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Abstract. In this paper, we introduce the notion of tripolar fuzzy set to be able to deal with tripolar information as a generalization of fuzzy set, bipolar fuzzy set and intuitionistic fuzzy set.The tripolar fuzzy set representation is very useful in discriminating relevant elements, irrelevent elements and contrary elements.We introduce the notion of tripolar fuzzy interior ideal of $\Gamma-$ semigroup and study some of their algebraic properties.

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Keywords: Fuzzy set, Bipolar fuzzy set, Tripolar fuzzy set, Tripolar fuzzy interior ideal.

1. Introduction

As a generalization of ring, the notion of a $\Gamma-$ring was introduced by Nobusawa [18] in 1964. In 1995, Murali Krishna Rao [9, 10, 11, 12] introduced the notion of a $\Gamma-$semiring as a generalization of $\Gamma-$ring, ring, ternary semiring and semiring. In 1981, Sen [20] introduced the notion of a $\Gamma-$semigroup as a generalization of semigroup. The notion of a ternary algebraic system was introduced by Lehmer [7] in 1932. The theory of fuzzy sets is the most appropriate theory for dealing with uncertainty was introduced by L.A. Zadeh [21] in 1965. There are many extensions of fuzzy sets, for example, intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets, bipolar fuzzy sets, cubic sets etc. The fuzzification of algebraic structure was introduced by Rosenfeld [19] and he introduced the notion of fuzzy subgroups in 1971. K. T. Attensov [1] studied intuitionistic fuzzy sets Mandal [8] studied fuzzy ideals and fuzzy interior ideals in ordered semiring. Murali Krishna Rao [13, 14, 15, 16, 17] studied fuzzy ideals, fuzzy soft ideals and fuzzy interior ideals in ordered $\Gamma-$semirings.

Bipolar fuzzy sets are an extension of fuzzy sets whose memberely degree range is $[-1, 1]$. In 1994, Zhang [22] initiated the concept of bipolar fuzzy set as a generalization of fuzzy set. In 2000,K.M. Lee [5, 6] used the term bipolar valued fuzzy sets and applied it to algebraic structure. Jun et al [2, 3] studied intutionistic fuzzy interior ideals in semigroups and introduced the notion of bipolar fuzzy ideals and bipolar fuzzy filters in CI-algebras. K. J. Lee [4] studied bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCIalgebras.

In this paper, we introduce the notion of tripolar fuzzy interior ideal of $\Gamma-$ semigroup. We study some of their algebraic properties and characterization of tripolar fuzzy interior ideals are given.We prove that for any homomorphism ϕ from a $\Gamma-$semigroup M to a $\Gamma-$semigroup N, if $A = (\mu_A, \lambda_A, \delta_A)$ is a tripolar fuzzy interior ideal of M then homomorphic image $\phi(A) = ((\phi(\mu_A), \phi(\lambda_A), \phi(\delta_A))$ is a tripolar fuzzy interior ideal of N and $B = (\mu_B, \lambda_B, \delta_B)$ is a tripolar fuzzy interior ideal of N then the pre image $\phi^{-1}(B) = \left(\phi^{-1}(\mu_B), \phi^{-1}(\lambda_B), \phi^{-1}(\delta_B)\right)$ is a tripolar fuzzy interior ideal of M.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. Let M and Γ be two non-empty sets. Then M is called a $\Gamma-$semigroup, if it satisfies

(i) $x\alpha y \in M$,

(ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M,\ \alpha, \beta \in \Gamma$.

Definition 2.2. Let M and N be $\Gamma-$ semigroups. A mapping $f : M \to N$ is called a homomorphism, if $f(a\alpha b) = f(a)\alpha f(b)$, for all $a, b \in M, \alpha \in \Gamma$.

Definition 2.3. A non-empty subset A of $\Gamma-$semigroup M is called

(i) a $\Gamma-$subsemigroup of M, if $A\Gamma A \subseteq A$,

(ii) a quasi ideal of M, if $A\Gamma M \cap M\Gamma A \subseteq A$ and $A\Gamma A \subseteq A$.

(iii) a bi-ideal of M, if $A\Gamma M\Gamma A \subseteq A$ and $A\Gamma A \subseteq A$.

(iv) an interior ideal of M, if $M\Gamma A\Gamma M \subseteq A$ and $A\Gamma A \subseteq A$.

