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Abstract. In this paper, our aim is to define the nearness semirings and to deal with their basic properties. Afterwards, we will study some properties of nearness semirings and ideals.

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1. Introduction

In 1982, Pawlak described the concept of rough set which is useful for modeling incompleteness and imprecision in information systems. The theory of rough sets is an extension of the set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. A basic notion in the Pawlak rough set model is an equivalence relation. An algebraic approach of rough sets has been given by Iwinski [9]. Afterwards, rough subgroups were introduced by Biswas and Nanda [1]. Kuroki in [10], introduced the notion of a rough ideal in a semigroup. Since then the subject has been investigated in many papers [2, 3, 11, 21].

In 2002, Peters introduced near sets theory, which is a generalization of rough set theory [15, 16]. In this theory, Peters defined an indiscernibility relation that depends on the features of the objects in order to define the nearness of the objects [19]. More recent work considers generalized approach theory in the study of the nearness of non-empty sets that resemble each other [17, 18, 20].

In 2012, İnan and Öztürk investigated the concept of nearness groups [4, 5]. Also, in 2015, Öztürk and İnan established nearness semigroups [6, 7] (and other algebraic approaches of near sets in [12]-[14], [8]).

In this paper, our aim is to define the nearness semirings and to deal with its basic properties. Afterwards, we will study some properties of nearness semirings and ideals.

2. Preliminaries

An object description is defined by means of a tuple of function values $\Phi(x)$ associated with an object $x \in X$. Assume that $B \subseteq \mathcal{F}$ is a given set of functions representing features of sample objects $X \subseteq \mathcal{O}$. Let $\varphi_i \in B$, where $\varphi_i : \mathcal{O} \to \mathbb{R}$. In combination, the functions representing object features provide a basis for an object description $\Phi : \mathcal{O} \to \mathbb{R}^L, \Phi(x) = (\varphi_1(x), \varphi_2(x), ..., \varphi_L(x))$ a vector containing measurements (returned values) associated with each functional value $\varphi_i(x)$, where the description length $\mid \Phi \mid = L$ ([17]).

The important thing to notice is the choice of functions $\varphi_i \in B$ used to describe an object of interest. Sample objects $X \subseteq \mathcal{O}$ are near each if and only if the objects have similar descriptions. Recall that each φ defines a descriptive form of an object. Then let $\triangle_{\varphi_i}$ denote $\triangle_{\Phi_i} = \mid \Phi_i(x') - \Phi_i(x) \mid$, where $x', x \in \mathcal{O}$. The difference φ leads to a description of the indiscernibility relation " $\sim_B$ " introduced by Peters in [17].

Definition 2.1 ([17])**.** Let $x, x' \in \mathcal{O}, B \subseteq \mathcal{F}$.

$$\sim_B = \{(x, x') \in \mathcal{O} \times \mathcal{O} \mid \triangle_{\varphi_i} = 0 \text{ for all } \varphi_i \in B\}$$

is called the indiscernibility relation on $\mathcal{O}$, where description length $i \leq \mid \Phi \mid$.

The basic idea in the near set approach to object recognition is to compare object descriptions. Sets of objects X, X' are considered near each other if the sets contain objects with at least partial matching descriptions.

Definition 2.2 ([17])**.** Let $X, X' \subseteq \mathcal{O}, B \subseteq \mathcal{F}$. Set X is called near X', if there exists $x \in X, x' \in X', \varphi_i \in B$ such that $x \sim_{\varphi_i} x'$.

Symbol	*Interpretation*
B	$B \subseteq \mathcal{F}$, set of probe functions,
r	$\binom{\lvert B\rvert}{r}$, i.e. , $\lvert B\rvert$ probe functions $\varphi_i \in B$ taken r at a time,
B_r	$r \leq \lvert B\rvert$ probe functions in B,
$\sim_{B_r}$	indiscernibility relation defined using B_r,
$[x]_{B_r}$	$[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}$, near equivalence class,
$\mathcal{O} / \sim_{B_r}$	$\mathcal{O} / \sim_{B_r} = \{[x]_{B_r} \mid x \in \mathcal{O}\} = \xi_{\mathcal{O}, B_r}$, quotient set,
$N_r(B)$	$N_r(B) = \{\xi_{\mathcal{O}, B_r} \mid B_r \subseteq B\}$,set of partitions,
ν_{N_r}	$\nu_{N_r} : \wp(\mathcal{O}) \times \wp(\mathcal{O}) \to [0, 1]$, overlap function,
$N_r(B)_* X$	$N_r(B)_* X = \bigcup_{[x]_{B_r} \subseteq X} [x]_{B_r}$, lower approximation,
$N_r(B)^* X$	$N_r(B)^* X = \bigcup_{[x]_{B_r} \cap X \neq \varnothing} [x]_{B_r}$, upper approximation,
$Bnd_{N_r(B)}(X)$	$N_r(B)^* X \diagdown N_r(B)_* X = \left\{x \in N_r(B)^* X \mid x \notin N_r(B)_* X\right\}$.

Table 1: Nearness Approximation Space Symbols

A nearness approximation space is a tuple $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ where the approximation space is defined with a set of perceived objects $\mathcal{O}$, set of probe functions $\mathcal{F}$ representing object features, $\sim_{B_r}$indiscernibility relation B_r defined relative to $B_r \subseteq B \subseteq \mathcal{F}$, collection of partitions (families of neighbour-hoods) $N_r(B)$, and overlap function ν_{N_r} ([17]).

Definition 2.3 ([6])**.** Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space and " $\cdot$ " be a binary operation defined on $\mathcal{O}$. A subset S of perceptual objects $\mathcal{O}$ is called a semigroup on nearness approximation space or shortly nearness semigroup, if the following properties are satisfied.

(i) $x \cdot y \in N_r(B)^* S$, for all $x, y \in S$,

(ii) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $N_r(B)^* S$, for all $x, y \in S$.

Definition 2.4 ([7])**.** Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space and " $+$ " and " $\cdot$ " be binary operations defined on $\mathcal{O}$. A subset R of the set of perceptual objects $\mathcal{O}$ is called a nearness ring, if the following properties are satisfied:

NR_1) R is an abelian near group on $\mathcal{O}$ with binary operation " $+$ ",

NR_2) R is a near semigroup on $\mathcal{O}$ with binary operation " $\cdot$ ",

NR_3) For all $x, y, z \in R$,

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z) \text{ and } (x + y) \cdot z = (x \cdot z) + (y \cdot z)$$

properties hold in $N_r(B)^* R$.

