Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 16, No. 1, (August 2018) pp. 133–155
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2018.16.1.133

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 16, No. 1, August 2018

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Received 15 April 2018; Accepted 31 May 2018

Abstract. We define the ordinary intuitionistic smooth topology and obtain some its basic properties. Next, we define the ordinary intuitionistic smooth neighborhood system and we show that an ordinary intuitionistic smooth neighborhood system has the same properties in a classical neighborhood system (See Theorem 4.5). Finally, we introduce the concepts of an ordinary intuitionistic smooth base and an ordinary intuitionistic smooth subbase, and obtain two characterization of an ordinary intuitionistic smooth base (See Theorems 5.3 and 5.4) and one characterization of an ordinary intuitionistic smooth subbase (See Theorem 5.12).

2010 AMS Classification: 54A40

Keywords: Ordinary intuitionistic smooth topology , Ordinary intuitionistic smooth cotopology, (λ, μ)-level, Strong (λ, μ)-level, Ordinary intuitionistic smooth neighborhood system, Ordinary intuitionistic smooth base, Ordinary intuitionistic smooth subbase

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1. Introduction

In 1975, Zadeh [45] introduced the idea of interval-valued fuzzy sets. In 1986, Atanassov [1] defined an intuitionistic fuzzy set as the generalization of a fuzzy set introduced by Zadeh [44]. After then, many researchers [1, 2, 3, 4, 6, 7] have worked mainly on operators and relations on intuitionistic fuzzy sets and iterval-valued intuitionistic fuzzy sets. In particular, In 2010, Cheong and Hur [12] introduced the concept of an intuitionistic interval-valued fuzzy set and studied it′s basic properties.

In 1986, Chang [8] was the first to introduce the notion of a fuzzy topology by using fuzzy sets. After that, many researchers [16, 17, 20, 21, 22, 29, 30, 31, 34, 35, 42] have investigated several properties in fuzzy topological spaces. In 1997, Çoker [13] introduced the idea of the topology of intuitionistic fuzzy sets. Moreover, Samanta and Mondal [38, 39] introduced the definitions of the topology of interval-valued fuzzy sets and the topology of interval-valued intuitionistic fuzzy sets, respectively. In 2012, Bayramov and Gunduz [5] studied intuitionistic fuzzy topology on function spaces.

However, in their definition of fuzzy topology, fuzziness in the notion of openness of a fuzzy set was absent. In 1992, Samanta et al. [10, 18] introduced the concept of gradation of openness(closecness) of fuzzy sets in X in two different ways, and gave definitions of a fuzzy topology on X. After that, some works have been done by Ramadan [36], Demirci [15], Chattopadhyay and Samanta [9] and Peters [32, 33]. In particular, Ying [43] introduced the concept of the topology (called a fuzzifying topology) considering the degree of openness of an ordinary subset of a set. In 2012, Hur et al. [27] studied general properties in ordinary smooth topological spaces. Also they [24, 25, 26] investigated closures interiors and compactness in ordinary smooth topological spaces. Moreover, Çoker and Demirci [14], and Samanta and Mondal [37, 40] defined intuitionistic gradation of openness (in short IGO) of fuzzy sets in Šostak′s sense [41] thereby gave the definition of an intuitionsitic fuzzy topology (in short, IFT). Also Hur et al. [11] studied an interval-valued smooth topology. They mainly dealt with intuitionistic gradation of openness of fuzzy sets in the sense of Chang. But in 2010, Lim et al. [28] investigated intuitionistic smooth topological spaces in Lowen′s sense. Recently, Kim et al. [23] studied continuities and neighborhood systems in intuitionistic smooth topological spaces.

In this paper, we define the ordinary intuitionistic smooth topology and obtain some its basic properties. Next, we define the ordinary intuitionistic smooth neighborhood system and we show that an ordinary intuitionistic smooth neighborhood system has the same properties in a classical neighborhood system (See Theorem 4.5). Finally, we introduce the concepts of an ordinary intuitionistic smooth base and an ordinary intuitionistic smooth subbase, and obtain two characterization of an ordinary intuitionistic smooth base (See Theorems 5.3 and 5.4) and one characterization of an ordinary intuitionistic smooth subbase (See Theorem 5.12).

2. Preliminaries

In this chapter, we list some concepts and results which are needed in later chapters. Throughout this paper, X, Y, Z, etc. always denote nonempty (ordinary) sets. We will write $I = [0, 1], I_0 = (0, 1]$ and $I_1 = [0, 1)$.

Definition 2.1 ([44])**.** A mapping $A : X \to I$ is called a fuzzy set in X. $\mathbf{0}$ and $\mathbf{1}$ are called the empty fuzzy set and the whole fuzzy set in X defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for each $x \in X$, respectively. The set $\{x \in X : A(x) > 0\}$ is called a support of A and is denoted by $S(A)$ or A_0.

We will denote the set of all fuzzy sets as I^X.

From [8], we can see that $(I^X, \cup, \cap, \mathbf{0}, \mathbf{1})$ is a complete distributive lattice satisfying the DeMorgan′s Laws with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$.

