Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 15, No. 3, (June 2018) pp. 207–226
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2018.15.3.207

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 15, No. 3, June 2018

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Received 6 December 2017; Revised 27 December 2017; Accepted 25 January 2018

Abstract. For an ITS (X, τ), we introduce an intuitionistic hyperspace $(2^{(X,\tau)}, \tau_v)$ [resp. $(2^{(X,\tau_I)}, \tau_{I,v})$ and $(2^{(X,\tau_{IV})}, \tau_{IV,v})$] of τ-type [resp. τ_I-type and τ_{IV}-type]. And we give some examples of each hyperspace and obtain some properties of the hyperspace $(2^{(X,\tau)}, \tau_v)$. Next, we find some relationships between openess in an ITS (X, τ) and its hyperspace $2^{(X,\tau)}$. Finally, we introduce an intuitionistic set-valued mapping and study its some continuities.

2010 AMS Classification: 54A40, 54B20

Keywords: Intuitionistic topological space, Intuitionistic locally compact space, Intuitionistic connected space, $T_3(i)$-space, Intuitionistic hyperspace, Intuitionistic set-valued mapping.

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1. Introduction

In 1983, Atanassove [1] introdued the concept of intuitionstic fuzzy sets as a generalization of a fuzzy set proposed by Zadeh [20]. In 1996, Coker [5] introduced the concept of an intuitionistic set (called an intuitionistic crisp set by Salama et al.[17]) as the generalzation of an ordinary set and the specialization of an intuitionistic fuzzy set. After that time, many researchers [3, 4, 6, 7, 8, 15, 16, 18, 19] applied the notion to topology. Recently, Kim et al. [10] studied the category **ISet** composed of intuitionistic sets and morphisms between them in the sense of a topological universe. Also, Kim et al. [11] studied some additional properties and give some examples related to intuitionistic closures and intuitionistic interiors in intuitionistic topological spaces. Lee et al. [14] introduced some types of continuities, open and closed mappings, and intuitionistic subspaces. In particular, Bavithra et al. [2] studied intuitionistic Fell topological spaces.

In this paper, first of all, we list some concepts related to intuitionistic sets and some results obtained by [5, 6, 7, 10, 11]. Second, for an ITS (X, τ), we introduce an intuitionistic hyperspace $(2^{(X,\tau)}, \tau_v)$ [resp. $(2^{(X,\tau_I)}, \tau_{I,v})$ and $(2^{(X,\tau_{IV})}, \tau_{IV,v})$] of τ-type [resp. τ_I-type and τ_{IV}-type]. And we give some examples of each hyperspace and obtain some properties of the hyperspace $(2^{(X,\tau)}, \tau_v)$. Third, we find some relationships between openess in an ITS (X, τ) and its hyperspace $2^{(X,\tau)}$. Finally, we introduce an intuitionistic set-valued mapping and study its some continuities.

2. Preliminaries

In this section, we list some concepts related to intuitionistic sets and some results obtained by [5, 6, 7, 10, 11].

Definition 2.1 ([5])**.** Let X be a non-empty set. Then A is called an intuitionistic set (in short, IS) of X, if it is an object having the form

$$A = (A_T, A_F),$$

such that $A_T \cap A_F = \phi$, where A_T [resp. A_F] is called the set of members [resp. nonmembers] of A.

In fact, A_T [resp. A_F] is a subset of X agreeing or approving [resp. refusing or opposing] for a certain opinion, view, suggestion or policy.

The intuitionistic empty set [resp. the intuitionistic whole set] of X, denoted by ϕ_I [resp. X_I], is defined by $\phi_I = (\phi, X)$ [resp. $X_I = (X, \phi)$].

In general, $A_T \cup A_F \neq X$.

We will denote the set of all ISs of X as $IS(X)$.

Definition 2.2 ([5])**.** Let $A, B \in IS(X)$ and let $(A_j)_{j\in J} \subset IS(X)$.

(i) We say that A is contained in B, denoted by $A \subset B$, if $A_T \subset B_T$ and $A_F \supset B_F$.

(ii) We say that A equals to B, denoted by $A = B$, if $A \subset B$ and $B \subset A$.

(iii) The complement of A denoted by A^c, is an IS of X defined as:

$$A^c = (A_F, A_T).$$

(iv) The union of A and B, denoted by $A \cup B$, is an IS of X defined as:

$$A \cup B = (A_T \cup B_T, A_F \cap B_F).$$

(v) The union of $(A_j)_{j\in J}$, denoted by $\bigcup_{j\in J} A_j$ (in short, $\bigcup A_j$), is an IS of X defined as:

$$\bigcup_{j\in J} A_j = (\bigcup_{j\in J} A_{j,T}, \bigcap_{j\in J} A_{j,F}).$$

(vi) The intersection of A and B, denoted by $A \cap B$, is an IS of X defined as:

$$A \cap B = (A_T \cap B_T, A_F \cup B_F).$$

(vii) The intersection of $(A_j)_{j\in J}$, denoted by $\bigcap_{j\in J} A_j$ (in short, $\bigcap A_j$), is an IS of X defined as:

$$\bigcap_{j\in J} A_j = (\bigcap_{j\in J} A_{j,T}, \bigcup_{j\in J} A_{j,F}).$$

(viii) $A - B = A \cap B^c$.

(ix) $[\ ]A = (A_T, {A_T}^c)$, $<> A = ({A_F}^c, A_F)$.

Result 2.3 ([10], Proposition 3.6)**.** *Let $A, B, C \in IS(X)$. Then*
(1) (Idempotent laws): $A \cup A = A$, $A \cap A = A$,
(2) (Commutative laws): $A \cup B = B \cup A$, $A \cap B = B \cap A$,
(3) (Associative laws): $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$,
(4) (Distributive laws): $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(5) (Absorption laws): $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$,
(6) (DeMorgan's laws): $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$,
(7) $(A^c)^c = A$,
(8) (8a) $A \cup \phi_I = A$, $A \cap \phi_I = \phi_I$,
(8b) $A \cup X_I = X_I$, $A \cap X_I = A$,
(8c) ${X_I}^c = \phi_I$, ${\phi_I}^c = X_I$,
(8d) *in general,* $A \cup A^c \neq X_I$, $A \cap A^c \neq \phi_I$.

We will denote the family of all ISs A in X such that $A_T \cup A_F = X$ as $IS_*(X)$, i.e.,

$$IS_*(X) = \{A \in IS(X) : A_T \cup A_F = X\}.$$

In this case, it is obvious that $A \cap A^c = \phi_I$ and $A \cup A^c = X_I$ and thus

$$(IS_*(X), \subset, \phi_I, X_I)$$

is a Boolean algebra. In fact, there is a one-to-one correspondence between $P(X)$ and $IS_*(X)$, where $P(X)$ denotes the power set of X. Moreover, for any $A, B \in IS_*(X)$,
$A = A_I = [\,]A = \langle\,\rangle A$ and $A \cup B, A \cap B, A - B \in IS_*(X)$.

Definition 2.4 ([5])**.** Let X be a non-empty set, $a \in X$ and let $A \in IS(X)$.

(i) The form $(\{a\}, \{a\}^c)$ [resp. $(\phi, \{a\}^c)$]is called an intuitionistic point [resp. vanishing point] of X and denoted by a_I [resp. a_{IV}].

(ii) We say that a_I [resp. a_{IV}] is contained in A, denoted by $a_I \in A$ [resp. $a_{IV} \in A$], if $a \in A_T$ [resp. $a \notin A_F$].

We will denote the set of all intuitionistic points or intuitionistic vanishing points in X as $IP(X)$.

Definition 2.5 ([6])**.** Let X be a non-empty set and let $\tau \subset IS(X)$. Then τ is called an intuitionistic topology (in short IT) on X, if it satisfies the following axioms:
(i) $\phi_I, X_I \in \tau$,
(ii) $A \cap B \in \tau$, for any $A, B \in \tau$,
(iii) $\bigcup_{j \in J} A_j \in \tau$, for each $(A_j)_{j \in J} \subset \tau$.

In this case, the pair (X, τ) is called an intuitionistic topological space (in short, ITS) and each member O of τ is called an intuitionistic open set (in short, IOS) in X. An IS F of X is called an intuitionistic closed set (in short, ICS) in X, if $F^c \in \tau$.

It is obvious that $\{\phi_I, X_I\}$ is the smallest IT on X and will be called the intuitionistic indiscreet topology and denoted by $\tau_{I,0}$. Also $IS(X)$ is the greatest IT on X and will be called the intuitionistic discreet topology and denoted by $\tau_{I,1}$. The pair $(X, \tau_{I,0})$ [resp. $(X, \tau_{I,1})$] will be called the intuitionistic indiscreet [resp. discreet] space.

We will denote the set of all ITs on X as $IT(X)$. For an ITS X, we will denote the set of all IOSs [resp. ICSs] on X as $IO(X)$ [resp. $IC(X)$].

Example 2.6. (1) ([6], Example 3.2) For any ordinary topological space (X, τ_o), let $\tau = \{(A, A^c) : A \in \tau_o\}$. Then clearly, (X, τ) is an ITS.

(2) ([6], Example 3.4) Let (X, τ) be an ordinary topological space such that τ is not indiscrete, where $\tau = \{\phi, X\} \cup \{G_j : j \in J\}$. Then there exist two ITs on X as follows: $\tau^1 = \{\phi_I, X_I\} \cup \{(G_j, \phi) : j \in J\}$ and $\tau^2 = \{\phi_I, X_I\} \cup \{(\phi, G_j^c) : j \in J\}$.

(3) ([11], Example 3.2 (4)) Let X be a set and let $A \in IS(X)$. Then A is said to be finite, if A_T is finite. Consider the family $\tau = \{U \in IS(X) : U = \phi_I$ or U^c is finite$\}$. Then we can easily show that τ is an IT on X.

In this case, τ will be called an intuitionistic cofinite topology on X and denoted by $ICof(X)$.

(4) ([11], Example 3.2 (5)) Let X be a set and let $A \in IS(X)$. Then A is said to be countable, if A_T is countable. Consider the family $\tau = \{U \in IS(X) : U = \phi_I$ or U^c is countable$\}$. Then we can easily show that τ is an IT on X.

In this case, τ will be called an intuitionistic cocountable topology on X and denoted by $ICoc(X)$.

Result 2.7 ([6], Proposition 3.5)**.** *Let (X, τ) be an ITS. Then the following two ITs on X can be defined by:*

$$\tau_{0,1} = \{[\ ]U : U \in \tau\}, \tau_{0,2} = \{<> U : U \in \tau\}.$$

Furthermore, the following two ordinary topologies on X can be defined by (See [3]):

$$\tau_1 = \{U_T : U \in \tau\},\ \tau_2 = \{U_F^c : U \in \tau\}.$$

Remark 2.8 ([11], Remark 3.4)**.** (1) Let (X, τ) be an ITS such that $\tau \subset IS_*(X)$. Then it is obvious that $\tau = \tau_{0,1} = \tau_{0,2}$.

