi : m(Z) \rightarrow m(Z')$ by $\psi(x) = \phi(y_x) \oplus' (-\phi(z_x))$. Then we can show that ψ is well defined. Also, we can show that ψ is bijective. Again, for any $u, v \in m(Z)$,

$$\begin{aligned}
\psi(u \oplus v) &= \psi[(y_u \oplus (-z_u)) \oplus (y_v \oplus (-z_v))] \\
&= \psi[(y_u \oplus y_v) \oplus (-(z_u \oplus z_v))] \\
&= \phi(y_u \oplus y_v) \oplus' (-\phi(z_u \oplus z_v)) \\
&= (\phi(y_u) \oplus' \phi(y_v)) \oplus' (-\phi(z_u) \oplus' (-\phi(z_v))) \\
&= (\phi(y_u) \oplus' (-\phi(z_u))) \oplus' (\phi(z_v) \oplus' (-\phi(z_v))) \\
&= \psi(u) \oplus' \psi(v).
\end{aligned}$$

Similarly, we can show that $\psi(u \odot v) = \psi(u) \odot' \psi(v)$.

Again, for any $u, v \in m(Z)$,

$$\begin{aligned}
u > v &\Rightarrow y_u \oplus (-z_u) > y_v \oplus (-z_v) \\
&\Rightarrow y_u \oplus z_v > y_v \oplus z_u \\
&\Rightarrow \phi(y_u \oplus z_v) >' \phi(y_v \oplus z_u) \\
&\Rightarrow \phi(y_u) \oplus' \phi(z_v) >' \phi(y_v) \oplus' \phi(z_u) \\
&\Rightarrow \phi(y_u) \oplus' (-\phi(z_u)) >' \phi(y_v) \oplus' (-\phi(z_v)) \\
&\Rightarrow \psi(u) >' \psi(v).
\end{aligned}$$

Thus ψ is an isomorphism from $(m(Z), \oplus, \odot, \geq)$ to $(m(Z'), \oplus', \odot', \geq')$.
So $(m(Z), \oplus, \odot, \geq) \cong (m(Z'), \oplus', \odot', \geq')$. Hence the uniqueness of the multi-integer system. □

Remark 3.51. Therefore, $(m(\widehat{Z}), \widehat{\oplus}, \widehat{\odot}, \widehat{\geq})$ is a multi-integer system. Also, multi-integer system is unique. So, from now on we shall abandon our multi-difference system and consider instead the multi-integer system $(m(\widehat{Z}), \widehat{\oplus}, \widehat{\odot}, \widehat{\geq})$. From now we will denote any multi-integer system by $(m(Z), \oplus, \odot, \geq)$. The copy of the multi-natural numbers embedded in $m(Z)$ will still denoted by $m(N)$ and it has all the properties that we have proven in paper [8], if we consider it in isolation.

Example 3.52. Consider three multi-integers Z_5^3, Z_3^4 and $Z_{-3}^{\frac{3}{5}}$.
Then $Z_3^7 \oplus Z_5^6 = Z_{3+5}^{7 \cdot 6} = Z_8^{42}$ and $Z_{-3}^{\frac{3}{5}} \odot Z_5^3 = Z_{(-3) \cdot 5}^{\frac{3}{5} \cdot 3} = Z_{-15}^{\frac{9}{5}}$.

4. Conclusion

In this paper, we have defined and studied multi-integer system as an extension of multi-natural number system. There is a huge scope of future research works in the field of multiset. Especially further study can be carried out in the following directions:

To study extension of multi-integer system towards multi-rational number system, multi-real number system etc.

To study throughly the properties of algebraic operations and order relations defined on them.

Also, to study the properties of general mset and multi-integral domain.