Let $I \oplus I = \{(a, b) \in I \times I : a + b \leq 1\}$, let $(a, b), (c, d) \in I \oplus I$ and let $\{(a_\alpha, b_\alpha)\}_{\alpha\in\Gamma} \subset I \oplus I$. We define the following(See [12]) :

(i) $(a, b) \leq (c, d)$ iff $a \leq c$ and $b \geq d$,

(ii) $(a, b) = (c, d)$ iff $(a, b) \leq (c, d)$ and $(c, d) \leq (a, b)$,

(iii) $(a, b)^c = (b, a)$, where $(a, b)^c$ denotes the complement of (a, b),

(iv) $\bigvee_{\alpha\in\Gamma}(a_\alpha, b_\alpha) = (\bigvee_{\alpha\in\Gamma} a_\alpha, \bigwedge_{\alpha\in\Gamma} b_\alpha)$,

(v) $\bigwedge_{\alpha\in\Gamma}(a_\alpha, b_\alpha) = (\bigwedge_{\alpha\in\Gamma} a_\alpha, \bigvee_{\alpha\in\Gamma} b_\alpha)$.

Each member (a, b) of $I \oplus I$ will be called an intuitionistic point. When the elements of $I \oplus I$ are denoted be capital letters $M, N, \cdots$, we will write $M = (\mu_M, \nu_M), N = (\mu_N, \nu_N), \cdots$, where μ_M and ν_M are the membership and the nonmembership points, respectively. Moreover, from Theorem 2.1 in [12], we can see that $(I \oplus I, \leq)$ is a complete distributive lattice with the greatest element $(1, 0)$ and the least element $(0, 1)$ satisfying DeMorgan's laws.

The following is the modification of the concept of the concept of intuitionistic fuzzy sets introduced by Atanassov (See [1]).

Definition 2.2 ([12])**.** A mapping $A : X \to I \oplus I$ is called an intuitionistic fuzzy set in X and we write $A(x) = (\mu_A(x), \nu_A(x))$ for each $x \in X$. $\widetilde{0}$ and $\widetilde{1}$ are the empty intuitionistic fuzzy set and the whole intuitionistic fuzzy set in X given by $\widetilde{0}(x) = (0, 1)$ and $\widetilde{1}(x) = (1, 0)$, respectively.

We will denote the set of all intuitionistic fuzzy sets in X as $(I \oplus I)^X$.

Definition 2.3 ([1])**.** Let $A, B \in (I \oplus I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset (I \oplus I)^X$. Then the union $\bigcup_{\alpha\in\Gamma} A_\alpha$, the intersection $\bigcap_{\alpha\in\Gamma} A_\alpha$, the complement A^c of A and the inclusion $A \subset B$ are defined as follows: for each $x \in X$,

(i) $(\bigcup_{\alpha\in\Gamma} A_\alpha)(x) = (\bigvee_{\alpha\in\Gamma} \mu_{A_\alpha}(x), \bigwedge_{\alpha\in\Gamma} \nu_{A_\alpha}(x))$,

(ii) $(\bigcap_{\alpha\in\Gamma} A_\alpha)(x) = (\bigwedge_{\alpha\in\Gamma} \mu_{A_\alpha}(x), \bigvee_{\alpha\in\Gamma} \nu_{A_\alpha}(x))$,

(iii) $A^c(x) = (\nu_A(x), \mu_A(x))$,

(iv) $A \subset B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

We can easily see that $((I \oplus I)^X, \cup, \cap)$ is a complete distributive lattice with the least element $\widetilde{0}$ and the greatest element $\widetilde{1}$ satisfying DeMorgan′s laws.

Definition 2.4 ([19])**.** Let δ and δ' be two Chang′s fuzzy topologies on a nonempty set X. Then the triple (X, δ, δ') is called a fuzzy bitopological space.

If $\tau, \tau' \in ST(X)$, then we will called the triple (X, τ, τ') as a smooth bitopological sapce (of fuzzy sets) and the ordered pair (τ, τ') will be called a smooth bitopology (of fuzzy sets) (in short, $SBTFS$) on X in Chang′s sense (See [40], p.324).

Definition 2.5 ([40])**.** Let $(X, (\tau, \tau'))$ be a smooth bitopological space in Chang′s sense. Then $(X, (\tau, \tau'))$ is said to be inclusive, if $\tau \subset \tau'$.

3. Ordinary intuitionistic smooth topological spaces

In this section, we define an ordinary intuitionistic smooth topological space and obtain some its properties. Throughout this paper, we denote the set of all subsets [resp. fuzzy subsets] of a set X as 2^X [resp. I^X].

Definition 3.1. Let X be a nonempty set. Then a mapping $\tau = (\mu_\tau, \nu_\tau) : 2^X \to I \oplus I$ is called an ordinary intuitionistic smooth topology (in short, *oist*) on X, if it satisfies the following axioms: for any $A, B \in 2^X$ and each $\{A_\alpha\}_{\alpha \in \Gamma} \subset 2^X$.