In addition,

NR_4) if $x \cdot y = y \cdot x$, for all $x, y \in R$, then R is said to be a commutative nearness ring,

NR_5) if $N_r(B)^* R$ contains an element 1_R such that $1_R \cdot x = x \cdot 1_R = x$, for all $x \in R$, then R is said to be a nearness ring with identity.

In [8], since $\nu_{N_r} : \wp(\mathcal{O}) \times \wp(\mathcal{O}) \to [0, 1]$ is not needed which is overlap function when algebraic structures are studied on the nearness approximation space $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$, the following definition was given.

Definition 2.5 ([8])**.** Let $\mathcal{O}$ be a set of perceived objects, $\mathcal{F}$ a set of the probe functions, $\sim_{B_r}$an indiscernibility relation, and N_r a collection of partitions. Then, $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ is called a weak nearness approximation space.

Theorem 2.6 ([8])**.** *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ be a weak nearness approximation space and $X, Y \subset \mathcal{O}$. Then the following statements hold:*

(1) $N_r(B)_* X \subseteq X \subseteq N_r(B)^* X$,

(2) $N_r(B)^* (X \cup Y) = N_r(B)^* X \cup N_r(B)^* Y$,

(3) $N_r(B)_* (X \cap Y) = N_r(B)_* X \cap N_r(B)_* Y$,

(4) $X \subseteq Y$ *implies* $N_r(B)_* X \subseteq N_r(B)_* Y$,

(5) $X \subseteq Y$ *implies* $N_r(B)^* X \subseteq N_r(B)^* Y$,

(6) $N_r(B)_* (X \cup Y) \supseteq N_r(B)_* X \cup N_r(B)_* Y$,

(7) $N_r(B)^* (X \cap Y) \subseteq N_r(B)^* X \cap N_r(B)^* Y$.

3. Nearness Semirings

Throughout this paper $\mathcal{O}$ denotes a $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ is weak near approximation spaces unless otherwise specified.

Definition 3.1. $(S,\cdot)$ is called a nearness monoid, if S is a nearness semigroup in which there exists an element $e \in N_r(B)^* S$ satisfying $x \cdot e = e \cdot x = x$, for all $x \in S$.

Definition 3.2. A nearness monoid $(S,\cdot)$ $((S,+))$ is called a commutative (abelian) ,if $x \cdot y = y \cdot x$ ($x + y = y + x$), for all $x, y \in S$.

Definition 3.3. A subset S of the weak near approximation spaces $\mathcal{O}$ is called a semiring on $\mathcal{O}$, if the following properties are satisfied:

NSR_1) $(S,+)$ is an abelian monoid on $\mathcal{O}$ with identity element 0,

NSR_2) $(S,\cdot)$ is a monoid on $\mathcal{O}$ with identity element 1,

NSR_3) for all $x, y, z \in S$,

$$x \cdot (y+z) = (x \cdot y) + (x \cdot z) \text{ and } (x+y) \cdot z = (x \cdot z) + (y \cdot z)$$

properties hold in $N_r(B)^* S$,

NSR_4) for all $x \in S$,

$$0 \cdot x = 0 = x \cdot 0$$

properties hold in $N_r(B)^* S$,

NSR_5) $1 \neq 0$.

Definition 3.4. A subset R of the weak near approximation spaces $\mathcal{O}$ is called a hemiring on $\mathcal{O}$, if the following properties are satisfied:

NHR_1) $(R,+)$ is an abelian monoid on $\mathcal{O}$ with identity element 0,

NHR_2) $(R,\cdot)$ is a semigroup on $\mathcal{O}$,

NHR_3) for all $x, y, z \in R$,

$$x \cdot (y+z) = (x \cdot y) + (x \cdot z) \text{ and } (x+y) \cdot z = (x \cdot z) + (y \cdot z)$$

properties hold in $N_r(B)^* R$,

NHR_4) for all $x \in R$,

$$0 \cdot x = 0 = x \cdot 0$$

properties hold in $N_r(B)^* R$.

Example 3.5. Let $\mathcal{O} = \{0, 1, a, b, c, d, e, f, g, h, i, j, k, l\}$ be a set of perceptual objects where

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$c = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, d = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

$$g = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, h = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, j = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$k = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, l = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, m = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, n = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

for $U = \{ [a_{ij}]_{2x2} \mid a_{ij} \in \mathbb{Z}_2\}$, $r = 1$, $B = \{\varphi_1, \varphi_2, \varphi_3\} \subseteq \mathcal{F}$ be a set of probe functions, and $S = \{a, b, c, e\} \subset \mathcal{O}$. Values of the probe functions

$$\varphi_1 : \mathcal{O} \to V_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5\},$$

$$\varphi_2 : \mathcal{O} \to V_2 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6\},$$

$$\varphi_3 : \mathcal{O} \to V_3 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$$

are given in *Table* 2.

	0	1	a	b	c	d	e	f	g	h	i	j	k	l
φ_1	α_1	α_2	α_3	α_2	α_1	α_3	α_2	α_1	α_1	α_1	α_2	α_1	α_5	α_5
φ_2	α_1	α_3	α_3	α_4	α_1	α_1	α_4	α_3	α_4	α_3	α_3	α_3	α_6	α_6
φ_3	α_3	α_3	α_1	α_1	α_4	α_4	α_5	α_1	α_3	α_3	α_4	α_3	α_5	α_6