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References

[1] S. Alkhazdeh, A. R. Salleh and N. Hassan, Soft multiset theory, Appl. Math. Sci. 5 (72) (2011) 3561–3573.
[2] L. Anderson, Combinatrics of finite sets, Clarendon Press, Oxford 1987.
[3] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets ans Systems 20 (1986) 87–96.
[4] Wayne D. Blizart, Multiset theory, Notre Dame Journal of formal logic 30 (1) (1989) 36–66.
[5] J. Casasnovas and F. Rossello, Scalar and fuzzy cardinalities of crisp and fuzzy multisets, Int. J. Intell. Syst. 24 (2009) 587–623.
[6] K. Chakraborty, R. Biswas and S. Nanda, On Yager's theory of bags and fuzzy bags, Comput. Artificial Intelligence 18 (1) (1999) 1–17.
[7] K. Chakraborty, On bags and fuzzy bags, Adv. Soft Comput. Techniq. Appl. 25 (2000) 201–212.
[8] Debjyoti Chatterjee and S. K. Samanta, An axiomatic development of multi-natural numbers, International Journal of Mathematics Trends and Technology (IJMTT) 48 (2) (2017) 136–146.
[9] G. F. Clements, On multiset k-families, Discrete Mathematics 69 (2) (1988) 153–164.
[10] A. Bronselaer, D. V. Birtosm and G. D. Tre, A framework for multiset merging, Fuzzy Sets and Systems 191 (2012) 1–20.
[11] N. G. De Bruijin, Denumerations of rooted trees and multisets, Discrete Appl.Math. 6 (1) (1983) 25–33.
[12] N. Dershowith and Z. Manna, Proving termination with multiset orderings, Comm. ACM 22 (1979) 465–476.
[13] K. P. Girish and S. J. John, Multiset topologies induced by multiset relations, Inform. Sci. 188 (2012) 298–313.
[14] K. P. Girish and S. J. John, On multiset topologies, Theory and Applications of Mathematics and Computer Science 2 (1) (2012) 37–52.
[15] K. P. Girish and S. J. John, Rough multisets and information multisystem, Advances in Decision Sciences 2011 (2011) 1–17.
[16] A. Hallez, A. Bronselaer and G. D. Tre, Comparison of sets and multisets, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems, 17 (Suppl. 1) (2009) 153–172.
[17] J. L. Hickman, Anote on the concept of multiset, Bull. Austral. Math. Soc. 22 (2) (1980) 211–217.

[18] A. M. Ibrahim, J. A. Awolola and A. J. Alkali, An extension of the concept of n-level sets to multisets, Ann. Fuzzy Math. Inform. 11 (6) (2016) 855–862.
[19] S. P. Jena, S. K. Ghosh and B. K. Tripathy, On the theory of bags and lists, Inform. Sci. 132 (2001) 241–254.
[20] A. Klausner and N. Goodman, Multirelations-Semantics and languages, in: Proceedings of the 11th Conference on Very Large Data Bases VLDB'85, 1985, pp. 251-258.
[21] W. A. Kosters and J. F. J. Laros, Metrices for mining multisets, in: 27th SGAI International Conference on Innovative Techniques and Applications of Artificial Intelligence, 2007, pp. 293-303.
[22] D. E. Knuth, The art of computer programming, Volume 2, p. 636 - Semi numerical Algorithms, Second Edition-Wesley, Reading, Mass. 1981.
[23] B. Li, W. Peizhang and L. Xihui, Fuzzy bags with set-valued statistics, Comput. Math. Appl. 15 (1988) 3–39.
[24] P. Majumdar and S. K. Samanta, Soft multisets, J. Math. Comput. Sci. 2 (6) (2012) 1700–1711.
[25] S. Miyamoto, Operations for real-valued bags and bag relations, in ISEA-EUSFLAT, (2009) 612-617.
[26] I. S. Mumick, H. Pirahesh and R. Ramakrishnan, The magic of duplicates and aggregates, in: Proceedings of the 16th Conference on a Very Large Data Bases VLDB'90, 1990, pp. 264-267.
[27] Sk. Nazmul, P. Majumdar and S. K. Samanta, On multisets and multigroups, Ann. Fuzzy Math. Inform. 6 (3) (2013) 643–656.
[28] G. Paun and M. J. Perez-Jimenez, Membrane computing: brief introduction, recent results and applications, Bio Systems 85 (2006) 11–22.
[29] T. K. Shinoj and Sunil Jacob John, Intuitionistic fuzzy multisets and its application in medical diagnosis, World Academy of Science, Engineering and Technology 6 (2012) 1–28.
[30] D. Singh, A. M. Ibrahim, T. Yohanna and J. N. Singh, An overview of the applications of multisets, NOVI SAD J. MATH. 37 (3) (2007) 73–92.
[31] D. Singhand A. M. Ibrahim, A systematization of fundamentals of multisets, Lecturas Matematicas 29 (2008) 33–48.
[32] R. R. Yagar, On the theory of bags, Int. General Syst. 13 (1986) 23–37.

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