(v) a left (right) ideal of M, if $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$.

(vi) an ideal of M, if $A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.

Definition 2.4. Let M be a non-empty set. A mapping $\mu : M \to [0, 1]$ is called a fuzzy subset of M.

Definition 2.5. If μ is a fuzzy subset of M, for $t \in [0, 1]$, then the set $\mu_t = \{x \in M \mid \mu(x) \geq t\}$ is called a level subset of M with respect to a fuzzy subset μ.

Definition 2.6. A fuzzy subset $\mu : M \to [0, 1]$ is a non-empty fuzzy subset, if μ is not a constant function.

Definition 2.7. For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$, for all $x \in M$.

Definition 2.8. A fuzzy subset μ of a $\Gamma-$semigroup M is called

(i) a fuzzy $\Gamma-$subsemigroup of M, if $\mu(x\alpha y) \geq min\ \{\mu(x), \mu(y)\}$, for all $x, y \in M,\ \alpha \in \Gamma$.

(ii) a fuzzy left (right) ideal of M, if $\mu(x\alpha y) \geq \mu(y)\ (\mu(x))$, for all $x, y \in M,\ \alpha \in \Gamma$.

(iii) a fuzzy ideal of M, if $\mu(x\alpha y) \geq max\ \{\mu(x), \mu(y)\}$, for all $x, y \in M,\ \alpha \in \Gamma$.
(iv) a fuzzy bi-ideal of M, if $\mu \circ \chi_M \circ \mu \subseteq \mu$.
(v) a fuzzy quasi -ideal of M, if $\mu \circ \chi_M \cap \chi_M \circ \mu \subseteq \mu$.

Definition 2.9. Let M and N be $\Gamma-$semigroups, $\phi : M \to N$ be a homomorphism of $\Gamma-$semigroups and f be a fuzzy subset of $\Gamma-$semigroup M. We define a fuzzy subset $\phi(f)$ of $\Gamma-$semigroup N by

$$\phi(f)(y) = \begin{cases} \sup\limits_{x \in \phi^{-1}(y)} f(x), & \text{if } \ \phi^{-1}(y) \neq \varnothing \\ 0, & \text{otherwise.} \end{cases}$$

We call $\phi(f)$ is the image of f under ϕ.

3. Tripolar fuzzy interior ideal

In this section, we introduce the notion of tripolar fuzzy set can deal with tripolar information as a generalization of fuzzy set, bipolar fuzzy set and intuitionistic fuzzy set. We also introduce the notion of tripolar fuzzy interior ideals of $\Gamma-$semigroup and study some of their algebraic properties

Definition 3.1. A fuzzy set A of a universe set X is said to be tripolar fuzzy set, if

$$A = \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in X \text{ and } 0 \leq \mu_A(x) + \lambda_A(x) \leq 1\},$$

where $\mu_A : X \to [0,1],\ \lambda_A : X \to [0,1],\ \delta_A : X \to [-1, 0]$.
The membership degree $\mu_A(x)$ characterizes the extent that the element x satisfies to the property corresponding to tripolar fuzzy set A,$\lambda_A(x)$ characterizes the extent that the element x satisfies to the not property(irrelevant) corresponding to tripolar fuzzy set A and $\delta_A(x)$ characterizes the extent that the element x satisfies to the implicit counter property of tripolar fuzzy set A. For simplicity $A = (\mu_A, \lambda_A, \delta_A)$ has been used for $A = \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in X, 0 \leq \mu_A(x) + \lambda_A(x) \leq 1\}$

We can think of a tripolar fuzzy set as simple as the following example.

Example 3.2. The taste of food stuffs is a tripolar fuzzy set and

$$A = \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in X\}.$$

Assuming the sweet taste of food stuff as a positive membership value $\mu_A(x)$ i.e. the element x is satisfying the sweet property. Then bitter taste of food stuff as a negative membership value $\delta_A(x)$ i.e. the element x is satisfying the bitter property, and the remaining tastes of food stuffs like acidic, chilly etc., as a non membership value $\lambda_A(x)$ i.e., the element is satisfying irrelevent to the sweet property. A tripolar fuzzy set A is a generalization of fuzzy set, bipolar fuzzy set and intuitionistic fuzzy set.