(OIST1) $\tau(\phi) = \tau(X) = (1, 0)$,

(OIST2) $\mu_\tau(A \cap B) \geq \mu_\tau(A) \wedge \mu_\tau(B)$ and $\nu_\tau(A \cap B) \leq \nu_\tau(A) \vee \nu_\tau(B)$,

(OIST3) $\mu_\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \mu_\tau(A_\alpha)$ and $\nu_\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \leq \bigvee_{\alpha\in\Gamma} \nu_\tau(A_\alpha)$.

The pair (X, τ) is called an ordinary intuitionistic smooth topological space (in short, *oists*).

We will denote the set of all ordinary intuitionistic smooth topologies on X as $OIST(X)$.

Let $2 = \{0, 1\}$ and let $\tau : 2^X \to 2 \oplus 2$ satisfy the axioms in Definition 3.1. Since we can consider as $(1, 0) = 1$ and $(0, 1) = 0$, $\tau \in T(X)$, where $T(X)$ denotes the set of all classical topologies on X. Moreover, $2^X \subset I^X$. Then we can see that $T(X) \subset OIST(X) \subset IST(X)$.

Remark 3.2. Let (X, τ) be an *oists*. Then (X, μ_τ) and (X, ν_τ^c) are ordinary smooth topological spaces. Moreover, $(X, \mu_\tau, \nu_\tau^c)$ is an inclusive smooth bitopological space, where $\nu_\tau^c = 1 - \nu_\tau$.

Example 3.3. (1) Let $X = \{a, b, c\}$. Then $2^X = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. We define the mapping $\tau : 2^X \to I \oplus I$ as follows:

$\tau(\phi) = \tau(X) = (1, 0)$,

$\tau(\{a\}) = (0.7, 0.2), \tau(\{b\}) = (0.4, 0.5), \tau(\{c\}) = (0.6, 0.3)$,

$\tau(\{a, b\}) = (0.6, 0.3), \tau(\{b, c\}) = (0.4, 0.6), \tau(\{a, c\}) = (0.3, 0.2)$.

Then we can easily see that $\tau \in OIST(X)$.

(2) Let X be a nonempty set. We define he mapping $\tau_\phi : 2^X \to I \oplus I$ as follows: for each $A \in 2^X$,

$$\tau_\phi(A) = \begin{cases} (1, 0) & \text{if either } A = \phi \text{ or } A = X, \\ (0, 1) & \text{otherwise.} \end{cases}$$

Then clearly, $\tau_\phi \in OIST(X)$.

In this case, τ_ϕ [resp. (X, τ_ϕ)] will be called the ordinary intuitionistic smooth indiscrete topology on X [resp. the ordinary intuitionistic smooth indiscrete space].

(3) Let X be a nonempty set. We define he mapping $\tau_X : 2^X \to I \oplus I$ as follows: for each $A \in 2^X$,

$$\tau_X(A) = (1, 0).$$

Then clearly, $\tau_X \in OIST(X)$.

In this case, τ_X [resp. (X, τ_X)] will be called the ordinary intuitionistic smooth discrete topology on X [resp. the ordinary intuitionistic smooth discrete space].

(4) Let X be a set and let $(r, s) \in I_1 \oplus I_0$ be fixed. We define the mapping $\tau : 2^X \to I \oplus I$ as follows: for each $A \in 2^X$,

$$\tau(A) = \begin{cases} (1, 0) & \text{if either } A = \phi \text{ or } A^c \text{ is finie,} \\ (r, s) & \text{otherwise.} \end{cases}$$

Then we can easily see that $\tau \in OIST(X)$.

In this case, τ will be called the (r, s)-ordinary intuitionistic smooth finite complement topology on X and will be denoted by $OISCof(X)$. $OISCof(X)$ is of interest only when X is an infinite set, because if X is finite, then $OISCof(X) = \tau_\phi$.

(5) Let X be an infinite set and let $(r,s) \in I_1 \oplus I_0$ be fixed. We define he mapping $\tau : 2^X \to I \oplus I$ as follows: for each $A \in 2^X$,

$$\tau(A) = \begin{cases} (1,0) & \text{if either } A = \phi \text{ or } A^c \text{ is countable,} \\ (r,s) & \text{otherwise.} \end{cases}$$

Then clearly, $\tau \in OIST(X)$.

In this case, τ will be called the (r,s)-ordinary intuitionistic smooth countable complement topology on X and will be denoted by $OISCoc(X)$.

(6) Let T be the topology generated by $\mathcal{S} = \{(a,b] : a,b \in \mathbb{R}, a < b\}$ as a subbase, let T_0 be the family of all open sets of $\mathbb{R}$ w.r.t. the usual topology of $\mathbb{R}$ and let $(r,s) \in I_1 \oplus I_0$ be fixed. We define the mapping $\tau : 2^{\mathbb{R}} \to I \oplus I$ as follows: for each $A \in I^{\mathbb{R}}$,

$$\tau(A) = \begin{cases} (1,0) & \text{if } A \in T_0, \\ (r,s) & \text{if } A \in T \setminus T_0, \\ (0,1) & \text{otherwise.} \end{cases}$$

Then we can easily see that $\tau \in OIST(X)$.