Table 2

Let us now determine the near equivalence classes according to the indiscernibility relation $\sim_{B_r}$ of elements in $\mathcal{O}$:

$$\begin{aligned}
[0]_{\varphi_1} &= \{\acute{x} \in \mathcal{O} \mid \varphi_1(\acute{x}) = \varphi_1(0) = \alpha_1\} = \{0, c, f, g, h, j\} \\
&= [c]_{\varphi_1} = [f]_{\varphi_1} = [g]_{\varphi_1} = [h]_{\varphi_1} = [j]_{\varphi_1}, \\
[1]_{\varphi_1} &= \{\acute{x} \in \mathcal{O} \mid \varphi_1(\acute{x}) = \varphi_1(1) = \alpha_2\} = \{1, b, e, i\} \\
&= [b]_{\varphi_1} = [e]_{\varphi_1} = [i]_{\varphi_1}, \\
[a]_{\varphi_1} &= \{\acute{x} \in \mathcal{O} \mid \varphi_1(\acute{x}) = \varphi_1(a) = \alpha_3\} = \{a, d\} \\
&= [d]_{\varphi_1}, \\
[k]_{\varphi_1} &= \{\acute{x} \in \mathcal{O} \mid \varphi_1(\acute{x}) = \varphi_1(k) = \alpha_5\} = \{k, l\}, \\
&= [l]_{\varphi_1}.
\end{aligned}$$

Then, we get that $\xi_{\varphi_1} = \left\{[0]_{\varphi_1}, [1]_{\varphi_1}, [a]_{\varphi_1}, [k]_{\varphi_1}\right\}$.

$$\begin{aligned}
[0]_{\varphi_2} &= \{\acute{x} \in \mathcal{O} \mid \varphi_2(\acute{x}) = \varphi_2(0) = \alpha_1\} = \{0, c, d\} \\
&= [c]_{\varphi_2} = [d]_{\varphi_2}, \\
[1]_{\varphi_2} &= \{\acute{x} \in \mathcal{O} \mid \varphi_2(\acute{x}) = \varphi_2(1) = \alpha_3\} = \{1, f, h, i, j\} \\
&= [a]_{\varphi_2} = [f]_{\varphi_2} = [h]_{\varphi_2} = [i]_{\varphi_2} = [j]_{\varphi_2}, \\
[b]_{\varphi_2} &= \{\acute{x} \in \mathcal{O} \mid \varphi_2(\acute{x}) = \varphi_2(\gamma) = \alpha_4\} = \{b, e, g\} \\
&= [e]_{\varphi_2} = [g]_{\varphi_2}, \\
[k]_{\varphi_2} &= \{\acute{x} \in \mathcal{O} \mid \varphi_2(\acute{x}) = \varphi_2(k) = \alpha_6\} = \{k, l\}, \\
&= [l]_{\varphi_2}.
\end{aligned}$$

Thus, we have that $\xi_{\varphi_2} = \left\{[0]_{\varphi_2}, [1]_{\varphi_2}, [b]_{\varphi_2}, [k]_{\varphi_2}\right\}$.

$$\begin{aligned}
[0]_{\varphi_3} &= \{x^{\prime} \in \mathcal{O} \mid \varphi_3(x^{\prime}) = \varphi_3(0) = \alpha_3\} = \{0, 1, g, h, j\} \\
&= [1]_{\varphi_3} = [g]_{\varphi_3} = [h]_{\varphi_3} = [j]_{\varphi_3}, \\
[a]_{\varphi_3} &= \{x^{\prime} \in \mathcal{O} \mid \varphi_3(x^{\prime}) = \varphi_3(a) = \alpha_1\} = \{a, b, f\} \\
&= [b]_{\varphi_3} = [f]_{\varphi_3}, \\
[c]_{\varphi_3} &= \{x^{\prime} \in \mathcal{O} \mid \varphi_3(x^{\prime}) = \varphi_3(c) = \alpha_4\} = \{c, d, i\} \\
&= [d]_{\varphi_3} = [i]_{\varphi_3}, \\
[e]_{\varphi_3} &= \{x^{\prime} \in \mathcal{O} \mid \varphi_3(x^{\prime}) = \varphi_3(e) = \alpha_5\} = \{e, k\} \\
&= [k]_{\varphi_3}, \\
[l]_{\varphi_3} &= \{x^{\prime} \in \mathcal{O} \mid \varphi_3(x^{\prime}) = \varphi_3(l) = \alpha_6\} = \{l\}.
\end{aligned}$$

From hence, we obtain that $\xi_{\varphi_3} = \left\{[0]_{\varphi_3}, [a]_{\varphi_3}, [c]_{\varphi_3}, [e]_{\varphi_3}, [l]_{\varphi_3}\right\}$. Therefore, for $r = 1$, a set of partitions of $\mathcal{O}$ is $N_r(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}, \xi_{\varphi_3}\}$. Then, we can write

$$\begin{aligned}
N_1(B)^* S &= \bigcup_{[x]_{\varphi_i} \cap S \neq \varnothing} [x]_{\varphi_i} \\
&= [0]_{\varphi_1} \cup [1]_{\varphi_1} \cup [b]_{\varphi_1} \cup [0]_{\varphi_2} \cup [b]_{\varphi_2} \cup [a]_{\varphi_3} \cup [c]_{\varphi_3} \cup [e]_{\varphi_3} \\
&= \{0, 1, a, b, c, d, e, f, g, h, i, j, k\}.
\end{aligned}$$

Considering the operation in $Table$ 3.

$+$	a	b	c	e
a	0	e	1	b
b	e	0	f	a
c	1	f	0	k
e	b	a	k	0

$Table$ 3

In that case, $(S, +)$ is an abelian monoid on $\mathcal{O}$ with identity element 0. Considering the operation in $Table$ 4.

$\cdot$	a	b	c	e
a	a	0	0	a
b	a	0	0	a
c	0	a	c	a
e	e	0	0	e

$Table$ 4

Then, $(S, \cdot)$ is a monoid on $\mathcal{O}$ with identity element 1. Moreover, $(S, +, \cdot)$ satisfies conditions (NSR_3), (NSR_4) and (NSR_5). Therefore, $(S, +, \cdot)$ is a semiring on the weak near approximation space $\mathcal{O}$, i. e. , $(S, +, \cdot)$ is a nearness semiring.

Example 3.6. Let $\mathcal{O} = \{0, 1, a, b, c, d, e, f, g, h, i, j, k, n\}$ be a set of perceptual objects where

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$
$$c = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, d = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$
$$g = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, h = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, j = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$
$$k = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, l = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, m = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, n = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

for $U = \{\ [a_{ij}]_{2x2} \mid a_{ij} \in \mathbb{Z}_2\}$, $r = 1$, $B = \{\varphi_1, \varphi_2, \varphi_3\} \subseteq \mathcal{F}$ be a set of probe functions, and $S = \{a, d, e, h\} \subset \mathcal{O}$. Values of the probe functions

$$\begin{aligned} \varphi_1 &: \mathcal{O} \to V_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}, \\ \varphi_2 &: \mathcal{O} \to V_2 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6\}, \\ \varphi_3 &: \mathcal{O} \to V_3 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} \end{aligned}$$

are given in *Table* 5.