Definition 3.3. A tripolar fuzzy set $A = (\mu_A, \lambda_A, \lambda_A)$ of $\Gamma-$semigroup M is called a tripolar fuzzy interior ideal of M, if
(i) $\mu_A(x\alpha y) \geq \min\{\mu_A(x), \mu_A(x)\}$,
(ii) $\lambda_A(x\alpha y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$,
(iii) $\delta_A(x\alpha y) \leq \max\{\delta_A(x), \delta_A(y)\}$,
(iv) $\mu_A(x\alpha z\beta y) \geq \mu_A(z)$,

(v) $\lambda_A(x\alpha z\beta y) \leq \lambda_A(z)$,
(vi) $\delta_A(x\alpha z\beta y) \leq \delta_A(z)$, for all $x, y, z \in M$, $\alpha, \beta \in \Gamma$.

Example 3.4. Let $M = \{a, b, c, d\}$ and $\Gamma = \{\beta, \alpha\}$. Then M is a $\Gamma-$semigroup if the ternary operation on M is defined as follows

β	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	a	a	a	a
d	a	a	a	d

and

α	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	a	a	a	a
d	a	a	a	d

.

Tripolar fuzzy set is defined as follows $A = (\mu_A, \lambda_A, \delta_A)$,
where $\mu_A = \{(a, 0.9), (b, 0.7), (c, 0.1), (d, 0.1)\}, \lambda_A = \{(a, 0.1), (b, 0.2), (c, 0.3), (d, 0.8)\}$, $\delta_A = \{(a, -0.7), (b, -0.8), (c, -0.3), (d, -0.1)\}$. Then tripolar fuzzy set A is a tripolar fuzzy interior ideal of $\Gamma-$semigroup M.

Theorem 3.5. *If a tripolar fuzzy set $A = (\mu_A, \lambda_A, \delta_A)$ of $\Gamma-$semigroup M is an interior ideal of $\Gamma-$semigroup M then $(\mu_A, \overline{\mu}_A, \delta_A)$ where $\overline{\mu}_A = 1 - \mu_A$, is a tripolar fuzzy interior ideal of $\Gamma-$semigroup M.*

Proof. Let $x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} \overline{\mu}_A(x\alpha y) &= 1 - \mu_A(x\alpha y) \leq 1 - \min\{\mu_A(x), \mu_A(y)\} \\ &= \max\{1 - \mu_A(x), 1 - \mu_A(y)\} \\ &= \max\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} \end{aligned}$$

and

$$\overline{\mu}_A(x\alpha z\beta y) = 1 - \mu_A(x\alpha z\beta y) \leq 1 - \mu_A(z) = \overline{\mu}_A(z).$$

Thus $(\mu_A, \overline{\mu}_A, \delta_A)$ is a tripolar fuzzy interior ideal of $\Gamma-$semigroup M. □

Definition 3.6. Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy set of $\Gamma-$semigroup M and $\alpha \in [0, 1]$. Then the sets $\mu_{A,\alpha} = \{x \in M \mid \mu_A(x) \geq \alpha\}$, $\lambda_{A,\alpha} = \{x \in M \mid \lambda_A(x) \leq \alpha\}$ and $\delta_{A,-\alpha} = \{x \in M \mid \delta_A(x) \leq -\alpha\}$ are called a $\mu-$level $\alpha-$cut,a $\lambda-$level $\alpha-$cut and a $\delta-$level $-\alpha-$cut of A respectively.

Theorem 3.7. *If $A = (\mu_A, \lambda_A, \delta_A)$ is a tripolar fuzzy interior ideal of $\Gamma-$semigroup M then $\mu-$level $t-$cut, $\lambda-$level $t-$cut and $\delta-$level $-t-$cut of A are interior ideals of $\Gamma-$semigroup M, for all $t \in Im(\mu_A) \cap Im(\lambda_A) \subseteq [0, 1]$ and $-t \in Im(\delta_A)$.*

Proof. Let $t \in Im(\mu_A) \cap Im(\lambda_A) \subseteq [0, 1]$ and $-t \in Im(\delta_A)$ and $x, y \in \mu_{A,t}$, $\alpha \in \Gamma$. Then $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$. Thus $\mu_A(x\alpha y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq t$. So $x\alpha y \in \mu_{A,t}$. Hence $\mu_{A,t}$ is a $\Gamma-$ subsemigroup of M.