(7) Let $T \in T(X)$. We define the mapping $\tau_T : 2^X \to I \oplus I$ as follows : for each $A \in 2^X$,

$$\tau_T(A) = \begin{cases} (1,0) & \text{if } A \in T, \\ (0,1) & \text{otherwise.} \end{cases}$$

Then it is easily seen that $\tau_T \in OIST(X)$. Moreover, we can see that if T is the classical indiscrete topology, then $\tau_T = \tau_\phi$ and if T is the classical discrete topology, then $\tau_T = \tau_X$.

Definition 3.4. Let X be a nonempty set. Then a mapping $\mathcal{C} = (\mu_{\mathcal{C}}, \nu_{\mathcal{C}}) : 2^X \to I \oplus I$ is called an ordinary intuitionistic smooth cotopology (in short, *oisct*) on X if it is satisfies the following conditions: for any $A, B \in 2^X$ and each $\{A_\alpha\}_{\alpha \in \Gamma} \subset 2^X$.

(OISCT1) $\mathcal{C}(\phi) = \mathcal{C}(X) = (1,0)$,

(OISCT2) $\mu_{\mathcal{C}}(A \cup B) \geq \mu_{\mathcal{C}}(A) \wedge \mu_{\mathcal{C}}(B)$ and $\nu_{\mathcal{C}}(A \cup B) \leq \nu_{\mathcal{C}}(A) \vee \nu_{\mathcal{C}}(B)$,

(OISCT3) $\mu_{\mathcal{C}}(\bigcap_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \mu_{\mathcal{C}}(A_\alpha)$ and $\nu_{\mathcal{C}}(\bigcap_{\alpha \in \Gamma} A_\alpha) \leq \bigvee_{\alpha \in \Gamma} \nu_{\mathcal{C}}(A_\alpha)$.

The pair $(X, \mathcal{C})$ is called an ordinary intuitionistic smooth cotopological space (in short, *oiscts*).

Remark 3.5. If $I = 2$, then we can think that Definition 3.1 also coincides with the known definition of classical topology.

Just as with ordinary topological spaces, the connection between intuitionistic smooth topologies and intuitionistic smooth cotopologies is a bijective one, and is given by means of complementation. Then we have the following result and its proof follows from Definitions 3.1 and 3.4.

Proposition 3.6. *We define two mappings $f : OIST(X) \to OISCT(X)$ and $g : OISCT(X) \to OIST(X)$ as follows, respectively:*

$$[f(\tau)](A) = \tau(A^c), \ \forall \ \tau \in OIST(X), \ \forall \ A \in 2^X$$

and

$$[g(\mathcal{C})](A) = \mathcal{C}(A^c), \ \forall \ \mathcal{C} \in OISCT(X), \ \forall \ A \in 2^X.$$

Then f and g are well-defined. Moreover, $g \circ f = 1_{OIST(X)}$ and $f \circ g = 1_{OISCT(X)}$.

Remark 3.7. For each $\tau \in OIST(X)$ and each $\mathcal{C} \in OISCT(X)$, let $f(\tau) = \mathcal{C}_\tau$ and $g(\mathcal{C}) = \tau_{\mathcal{C}}$. Then, from Proposition 3.6, we can see that $\tau_{\mathcal{C}_\tau} = \tau$ and $\mathcal{C}_{\tau_{\mathcal{C}}} = \mathcal{C}$.

Definition 3.8. Let $\tau_1, \tau_2 \in OIST(X)$ and let $\mathcal{C}_1, \mathcal{C}_2 \in OISCT(X)$.

(i) We say that τ_1 is finer than τ_2 or τ_2 is coarser than τ_1, denoted by $\tau_2 \preceq \tau_1$, if $\tau_2(A) \leq \tau_1(A)$, i.e., $\mu_{\tau_2}(A) \leq \mu_{\tau_1}(A)$ and $\nu_{\tau_2}(A) \geq \nu_{\tau_1}(A)$, for each $A \in 2^X$.

(ii) We say that $\mathcal{C}_1$ is finer than $\mathcal{C}_2$ or $\mathcal{C}_2$ is coarser than $\mathcal{C}_1$, denoted by $\mathcal{C}_2 \preceq \mathcal{C}_1$, if $\mathcal{C}_2(A) \leq \mathcal{C}_1(A)$, i.e., $\mu_{\mathcal{C}_2}(A) \leq \mu_{\mathcal{C}_1}(A)$ and $\nu_{\mathcal{C}_2}(A) \geq \nu_{\mathcal{C}_1}(A)$, for each $A \in 2^X$.

We can easily see that τ_1 is finer than τ_2 if and only if $\mathcal{C}_{\tau_1}$ is finer than $\mathcal{C}_{\tau_2}$, and $(OIST(X), \preceq)$ and $(OISCT(X), \preceq)$ are posets, respectively.

From Example 3.3 (2) and (3), it is obvious that τ_ϕ is the coarest ordinary intuitionistic smooth topology on X and τ_X is the finest ordinary intuitionistic smooth topology on X.

Proposition 3.9. *If* $\{\tau_\alpha\}_{\alpha\in\Gamma} \subset OIST(X)$, *then* $\bigcap_{\alpha\in\Gamma} \tau_\alpha \in OIST(X)$, *where* $[\bigcap_{\alpha\in\Gamma} \tau_\alpha](A) = (\bigwedge_{\alpha\in\Gamma} \mu_{\tau_\alpha}(A), \bigvee_{\alpha\in\Gamma} \nu_{\tau_\alpha}(A)),\ \forall\ A \in 2^X$.