(2) For an IT τ on a set X, we will denote two ITs $\tau_{0,1}$ and $\tau_{0,2}$ defined in Result 2.7 as $\tau_{0,1} = [\ ]\tau$ and $\tau_{0,2} =<> \tau$, respectively.

(3) For an IT τ on a set X, let τ_1 and τ_2 be ordinary topologies on X defined in Result 2.7. Then (X, τ_1, τ_2) is a bitopological space by Kelly [9] (Also see Proposition 3.1 in [4]).

Definition 2.9 ([6])**.** Let (X, τ) be an ITS.

(i) A subfamily β of τ is called an intuitionistic base (in short, IB) for τ, if for each $A \in \tau$, $A = \phi_I$ or there exists $\beta' \subset \beta$ such that $A = \bigcup \beta'$.

(ii) A subfamily σ of τ is called an intuitionistic subbase (in short, ISB) for τ, if the family $\beta = \{\bigcap \sigma' : \sigma'$ is a finite subset of $\sigma\}$ is a base for τ.

In this case, the IT τ is said to be generated by σ. In fact, $\tau = \{\phi_I\} \cup \{\bigcup \beta' : \beta' \subset \beta\}$.

Definition 2.10 ([7])**.** Let X be an ITS, $p \in X$ and let $N \in IS(X)$. Then

(i) N is called a neighborhood of p_I, if there exists an IOS G in X such that

$$p_I \in G \subset N, \text{ i.e., } p \in G_T \subset N_T \text{ and } G_F \supset N_F,$$

(ii) N is called a neighborhood of p_{IV}, if there exists an IOS G in X such that

$$p_{IV} \in G \subset N, \text{ i.e., } G_T \subset N_T \text{ and } p \notin G_F \supset N_F.$$

We will denote the set of all neighborhoods of p_I [resp. p_{IV}] by $N(p_I)$ [resp. $N(p_{IV})$].

Result 2.11 ([7], Proposition 3.4). *Let (X, τ) be an ITS. We define the families*

$$\tau_I = \{G : G \in N(p_I), \text{ for each } p_I \in G\}$$

and

$$\tau_{IV} = \{G : G \in N(p_{IV}), \text{ for each } p_{IV} \in G\}.$$

Then $\tau_I, \tau_{IV} \in IT(X)$.

In fact, from Remark 4.5 in [11], we can see that for an IT τ on a set X and each $U \in \tau$,

$$\tau_I = \tau \cup \{(U_T, S_U) : S_U \subset U_F\} \cup \{(\phi, S) : S \subset X\}$$

and

$$\tau_{IV} = \tau \cup \{(S_U, U_F) : S_U \supset U_T \text{ and } S_U \cap U_F = \phi\}.$$

Result 2.12 ([7], Proposition 3.5). *Let (X, τ) be an ITS. Then $\tau \subset \tau_I$ and $\tau \subset \tau_{IV}$.*

Result 2.13 ([11], Corollary 4.8). *Let (X, τ) be an ITS and let IC_τ [resp. IC_{τ_I} and $IC_{\tau_{IV}}$] be the set of all ICSs w.r.t. τ[resp. τ_I and τ_{IV}]. Then*

$$IC_\tau(X) \subset IC_{\tau_I}(X) \text{ and } IC_\tau(X) \subset IC_{\tau_{IV}}(X).$$

Result 2.14 ([7], Proposition 3.9). *Let (X, τ) be an ITS. Then $\tau = \tau_I \cap \tau_{IV}$.*

Result 2.15 ([11], Corollary 4.13). *Let (X, τ) be an ITS and let IC_τ]. Then*

$$IC_\tau(X) = IC_{\tau_I}(X) \cap IC_{\tau_{IV}}(X).$$

Definition 2.16 ([6]). Let (X, τ) be an ITS and let $A \in IS(X)$.

(i) The intuitionistic closure of A w.r.t. τ, denoted by $Icl(A)$, is an IS of X defined as:

$$Icl(A) = \bigcap\{K : K^c \in \tau \text{ and } A \subset K\}.$$

(ii) The intuitionistic interior of A w.r.t. τ, denoted by $Iint(A)$, is an IS of X defined as:

$$Iint(A) = \bigcup\{G : G \in \tau \text{ and } G \subset A\}.$$

Result 2.17 ([6], Proposition 3.15). *Let (X, τ) be an ITS and let $A \in IS(X)$. Then*

$$Iint(A^c) = (Icl(A))^c \text{ and } Icl(A^c) = (Iint(A))^c.$$

3. Intuitionistic hyperspaces

In this section, for an ITS (X, τ), we introduce an intuitionistic hyperspace $(2^{(X,\tau)}, \tau_v)$ [resp. $(2^{(X,\tau_I)}, \tau_{I,v})$ and $(2^{(X,\tau_{IV})}, \tau_{IV,v})$] of τ-type [resp. τ_I-type and τ_{IV}-type]. And we give some examples of each hyperspace and obtain some properties of the hyperspace $(2^{(X,\tau)}, \tau_v)$.

Notation 3.1. Let (X, τ) be an ITS. Then

(1) $2^{(X,\tau)} = \{E \in IS(X) : \phi_I \neq E \in IC_\tau(X)\}$,
(2) $2^{(X,\tau_I)} = \{E \in IS(X) : \phi_I \neq E \in IC_{\tau_I}(X)\}$,
(3) $2^{(X,\tau_{IV})} = \{E \in IS(X) : \phi_I \neq E \in IC_{\tau_{IV}}(X)\}$,
(4) $\mathfrak{F}_{2^{(X,\tau)},n}(X) = \{E \in 2^{(X,\tau)} : E_T \text{ has at most n elements}\}$,

(5) $\mathfrak{F}_{2^{(X,\tau)}}(X) = \{E \in 2^{(X,\tau)} : E_T \text{ is finite}\}$,
(6) $\mathfrak{K}_{2^{(X,\tau)}}(X) = \{E \in 2^{(X,\tau)} : E \text{ is compact}\}$,
(7) $\mathfrak{C}_{2^{(X,\tau)}}(X) = \{E \in 2^{(X,\tau)} : E \text{ is connected}\}$,
(8) $\mathfrak{C}_{2^{(X,\tau)},K}(X) = \mathfrak{K}_{2^{(X,\tau)}}(X) \cap \mathfrak{C}(_{2^{(X,\tau)}}(X)$.

The following is the immediate result of Notation 3.1, and Results 2.12 and 2.14.

Proposition 3.2. *Let (X, τ) be an ITS. Then*

$$2^{(X,\tau)} \subset 2^{(X,\tau_I)} \text{ and } 2^{(X,\tau)} \subset 2^{(X,\tau_{IV})}.$$

Moreover, $2^{(X,\tau)} = 2^{(X,\tau_I)} \cap 2^{(X,\tau_{IV})}$.

Example 3.3. Let $X = \{a, b, c\}$ and let τ be the IT on X given by:

$$\tau = \{\phi_I, X_I, A_1, A_2, A_3, A_4\},$$

where $A_1 = (\{a\}, \{b\}),\ A_2 = (\{b\}, \{c\}),\ A_3 = (\{a, b\}, \phi),\ A_4 = (\phi, \{b, c\})$.
Then $\tau_I = \tau \cup \{A_5, A_6, A_7, A_8, A_9\}$ and $\tau_I = \tau \cup \{A_{10}, A_{11}, A_{12}\}$,
where $A_5 = (\phi, \{a\}),\ A_6 = (\phi, \{b\}),\ A_7 = (\phi, \{c\}),\ A_8 = (\phi, \{a, b\})$,
$A_9 = (\phi, \{a, c\}),\ A_{10} = (\{a, c\}, \{b\}),\ A_{11} = (\{a, b\}, \{c\}),\ A_{12} = (\{a\}, \{b, c\})$.
Thus $IC_\tau(X) = \{\phi_I, X_I, F_1, F_2, F_3, F_4\}$,
$IC_{\tau_I}(X) = IC_\tau(X) \cup \{F_5, F_6, F_7, F_8, F_9\}$
and
$IC_{\tau_{IV}}(X) = IC_\tau(X) \cup \{F_{10}, F_{11}, F_{12}\}$,
where $F_1 = (\{b\}, \{a\}),\ F_2 = (\{c\}, \{b\}),\ F_3 = (\phi, \{a, b\}),\ F_4 = (\{b, c\}, \phi)$,
$F_5 = (\{a\}, \phi),\ F_6 = (\{b\}, \phi),\ F_7 = (\{c\}, \phi),\ F_8 = (\{a, b\}, \phi)$,
$F_9 = (\{a, c\}, \phi),\ F_{10} = (\{b\}, \{a, c\}),\ F_{11} = (\{c\}, \{a, b\}),\ F_{12} = (\{b, c\}, \{a\})$.
So $2^{(X,\tau)} = \{X_I, F_1, F_2, F_3, F_4\}$,
$2^{(X,\tau_I)} = 2^{(X,\tau)} \cup \{F_5, F_6, F_7, F_8, F_9\}$,
$2^{(X,\tau_{IV})} = 2^{(X,\tau)} \cup \{F_{10}, F_{11}, F_{12}\}$.
In fact, we can confirm that Proposition 3.2 holds.

Proposition 3.4. *Let (X, τ) be an ITS and let*

$$\beta_{\tau,v} = \{< U_1, U_2, ..., U_n >_{\tau,v}: U_j \in \tau \text{ for } j = 1, ..., n\},$$

$$\beta_{\tau_I,v} = \{< U_1, U_2, ..., U_n >_{\tau_I,v}: U_j \in \tau \text{ for } j = 1, ..., n\},$$

$$\beta_{\tau_{IV},v} = \{< U_1, U_2, ..., U_n >_{\tau_{IV},v}: U_j \in \tau \text{ for } j = 1, ..., n\},$$

where $< U_1, U_2, ..., U_n >_{\tau,v}$
$= \{E \in 2^{(X,\tau)} : E \subset \bigcup_{j=1}^n U_j \text{ and } E \cap U_j \neq \phi_I \text{ for } j = 1, ..., n\}$,
$< U_1, U_2, ..., U_n >_{\tau_I,v}$
$= \{E \in 2^{(X,\tau_I)} : E \subset \bigcup_{j=1}^n U_j \text{ and } E \cap U_j \neq \phi_I \text{ for } j = 1, ..., n\}$,
$< U_1, U_2, ..., U_n >_{\tau_{IV},v}$
$= \{E \in 2^{(X,\tau_{IV})} : E \subset \bigcup_{j=1}^n U_j \text{ and } E \cap U_j \neq \phi_I \text{ for } j = 1, ..., n\}$,
Then there exists a unique topology τ_v [resp. $\tau_{I,v}$ and $\tau_{IV,v}$] on $2^{(X,\tau)}$ [resp. $2^{(X,\tau_I)}$ and $2^{(X,\tau_{IV})}$] such that $\beta_{\tau,v}$ [resp. $\beta_{\tau_I,v}$ and $\beta_{\tau_{IV},v}$] is a base for τ_v [resp. $\tau_{I,v}$ and $\tau_{IV,v}$].