Let $x, y \in M, z \in \mu_{A,t}$ and $\alpha, \beta \in M$. Then $\mu_A(x\alpha z\beta y) \geq \mu_A(z) \geq t$. Thus $x\alpha z\beta y \in \mu_{A,t}$. So $\mu_{A,t}$ is an interior ideal of $\Gamma-$semigroup M.

Suppose $x, y \in \lambda_{A,t}$ and $\alpha \in \Gamma$. Then $\lambda_A(x) \leq t$ and $\lambda_A(y) \leq t$. Thus

$$\lambda_A(x\alpha y) \leq \max\{\lambda_A(x), \lambda_A(y)\} \leq t. \text{Therefore } x\alpha y \in \lambda_{A,t}.$$

So $\lambda_{A,t}$ is a $\Gamma-$ subsemigroup of M.

Let $x, y \in M, z \in \lambda_{A,t}$ and $\alpha, \beta \in \Gamma$. Then $\lambda_A(x\alpha z\beta y) \leq \lambda_A(z) \leq t$. Thus $x\alpha z\beta y \in \lambda_{A,t}$.

Suppose $x, y \in \delta_{A,-t}$ and $\alpha \in \Gamma$. Then $\delta_A(x) \leq -t, \delta_A(y) \leq -t$. Thus $\delta_A(x\alpha y) \leq \max\{\delta_A(x), \delta_A(y)\} \leq -t$, Thus $x\alpha y \in \delta_{A,-t}$.

Let $x, y \in M, z \in \delta_{A,-t}$ and $\alpha \in \Gamma$. Then $\delta_A(x\alpha z\beta y) \leq \delta_a(z) \leq -t$, Thus $x\alpha z\beta y \in \delta_{A,-t}$. So $\delta_{A,-t}$ is an interior ideal of $\Gamma-$semigroup M. □

Theorem 3.8. *If $A = (\mu_A, \lambda_A, \delta_A)$ is a tripolar fuzzy interior ideal of $\Gamma-$semigroup M then*
(1) $\mu_A(x) = \sup\{\alpha \in [0, 1] \mid x \in \mu_{A,\alpha}\}$
(2) $\lambda_A(x) = \inf\{\alpha \in [0, 1] \mid x \in \lambda_{A,\alpha}\}$
(3) $\delta_A(x) = \inf\{\alpha \in [-1, 0] \mid x \in \delta_{A,\alpha}\}$, *for all* $x \in M, \alpha \in \Gamma$.

Proof. Proofs of (1) and (2) are similar to proof of Theorem 3.12 in [3], so we omit the proof. Let $\eta = \inf\{\alpha \in [-1, 0] \mid x \in \delta_{A,\alpha}\}$. Then $\inf\{\alpha \in [-1, 0] \mid x \in \delta_{A,\alpha}\} < \eta + \epsilon$, for any $\epsilon > 0$, $\Rightarrow \ \alpha < \eta + \epsilon$, for some $\alpha \in [-1, 0]$, $x \in \delta_{A,\alpha}$, $\Rightarrow \delta_A(x) \leq \eta$, since $\delta_A(x) \leq \alpha$. Let $\delta_A(x) = \beta$. Thus $x \in \delta_{A,\beta} \Rightarrow \beta \in \{\alpha \in [-1, 0] \mid x \in \delta_{A,\alpha}\}, \Rightarrow \inf\{\alpha \in [-1, 0] \mid x \in \delta_{A,\alpha}\} \leq \beta, \Rightarrow \eta \leq \beta = \delta_A(x)$. $\Rightarrow \delta_A(x) = \eta$. So $\delta_A(x) = \inf\{\alpha \in [-1, 0] \mid x \in \delta_{A,\alpha}\}$. Hence the theorem holds. □

The following proof of the theorem is similar to proof of the Theorem 3.9 in [3]. Hence we omit the proof of the following theorem.

Theorem 3.9. *Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy set in $\Gamma-$semigroup M such that non-empty sets $\mu_{A,\alpha}, \lambda_{A,\alpha}, \delta_{A,-\alpha}$ are interior ideals of M, for all $\alpha \in [0, 1]$. Then A is a tripolar fuzzy interior ideal of M.*

Theorem 3.10. *A tripolar fuzzy set $A = (\mu_A, \lambda_A, \delta_A)$ is a fuzzy interior ideal of $\Gamma-$semigroup M if and only if fuzzy subsets $\mu_A, \overline{\lambda}_A, \delta_A$ are fuzzy interior ideals of $\Gamma-$semigroup M.*

Proof. Suppose $A = (\mu_A, \lambda_A, \delta_A)$ is a tripolar fuzzy interior ideal of $\Gamma-$semigroup M. Then obviously, μ_A, δ_A are fuzzy interior ideals of M.