Proof. Let $\tau = \bigcap_{\alpha\in\Gamma} \tau_\alpha$ and let $A \in 2^X$. Since $\tau_\alpha \in OIST(X)$, for each $\alpha \in \Gamma$, $\mu_{\tau_\alpha}(A) + \nu_{\tau_\alpha}(A) \leq 1$. Then $\mu_{\tau_\alpha}(A) \leq 1 - \nu_{\tau_\alpha}(A)$. Thus $\bigwedge_{\alpha\in\Gamma} \mu_{\tau_\alpha}(A) \leq \bigwedge_{\alpha\in\Gamma} (1 - \nu_{\tau_\alpha}(A)) = 1 - \bigvee_{\alpha\in\Gamma} \nu_{\tau_\alpha}(A)$. So $\bigwedge_{\alpha\in\Gamma} \mu_{\tau_\alpha}(A) + \bigvee_{\alpha\in\Gamma} \nu_{\tau_\alpha}(A) \leq 1$. Hence $\mu_\tau(A) + \nu_\tau(A) \leq 1$ and thus $\tau : 2^X \to I \oplus I$ is a mapping. Therefore the condition (OIST1) holds.

Let $A, B \in 2^X$. Then

$$\begin{aligned}
\mu_\tau(A \cap B) &= \textstyle\bigwedge_{\alpha\in\Gamma} \mu_{\tau_\alpha}(A \cap B) \text{ [By the definition of } \tau] \\
&\geq \textstyle\bigwedge_{\alpha\in\Gamma} (\mu_{\tau_\alpha}(A) \wedge \mu_{\tau_\alpha}(B)) \text{ [Since } \tau_\alpha \in OIST(X)] \\
&= (\textstyle\bigwedge_{\alpha\in\Gamma} \mu_{\tau_\alpha}(A)) \wedge (\textstyle\bigwedge_{\alpha\in\Gamma} \mu_{\tau_\alpha}(B)) \\
&= \mu_\tau(A) \wedge \mu_\tau(B) \text{ [By the definition of } \tau]
\end{aligned}$$

and

$$\begin{aligned}
\nu_\tau(A \cap B) &= \textstyle\bigvee_{\alpha\in\Gamma} \nu_{\tau_\alpha}(A \cap B) \\
&\leq \textstyle\bigvee_{\alpha\in\Gamma} (\nu_{\tau_\alpha}(A) \vee \nu_{\tau_\alpha}(B)) \\
&= (\textstyle\bigvee_{\alpha\in\Gamma} \nu_{\tau_\alpha}(A)) \vee (\textstyle\bigvee_{\alpha\in\Gamma} \nu_{\tau_\alpha}(B)) \\
&= \nu_\tau(A) \vee \nu_\tau(B).
\end{aligned}$$

So the condition (OIST2) holds.

Now let $\{A_j\}_{j\in J} \subset 2^X$. Then

$$\begin{aligned}
\mu_\tau(\textstyle\bigcup_{j\in J} A_j) &= \textstyle\bigwedge_{\alpha\in\Gamma} \mu_{\tau_\alpha}(\textstyle\bigcup_{j\in J} A_j) \text{ [By the definition of } \tau] \\
&\geq \textstyle\bigwedge_{\alpha\in\Gamma} (\textstyle\bigwedge_{j\in J} \mu_{\tau_\alpha}(A_j)) \text{ [Since } \tau_\alpha \in OIST(X)] \\
&= \textstyle\bigwedge_{j\in J} (\textstyle\bigwedge_{\alpha\in\Gamma} \mu_{\tau_\alpha}(A_j)) \\
&= \textstyle\bigwedge_{j\in J} [\textstyle\bigcap_{\alpha\in\Gamma} \mu_{\tau_\alpha}](A_j) \text{ [By the definition of } \tau] \\
&= \textstyle\bigvee_{j\in J} \mu_\tau(A_j)
\end{aligned}$$

and

$$\begin{aligned}
\nu_\tau(\textstyle\bigcup_{j\in J} A_j) &= \textstyle\bigvee_{\alpha\in\Gamma} \nu_{\tau_\alpha}(\textstyle\bigcup_{j\in J} A_j) \text{ [By the definition of } \tau] \\
&\leq \textstyle\bigvee_{\alpha\in\Gamma} (\textstyle\bigvee_{j\in J} \nu_{\tau_\alpha}(A_j)) \text{ [Since } \tau_\alpha \in OIST(X)] \\
&= \textstyle\bigvee_{j\in J} (\textstyle\bigvee_{\alpha\in\Gamma} \nu_{\tau_\alpha}(A_j)) \\
&= \textstyle\bigvee_{j\in J} [\textstyle\bigcup_{\alpha\in\Gamma} \nu_{\tau_\alpha}](A_j) \text{ [By the definition of } \tau] \\
&= \textstyle\bigvee_{j\in J} \nu_\tau(A_j).
\end{aligned}$$

Thus the condition (OIST3) holds. This completes the proof. □

From Definition 3.8 and Proposition 3.9, we have the following.