	0	1	a	b	c	d	e	f	g	h	i	j	k	n
φ_1	α_1	α_2	α_1	α_3	α_1	α_3	α_4	α_3	α_4	α_3	α_1	α_4	α_5	α_5
φ_2	α_3	α_3	α_4	α_3	α_1	α_1	α_4	α_3	α_4	α_4	α_3	α_4	α_6	α_6
φ_3	α_3	α_3	α_1	α_1	α_4	α_4	α_6	α_1	α_3	α_6	α_3	α_6	α_5	α_6

Table 5

Let us now determine the near equivalence classes according to the indiscernibility relation $\sim_{B_r}$ of elements in $\mathcal{O}$:

$$\begin{aligned} [0]_{\varphi_1} &= \{\acute{x} \in \mathcal{O} \mid \varphi_1(\acute{x}) = \varphi_1(0) = \alpha_1\} = \{0, a, c, i\} \\ &= [a]_{\varphi_1} = [c]_{\varphi_1}, \\ [1]_{\varphi_1} &= \{\acute{x} \in \mathcal{O} \mid \varphi_1(\acute{x}) = \varphi_1(1) = \alpha_2\} = \{1\}, \\ [b]_{\varphi_1} &= \{\acute{x} \in \mathcal{O} \mid \varphi_1(\acute{x}) = \varphi_1(a) = \alpha_3\} = \{b, d, f, h\} \\ &= [d]_{\varphi_1} = [f]_{\varphi_1} = [h]_{\varphi_1}, \\ [e]_{\varphi_1} &= \{\acute{x} \in \mathcal{O} \mid \varphi_1(\acute{x}) = \varphi_1(e) = \alpha_4\} = \{e, g, j\}, \\ &= [g]_{\varphi_1} = [j]_{\varphi_1}, \\ [k]_{\varphi_1} &= \{\acute{x} \in \mathcal{O} \mid \varphi_1(\acute{x}) = \varphi_1(e) = \alpha_5\} = \{k, n\} \\ &= [n]_{\varphi_1}. \end{aligned}$$

Then, we obtain $\xi_{\varphi_1} = \left\{ [0]_{\varphi_1}, [1]_{\varphi_1}, [b]_{\varphi_1}, [e]_{\varphi_1} [k]_{\varphi_1} \right\}$.

$$\begin{aligned}
[0]_{\varphi_2} &= \{ \acute{x} \in \mathcal{O} \mid \varphi_2(\acute{x}) = \varphi_2(0) = \alpha_3 \} = \{0, 1, b, f, i\} \\
&= [1]_{\varphi_2} = [b]_{\varphi_2} = [f]_{\varphi_2} = [i]_{\varphi_2}, \\
[a]_{\varphi_2} &= \{ \acute{x} \in \mathcal{O} \mid \varphi_2(\acute{x}) = \varphi_2(a) = \alpha_4 \} = \{a, e, g, h, j\} \\
&= [e]_{\varphi_2} = [g]_{\varphi_2} = [h]_{\varphi_2} = [j]_{\varphi_2}, \\
[c]_{\varphi_2} &= \{ \acute{x} \in \mathcal{O} \mid \varphi_2(\acute{x}) = \varphi_2(\gamma) = \alpha_1 \} = \{c, d\} \\
&= [d]_{\varphi_2}, \\
[k]_{\varphi_2} &= \{ \acute{x} \in \mathcal{O} \mid \varphi_2(\acute{x}) = \varphi_2(k) = \alpha_6 \} = \{k, n\}, \\
&= [n]_{\varphi_2}.
\end{aligned}$$

Thus, we have that $\xi_{\varphi_2} = \left\{ [0]_{\varphi_2}, [a]_{\varphi_2}, [c]_{\varphi_2} [k]_{\varphi_2} \right\}$.

$$\begin{aligned}
[0]_{\varphi_3} &= \{ \acute{x} \in \mathcal{O} \mid \varphi_3(\acute{x}) = \varphi_3(0) = \alpha_3 \} = \{0, 1, g, i\} \\
&= [1]_{\varphi_3} = [g]_{\varphi_3} = [h]_{\varphi_3} = [i]_{\varphi_3}, \\
[a]_{\varphi_3} &= \{ \acute{x} \in \mathcal{O} \mid \varphi_3(\acute{x}) = \varphi_3(a) = \alpha_1 \} = \{a, b, f\} \\
&= [b]_{\varphi_3} = [f]_{\varphi_3}, \\
[c]_{\varphi_3} &= \{ \acute{x} \in \mathcal{O} \mid \varphi_3(\acute{x}) = \varphi_3(c) = \alpha_4 \} = \{c, d\} \\
&= [d]_{\varphi_3}, \\
[e]_{\varphi_3} &= \{ \acute{x} \in \mathcal{O} \mid \varphi_3(\acute{x}) = \varphi_3(e) = \alpha_6 \} = \{e, h, j, n\} \\
&= [h]_{\varphi_2} = [j]_{\varphi_3} = [n]_{\varphi_3}, \\
[k]_{\varphi_3} &= \{ \acute{x} \in \mathcal{O} \mid \varphi_3(\acute{x}) = \varphi_3(k) = \alpha_5 \} = \{k\}.
\end{aligned}$$

From hence, we obtain that $\xi_{\varphi_3} = \left\{ [0]_{\varphi_3}, [a]_{\varphi_3}, [c]_{\varphi_3}, [e]_{\varphi_3}, [k]_{\varphi_3} \right\}$. Therefore, for $r = 1$, a set of partitions of $\mathcal{O}$ is $N_r(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}, \xi_{\varphi_3}\}$. Then, we can write

$$\begin{aligned}
N_1(B)^* S &= \bigcup_{[x]_{\varphi_i} \cap S \neq \varnothing} [x]_{\varphi_i} \\
&= [0]_{\varphi_1} \cup [a]_{\varphi_1} \cup [e]_{\varphi_1} \cup [a]_{\varphi_2} \cup [c]_{\varphi_2} \cup [a]_{\varphi_3} \cup [c]_{\varphi_3} \cup [e]_{\varphi_3} \\
&= \{0, a, b, c, d, e, f, g, h, i, j, n\}.
\end{aligned}$$

Considering the operation in $Table$ 6.