Proof. Clearly, $X_I \in \tau$ and $< X_I >_{\tau,v} \in \beta_{\tau,v}$. Then $\bigcup \beta_{\tau,v} =< X_I >_{\tau,v}= 2^{(X,\tau)}$.

Let $< U_1, U_2, ..., U_n >_{\tau,v}, < V_1, V_2, ..., V_m >_{\tau,v} \in \beta_{\tau,v}$ and let $U = \bigcup_{i=1}^{n} U_i, V = \bigcup_{j=1}^{m} V_j$. Let $\mathbf{B}_{\tau,\mathbf{v}} =< U_1 \cap V, U_2 \cap V, ..., U_n \cap V, U \cap V_1, U \cap V_2, ..., U \cap V_m >_{\tau,v}$. Let $E \in \mathbf{B}_{\tau,\mathbf{v}}$. Then $E \subset \bigcup_{i=1}^{n}[(U_i \cap V)] \cup \bigcup_{j=1}^{m}[(U \cap V_j)]$, $E \cap U_i \cap V \neq \phi_I$, for $i = 1, ..., n$ and $E \cap U \cap V_j \neq \phi_I$, for $j = 1, ..., m$. Thus

$$F \in \mathbf{B}_{\tau,\mathbf{v}} =< U_1, U_2, ..., U_n >_{\tau,v} \cap < V_1, V_2, ..., V_m >_{\tau,v} .$$

So $\beta_{\tau,v}$ generates the unique topology τ_v on $2^{(X,\tau)}$ such that $\beta_{\tau,v}$ is a base for τ_v.

Similarly, we can show that $\beta_{\tau_I,v}$ and $\beta_{\tau_{IV},v}$ generate the unique topologies $\tau_{\tau_I,v}$ and $\tau_{\tau_{IV},v}$ on $2^{(X,\tau_I)}$ and $2^{(X,\tau_{IV})}$ such that $\beta_{\tau_I,v}$ and $\beta_{\tau_{IV},v}$ are bases for $\tau_{\tau_I,v}$ and $\tau_{\tau_{IV},v}$, respectively. □

In the above Proposition, the topology τ_v [resp. $\tau_{I,v}$ and $\tau_{IV,v}$] on $2^{(X,\tau)}$ [resp. $2^{(X,\tau_I)}$ and $2^{(X,\tau_{IV})}$] induced by $\beta_{\tau,v}$ [resp. $\beta_{\tau_I,v}$ and $\beta_{\tau_{IV},v}$] will be called the intuitionistic Vietories topology (in short, IVT) on $2^{(X,\tau)}$ [resp. $2^{(X,\tau_I)}$ and $2^{(X,\tau_{IV})}$]. The pair $(2^{(X,\tau)}, \tau_v)$ [resp. $(2^{(X,\tau_I)}, \tau_{I,v})$ and $(2^{(X,\tau_{IV})}, \tau_{IV,v})$] will be called an intuitionistic hyperspace of τ-type [resp. τ_I-type and τ_{IV}-type].

The following is the immediate result of Proposition 3.4, and Results 2.12 and 2.14.

Proposition 3.5. *Let (X, τ) be an ITS. Then $\tau_v \subset \tau_{I,v}$ and $\tau_v \subset \tau_{IV,v}$. Moreover,*

$$\tau_v = \tau_{I,v} \cap \tau_{IV,v}.$$

Example 3.6. Let (X, τ) be the ITS in Example 3.3. Then we can easily check the followings:

$$\begin{aligned}
\tau_v = \{&\phi, \{F_1\}, \{F_3\}, \{F_1, F_3\}, \{F_2, F_4, X_I\}, \{F_1, F_2, F_4, X_I\}, \{F_2, F_3, F_4, X_I\}, 2^{(X,\tau)}\},\\
\tau_{I,v} = \{&\phi, \{F_1\}, \{F_3\}, \{F_5\}, \{F_1, F_3\}, \{F_1, F_5\}, \{F_1, F_6\}, \{F_3, F_5\}, \{F_5, F_8\},\\
&\{F_1, F_3, F_5\}, \{F_1, F_3, F_6\}, \{F_1, F_5, F_8\}, \{F_5, F_6, F_8\}, \{F_1, F_5, F_6, F_8\},\\
&\{F_1, F_3, F_5, F_6\}, \{F_1, F_3, F_5, F_8\}, \{F_3, F_5, F_6, F_8\}, \{F_1, F_3, F_5, F_6, F_8\},\\
&\{F_2, F_4, X_I\}, \{F_1, F_2, F_4, X_I\}, \{F_2, F_3, F_4, X_I\}, 2^{(X,\tau)}\},\\
&\{F_1, F_4, F_5, F_6, F_7, F_8, F_9, X_I\}, \{F_1, F_3, F_4, F_5, F_6, F_7, F_8, F_9, X_I\},\\
&\{F_1, F_2, F_4, F_5, F_6, F_7, F_8, F_9, X_I\}, 2^{(X,\tau_I)}\},\\
\tau_{IV,v} = \{&\phi, \{F_1\}, \{F_2\}, \{F_3\}, \{F_{10}\}, \{F_1, F_2\}, \{F_1, F_3\}, \{F_1, F_{10}\}, \{F_2, F_3\}, \{F_2, F_{10}\},\\
&\{F_3, F_{10}\}, \{F_1, F_2, F_3\}, \{F_1, F_3, F_{10}\}, \{F_2, F_3, F_{10}\}, \{F_1, F_2, F_3, F_{10}\},\\
&\{F_2, F_4, X_I\}, \{F_1, F_2, F_4, X_I\}, \{F_2, F_3, F_4, X_I\}, 2^{(X,\tau)}\},\\
&\{F_1, F_2, F_4, F_{10}, F_{12}, X_I\}, \{F_1, F_2, F_2, F_3, F_{11}, F_{12}, X_I\}, 2^{(X,\tau_{IV})}\}.
\end{aligned}$$

In fact, we can confirm that Proposition 3.5 holds.

Proposition 3.7. *Let (X, τ) be an ITS. Then the following two subfamilies $\beta_{\tau_{0,1}}$ and $\beta_{\tau_{0,2}}$ of $2^{(X,\tau)}$, respectively can be defined by:*

$$\beta_{\tau_{0,1}} = \{< [\ ]U_1, \cdots , [\ ]U_n >_{\tau_{0,1}} : U_j \in \tau \text{ for } j = 1, ..., n\}$$

and

$$\beta_{\tau_{0,2}} = \{<< > U_1, \cdots , < > U_n >_{\tau_{0,2}} : U_j \in \tau \text{ for } j = 1, ..., n\},$$

where $< [\ ]U_1, \cdots , [\ ]U_n >_{\tau_{0,1}}$
$= \{[\ ]E \in 2^{(X,\tau_{0,1})} : [\ ]E \subset \bigcup_{j=1}^{n}[\ ]U_j,\ [\ ]E \cap [\ ]U_j \neq \phi_I, \text{ for } j = 1, ..., n,$

$E^c \in \tau\}$

and

$<< > U_1, \cdots, < > U_n >_{\tau_{0,2}}$
$= \{< > E \in 2^{(X,\tau_{0.2})} :< > E \subset \bigcup_{j=1}^{n} < > U_j,\ < > E\cap < > U_j \neq \phi_I,$
for $j = 1, ..., n, E^c \in \tau\}$.

Furthermore, $\beta_{\tau_{0,1}}$ *and* $\beta_{\tau_{0,2}}$ *generate unique topologies* $(\tau_{0,1})_v$ *and* $(\tau_{0,2})_v$ *on* $2^{(X,\tau)}$.

In this case, the pair $(2^{(X,\tau)}, (\tau_{0,1})_v)$ [resp. $(2^{(X,\tau)}, (\tau_{0,2})_v)$] will be called an intuitionistic hyperspace of $\tau_{0,1}$-type [resp. $\tau_{0,2}$-type] and simply, will be denoted $2^{(X,\tau_{0,1})}$ [resp. $2^{(X,\tau_{0,2})}$].

Proof. The proofs are easy. □

Example 3.8. Let (X, τ) be the ITS in Example 3.3. Then

$[\ ]A_1 = (\{a\}, \{b, c\}),\ [\ ]A_2 = (\{b\}, \{a, c\}),\ [\ ]A_3 = (\{a, b\}, \{c\})$

and

$< > A_1 = (\{a, c\}, \{b\}),\ < > A_2 = (\{a, b\}, \{c\}),\ < > A_3 = (\{a\}, \{b, c\})$.

Thus

$$IC_{\tau_{0,1}}(X) = \{\phi_I, X_I, [\ ]F_1, [\ ]F_2, [\ ]F_4\}$$

and

$$IC_{\tau_{0,2}}(X) = \{\phi_I, X_I, < > F_1, < > F_2, < > F_3\},$$

where $[\ ]F_1 = (\{b\}, \{a, c\}), [\ ]F_2 = (\{c\}, \{a, b\}), [\ ]F_4 = (\{b, c\}, \{a\})$

and

$< > F_1 = (\{b, c\}, \{a\}),\ < > F_2 = (\{a, c\}, \{b\}),\ < > F_3 = (\{c\}, \{a, b\})$.

So $(\tau_{0,1})_v = \{\phi, \{X_I\}, \{[\ ]F_1, [\ ]F_4, X_I\}, 2^{(X,\tau_{0,1}}\}$

and

$(\tau_{0,2})_v = \{\phi, \{< > F_2\}, \{< > F_2, < > F_3\}, \{< > F_2, X_I\},$
$\{< > F_1, < > F_2, X_I\}, \{< > F_2, < > F_3, X_I\}, 2^{(X,\tau_{0,2}}\}$.