Proposition 3.10. *$(OIST(X), \preceq)$ is a meet complete lattice with the least element τ_ϕ and the greatest element τ_X.*

Definition 3.11. Let (X, τ) be an *oists* and let $(\lambda, \mu) \in I \oplus I$. We define $[\tau]_{(\lambda,\mu)}$ and $[\tau]^*_{(\lambda,\mu)}$ as follows, respectively:

(i) $[\tau]_{(\lambda,\mu)} = \{A \in 2^X : \mu_\tau(A) \geq \lambda,\ \nu_\tau(A) \leq \mu\}$,

(ii) $[\tau]^*_{(\lambda,\mu)} = \{A \in 2^X : \mu_\tau(A) > \lambda,\ \nu_\tau(A) < \mu\}$.

$[\tau]_{(\lambda,\mu)}$ [resp. $[\tau]^*_{(\lambda,\mu)}$] is called the (λ, μ)-level [resp. strong (λ, μ)-level] of τ. If $(\lambda, \mu) = (0, 1)$, then $[\tau]_{(0,1)} = 2^X$, i.e., $[\tau]_{(0,1)}$ is the classical discrete topology on X and if $(\lambda, \mu) = (1, 0)$, then $[\tau]^*_{(1,0)} = \varnothing$. Moreover, we can easily see that for any $(\lambda, \mu) \in I \oplus I$, $[\tau]^*_{(\lambda,\mu)} \subset [\tau]_{(\lambda,\mu)}$.

Lemma 3.12. *Let $\tau \in OIST(X)$.*

(1) *For each $(\lambda, \mu) \in I \oplus I$, $[\tau]_{(\lambda,\mu)} \in T(X)$.*

(2) *If $(\lambda_1, \mu_1) \leq (\lambda_2, \mu_2)$ in $I \oplus I$, then $[\tau]_{(\lambda_2,\mu_2)} \subset [\tau]_{(\lambda_1,\mu_1)}$.*

(3) *For each $(\lambda, \mu) \in I_0 \oplus I_1$, $[\tau]_{(\lambda,\mu)} = \bigcap_{(\lambda',\mu')<(\lambda,\mu)} [\tau]_{(\lambda',\mu')}$.*

$(1)'$ *For each $(\lambda, \mu) \in I_1 \oplus I_0$, $[\tau]^*_{(\lambda,\mu)} \in T(X)$.*

$(2)'$ *If $(\lambda_1, \mu_1) \leq (\lambda_2, \mu_2)$ in $I \oplus I$, then $[\tau]^*_{(\lambda_2,\mu_2)} \subset [\tau]^*_{(\lambda_1,\mu_1)}$.*

$(3)'$ *For each $(\lambda, \mu) \in I_1 \oplus I_0$, $[\tau]^*_{(\lambda,\mu)} = \bigcup_{(\lambda',\mu')>(\lambda,\mu)} [\tau]^*_{(\lambda',\mu')}$.*

Proof. The proofs of (1), $(1)'$, (2) and $(2)'$ are obvious from Definitions 3.1 and 3.11.

(3) From (2), $\{[\tau]_{(\lambda,\mu)}\}_{(\lambda,\mu)\in I_0\oplus I_1}$ is a descending family of classical topologies on X. Then clearly, $[\tau]_{(\lambda,\mu)} \subset \bigcap_{(\lambda',\mu')<(\lambda,\mu)} [\tau]_{(\lambda',\mu')}$, for each $(\lambda, \mu) \in I_0 \oplus I_1$.

Suppose $A \notin [\tau]_{(\lambda,\mu)}$. Then $\mu_\tau(A) < \lambda$ or $\nu_\tau(A) > \mu$. Thus

$$\exists\ s \in I_0 \text{ such that } \mu_\tau(A) < s < \lambda$$

or

$$\exists\ t \in I_1 \text{ such that } \nu_\tau(A) > t > \mu.$$

So $A \notin [\tau]_{(s,t)}$, for some $(s, t) < (\lambda, \mu)$, i.e., $A \notin \bigcap_{(\lambda',\mu')<(\lambda,\mu)} [\tau]_{(\lambda',\mu')}$. Hence $\bigcap_{(\lambda',\mu')<(\lambda,\mu)} [\tau]_{(\lambda',\mu')} \subset [\tau]_{(\lambda,\mu)}$. Therefore $[\tau]_{(\lambda,\mu)} = \bigcap_{(\lambda',\mu')<(\lambda,\mu)} [\tau]_{(\lambda',\mu')}$.

$(3)'$ The proof is similar to (3). □

Remark 3.13. From (1) and (2) in Lemma 3.12, we can see that for each $\tau \in OIST(X)$, $\{[\tau]_{(\lambda,\mu)}\}_{(\lambda,\mu)\in I\oplus I}$ is a family of descending classical topologies called the (λ, μ)-level classical topologies on X w.r.t. τ.

The following is the immediate result of Lemma 3.12.