$+$	a	d	e	h
a	0	h	b	d
d	h	0	j	a
e	b	j	0	i
h	d	a	i	0

$Table$ 6

In that case, $(S,+)$ is an abelian monoid on $\mathcal{O}$ with identity element 0. Considering the operation in *Table* 7.

$\cdot$	a	d	e	h
a	a	d	a	h
d	0	0	a	0
e	e	g	e	n
h	a	0	e	0

Table 7

Then, $(S,\cdot)$ is a semigroup on $\mathcal{O}$. Moreover, $(S,+,\cdot)$ satisfies conditions (NHR_3) and (NHR_4). Therefore, $(S,+,\cdot)$ is a hemiring on the weak near approximation space $\mathcal{O}$, i. e. , $(S,+,\cdot)$ is a nearness hemiring.

Considering the operations in *Table* 8 and *Table* 9.

$+$	0	a	b	c	d	e	f	g	h	i	j	n
0	0	a	b	c	d	e	f	g	h	i	j	n
a	a	0	e	1	h	b	k	m	d	j	i	l
b	b	e	0	f	i	a	c	l	j	d	h	m
c	c	1	f	0	g	k	b	d	m	l	n	j
d	d	h	i	g	0	j	l	c	a	b	e	k
e	e	b	a	k	j	0	1	n	i	h	d	g
f	f	k	c	b	l	1	0	i	n	g	m	h
g	g	m	l	d	l	n	i	0	1	f	k	e
h	h	d	j	m	a	i	n	1	0	e	b	f
i	i	j	d	l	b	h	f	f	e	0	a	1
j	j	i	h	n	e	d	m	k	b	a	0	c
n	n	l	m	j	k	g	h	e	f	1	c	0

Table 8

and

$\cdot$	0	a	b	c	d	e	f	g	h	i	j	n
0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	a	0	0	d	a	0	d	h	d	h	h
b	0	b	0	0	c	b	0	c	f	c	f	f
c	0	0	b	c	0	b	f	c	0	b	b	f
d	0	0	a	d	0	a	h	d	0	a	a	h
e	0	e	0	0	g	e	0	g	n	g	n	n
f	0	b	b	c	c	0	f	0	f	f	c	0
g	0	0	b	g	0	e	n	g	0	e	e	n
h	0	a	e	g	0	e	n	g	0	h	e	n
i	0	b	a	d	c	e	h	g	f	1	k	n
j	0	e	a	d	g	b	h	c	n	m	l	f
n	0	e	e	g	g	0	n	0	n	n	g	0

Table 9

(NSR_1), (NSR_2), (NSR_3), and (NSR_4) properties have to hold in $N_r(B)^* S$ for all elements of S. However, sum or multiplying of elements in $N_r(B)^* S$ may not always belongs to $N_r(B)^* S$ (or $\mathcal{O}$). For instance, $d+f = l \notin \mathcal{O}$ for $d, f \in N_r(B)^* S$, $a + c = 1 \notin N_r(B)^* S$ for $a, c \in N_r(B)^* S$, $j \cdot j = l \notin N_r(B)^* S$ for $j \in N_r(B)^* S$.

An element x in nearness semiring S is said to be *left* (resp. *right*) *invertible* if there exists $y \in S$ (resp. $z \in S$) such that $y \cdot x = 1_S \in N_r(B)^* S$ (resp. $x \cdot z = 1_S \in N_r(B)^* S$). The element y (resp. z) is called a *left* (resp. *right*) *inverse* of x. If $x \in S$ is both left and right invertible, then x is said to be *nearness invertible* or *nearness unit*.

Some elementary properties of elements in nearness semirings S are not always provided as in semirings S. If we consider $N_r(B)^* S$ as a semiring, then elementary properties of elements in nearness semiring S are provided.

Definition 3.7. Let S be a semiring on $\mathcal{O}$, $B_r \subseteq \mathcal{F}$ where $r \leq \mid B \mid$ and $B \subseteq \mathcal{F}$, $\sim_{B_r}$be a indiscernibility relation on $\mathcal{O}$. Then, $\sim_{B_r}$ is called a congruence indiscernibility relation on nearness semiring S, if $x \sim_{B_r} y$, where $x, y \in S$ implies $x + a \sim_{B_r} y + a$, $a + x \sim_{B_r} a + y$ $xa \sim_{B_r} ya$ and $ax \sim_{B_r} ay$, for all $a \in S$.

Lemma 3.8. *Let S be a nearness semiring. If $\sim_{B_r}$ is a congruence indiscernibility relation on S, then $[x]_{B_r} + [y]_{B_r} \subseteq [x+y]_{B_r}$ and $[x]_{B_r}[y]_{B_r} \subseteq [xy]_{B_r}$, for all $x, y \in S$.*

Proof. Let $z \in [x]_{B_r} + [y]_{B_r}$. In his case, $z = a + b; a \in [x]_{B_r}, b \in [y]_{B_r}$. From here $x \sim_{B_r} a$, and $y \sim_{B_r} b$, and so, we have $x + y \sim_{B_r} a + y$, and $a + y \sim_{B_r} a + b$ by hypothesis. Thus, $x + y \sim_{B_r} a + b \Rightarrow z = a + b \in [x + y]_{B_r}$. Similarly, $[x]_{B_r}[y]_{B_r} \subseteq [xy]_{B_r}$ is obtained. □

Definition 3.9. Let S be a nearness semiring, $B_r \subseteq \mathcal{F}$ where $r \leq \mid B \mid$ and $B \subseteq \mathcal{F}$, $\sim_{B_r}$be a indiscernibility relation on $\mathcal{O}$. Then, $\sim_{B_r}$ is called a complete congruence indiscernibility relation on nearness semiring S, if $[x]_{B_r} + [y]_{B_r} = [x + y]_{B_r}$ and $[x]_{B_r}[y]_{B_r} = [xy]_{B_r}$, for all $x, y \in S$.