Proposition 3.9. *Let* (X, τ) *be an ITS. Then the following two ordinary subfamilies* β_{τ_1} *and* β_{τ_2} *of* $2^{(X,\tau)}$*, respectively can be defined by:*

$\beta_{\tau_1} = \{< U_{1,T}, \cdots, U_{n,T} >_{\tau_1}: U_j \in \tau$ for $j = 1, ..., n\}$

and

$\beta_{\tau_2} = \{< U^c_{1,F}, \cdots, U^c_{n,F} >_{\tau_2}: U_j \in \tau$ for $j = 1, ..., n\}$,

where $< U_{1,T}, \cdots, U_{n,T} >_{\tau_1}$
$= \{E \in 2^{(X,\tau_1)} : E \subset \bigcup_{j=1}^{n} U_{j,T}$ and $E \cap U_{j,T} \neq \phi$ for $j = 1, ..., n\}$

and

$< U^c_{1,F}, \cdots, U^c_{n,F} >_{\tau_2}$
$= \{E \in 2^{(X,\tau_2)} : E \subset \bigcup_{j=1}^{n} U^c_{j,F}$ and $E \cap U^c_{j,F} \neq \phi$ for $j = 1, ..., n\}$.

Furthermore, β_{τ_1} *and* β_{τ_2} *generate unique ordinary Vietories topologies* $\tau_{1,v}$ *and* $\tau_{2,v}$ *on* 2^X.

In this case, the pair $(2^{(X,\tau)}, \tau_{1,v}))$ [resp. $(2^{(X,\tau)}, \tau_{2,v}))$] will be called an ordinary hyperspace of τ_1-type [resp. τ_2-type] and simply, will be denoted $2^{(X,\tau_1)}$ [resp.

$2^{(X,\tau_2)}$], and the triple $(2^{(X,\tau)}, \tau_{1,v}, \tau_{2,v})$ will be called an ordinary bihyperspace induced by (X, τ).

Proof. The proofs are easy. □

Example 3.10. Let $X = \{a, b, c\}$ and let τ be the IT on X given by:

$$\tau = \{\phi_I, X_I, A_1, A_2, A_3, A_4, A_5\},$$

where $A_1 = (\{a, b\}, \{c\})$, $A_2 = (\{b, c\}, \{a\})$, $A_3 = (\{a\}, \{c\})$
$A_4 = (\{b\}, \{a, c\})$, $A_5 = (\phi, \{a, c\})$.
Then

$$\tau_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$$

and

$$\tau_2 = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}.$$

Thus $\tau_1^c = \{\phi, X, \{a\}, \{c\}, \{b, c\}, \{a, c\}\}$ and $\tau_2^c = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$.
where τ_1^c and τ_2^c denote the families of closed sets in (X, τ_1) and (X, τ_2), respectively.
So $\tau_{1,v} = \{\{\phi\}, \{\{a\}\}, \{\{c\}\}, \{\{b, c\}\}, \{\{a, c\}\}, \{\{b, c\}, \{a, c\}\}, 2^{(X,\tau_1)}\}$
and

$$\tau_{2,v} = \{\{\phi\}, \{\{a\}\}, \{\{c\}\}, \{\{a, c\}\}, 2^{(X,\tau_2)}\}.$$

Proposition 3.11. *Let X be an ITS, $A, B \in IS(X)$ and let $(A_\alpha)_{\alpha\in\Gamma} \subset IS(X)$. Then $2^{A\cap B} = 2^A \cap 2^B$ and generally, $2^{\bigcap_{\alpha\in\Gamma} A_\alpha} = \bigcap_{\alpha\in\Gamma} A_\alpha$,
where $2^A = \{E \in 2^{(X,\tau)} : E \subset A\}$.*

Proof. $E \in 2^{A\cap B} \Leftrightarrow E \in 2^{(X,\tau)}$ such that $E \subset A \cap B$
$\Leftrightarrow E \in 2^{(X,\tau)}$ such that $E \subset A$ and $E \subset B$
$\Leftrightarrow E \in 2^A$ and $E \in 2^B$, i.e., $E \in 2^A \cap 2^B$.

On the other hand,
$E \in 2^{\bigcap_{\alpha\in\Gamma} A_\alpha} \Leftrightarrow E \in 2^{X_I}$ such that $E \subset \bigcap_{\alpha\in\Gamma} A_\alpha$
$\Leftrightarrow E \in 2^{X_I}$ such that $E \subset A_\alpha$, for each $\alpha \in \Gamma$
$\Leftrightarrow E \in 2^{X_I}$, for each $\alpha \in \Gamma$
$\Leftrightarrow E \in \bigcap_{\alpha\in\Gamma} 2^{A_\alpha}$.

□

Definition 3.12 ([3])**.** An ITS X is said to be a:

(i) $T_1(i)$-space, if for any $x \neq y \in X$, there exist $U, V \in IO(X)$ such that

$$x_I \in U, y_I \notin U \text{ and } x_I \notin V, y_I \in V,$$

(ii) $T_1(ii)$-space, if for any $x \neq y \in X$, there exist $U, V \in IO(X)$ such that

$$x_{IV} \in U, y_{IV} \notin U \text{ and } x_{IV} \notin V, y_{IV} \in V,$$

(iii) $T_1(iii)$-space, if for any $x \neq y \in X$, there exist $U, V \in IO(X)$ such that

$$x_I \in U \subset y_I^c \text{ and } y_I \in V \subset x_I^c,$$

(iv) $T_1(iv)$-space, if for any $x \neq y \in X$, there exist $U, V \in IO(X)$ such that

$$x_{IV} \in U \subset y_{IV}^c \text{ and } y_{IV} \in V \subset x_{IV}^c,$$

(v) $T_1(v)$-space, if for any $x \neq y \in X$, there exist $U, V \in IO(X)$ such that

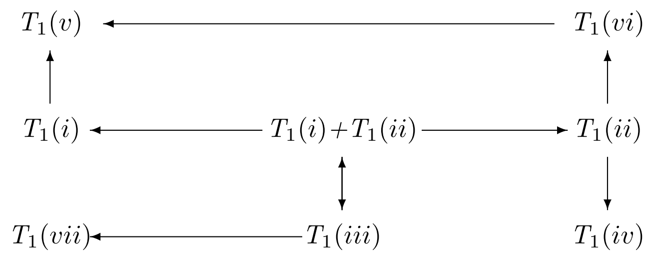
$$y_I \notin U \text{ and } x_I \notin V,$$

(vi) $T_1(vi)$-space, if for any $x \neq y \in X$, there exist $U, V \in IO(X)$ such that

$$y_{IV} \notin U \text{ and } x_{IV} \notin V,$$

(vii) $T_1(vii)$-space, if for each $x \in X$, $x_I \in IC(X)$,
(viii) $T_1(viii)$-space, if for each $x \in X$, $x_{IV} \in IC(X)$.

Result 3.13 ([3], Theorem 3.2)**.** *Let (X, τ) be an ITS. Then the following implications are true:*



Result 3.14 ([3], Proposition 3.11)**.** *Let (X, τ) be an ITS. Then*
(1) *(X, τ) is $T_1(i)$ if and only if (X, τ_1) is T_1,*
(2) *(X, τ) is $T_1(ii)$ if and only if (X, τ_2) is T_1,*
(3) *(X, τ) is $T_1(i)$ if and only if $(X, \tau_{0,1})$ is $T_1(i)$,*
(4) *(X, τ) is $T_1(ii)$ if and only if $(X, \tau_{0,2})$ is $T_1(ii)$.*

Proposition 3.15. *Let (X, τ) be an ITS such that $\tau \subset IS_*(X)$. Then*
(1) *(X, τ) is $T_1(vii)$ if and only if $(X, \tau_{0,1})$ is $T_1(vii)$,*
(2) *(X, τ) is $T_1(viii)$ if and only if $(X, \tau_{0,1})$ is $T_1(viii)$.*

Proof. For any $A \in IS_*(X)$, we can easily see that $[\ ]A^c = ([\ ]A)^c$. Then from this fact and Definition 2.16 (i), we can prove that (1) and (2) hold. □

Proposition 3.16. *Let (X, τ) be an ITS.*
(1) *If (X, τ) is $T_1(vii)$, then (X, τ_1) is T_1, i.e., $\{x\}$ is closed in (X, τ_1), for each $x \in X$.*
(2) *If (X, τ) is $T_1(viii)$, then (X, τ_2) is T_1, i.e., $\{x\}$ is closed in (X, τ_2), for each $x \in X$.*

Proof. (1) Suppose (X, τ) is $T_1(vii)$ and let $x \neq y \in X$. Then clearly, x_I, $y_I \in IC(X)$. Thus x_I^c, $y_I^c \in \tau$. Moreover, $x_I \notin x_I^c$, $x_I \in y_I^c$ and $y_I \in x_I^c$, $y_I \notin y_I^c$. So (X, τ) is $T_1(i)$. Hence by Result 3.14 (1), (X, τ_1) is T_1.
(2) The proof is similar. □

Theorem 3.17. *Let X be $T_1(iii)$ [resp. $T_1(viii)$]. Then $A \subset B$ if and only if $2^A \subset 2^B$ and thus $A = B$ if and only if $2^A = 2^B$.*

Proof. ($\Rightarrow$): It is obvious.
($\Leftarrow$): Suppose $2^A \subset 2^B$ and let $p_I \in A$. Since X is $T_1(iii)$, by Result 3.13, it is $T_1(vii)$. Then $p_I \in IC(X)$ and $p_I \subset A$. Thus $p_I \in 2^A$. By the hypothesis, $p_I \in 2^B$, i.e., $p_I \subset B$. So $p_I \in B$. Hence $A \subset B$.

Now let $p_{IV} \in A$. Since X is $T_1(viii)$, by Definition 3.12, $p_{IV} \in IC(X)$. Then $p_{IV} \in 2^A$. Thus by the hypothesis, $p_I \in 2^B$, i.e., $p_I \subset B$. So $p_I \in B$. Hence $A \subset B$. This completes the proof. □

Proposition 3.18. *Let (X, τ) be an ITS. Then*

$$(2^{A^c})^c = 2^{X_I} - 2^{A^c} = \{E \in 2^{(X,\tau)} : E \cap A \neq \phi_I\}.$$

Proof. $E \in (2^{A^c})^c \Leftrightarrow E \notin 2^{A^c} \Leftrightarrow E \not\subset A^c \Leftrightarrow E_T \not\subset A_F$ or $E_F \not\supset A_T$
$\Leftrightarrow E_T \cap A_T \not\subset A_F \cap A_T = \phi$ or $E_F \cup A_T \not\supset A_T \cup A_T = A_T$
$\Leftrightarrow E \cap A \neq \phi_I$. □

Theorem 3.19. *Let (X, τ) be a $T_1(iii)$-space and let $A \in IS(X)$. Then*

$$2^{Icl(A)} = cl(2^A),$$

where $cl(2^A)$ denotes the closure of 2^A in $2^{(X,\tau)}$.