Let $x, y \in M,\ \alpha \in \Gamma$. Then for all $x, y \in M,\ \alpha \in \Gamma$,

$$\begin{aligned}\overline{\lambda}_A(x\alpha y) &= 1 - \lambda_A(x\alpha y) \geq 1 - \max\{\lambda_A(x), \lambda_A(y)\} \\ &= \min\{1 - \lambda_A(x), 1 - \lambda_A(y)\} = \min\{\overline{\lambda}_A(x), \overline{\lambda}_A(y)\}.\end{aligned}$$

Suppose $x, y, z \in M, \alpha, \beta \in \Gamma$. Then

$$\overline{\lambda}_A(x\alpha z\beta y) = 1 - \lambda_A(x\alpha z\beta y) \geq 1 - \lambda_A(z) = \overline{\lambda}_A(z).$$

Thus $\overline{\lambda}$ is an fuzzy interior ideal of M.

Conversely, suppose that $\mu_A, \overline{\lambda}_A, \delta_A$ are fuzzy interior ideals of $\Gamma-$semigroup M. Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned}\lambda_A(x\alpha y) &= 1 - \overline{\lambda}_A(x\alpha y) \\ &\geq \max\{1 - \overline{\lambda}_A(x), 1 - \overline{\lambda}_A(y)\} \\ &= \max\{\lambda_A(x), \lambda_A(y)\}.\end{aligned}$$

Thus $\overline{\lambda}_A(x\alpha z\beta y) \geq \overline{\lambda}_A(z)$. So $1 - \lambda_A(x\alpha z\beta y) \geq 1 - \lambda_A(z)$. Hence $\lambda_A(x\alpha z\beta y) \leq \lambda_A(z)$. Therefore this completes the proof. □

Corollary 3.11. *A tripolar fuzzy set $A = (\mu_A, \lambda_A, \delta_A)$ is a tripolar fuzzy interior ideal of $\Gamma-$semigroup M if and only if the tripolar fuzzy sets $(\mu_A, \overline{\mu}_A, \delta_A)$ and $(\overline{\lambda}_A, \lambda_A, \delta_A)$ are tripolar fuzzy interior ideals of $\Gamma-$semigroup M.*

Definition 3.12. Let $f : X \to Y$ be a map and let $A = (\mu_A, \lambda_A, \delta_A)$ and $B = (\mu_B, \lambda_B, \delta_B)$ be tripolar fuzzy sets in X and Y, respectively. Then pre-image of B under f, denoted by $f^{-1}(B)$ is a tripolar fuzzy set in X defined by:

$$f^{-1} = \big(f^{-1}(\mu_B), f^{-1}(\lambda_B), f^{-1}(\delta_B)\big),$$

where $f^{-1}(\mu_B) = \mu_B(f), f^{-1}(\lambda_B) = \lambda_B(f)$ and $f^{-1}(\delta_B) = \delta_B(f)$.

Theorem 3.13. *Let $f : M \to N$ be a homomorphism of $\Gamma-$semigroups. If $B = (\mu_b, \lambda_B, \delta_B)$ is a tripolar fuzzy interior ideal of $\Gamma-$semigroup N then $f^{-1}(B) = \big(f^{-1}(\mu_B), f^{-1}(\lambda_B), f^{-1}(\delta_B)\big)$ is a tripolar fuzzy interior ideal of $\Gamma-$semigroup M.*

Proof. Suppose $B = (\mu_B, \lambda_B, \delta_B)$ is a tripolar fuzzy interior ideal of $\Gamma-$semigroup N and $x, y \in M, \alpha \in \Gamma$. Then

$$\begin{aligned} f^{-1}\big(\mu_B(x\alpha y)\big) &= \mu_B(f(x\alpha y)) = \mu_B\big(f(x)\alpha f(y)\big) \\ &\geq \min\{\mu_B(f(x)), \mu_B(f(y))\} \\ &= \min\{f^{-1}(\mu_B(x)), f^{-1}(\mu_B(y))\}. \end{aligned}$$