Let S be a nearness semiring. Let $X + Y = \{x + y \mid x \in X \text{ and } y \in Y\}$ and $X \cdot Y = \{\sum\limits_{finite} x_i y_i \mid x_i \in X \text{ and } y_i \in Y\}$, where subsets X and Y of S.

Lemma 3.10. *Let S be a nearness semiring. The following properties hold:*
(1) *if $X, Y \subseteq S$, then $(N_r(B)^* X) + (N_r(B)^* Y) \subseteq N_r(B)^* (X + Y)$,*
(2) *if $X, Y \subseteq S$, then $(N_r(B)^* X) \cdot (N_r(B)^* Y) \subseteq N_r(B)^* (X \cdot Y)$.*

Proof. (1) Let $x \in (N_r(B)^* X) + (N_r(B)^* Y)$. We have $x = a + b; a \in N_r(B)^* X, b \in N_r(B)^* Y$. $a \in N_r(B)^* X \Rightarrow [a]_{B_r} \cap X \neq \varnothing \Rightarrow \exists y \in [a]_{B_r} \cap X \Rightarrow y \in [a]_{B_r}$ and $y \in X$. Likewise, $b \in N_r(B)^* Y \Rightarrow [b]_{B_r} \cap Y \neq \varnothing \Rightarrow \exists z \in [b]_{B_r} \cap Y \Rightarrow z \in [b]_{B_r}$ and $z \in Y$. Since $w = y + z \in [a]_{B_r} + [b]_{B_r} \subseteq [a + b]_{B_r}$, we get $w \in [a + b]_{B_r}$ and $w \in X + Y$. Thus, $w \in [a + b]_{B_r} \cap (X + Y) \Rightarrow [a + b]_{B_r} \cap (X + Y) \neq \varnothing$, and so $a + b = x \in N_r(B)^* (X + Y)$.

(2) Let $x \in (N_r(B)^* X) \cdot (N_r(B)^* Y)$. Then $x = \sum\limits_{i=1} a_i b_i$, where $a_i \in N_r(B)^* X$ and $b_i \in N_r(B)^* Y$, $1 \leq i \leq n$. Thus, $[a_i]_{B_r} \cap X \neq \varnothing$ and $[b_i]_{B_r} \cap Y \neq \varnothing$. So, there exists elements $x_i \in [a_i]_{B_r}$, $x_i \in X$ and $y_i \in [b_i]_{B_r}$, $y_i \in Y$, $1 \leq i \leq n$. Hence, $x_i y_i \in [a_i]_{B_r}[b_i]_{B_r} \subseteq [a_i b_i]_{B_r}$, $1 \leq i \leq n$, by Lemma 3.8. Therefore, we get

$\sum\limits_{i=1} x_iy_i \in [\sum\limits_{i=1} a_ib_i]_{B_r} = [x]_{B_r}$ and $\sum\limits_{i=1} x_iy_i \in X \cdot Y$. In this case, $[x]_{B_r} \cap (X \cdot Y) \neq \varnothing$, which implies that $x \in N_r(B)^*(X \cdot Y)$. □

Theorem 3.11. *Let S be a nearness semiring, $\sim_{B_r}$ a complete congruence indiscernibility relation on S, and X, Y two non-empty subsets of S. The following properties hold:*

(1) $(N_r(B)^* X) + (N_r(B)^* Y) = N_r(B)^*(X+Y)$,
(2) $(N_r(B)^* X) \cdot (N_r(B)^* Y) = N_r(B)^*(X \cdot Y)$,
(3) $(N_r(B)_* X) + (N_r(B)_* Y) \subseteq N_r(B)_*(X+Y)$,
(4) $(N_r(B)_* X) \cdot (N_r(B)_* Y) \subseteq N_r(B)_*(X \cdot Y)$.

Proof. The proof of (1) and (2) is straightforward by the similar way to the proof of Lemma 3.10.

(3) Let $x \in (N_r(B)_* X) + (N_r(B)_* Y)$. We have $x = a + b$; $a \in N_r(B)_* X, b \in N_r(B)_* Y$. In this case, $a \in N_r(B)_* X \Rightarrow [a]_{B_r} \subseteq X$ and $b \in N_r(B)_* Y \Rightarrow [b]_{B_r} \subseteq Y$, so, we obtain $[a]_{B_r} + [a]_{B_r} \subseteq X + Y$. On the other hand, since $[a+b]_{B_r} = [a]_{B_r} + [b]_{B_r} \subseteq X + Y$. Thus, $[a+b]_{B_r} \subseteq X+Y$, and so $a+b = x \in N_r(B)_*(X+Y)$.

(4) Let $x \in (N_r(B)_* X) \cdot (N_r(B)_* Y)$. Then, we have $x = \sum\limits_{i=1} a_ib_i$ such that $a_i \in N_r(B)_* X$ and $b_i \in N_r(B)_* Y$, $1 \leq i \leq n$. Thus, $[a_i]_{B_r} \subseteq X$ and $[b_i]_{B_r} \subseteq Y$. So, there exists elements $x_i \in [a_i]_{B_r}$ and $y_i \in [b_i]_{B_r}$, $1 \leq i \leq n$. Hence, $x_iy_i \in [a_ib_i]_{B_r} = [a_i]_{B_r}[b_i]_{B_r}$, $1 \leq i \leq n$, by Definition 3.9. Therefore, we get $\sum\limits_{i=1} x_iy_i \in [\sum\limits_{i=1} a_ib_i]_{B_r} = [x]_{B_r} \subseteq X \cdot Y$, and hence $x \in N_r(B)_*(X \cdot Y)$. □

Definition 3.12. Let S be a nearness semiring, and A a non-empty subset of S.

(1) A is called a subsemiring of S, if $A + A \subseteq N_r(B)^* A$ and $A \cdot A \subseteq N_r(B)^* A$.

(2) A is called a upper-near subsemiring of S, if $(N_r(B)^* A) + (N_r(B)^* A) \subseteq N_r(B)^* A$ and $(N_r(B)^* A) \cdot (N_r(B)^* A) \subseteq N_r(B)^* A$.

Now, let's give an example to $N_r(B)^*(N_r(B)^* S) = N_r(B)^* S$ where S is a subset of perceptual objects set $\mathcal{O}$.