Proof. It is clear that $A \subset Icl(A)$. Then $2^A \subset 2^{Icl(A)}$.

Let $E \in 2^{Icl(A)}$, i.e., $E \subset Icl(A)$. Let $< U_1, ..., U_n >_{\tau_v}$ containing E. Then $E \subset \bigcup_{j=1}^n U_j$ and $E \cap U_j \neq \phi_I$, for $j = 0, 1, 2, ..., n$. Since $E \subset Icl(A)$, there exists $p_{j,I} \in A \cap U_j$, for $j = 1, 2, ..., n$. Let $F = \bigcup\{p_{1,I}, ..., p_{n,I}\}$. Since (X, τ) is a $T_1(iii)$-space, by Definition 3.12 and Result 3.13, $p_{j,I} \in IC(X)$, for $j = 1, 2, ..., n$. Thus $F \in IC(X)$. So $F \in 2^A \cap < U_1, ..., U_n >_{\tau_v}$. Hence $E \in cl(2^A)$, i.e., $2^A \subset 2^{Icl(A)} \subset cl(2^A)$. Therefore $2^{Icl(A)} = cl(2^A)$. □

The following is the immediate result of Theorem 3.19.

Corollary 3.20. *Let (X, τ) be a $T_1(iii)$-space and let $A \in IC(X)$. Then 2^A is closed in $2^{(X,\tau)}$.*

Proof. Since $A \in IC(X)$, $Icl(A) = A$. Then by 3.19, $cl(2^A) = 2^{Icl(A)} = 2^A$. Thus 2^A is closed in $2^{(X,\tau)}$. □

Theorem 3.21. *Let (X, τ) be a $T_1(iii)$-space and let $A \in IS(X)$. Then*

$$2^{Iint(A)} = int(2^A),$$

where $int(2^A)$ denotes the interior of 2^A in $2^{(X,\tau)}$.

Proof. It is clear that $Iint(A) \subset A$. Then $2^{Iint(A)} \subset 2^A$.

Assume that $E \notin 2^{Iint(A)}$. Then $E \not\subset Iint(A)$. Thus there exists $a \in X$ such that $a_I \in E$ but $a_I \notin Iint(A)$. Let $E \in < U_1, ..., U_n >_{\tau_v}$. Then $E \subset \bigcup_{j=1}^n U_j$ and $E \cap U_j \neq \phi_I$, for $j = 1, 2, ..., n$. Since $a_I \in U_j \in \tau$, for some j and $a_I \notin Iint(A)$, $U_j \not\subset Iint(A)$. Thus there exists $b_j \in X$ such that $b_{j,I} \in U_j$ but $b_{j,I} \notin A$. Since (X, τ) is a $\text{T}_1(iii)$-space, $b_{j,I} \in IC(X)$. Let $F = E \cup b_{j,I}$. Then clearly, $F \not\subset A$. Furthermore, $F \subset \bigcup_{j=1}^n U_j$ and $F \cap U_j \neq \phi_I$, for $j = 1, 2, ..., n$. Thus $F \in < U_1, ..., U_n >_{\tau_v}$. So each neighbourhood of E in $2^{(X,\tau)}$ contains an F such that $F \not\subset A$, i.e., $F \in (2^A)^c$. Hence $F \in cl((2^A)^c)$, i.e., $F \notin int(2^A)$, i.e., $int(2^A) \subset 2^{Iint(A)}$. Therefore $2^{Iint(A)} = int(2^A)$. □

The following is the immediate result of Result 2.17 and Theorems 3.21.

Corollary 3.22. *Let (X, τ) be a $T_1(iii)$-space and let $A \in IC(X)$. Then $(2^{A^c})^c$ is closed in $2^{(X,\tau)}$.*

Proof.
$$\begin{aligned} cl((2^{A^c})^c) &= [int(2^{A^c})]^c \\ &= (2^{IintA^c})^c \text{ [By Theorem 3.21]} \\ &= [(2^{(Icl(A)^c}]^c \text{ [By Result 2.17]} \\ &= (2^{A^c})^c. \text{ [Since } A \in IC(X)] \end{aligned}$$
Then $(2^{A^c})^c$ is closed in $2^{(X,\tau)}$. □

Theorem 3.23. *Let* (X, τ) *be* $T_1(iii)$ *[resp.* $T_1(viii)$*].*

(1) $< U_1, \cdots, U_n > \subset < V_1, \cdots, V_m >$ *if and only if* $\bigcup_{i=1}^n U_i \subset \bigcup_{j=1}^m V_j$ *and there is* U_i *such that* $U_i \subset V_j$*, for each* V_j.

(2) $cl(< U_1, \cdots, U_n >) =< Icl(U_1), \cdots, Icl(U_n) >$*, where* $\tau \subset IS_*(X)$.

Proof. (1) $\mathfrak{U} =< U_1, \cdots, U_n >$ and $\mathfrak{V} =< V_1, \cdots, V_m >$. Suppose $\mathfrak{U} \subset \mathfrak{V}$ and assume that $\bigcup_{i=1}^n U_i \not\subset \bigcup_{j=1}^m V_j$, say $x_{n+1,I} \in \bigcup_{i=1}^n U_i$ but $x_{n+1,I} \notin \bigcup_{j=1}^m V_j$. Let $x_{i,I} \in U_i$, for each $i = 1, \cdots, n$ and let $E = \cup\{x_{i,I} : i = 1, \cdots, n+1\}$. Since (X, τ) is $T_1(iii)$, by Result 3.13, $x_{i,I} \in IC(X)$, for each $i = 1, \cdots, n+1$. Then $E \in IC(X)$. Thus $E \in \mathfrak{U} - \mathfrak{V}$. This contradicts the fact that $\mathfrak{U} \subset \mathfrak{V}$. So $\bigcup_{i=1}^n U_i \subset \bigcup_{j=1}^m V_j$. Now assume that there is V_j such that $U_i - V_j \neq \phi$, for all $i = 1, \cdots, n$ and let $x_{i,I} \in U_i - V_j$. Let $F = \cup\{x_{i,I} : i = 1, \cdots, n\}$. Then by 3.13, $x_{i,I} \in IC(X)$, for each $i = 1, \cdots, n$. Thus $F \in IC(X)$. So $F \in \mathfrak{U} - \mathfrak{V}$. This contradicts the fact that $\mathfrak{U} \subset \mathfrak{V}$. Hence there is U_i such that $U_i \subset V_j$, for each V_j.

Suppose $\mathfrak{U} \subset \mathfrak{V}$ and assume that $\bigcup_{i=1}^n U_i \not\subset \bigcup_{j=1}^m V_j$, say $x_{n+1,IV} \in \bigcup_{i=1}^n U_i$ but $x_{n+1,IV} \notin \bigcup_{j=1}^m V_j$. Let $x_{i,IV} \in U_i$, for each $i = 1, \cdots, n$ and let $E = \cup\{x_{i,IV} : i = 1, \cdots, n+1\}$. Since (X, τ) is $T_1(viii)$, by Definition 3.12, $x_{i,IV} \in IC(X)$, for each $i = 1, \cdots, n+1$. Then $E \in IC(X)$. Thus $E \in \mathfrak{U} - \mathfrak{V}$. This contradicts the fact that $\mathfrak{U} \subset \mathfrak{V}$. So $\bigcup_{i=1}^n U_i \subset \bigcup_{j=1}^m V_j$. Now assume that there is V_j such that $U_i - V_j \neq \phi$, for all $i = 1, \cdots, n$ and let $x_{i,IV} \in U_i - V_j$. Let $F = \cup\{x_{i,IV} : i = 1, \cdots, n\}$. Then by Definition 3.12, $x_{i,IV} \in IC(X)$, for each $i = 1, \cdots, n$. Thus $F \in IC(X)$. So $F \in \mathfrak{U} - \mathfrak{V}$. This contradicts the fact that $\mathfrak{U} \subset \mathfrak{V}$. Hence there is U_i such that $U_i \subset V_j$, for each V_j.

Conversely, suppose the necessary conditions hold, and let $E \in 2^{(X,\tau)}$ and let $E \in \mathfrak{U}$. Then clearly, $E \subset \bigcup_{i=1}^n U_i$. Thus by the hypothesis, $E \subset \bigcup_{j=1}^m V_j$. Now let U_i be such that $U_i \subset V_j$. Since $E \cap U_i \neq \phi_I$ and $E \cap V_j \neq \phi_I$, $E \cap V_j \neq \phi_I$, for each j. So $E \in \mathfrak{V}$. Hence $\mathfrak{U} \subset \mathfrak{V}$.

(2) Let $E \in < Icl(U_1), \cdots, Icl(U_n) >$, let $\mathfrak{V} =< V_1, \cdots, V_m > \in N_{\tau_v}(E)$, and let $U = \bigcup_{i=1}^n U_i$ and $V = \bigcup_{j=1}^m V_i$. Since $\mathfrak{V} \in N_{\tau_v}(E)$, $E \in \mathfrak{V}$, i.e., $E \subset V$. Thus $E \subset Icl(V)$. Moreover, $E \cap Icl(U_i) \neq \phi_I$, for $i = 1, \cdots, n$ and $E \cap V_i \neq \phi_I$, for $j = 1, \cdots, m$. So $V \cap Icl(U_i) \neq \phi_I \neq V_j \cap Icl(U)$ imply that $V \cap U_i \neq \phi_I \neq V_j \cap U$, for $i = 1, \cdots, n$ and $j = 1, \cdots, m$. Choose $x_{i,I} \in V \cap U_i$ [resp. $x_{i,IV} \in V \cap U_i$], for $i = 1, \cdots, n$ and $y_{j,I} \in V_j \cap U$ [resp. $y_{j,IV} \in V_j \cap U$], for $j = 1, \cdots, m$ and let $F = [\bigcup_{i=1}^n x_{i,I}] \cup [\bigcup_{j=1}^m y_{j,I}]$ [resp. $F = [\bigcup_{i=1}^n x_{i,IV}] \cup [\bigcup_{j=1}^m y_{j,IV}]$]. Since (X, τ) be both $T_1(iii)$ and $T_1(viii)$, by Result 3.13 [resp. Definition 3.12], $F \in IC(X)$. Moreover, $F \in \mathfrak{U} \cap \mathfrak{V} \neq \phi$. So E is a limit point of $\mathfrak{U}$, i.e., $E \in cl(\mathfrak{U})$. Hence $< Icl(U_1), \cdots, Icl(U_n) > \subset cl < U_1, \cdots, U_n >$.