Suppose $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then we have

$$\begin{aligned} f^{-1}\big(\mu_B(x\alpha z\beta y)\big) &= \mu_B(f(x\alpha z\beta y)) = \mu_B\big(f(x)\alpha f(z)\beta f(y)\big) \\ &\geq \mu_B f(z) = f^{-1}(\mu_B f(z)), \\ f^{-1}\big(\lambda_B(x\alpha y)\big) &= \lambda_B(f(x\alpha y)) = \lambda_B\big(f(x)\alpha f(y)\big) \\ &\leq \max\{\lambda_B(f(x)), \lambda_B(f(y))\} \\ &= \max\{f^{-1}(\lambda_B(x)), f^{-1}(\lambda_B(y))\}, \\ f^{-1}\big(\lambda_B(x\alpha z\beta y)\big) &= \lambda_B(f(x\alpha z\beta y)) \\ &= \lambda_B\big(f(x)\alpha f(z)\beta f(y)\big) \\ &\leq \lambda_B(f(z)) = f^{-1}(\lambda_B(f(z))), \\ f^{-1}\big(\delta_B(x\alpha y)\big) &= \delta_B(f(x\alpha y)) = \delta_B\big(f(x)\alpha f(y)\big) \\ &\leq \max\{\delta_B(f(x)), \delta_B(f(y))\} \\ &= \max\{f^{-1}(\delta_B(x)), f^{-1}(\delta_B(y))\}, \\ f^{-1}\big(\delta_B(x\alpha z\beta y)\big) &= \delta_B(f(x\alpha z\beta y)) = \delta_B\big(f(x)\alpha f(z)\beta f(y)\big) \\ &\leq \delta_B(f(z)) = f^{-1}(\delta_B f(z)). \end{aligned}$$

Thus $f^{-1}(B) = \big(f^{-1}(\mu_B), f^{-1}(\lambda_B), f^{-1}(\delta_B)\big)$ is a tripolar fuzzy interior ideal of $\Gamma-$semigroup M. □

Theorem 3.14. *Let I be an interior ideal of $\Gamma-$semigroup M and $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy set of M defined by:*

$$\mu_A(x) = \begin{cases} \alpha_0, & \text{if } x \in I \\ \alpha_1, & \text{otherwise,} \end{cases}$$

$$\lambda_A(x) = \begin{cases} \beta_0, & \text{if } x \in I \\ \beta_1, & \text{otherwise,} \end{cases}$$

$$\delta_A(x) = \begin{cases} \gamma_0, & \text{if } x \in I \\ \gamma_1, & \text{otherwise,} \end{cases}$$

for all $x \in M$ and $\alpha_i, \beta_i \in [0,1]$ such that $\alpha_0 > \alpha_1, \beta_0 > \beta_1$ and $\alpha_i + \beta_i \leq 1$, for $i = 0, 1, \gamma_0 > \gamma_1, \gamma_0, \gamma_1 \in [-1, 0]$.

Then $A = (\mu_A, \lambda_A, \delta_A)$ *is a tripolar fuzzy interior ideal of* $\Gamma-$*semigroup* M *and* $\mu_{A,\alpha_0} = \lambda_{A,\beta_0} = \delta_{A,\gamma_0} = I$.

Proof. Obviously, $A = (\mu_A, \lambda_A, \delta_A)$ is tripolar fuzzy $\Gamma-$subsemigroup of M. Suppose $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

case (i) If $z \notin I$, then $\mu_A(x\alpha z\beta y) \geq \alpha_1 = \mu_A(z)$, $\lambda_A(x\alpha z\beta y) \leq \beta_1 = \lambda_A(z)$, $\delta_A(x\alpha z\beta y) \leq \gamma_1 = \delta_A(z)$.

case (ii) If $z \in I$, then $\mu_A(x\alpha z\beta y) = \alpha_0 = \mu_A(z)$, $\lambda_A(x\alpha z\beta y) = \beta_0 = \lambda_A(z)$, $\delta_A(x\alpha z\beta y) = \gamma_0 = \delta_A(z)$.