Example 3.13. Let $\acute{\mathcal{O}} = \{0, 1, a, b, c, d, e, f, g, h, i, j, k, l\}$ be a subset of perceptual objects set $\mathcal{O}$ in Example 3.5, $r = 2$, and $B = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \subseteq \mathcal{F}$ be a set of probe functions. Values of the probe functions

$$\varphi_1 : \acute{\mathcal{O}} \to V_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\},$$
$$\varphi_2 : \acute{\mathcal{O}} \to V_2 = \{\alpha_3, \alpha_4, \alpha_5\},$$
$$\varphi_3 : \acute{\mathcal{O}} \to V_3 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5\},$$
$$\varphi_4 : \acute{\mathcal{O}} \to V_4 = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5\}$$

are given in *Table* 10.

	a	b	c	d	e	f	g	h	i	j
φ_1	α_3	α_1	α_2	α_2	α_5	α_3	α_4	α_4	α_4	α_3
φ_2	α_3	α_3	α_4	α_3	α_5	α_4	α_5	α_3	α_5	α_5
φ_3	α_2	α_1	α_3	α_5	α_1	α_2	α_3	α_5	α_1	α_1
φ_4	α_2	α_1	α_4	α_5	α_5	α_2	α_4	α_4	α_4	α_4

Table 10

In this case,

$$[a]_{\{\varphi_1,\varphi_2\}} = \{x' \in \mathcal{O}' \mid \varphi_1(x') = \varphi_2(x') = \varphi_1(a) = \varphi_2(a) = \alpha_3\} = \{a\}$$
$$[e]_{\{\varphi_1,\varphi_2\}} = \{x' \in \mathcal{O}' \mid \varphi_1(x') = \varphi_2(x') = \varphi_1(e) = \varphi_2(e) = \alpha_5\} = \{e\}.$$

Then, we have that $\xi_{\{\varphi_1,\varphi_2\}} = \left\{[a]_{\{\varphi_1,\varphi_2\}}, [e]_{\{\varphi_1,\varphi_2\}}\right\}$.

$$[b]_{\{\varphi_1,\varphi_3\}} = \{x' \in \mathcal{O}' \mid \varphi_1(x') = \varphi_3(x') = \varphi_1(b) = \varphi_2(b) = \alpha_1\} = \{b\}.$$

We get $\xi_{\{\varphi_1,\varphi_3\}} = \left\{[b]_{\{\varphi_1,\varphi_3\}}\right\}$.

$$\begin{aligned}
[b]_{\{\varphi_1,\varphi_4\}} &= \{x' \in \mathcal{O}' \mid \varphi_1(x') = \varphi_4(x') = \varphi_1(b) = \varphi_4(b) = \alpha_1\} = \{b\},\\
[e]_{\{\varphi_1,\varphi_4\}} &= \{x' \in \mathcal{O}' \mid \varphi_1(x') = \varphi_4(x') = \varphi_1(e) = \varphi_4(e) = \alpha_5\} = \{e\},\\
[g]_{\{\varphi_1,\varphi_4\}} &= \{x' \in \mathcal{O}' \mid \varphi_1(x') = \varphi_4(x') = \varphi_1(g) = \varphi_4(g) = \alpha_4\} = \{g, h, i\}\\
&= [h]_{\{\varphi_1,\varphi_4\}} = [i]_{\{\varphi_1,\varphi_4\}}.
\end{aligned}$$

Thus, $\xi_{\{\varphi_1,\varphi_4\}} = \left\{[b]_{\{\varphi_1,\varphi_3\}}, [e]_{\{\varphi_1,\varphi_4\}}, [g]_{\{\varphi_1,\varphi_4\}}\right\}$.

$$\begin{aligned}
[c]_{\{\varphi_2,\varphi_4\}} &= \{x' \in \mathcal{O}' \mid \varphi_2(x') = \varphi_4(x') = \varphi_2(c) = \varphi_4(c) = \alpha_4\} = \{c\},\\
[e]_{\{\varphi_2,\varphi_4\}} &= \{x' \in \mathcal{O}' \mid \varphi_2(x') = \varphi_4(x') = \varphi_2(e) = \varphi_4(e) = \alpha_5\} = \{e\}.
\end{aligned}$$

We get that $\xi_{\{\varphi_2,\varphi_4\}} = \left\{[c]_{\{\varphi_2,\varphi_4\}}, [e]_{\{\varphi_2,\varphi_4\}}\right\}$.

$$\begin{aligned}
[a]_{\{\varphi_3,\varphi_4\}} &= \{x' \in \mathcal{O}' \mid \varphi_3(x') = \varphi_4(x') = \varphi_3(a) = \varphi_4(a) = \alpha_2\} = \{a, f\}\\
&= [f]_{\{\varphi_3,\varphi_4\}},\\
[b]_{\{\varphi_2,\varphi_4\}} &= \{x' \in \mathcal{O}' \mid \varphi_3(x') = \varphi_4(x') = \varphi_3(b) = \varphi_4(b) = \alpha_1\} = \{b\},\\
[d]_{\{\varphi_2,\varphi_4\}} &= \{x' \in \mathcal{O}' \mid \varphi_3(x') = \varphi_4(x') = \varphi_3(d) = \varphi_4(d) = \alpha_5\} = \{d\}.
\end{aligned}$$

From hence, we obtain that $\xi_{\{\varphi_3,\varphi_4\}} = \left\{[a]_{\{\varphi_3,\varphi_4\}}, [b]_{\{\varphi_2,\varphi_4\}}, [d]_{\{\varphi_2,\varphi_4\}}\right\}$. Therefore, for $r = 2$, a set of partitions of $\mathcal{O}'$ is

$$N_r(B) = \left\{\xi_{\{\varphi_1,\varphi_2\}}, \xi_{\{\varphi_1,\varphi_3\}}, \xi_{\{\varphi_1,\varphi_4\}}, \xi_{\{\varphi_2,\varphi_4\}}, \xi_{\{\varphi_3,\varphi_4\}}\right\}.$$