On the other hand, we can easily that

$$< Icl(U_1), \cdots, Icl(U_n) >= (\bigcap_{i=1}^n \{E \in 2^{(X,\tau)} : E \cap Icl(U_i) \neq \phi_I\}) \cap < Icl(U) > .$$

Then by Corollary 3.22, $\{E \in 2^{(X,\tau)} : E \cap Icl(U_i) \neq \phi_I\}$ is closed in $2^{(X,\tau)}$. Thus $(\bigcap_{i=1}^n \{E \in 2^{(X,\tau)} : E \cap Icl(U_i) \neq \phi_I\}) \cap < Icl(U) >$ is closed in $2^{(X,\tau)}$. So $< Icl(U_1), \cdots, Icl(U_n) >$ is closed in $2^{(X,\tau)}$ and $\mathfrak{V} \subset < Icl(U_1), \cdots, Icl(U_n) >$. Hence $cl(\mathfrak{U}) \subset < Icl(U_1), \cdots, Icl(U_n) >$. This completes the proof. □

4. The relationships between openess in ITS (X, τ) and its hyperspace $2^{(X,\tau)}$

In this section, we find some relationships between openess in an ITS (X, τ) and its hyperspace $2^{(X,\tau)}$.

Result 4.1 ([11], Proposition 3.16)**.** *Let (X, τ) be a ITS such that $\tau \subset IS_*(X)$ and let $A \in IS_*(X)$.*

(1) *If there is $U \in \tau$ such that $a_I \in U \subset A$, for each $a_I \in A$, then $A \in \tau$.*

(2) *If there is $U \in \tau$ such that $a_{IV} \in U \subset A$, for each $a_{IV} \in A$, then $A \in \tau$.*

Proposition 4.2. *Let (X, τ) be $T_1(iii)$ [resp. $T_1(viii)$].*

(1) *If $\{U_j\}_{j\in J}$ is a neighborhood base at x_I [resp. x_{IV}] , then $\{< U_j >\}_{j\in J}$ is a neighborhood base at $\{x_I\}$ [resp. $\{x_{IV}\}$] in $2^{(X,\tau)}$.*

(2) *If $\mathfrak{O}$ is open in $2^{(X,\tau)}$, then $\cup\mathfrak{O} \in \tau$, where $\tau \subset IS_*(X)$.*

(3) *If $U \in \tau$, then $2^U =< U >$ is open in $2^{(X,\tau)}$, where $\tau \subset IS_*(X)$.*

Proof. (1) It is clear that $\{x_I\} \in 2^{(X,\tau)}$ [resp. $\{x_{IV}\} \in 2^{(X,\tau)}$]. Let $\mathfrak{U}$, $\mathfrak{V} \in \{< U_j >\}_{j\in J}$ such that $\{x_I\} \in \mathfrak{U} \cap \mathfrak{V}$ [resp. $\{x_{IV}\} \in \mathfrak{U} \cap \mathfrak{V}$]. Then there are i, $j \in J$ such that $\mathfrak{U} =< U_i >$, $\mathfrak{V} =< V_j >$. Since $\{x_I\} \in \mathfrak{U} \cap \mathfrak{V}$ [resp. $x_{IV} \in \mathfrak{U} \cap \mathfrak{V}$], $\{x_I\} \in< U_i >$ and $\{x_I\} \in< U_j >$ [resp. $x_{IV} \in< U_i >$ and $x_{IV} \in< U_j >$]. Thus $\{x_I\} \subset U_i$ and $\{x_I\} \subset U_j$ [resp. $\{x_{IV}\} \subset U_i$ and $\{x_{IV}\} \subset U_j$], i.e., $x_I \in U_i$ and $x_I \in U_j$ [resp. $x_{IV} \in U_i$ and $x_{IV} \in U_j$]. So by the hypothesis, there is $k \in J$ such that $x_I \in U_k \subset U_i \cap U_j$ [resp. $x_{IV} \in U_k \subset U_i \cap U_j$]. Hence $\{x_I\} \in< U_k >\subset< U_i > \cap < U_j >$ [resp. $\{x_{IV}\} \in< U_k >\subset< U_i > \cap < U_j >$]. This completes the proof.

(2) It is sufficient to show that for each base element $\mathfrak{U} =< U_1, \cdots, U_n >$, $\bigcup\mathfrak{U} \in \tau$. Let $U = \bigcup\mathfrak{U}$ and let $x_I \in U$ [resp. $x_{IV} \in U$]. Let $O \in \tau$ such that $x_I \in O \subset \bigcup_{i=1}^n U_i$ [resp. $x_{IV} \in O \subset \bigcup_{i=1}^n U_i$] and let $y_I \in O$ [resp. $y_{IV} \in O$]. Choose $x_{i,I} \in U_i$ [resp. $x_{i,IV} \in U_i$], for for $i = 1, \cdots, n$ and let $E = \bigcup\{x_{1,I}, \cdots, x_{n,I}, y_I\}$ [resp. $E = \bigcup\{x_{1,V}, \cdots, x_{n,IV}, y_{IV}\}$]. Since (X, τ) is $T_1(iii)$ [resp. $T_1(viii)$], by Result 3.13 [resp. Definition 3.12], $E \in IC(X)$. Moreover, $E \subset \bigcup_{i=1}^n U_i$ and $E \cap U_i \neq \phi_I$. Then $y_I \in E \in \mathfrak{U}$ [resp. $y_{IV} \in E \in \mathfrak{U}$]. So $y_I \in U$. Hence $O \subset U$, i.e., $x_I \in O \subset U$ [resp. $x_{IV} \in O \subset U$]. Therefore by Result 4.1, $U = \bigcup\mathfrak{U} \in \tau$.

(3) By Theorem 3.21, $2^U = 2^{Iint(U)} = int(2^U)$. Then 2^U is open in $2^{(X,\tau)}$. □

The followings are immediate results of Propositions 3.15 and 4.2.

Corollary 4.3. *Let (X, τ) be $T_1(iii)$ [resp. $T_1(viii)$] such that $\tau \subset IS_*(X)$.*

(1) *If $\{U_j\}_{j\in J}$ is a neighborhood base at x_I [resp. x_{IV}] , then $\{< [\ ]U_j >\}_{j\in J}$ [resp. $\{<< > U_j >\}_{j\in J}$ is a neighborhood base at $\{x_I\}$ [resp. $\{x_{IV}\}$] in $2^{(X,\tau_{0,1})}$ [resp. $2^{(X,\tau_{0,2})}$].*

(2) *If $\mathfrak{O}$ is open in $2^{(X,\tau_{0,1})}$ [resp. $2^{(X,\tau_{0,2})}$], then $\cup\mathfrak{O} \in \tau_{0,1}$ [resp. $\cup\mathfrak{O} \in \tau_{0,2}$].*

(3) *If $U \in \tau_{0,1}$ [resp. $U \in \tau_{0,2}$], then $2^U =< U >$ is open in $2^{(X,\tau_{0,1})}$ [resp. $2^{(X,\tau_{0,2})}$].*

The followings are immediate results of Proposition 4.2 and Result 3.14.

Corollary 4.4. *Let (X, τ) be $T_1(iii)$ [resp. $T_1(viii)$].*

(1) *If $\{U_j\}_{j\in J}$ is a neighborhood base at x_I [resp. x_{IV}] , then $\{< U_{j,T} >\}_{j\in J}$ [resp. $\{< U^c_{j,F} >\}_{j\in J}$ is a neighborhood base at $\{x\}$ in $2^{(X,\tau_1)}$ [resp. $2^{(X,\tau_2)}$].*

(2) *If $\mathfrak{O}$ is open in $2^{(X,\tau_1)}$ [resp. $2^{(X,\tau_2)}$], then $\cup\mathfrak{O} \in \tau_1$ [resp. $\cup\mathfrak{O} \in \tau_2$].*

(3) *If $U \in \tau_1$ [resp. $U \in \tau_2$], then $2^U =< U >$ is open in $2^{(X,\tau_1)}$ [resp. $2^{(X,\tau_2)}$].*

Definition 4.5 ([6]). Let (X, τ) be an ITS and let $A \in IS(X)$.

(i) $\mathfrak{A} \subset IS(X)$ is called a cover of A, if $A \subset \bigcup_{A\in\mathfrak{A}} A$.

(ii) The cover $\mathfrak{A}$ of A is called an open cover, if $A \in \tau$, for each $A \in \mathfrak{A}$.

In particular, $\mathfrak{A}$ is called an open cover of X, if $\mathfrak{A} \subset \tau$ and $A \subset \bigcup\mathfrak{A}$.

(iii) A is called an intuitionistic compact subset of X, if every open cover of A has a finite subcover.

(iv) (X, τ) is said to be compact, if every open cover of X has a finite subcover.

(v) A family $\mathfrak{A} \subset IS(X)$ satisfies the finite intersection property (in short, FIP), if for each finite subfamily $\mathfrak{A}'$, $\bigcap\mathfrak{A}' \neq \phi_I$.

Result 4.6 ([6], Proposition 5.4). *Let (X, τ) be an ITS. Then (X, τ) is compact if and only if $(X, \tau_{0,1})$ is compact. In fact, (X, τ) is compact if and only if (X, τ_1) is compact.*

Proposition 4.7. *Let (X, τ) be $T_1(iii)$ such that $\tau \subset IS_*(X)$. If $\mathfrak{U}$ is open in the subspace $\mathfrak{K}_{2^{(X,\tau)}}(X)$, then $\bigcup\mathfrak{U} \in \tau$.*

Proof. Without loss of generality, let $\mathfrak{U} =< U_1, \cdots, U_n > \cap \mathfrak{K}_{2^{(X,\tau)}}(X)$ and let $U = \bigcup\mathfrak{U} = \{A : A \in \mathfrak{U}\}$. Let $x_I \in U$. Then there is j such that $x_I \in U_j$. Let us take $x_{i,I} \in U_i$, for each $i \neq j$. For each $y_I \in U_i$, let

$$E_{y_I} = \bigcup\{x_{1,I}, \cdots, x_{i-1,I}, y_I, x_{i+1,I}, \cdots, x_{n,I}\}.$$

Then by Result 3.13, $E_{y_I} \in \mathfrak{U}$. Thus $y_I \in E_{y_I} \subset U$. So $x_I \in U_j \subset U$. Hence by Result 4.1, $\bigcup\mathfrak{U} \in \tau$. □

The followings are immediate results of Proposition 4.7 and Results 3.13 and 4.6.