Corollary 3.14. *Let (X, τ) be an oists. Then*

$$[\tau]_{(\lambda,1-\lambda)} = \bigcap_{\lambda'<\lambda} [\tau]_{(\lambda',1-\lambda')},\ \forall\ \lambda \in I_0.$$

Lemma 3.15. (1) *Let $\{T_{(\lambda,\mu)}\}_{(\lambda,\mu)\in I\oplus I}$ be a descending family of classical topologies on X such that $T_{(0,1)}$ is the classical discrete topology on X. We define the mapping $\tau : 2^X \to I \oplus I$*

as follows: for each $A \in 2^X$,

$$\tau(A) = (\bigvee_{A \in T_{(\lambda,\mu)}} \lambda, \bigwedge_{A \in T_{(\lambda,\mu)}} \mu).$$

Then $\tau \in OIST(X)$.

(2) *If* $T_{(\lambda,\mu)} = \bigcap_{(\lambda',\mu')<(\lambda,\mu)} \delta_{(\lambda',\mu')}$, *for each* $(\lambda,\mu) \in I_0 \oplus I_1$, *then* $[\tau]_{(\lambda,\mu)} = T_{(\lambda,\mu)}$.

(3) *If* $T_{(\lambda,\mu)} = \bigcup_{(\lambda',\mu')>(\lambda,\mu)} T_{(\lambda',\mu')}$, *for each* $(\lambda,\mu) \in I_1 \oplus I_0$, *then* $[\tau]^*_{(\lambda,\mu)} = T_{(\lambda,\mu)}$.

Proof. The proof is similar to Lemma 3.9 in [28]. □

The following is the immediate result of Lemma 3.15.

Corollary 3.16. *Let* $\{T_{(\lambda,1-\lambda)}\}_{\lambda \in I_0}$ *be a descending family of classical topologies on* X *such that* $T_{(0,1)}$ *is the classical discrete topology on* X. *We define the mapping* $\tau : 2^X \to I \oplus I$ *as follows: for each* $A \in 2^X$,

$$\tau(A) = (\bigvee_{A \in T_{(\lambda,1-\lambda)}} \lambda, \bigwedge_{A \in T_{(\lambda,1-\lambda)}} (1-\lambda)).$$

Then (1) $\tau \in OIST(X)$,

(2) $[\tau]_{(\lambda,1-\lambda)} = \bigcap_{\lambda'<\lambda} T_{(\lambda',1-\lambda')} = T_{(\lambda,1-\lambda)}, \ \forall\ \lambda \in I_0$.

From Lemmas 3.12 and 3.15, we have the following result.

Proposition 3.17. *Let* $\tau \in OIST(X)$ *and let* $[\tau]_{(\lambda,\mu)}$ *be the* (λ,μ)*-level classical topology on* X *w.r.t.* τ. *We define the mapping* $\eta : 2^X \to I \oplus I$ *as follows: for each* $A \in 2^X$,

$$\eta(A) = (\bigvee_{A \in [\tau]_{(\lambda,\mu)}} \lambda, \bigwedge_{A \in [\tau]_{(\lambda,\mu)}} \mu).$$

Then $\eta = \tau$.

The fact that an ordinary intuitionistic smooth topological space fully determined by it′s decomposition in classical topologies is restated in the following Theorem.

Theorem 3.18. *Let* $\tau_1, \tau_2 \in IST(X)$. *Then* $\tau_1 = \tau_2$ *if and only if* $[\tau_1]_{(\lambda,\mu)} = [\tau_2]_{(\lambda,\mu)}$, *for each* $(\lambda,\mu) \in I \oplus I$, *or, alternatively, if and only if* $[\tau_1]^*_{(\lambda,\mu)} = [\tau_2]^*_{(\lambda,\mu)}$, *for each* $(\lambda,\mu) \in I \oplus I$.

Remark 3.19. In a similar way, we can construct an ordinary intuitionistic smooth cotopology $\mathcal{C}$ on a set X, by using the (λ,μ)-levels,

$$[\mathcal{C}]_{(\lambda,\mu)} = \{A \in I^X : \mu_c(A) \geq \lambda \text{ and } \nu_c(A) \leq \mu\}$$

and

$$[\mathcal{C}]^*_{(\lambda,\mu)} = \{A \in I^X : \mu_c(A) > \lambda \text{ and } \nu_c(A) < \mu\},$$

for each $(\lambda,\mu) \in I \oplus I$.

Definition 3.20. Let $T \in T(X)$ and let $\tau \in OIST(X)$. Then τ is said to the compatible with T, if $T = S(\tau)$, where $S(\tau) = \{A \in 2^X : \mu_\tau(A) > 0 \text{ and } \nu_\tau(A) < 1\}$.

Example 3.21. (1) Let τ_ϕ be the ordinary intuitionistic smooth indiscrete topology on a nonempty set X and let T_0 be the classical indiscrete topology on X. Then clearly,

$$S(\tau_\phi) = \{A \in 2^X : \mu_{\tau_\phi}(A) > 0 \text{ and } \nu_{\tau_\phi}(A) < 1\} = \{\phi, X\} = T_0.$$

Thus τ_ϕ is compatible with T_0.