Thus $A = (\mu_A, \lambda_A, \delta_A)$ is a tripolar fuzzy interior ideal of $\Gamma-$semigroup M. Obviously, by definitions of μ_A, λ_A and δ_A, $\mu_{A,\alpha_0} = \lambda_{A,\beta_0} = \delta_{A,\gamma_0} = I$. So the theorem holds. □

Corollary 3.15. *Let* χ_I *be the characteristic function of an interior ideal* I *of* $\Gamma-$*semigroup* M. *Then tripolar fuzzy set* $(\chi_I, \overline{\chi}_I, \delta_I)$, *where*

$$\delta_I(x) = \begin{cases} 0, & \text{if } x \in I \\ -1, & \text{otherwise} \end{cases}$$

is tripolar interior fuzzy ideal of $\Gamma-$*semigroup* M.

Definition 3.16. Let $\phi : M \to N$ be a homomorphism of $\Gamma-$semigroups M, N and $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy set of $\Gamma-$semigroup M. Then A is said to be ϕ homomorphism invariant, provided that if $\phi(x) = \phi(y)$, then

(i) $\mu_A(x) = \mu_A(y)$, (ii) $\lambda_A(x) = \lambda_A(y)$, (iii) $\delta_A(x) = \delta_A(y)$, for all $x, y \in M$.

Theorem 3.17. *Let* M *and* N *be* $\Gamma-$*semigroups and* $\phi : M \to N$ *be an onto homomorphism. If* A *is a homomorphism* ϕ *invariant tripolar interior ideal of* $\Gamma-$*semigroup* M *then image of* A *under homomorphism* ϕ *is a tripolar fuzzy interior ideal of* $\Gamma-$*semigroup* N.

Proof. Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar interior ideal of $\Gamma-$semigroup M and $x, y \in N, \alpha \in \Gamma$. Then there exist $a, b \in M$ such that $\phi(a) = x, \phi(b) = y$, $\phi(\mu_A(x\alpha y)) = \mu_A(a\alpha b) \geq \min\{\mu_A(a), \mu_A(b)\} = \min\{\phi(\mu_A)(x), \phi(\mu_A)(y)\}$.

Suppose $x, y, z \in N, \alpha, \beta \in \Gamma$. Then there exist $a, b, c \in M$ such that $\phi(a) = x, \phi(b) = y$ and $\phi(c) = z$. Thus $\phi(\mu_A(x\alpha z\beta y)) = \mu_A(a\alpha c\beta b) \geq \mu_A(c) = \phi(\mu_A(z))$. So $\phi(\mu_A)$ is a fuzzy interior ideal of $\Gamma-$semigroup M. On one hand,

$\phi(\lambda_A(x\alpha y)) = \lambda_A(a\alpha b) \leq \min\{\lambda_A(x), \lambda_A(b)\} = \min\{\phi(\lambda_A)(x), \phi(\lambda_A)(y)\}$

and

$\phi(\lambda_A(x\alpha z\beta y) = \lambda_A(a\alpha c\beta b) \leq \lambda_A(c) = \phi(\lambda_A(z))$.

Hence $\phi(\lambda_A)$ is a fuzzy interior ideal of $\Gamma-$semigroup N. On the other hand,

$\phi(\delta_A(x\alpha y)) = \delta_A(a\alpha b) \leq \min\{\delta_A(x), \delta_A(b)\} = \min\{\phi(\delta_A)(x), \phi(\delta_A)(y)\}$

and

$\phi(\delta_A(x\alpha z\beta y) = \delta_A(a\alpha c\beta b) \leq \delta_A(c) = \phi(\delta_A(z))$.

Therefore $\phi(\delta_A)$ is a fuzzy interior ideal of $\Gamma-$semigroup N. Hence $\phi(A)$ is a tripolar fuzzy interior ideal of $\Gamma-$semigroup N. □

4. Conclusion

In this paper, we introduced the notion of tripolar fuzzy set to be able to deal with tripolar information as a generalization of fuzzy set, bipolar fuzzy set and intuitionistic fuzzy set. We also introduced the notion of tripolar fuzzy interior ideals of $\Gamma-$semigroup. We proved that for any homomorphism ϕ from a $\Gamma-$semigroup M to a $\Gamma-$semigroup N, if A is a tripolar fuzzy interior ideal of M then homomorphic image $\phi(A)$ is a tripolar fuzzy interior ideal of N and B is a tripolar fuzzy interior ideal of N then the pre image $\phi^{-1}(B)$ is a tripolar fuzzy interior ideal of M.

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