Corollary 4.8. *Let (X, τ) be $T_1(iii)$.*

(1) *If $\mathfrak{U}$ is open in the subspace $\mathfrak{K}_{2^{(X,\tau_{0,1})}}(X)$, then $\bigcup\mathfrak{U} \in \tau_{0,1}$.*

(2) *If $\mathfrak{U}$ is open in $\mathfrak{K}_{2^{(X,\tau_1)}}(X)$, then $\cup\mathfrak{U} \in \tau_1$.*

Proposition 4.9. *Let (X, τ) be $T_1(iii)$ such that $\tau \subset IS_*(X)$. If $\mathfrak{U}$ is open in the subspace $\mathfrak{F}_{2^{(X,\tau)},n}(X)$, then $\bigcup\mathfrak{U} \in \tau$.*

Proof. Let $U = \bigcup\mathfrak{U}$ and let $x_{1,I} \in U$. Then there is $E \in \mathfrak{U}$ such that $x_{1,I} \in U \in \mathfrak{U}$. Let $E = \bigcup\{x_{1,I}, \cdots, x_{m,I}\}$, $m \leq n$. Since $\mathfrak{U}$ is open in $\mathfrak{F}_{2^{(X,\tau)},n}(X)$, there is a basic open set $< U_1, \cdots, U_k > \cap \mathfrak{K}_{2^{(X,\tau)},n}(X)$ such that $E \in< U_1, \cdots, U_k > \cap \mathfrak{K}_{2^{(X,\tau)},n}(X) \in \mathfrak{U}$. We may assume that $x_{i,I} \in U_1$. Let $\mathfrak{F} = \{U_1, \cdots, U_k\}$. For each $x_{i,I} \in E$, let $\mathfrak{F}_i = \{U_j \in \mathfrak{F} : x_{i,I} \in U_j\}$ and let $W_i = \bigcap\mathfrak{F}_i$. Then by Theorem 3.23 (1),

$$E \in< W_1, \cdots, W_m > \cap \mathfrak{F}_{2^{(X,\tau)},n}(X) \subset< U_1, \cdots, U_k > \cap \mathfrak{F}_{2^{(X,\tau)},n}(X).$$

Let $y_{1,I} \in W_1$. Then

$$E_{y,I} = \{y_{1,I}, x_2, \cdots, x_m\} \in < W_1, \cdots, W_m > \cap \mathfrak{F}_{2^{(X,\tau)},n}(X)$$

Thus $E_{y,I} \in \mathfrak{U}$. So $E_{y,I} \subset U$. It follows that $x_{1,I}, y_I \in W_1 \subset U$. Hence by Result 4.1, $\bigcup \mathfrak{U} \in \tau$. □

The following is the immediate result of Proposition 4.9.

Corollary 4.10. *Let (X, τ) be $T_1(iii)$ such that $\tau \subset IS_*(X)$. If $\mathfrak{U}$ is open in the subspace $\mathfrak{F}_{2^{(X,\tau)}}(X)$, then $\bigcup \mathfrak{U} \in \tau$.*

Definition 4.11 ([13])**.** An ITS X is said to be connected, if it cannot be expressed as the union of two non-empty, disjoint open sets in X.

Definition 4.12 ([13])**.** (X, τ) be an ITS and let $A, B \in IS(X)$.
(i) A and B are said to be separated in X, if $Icl(A) \cap B = A \cap Icl(B) = \phi_I$.
(ii) A and B are said to form a separation of X, if A and B are said to be separated in X and $A \cup B = X_I$.

Result 4.13 ([13], Theorem 3.4)**.** *(X, τ) be an ITS such that $\tau \subset IS_*(X)$. Then the followings are equivalent:*
(1) *(X, τ) is connected,*
(2) *$(X, \tau_{0,1})$ is connected,*
(3) *(X, τ_1) is connected.*

Definition 4.14 ([13])**.** Let (X, τ) be an ITS. Then X is said to be:
(i) locally connected at $p_I \in X_I$, if for each $U \in N(p_I)$, there is a connected $V \in N(p_I)$ such that $V \subset U$,
(ii) locally connected, if it is locally connected at each $p_I \in X_I$.

Definition 4.15 ([12])**.** (i) A $T_1(i)$-space X is called a $T_3(i)$-space, if the following conditions:
[the regular axiom (i)] for any $F \in IC(X)$ such that $x_I \in F^c$, there exist $U, V \in IO(X)$ such that $F \subset U$, $x_I \in V$ and $U \cap V = \phi_I$.
(ii) A $T_1(ii)$-space X is called a $T_3(ii)$-space, if the following conditions:
[the regular axiom (ii)] for any $F \in IC(X)$ such that $x_{IV} \in F^c$, there exist $U, V \in IO(X)$ such that $F \subset U$, $x_{IV} \in V$ and $U \cap V = \phi_I$.

Result 4.16 ([12], Theorem 4.4)**.** *Let (X, τ) be an ITS such that $\tau \subset IS_*(X)$. Then*
(1) *(X, τ) is $T_3(i)$ if and only if (X, τ_1) is T_3,*
(2) *(X, τ) is $T_3(ii)$ if and only if (X, τ_2) is T_3.*

Result 4.17 ([12], Theorem 4.7)**.** *Let (X, τ) be an ITS such that $\tau \subset IS_*(X)$. Then*
(1) *(X, τ) is $T_3(i)$ if and only $(X, \tau_{0,1})$ is $T_3(i)$,*
(2) *(X, τ) is $T_3(ii)$ if and only $(X, \tau_{0,2})$ is $T_3(ii)$.*

Proposition 4.18. *Let (X, τ) be locally connected both $T_1(iii)$ and $T_3(i)$ such that $\tau \subset IS_*(X)$. If $\mathfrak{U}$ is open in the subspace $\mathfrak{C}_{2^{(X,\tau)}}(X)$, then $\bigcup \mathfrak{U} \in \tau$.*

Proof. Let $x_I \in U = \bigcup \mathfrak{U}$. Without loss of generality, let

$$\mathfrak{U} = < U_1, \cdots, U_n > \cap \mathfrak{C}_{2^{(X,\tau)}}(X).$$

Then there is $E \in \mathfrak{U}$ such that $x_I \in E$. Since $x_I \in U = \bigcup \mathfrak{U}$, there is i such that $x_I \in U_i$. Since (X, τ) is locally connected both $\mathrm{T}_1(iii)$ and $\mathrm{T}_3(i)$, by Definitions 4.14 and 4.15, there is a connected set $V \in \tau$ such that $x_I \in V \subset Icl(V) \subset U_i$. Thus $E \cup Icl(V) \in \mathfrak{U}$. So $V \subset E \cup Icl(V) \subset U$. Hence by Result 4.1 (1), $\bigcup \mathfrak{U} \in \tau$. □

The followings are immediate results of Proposition 4.18 and Result 4.17.

Corollary 4.19. *Let (X, τ) be locally connected both $T_1(iii)$ and $T_3(i)$ such that $\tau \subset IS_*(X)$. If $\mathfrak{U}$ is open in the subspace $\mathfrak{C}_{2^{(X,\tau_{0,1})}}(X)$, then $\bigcup \mathfrak{U} \in \tau_{0,1}$.*

5. Intuitionistic continuous set-valued mappings

In this section, we introduce an intuitionistic set-valued mapping and study its some continuities.

Definition 5.1 ([5]). Let $f : X \to Y$ be a mapping, and let $A \in IS(X)$ and $B \in IS(Y)$. Then

(i) the image of A under f, denoted by $f(A)$, is an IS in Y defined as:

$$f(A) = (f(A)_T, f(A)_F),$$

where $f(A)_T = f(A_T)$ and $f(A)_F = (f(A_F^c))^c$.

(ii) the preimage of B, denoted by $f^{-1}(B)$, is an IS in X defined as:

$$f^{-1}(B) = (f^{-1}(B)_T, f^{-1}(B)_F),$$

where $f^{-1}(B)_T = f^{-1}(B_T)$ and $f^{-1}(B)_F = f^{-1}(B_F)$.

Result 5.2. (See [5], Corollary 2.11) *Let $f : X \to Y$ be a mapping and let $A, B, C \in IS(X)$, $(A_j)_{j \in J} \subset IS(X)$ and $D, E, F \in IS(Y)$, $(D_k)_{k \in K} \subset IS(Y)$. Then the followings hold:*

(1) *if $B \subset C$, then $f(B) \subset f(C)$ and if $E \subset F$, then $f^{-1}(E) \subset f^{-1}(F)$.*

(2) *$A \subset f^{-1}f(A))$ and if f is injective, then $A = f^{-1}f(A))$,*

(3) *$f(f^{-1}(D)) \subset D$ and if f is surjective, then $f(f^{-1}(D)) = D$,*

(4) *$f^{-1}(\bigcup D_k) = \bigcup f^{-1}(D_k)$, $f^{-1}(\bigcap D_k) = \bigcap f^{-1}(D_k)$,*

(5) *$f(\bigcup A_j) = \bigcup f(A_j)$, $f(\bigcap A_j) \subset \bigcap f(A_j)$,*

(6) *$f(A) = \phi_N$ if and only if $A = \phi_N$ and hence $f(\phi_N) = \phi_N$, in particular if f is surjective, then $f(X_N) = Y_N$,*

(7) *$f^{-1}(Y_N) = Y_N$, $f^{-1}(\phi_N) = \phi$.*

(8) *if f is surjective, then $f(A)^c \subset f(A^c)$ and furthermore, if f is injective, then $f(A)^c = f(A^c)$,*

(9) *$f^{-1}(D^c) = (f^{-1}(D))^c$.*

Definition 5.3. Let X, Y be non-empty sets. Then a mapping $F : Y \to IS(X)$ is called an intuitionistic set-valued mapping.

Example 5.4. (1) Let $X = \{a, b, c\}, Y = \{1, 2\}$ and let $F : Y \to ISX$ be given by $F(1) = (\{a, b\}, \{c\})$ and $F(2) = (\{a\}, \{b\})$. Then F is an intuitionistic crisp set-valued mapping. In particular, if $A = (\{a, b\}, \{c\})$, then

$$2^A = \{\phi_I, (\{a\}, \{c\}), (\{a\}, \{b, c\}), (\{b\}, \{c\}), (\{b\}, \{a, c\}), (\phi, \{c\}), (\phi, \{b, c\}), (\phi, \{a, c\})\}.$$

(2) (See Definition 5.1) Let X, Y be non-empty sets, let $f : X \to Y$ be a mapping. We define two mappings $f_* : IS(X) \to IS(Y)$ and $f_*^{-1} : 2^{Y_I} \to 2^{X_I}$ as follows:

(i) for each $A \in IS(X)$, $f_*(A) = f(A) = (f(A_T), (f(A_F^c))^c)$,
(ii) for each $B \in IS(Y)$, $f_*^{-1}(B) = f^{-1}(B) = (f^{-1}(B_T), f^{-1}(B_F))$.
Then f_* and f_*^{-1} are intuitionistic set-valued mappings.