If $S = \{e, f, g\}$, then we can write

$$\begin{aligned}
N_2(B)^* S &= \bigcup_{[x]_{\{\varphi_i,\varphi_j\}} \cap S \neq \varnothing} [x]_{\{\varphi_i,\varphi_j\}}\\
&= [e]_{\{\varphi_1,\varphi_2\}} \cup [e]_{\{\varphi_1,\varphi_4\}} \cup [g]_{\{\varphi_1,\varphi_4\}} \cup [e]_{\{\varphi_2,\varphi_4\}} \cup [a]_{\{\varphi_3,\varphi_4\}}\\
&= \{e\} \cup \{e\} \cup \{g, h, i\} \cup \{e\} \cup \{a, f\}\\
&= \{a, e, f, g, h, i\}
\end{aligned}$$

and also

$$\begin{aligned} N_2(B)^*(N_2(B)^*S) &= \bigcup_{[x]_{\{\varphi_i,\varphi_j\}}\cap N_2(B)^*S\neq\varnothing}[x]_{\{\varphi_i,\varphi_j\}} \\ &= \{a\}\cup\{e\}\cup\{e\}\cup\{g,h,i\}\cup\{e\}\cup\{a,f\} \\ &= \{a,e,f,g,h,i\}. \end{aligned}$$

In that case, $N_2(B)^*(N_2(B)^*S) = N_2(B)^*S$ is obtained.

Theorem 3.14. *Let S be a nearness semiring. The following properties hold:*
(1) *if $\varnothing \neq A \subseteq S$, $A+A \subseteq A$ and $A\cdot A\subseteq A$, then A is a upper-near subsemiring of S.*
(2) *if A is a subsemiring of S, and $N_r(B)^*(N_r(B)^*A) = N_r(B)^*A$, then A is a upper-near subsemiring of S.*

Proof. (1) Let $\varnothing \neq A \subseteq S$, $A+A\subseteq A$ and $A\cdot A\subseteq A$. Then, From (1) and (2) of Lemma 3.10, we have

$$(N_r(B)^*A)+(N_r(B)^*A)\subseteq N_r(B)^*(A+A)$$

and

$$(N_r(B)^*A)\cdot(N_r(B)^*A)\subseteq N_r(B)^*(A\cdot A).$$

On the other hand, from Theorem 2.6 (5), we have that $N_r(B)^*(A+A)\subseteq N_r(B)^*A$ and $N_r(B)^*(A\cdot A)\subseteq N_r(B)^*A$. In this case,

$$(N_r(B)^*A)+(N_r(B)^*A)\subseteq N_r(B)^*A$$

and

$$(N_r(B)^*A)\cdot(N_r(B)^*A)\subseteq N_r(B)^*A$$

is obtained. Thus, A is a upper-near subsemiring of S.

(2) Since A is a subsemiring of S, $A+A\subseteq N_r(B)^*A$, and $A\cdot A\subseteq N_r(B)^*A$. Then, by Theorem 2.6 (5) and hypothesis, we have

$$N_r(B)^*(A+A)\subseteq N_r(B)^*(N_r(B)^*A) = N_r(B)^*A$$

and

$$N_r(B)^*(A\cdot A)\subseteq N_r(B)^*(N_r(B)^*A) = N_r(B)^*A.$$

Thus, by combining this and Lemma 3.10,

$$(N_r(B)^*A)+(N_r(B)^*A)\subseteq N_r(B)^*A$$

and

$$(N_r(B)^*A)\cdot(N_r(B)^*A)\subseteq N_r(B)^*A.$$

So, A is a upper-near subsemiring of S. □

Definition 3.15. Let S be a nearness semiring, and A a subsemiring of S, where $A\neq S$.
(1) A is called a right (left) ideals of S, if $A\cdot S\subseteq N_r(B)^*A$ ($S\cdot A\subseteq N_r(B)^*A$).
(2) A is called a upper-near right (left) ideals of S, if $(N_r(B)^*A)\cdot S\subseteq N_r(B)^*A$ ($S\cdot(N_r(B)^*A)\subseteq N_r(B)^*A$).

Theorem 3.16. *Let S be a nearness semiring. The following properties hold:*

(1) *if $\varnothing \neq A \subseteq S$, $A + A \subseteq A$ and $A \cdot A \subseteq A$, then A is a upper-near right (left) ideal of S,*

(2) *if A is a right (left) ideal of S, and $N_r(B)^*(N_r(B)^* A) = N_r(B)^* A$, then A is a upper-near right (left) ideal of S.*

Proof. It is similar to the proof of Theorem 3.14. □

Theorem 3.17. *Let $\{A_i \mid i \in I\}$ be a set of ideals of the nearness semiring S where an arbitrary index set I.*

(1) *If $N_r(B)^* \left(\bigcap\limits_{i\in I} A_i\right) = \bigcap\limits_{i\in I} N_r(B)^* A_i$, then $\bigcap\limits_{i\in I} A_i$ is a ideal of S.*

(2) *$\bigcup\limits_{i\in I} A_i$ is a ideal of S.*

Proof. (1) Let $x, y \in \bigcap\limits_{i\in I} A_i$. Then $x, y \in A_i$, for all $i \in I$. Thus $x + y \in N_r(B)^* A_i$, for all $i \in I$. So $x + y \in \bigcap\limits_{i\in I} N_r(B)^* A_i = N_r(B)^* \left(\bigcap\limits_{i\in I} A_i\right)$. Similarly, we have that $x \cdot s$, $s \cdot x \in N_r(B)^* \left(\bigcap\limits_{i\in I} A_i\right)$, for all $x \in \bigcap\limits_{i\in I} A_i$, $s \in S$. Hence, $\bigcap\limits_{i\in I} A_i$ is a ideal of S.

(2) Let $x, y \in \bigcup\limits_{i\in I} A_i$. Then there is at least one $i \in I$ such that $x \in A_i$ and $j \in I$ such that $y \in A_j$. Since A_i and A_j are ideals of S, for $i, j \in I$ $(i \neq j)$, we get that either $x + y \in N_r(B)^* A_i$ or $x + y \in N_r(B)^* A_j$. From here, $x + y \in \bigcup\limits_{i\in I} N_r(B)^* A_i$. Thus, from Theorem 2.6 (2), $x + y \in N_r(B)^* \left(\bigcup\limits_{i\in I} A_i\right)$. Similarly, we have that $x \cdot s$, $s \cdot x \in N_r(B)^* \left(\bigcup\limits_{i\in I} A_i\right)$, for all $x \in \bigcup\limits_{i\in I} A_i$, $s \in S$. So, $\bigcup\limits_{i\in I} A_i$ is a ideal of S. □

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