Definition 5.5. Let X, Y be non-empty sets, let $F, G : Y \to IS(X)$ be intuitionistic crisp set-valued mappings and let $\{F_\alpha\}_{\alpha\in\Gamma}$, where $F_\alpha : Y \to IS(X)$ is an intuitionistic crisp set-valued mappings, for each $\alpha \in \Gamma$.
(i) $F \subset G$ if and only if $F(y) \subset G(y)$, for each $y \in Y$,
(ii) $(F \cup G)(y) = F(y) \cup G(y)$, for each $y \in Y$,
(iii) $(F \cap G)(y) = F(y) \cap G(y)$, for each $y \in Y$,
(iv) $(\bigcup_{\alpha\in\Gamma} F_\alpha)(y) = \bigcup_{\alpha\in\Gamma} F_\alpha$, for each $y \in Y$,
(v) $(\bigcap_{\alpha\in\Gamma} F_\alpha)(y) = \bigcap_{\alpha\in\Gamma} F_\alpha$, for each $y \in Y$.

Proposition 5.6. *Let $F, G : Y \to IS(X)$ be intuitionistic set-valued mappings and let $\{F_\alpha\}_{\alpha\in\Gamma}$, where $F_\alpha : Y \to IS(X)$ is an intuitionistic set-valued mappings, for each $\alpha \in \Gamma$ and let $2_*^A = \{B \in IS(X) : B \subset A\}$, for each $A \in IS(X)$.*
(1) If $F \subset G$, then $G^{-1}(2_*^A) \subset F^{-1}(2_*^A)$.
(2) $(F \cup G)^{-1}(2_*^A) = F^{-1}(2_*^A) \cap G^{-1}(2_*^A)$,
in general, $(\bigcup_{\alpha\in\Gamma} F_\alpha)^{-1}(2_*^A) = \bigcap_{\alpha\in\Gamma} F_\alpha^{-1}(2_*^A)$.
(3) $F^{-1}(2_*^A) \cup G^{-1}(2_*^A) \subset (F \cap G)^{-1}(2_*^A)$,
in general, $\bigcup_{\alpha\in\Gamma} F_\alpha^{-1}(2_*^A) \subset (\bigcap_{\alpha\in\Gamma} F_\alpha)^{-1}(2_*^A)$.

Proof. (1) Let $y \in G^{-1}(2_*^A)$. Then $G(y) \in 2_*^A$. Thus $G(y) \subset A$. Since $F \subset G$, $F(y) \subset G(y)$. So $F(y) \subset A$, i.e., $F(y) \in 2_*^A$. Hence $y \in F^{-1}(2_*^A)$. Therefore $G^{-1}(2^A) \subset F^{-1}(2_*^A)$.

(2) Let $y \in (F\cup G)^{-1}(2_*^A) = F^{-1}(2_*^A) \cap G^{-1}(2_*^A)$. Then $(F\cup G)(y) = F(y)\cup G(y) \in 2_*^A$, i.e., $F(y)\cup G(y) = (F(y)_T \cup G(y)_T, F(y)_F \cap G(y)_F) \subset A$. Thus $F(y)_T \cup G(y)_T \subset A_T$ and $F(y)_F \cap G(y)_F \supset A_F$. So $F(y)_T \subset A_T$, $G(y)_T \subset A_T$ and $F(y)_F \supset A_F$, $G(y)_F \supset A_F$, i.e., $F(y) \subset A$ and $G(y) \subset A$, i.e., $F(y) \in 2_*^A$ and $G(y) \in 2_*^A$. Hence $y \in F^{-1}(2_*^A)$ and $y \in G^{-1}(2_*^A)$, i.e., $y \in F^{-1}(2_*^A) \cap G^{-1}(2_*^A)$. The converse inclusion is proved similarly.

The proof of the second part is similar.

(3) Let $y \in F^{-1}(2_*^A) \cup G^{-1}(2_*^A)$. Then $y \in F^{-1}(2_*^A)$ or $y \in G^{-1}(2_*^A)$, i.e., $F(y) \subset A$ or $G(y) \subset A$. Then $F(y) \cap G(y) \subset A$. Thus $(F \cap G)(y) \subset A$, i.e., $(F \cap G)(y) \in 2_*^A$. So $y \in (F \cap G)^{-1}(2_*^A)$. Hence the result holds.

The proof of the second part is similar. □

Theorem 5.7. *Let (X, τ) be an ITS and let (Y, σ) be an ordinary topological space and let $F : (Y, \sigma) \to 2^{(X,\tau)}$ be an intuitionistic set-valued mapping. Then F is continuous if and only if the set*

$$F^{-1}(2^A) = \{y \in Y : F(y) \in 2^A\} = \{y \in Y : F(y) \subset A\} \tag{5.5.1}$$

is open in Y, whenever $A \in \tau$, and is closed in Y, whenever $A \in IC(X)$.

Equivalently, for each $A \in IC(X)$ [resp. $A \in \tau$], the set

$$Y - F^{-1}(A^c) = \{y \in Y : F(y) \cap A \neq \phi_I\} \tag{5.5.2}$$

is open [resp. closed] in Y.

More precisely, F is continuous at $y \in Y$ if and only if both implication hold:

$$y \in F^{-1}(2^G) \Rightarrow y \in int(F^{-1}(2^G)), \text{ whenever } G \in \tau \tag{5.5.3}$$

and

$$y \in cl(F^{-1}(2^K)) \Rightarrow y \in F^{-1}(2^K), \text{ whenever } K \in IC(X). \tag{5.5.4}$$

Proof. Suppose F is continuous at $y_0 \in Y$. Let G be open in $2^{(X,\tau)}$ and suppose $y \in F^{-1}(G)$. Then $F(y) \in G$. Since G is open in $2^{(X,\tau)}$, G is a neighbourhood of $F(y_0)$. Thus there exists $U \in \tau_v$ such that $F(y_0) \in F(U) \subset G$. So $y_0 \in U \subset F^{-1}(G)$. Hence $y_0 \in int(F^{-1}(G))$.

Now let K be closed in $2^{(X,\tau)}$ and suppose $y_0 \in cl(F^{-1}(K))$. By result 5.2 (9),

$$cl(F^{-1}(K)) = cl(F^{-1}((K^c)^c) = cl(F^{-1}(K^c))^c = (int(F^{-1}(K^c)))^c.$$

Then $y_0 \in (int(F^{-1}(K^c)))^c$. Thus $y_0 \notin int(F^{-1}(K^c)) = int((F^{-1}(K))^c)$. Since $int((F^{-1}(K)^c) \subset (F^{-1}(K))^c$, $y_0 \notin (F^{-1}(K))^c$. So $y_0 \in F^{-1}(K)$. Hence the following implications:

$$y_0 \in F^{-1}(G) \Rightarrow y_0 \in int(F^{-1}(G)), \text{ whenever } G \text{ is open in } 2^{(X,\tau)} \tag{5.5.5}$$

and

$$y_0 \in cl(F^{-1}(K)) \Rightarrow y_0 \in F^{-1}(K), \text{ whenever } K \text{ is closed in } 2^{(X,\tau)}. \tag{5.5.6}$$

Therefore by replacing G by 2^G for $G \in \tau$, and K by 2^K for $K \in IC(X)$, we can obtain two implications (5.5.3) and (5.5.4).

Conversely, suppose the implication (5.5.5) holds. Then we can easily see that F is continuous at $y_0 \in Y$. If the implication (5.5.6) holds, then we can easily see that F is continuous at $y_0 \in Y$. Moreover, since the range of G can be restricted to a subbase of $2^{(X,\tau)}$, we may assume that $G = 2^A$ or $G = (2^{A^c})^c$ with $A \in \tau$. In the first case, (5.5.5) follows directly from (5.5.3). In the second case, (5.5.6) can be deduced from (5.5.4). □

Definition 5.8 ([6])**.** Let X, Y be an ITSs. Then a mapping $f : X \to Y$ is said to be continuous, if $f^{-1}(V) \in IO(X)$, for each $V \in IO(Y)$.

Definition 5.9. Let X, Y be ITSs. Then a mapping $f : X \to Y$ is said to be:
(i) open [6], if $f(A) \in IO(Y)$, for each $A \in IO(X)$,
(ii) closed [15], if $f(F) \in IC(Y)$, for each $F \in IC(X)$.

Theorem 5.10. *Let $(X,\tau), (Y,\sigma)$ be $T_1(iii)$-spaces such that $\tau \subset IS_*(X)$ and $\sigma \subset IS_*(Y)$, and let $f : X \to Y$ be intuitionistic continuous. Then the mapping $f_*^{-1} : 2^{(Y,\sigma)} \to 2^{(X,\tau)}$ is continuous if and only if f is both intuitionistic open and closed.*

Proof. Suppose $f_*^{-1} : 2^{Y_I} \to 2^{X_I}$ is continuous and let $G \in \tau$. Since X is a $T_1(iii)$-space, by Proposition 4.2 (3), 2^G is open in $2^{(X,\tau)}$. Then by the hypothesis and (5.5.1), $(f_*^{-1})^{-1}(2^G) = (f^{-1})^{-1}(2^G) = f(2^G)$ is open in $2^{(Y,\sigma)}$. Thus

$$f(2^G) = \{f(A) \in IS(Y) : A \in 2^G\} = \{f(A) \in IS(Y) : A \subset G\} = 2^{f(G)}$$

is open in $2^{(Y,\sigma)}$. So by Theorem 3.21, $f(G) \in \sigma$, i.e., f is intuitionistic open.

Now let $F \in IC(X)$. Then by Corollary 3.20, 2^F is closed in $2^{(X,\tau)}$. Since f_*^{-1} is continuous, $(f_*^{-1})^{-1}(2^F) = (f^{-1})^{-1}(2^F) = f(2^F) = 2^{f(F)}$ is closed in $2^{(Y,\sigma)}$. Thus by Theorem 3.19, $f(F) \in IC(Y)$. So f is intuitionistic closed. Hence f is both intuitionistic closed. Therefore f is both intuitionistic open and closed.

The converse can be easily proved. □

The following is the immediate result of Proposition 5.6 (2) and Theorem 5.7.

Proposition 5.11. *Let (X, τ) be an ITS and (Y, σ) be an ordinary topological space and let $F, G : (Y, \sigma) \to 2^{(X,\tau)}$ be intuitionistic set-valued mappings. If F and G are continuous, then $F \cup G$ is continuous.*

6. Conclusions

We introduced three types intuitionistic hyperspaces and obtained their some properties. In the future, we expect that we will find some relationships between separation axioms T_0, T_1, T_2, T_3 and T_4 in ITSs and intuitionistic hyperspaces. Also we will deal with separability and axioms of countability between an ITS and its hyperspace.

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