(2) Let τ_X be the ordinary intuitionistic smooth discrete topology on a nonempty set X and let T_1 be the classical discrete topology on X. Then clearly,

$$S(\tau_X) = \{A \in 2^X : \mu_{\tau_X}(A) > 0 \text{ and } \nu_{\tau_X}(A) < 1\} = 2^X = T_1.$$

Thus τ_X is compatible with T_1.

(3) Let X be a nonempty set and let $(r, s) \in I_0 \oplus I_1$ be fixed. We define he mapping $\tau : 2^X \to I \oplus I$ as follows: for each $A \in 2^X$,

$$\tau(A) = \begin{cases} (1,0) & \text{if either } A = \phi \text{ or } A = X, \\ (r,s) & \text{otherwise.} \end{cases}$$

Then clearly, $\tau \in OIST(X)$ and τ is compatible with T_1.

Furthermore, every classical topology can be considered as an ordinary intuitionistic smooth topology in the sense of the following result.

Proposition 3.22. *Let (X, T) be a classical topological space and and let $(\lambda, \mu) \in I_0 \oplus I_1$ be fixed. Then there exists $T^{(\lambda,\mu)} \in OIST(X)$ such that $T^{(\lambda,\mu)}$ is compatible with T. Moreover, $[T^{(\lambda,\mu)}]_{(\lambda,\mu)} = T$.*

In this case, $T^{(\lambda,\mu)}$ is called (λ, μ)-th ordinary intuitionistic smooth topology on X and $(X, T^{(\lambda,\mu)})$ is called a (λ, μ)-th ordinary intuitionistic smooth topological space.

Proof. Let $(\lambda, \mu) \in I_0 \oplus I_1$ be fixed and we the mapping $T^{(\lambda,\mu)} : 2^X \to I \oplus I$ as follows: for each $A \in 2^X$,

$$T^{(\lambda,\mu)}(A) = \begin{cases} (1,0) & \text{if either } A = \phi \text{ or } A = X, \\ (\lambda,\mu) & \text{if } A \in T \setminus \{\phi, X\}, \\ (0,1) & \text{otherwise.} \end{cases}$$

Then we can easily see that $T^{(\lambda,\mu)} \in OIST(X)$ and $[T^{(\lambda,\mu)}]_{(\lambda,\mu)} = T$. Moreover, by the definition of $T^{(\lambda,\mu)}$,

$$S(T^{(\lambda,\mu)}) = \{A \in 2^X : \mu_{T^{(\lambda,\mu)}}(A) > 0 \text{ and } \nu_{T^{(\lambda,\mu)}}(A) < 1\} = T.$$

Thus $T^{(\lambda,\mu)}$ is compatible with T. □

Proposition 3.23. *Let (X, T) be a classical topological space and let $C(T)$ be the set of all oists on X compatible with T. Then there is a one-to-one correspondence between $C(T)$ and the set $(I_0 \oplus I_1)^{\widetilde{T}}$, where $\widetilde{T} = T \setminus \{\phi, X\}$.*

Proof. We define the mapping $F : (I_0 \oplus I_1)^{\widetilde{T}} \to C(T)$ as follows: for each $f \in (I_0 \oplus I_1)^{\widetilde{T}}$,

$$F(f) = \tau_f,$$

where $\tau_f : 2^X \to I \oplus I$ is the mapping defined by: for each $A \in 2^X$,

$$\tau_f(A) = \begin{cases} (1,0) & \text{if either } A = \phi \text{ or } A = X, \\ f(A) & \text{if } A \in \widetilde{T}, \\ (0,1) & \text{otherwise.} \end{cases}$$

Then we easily see that $\tau_f \in C(T)$.

Now we define the mapping $G : C(T) \to (I_0 \oplus I_1)^{\widetilde{T}}$ as follows: for each $\tau \in C(T)$,

$$G(\tau) = f_\tau,$$

where $f_\tau : \widetilde{T} \to I_0 \oplus I_1$ is the mapping defined by: for each $A \in \widetilde{T}$,

$$f_\tau(A) = \tau(A).$$

Then clearly, $f_\tau \in (I_0 \oplus I_1)^{\widetilde{T}}$.

Furthermore, we can see that $F \circ G = id_{C(T)}$ and $G \circ F = id_{(I_0 \oplus I_1)^{\widetilde{T}}}$. Thus $C(T)$ is equipotent to $(I_0 \oplus I_1)^{\widetilde{T}}$. This completes the proof. □

Proposition 3.24. *Let (X, τ) be an oists and let $A \subset X$. We define the mapping $\tau_A : 2^A \to I \oplus I$ as follows: for each $B \in 2^A$,*

$$\tau_A(B) = (\mu_{\tau_A}(B), \nu_{\tau_A}(B)) = (\bigvee_{C \in 2^X,\ B = C \cap A} \mu_\tau(C), \bigwedge_{C \in 2^X,\ B = C \cap A} \nu_\tau(C)).$$

Then $\tau_A \in OIST(A)$, and $\mu_{\tau_A}(B) \geq \mu_\tau(B)$ and $\nu_{\tau_A}(B) \leq \nu_\tau(B)$, for each $B \in 2^A$.

In this case, (A, τ_A) is called an ordinary intuitionistic smooth subspace of (X, τ) and τ_A is called the induced ordinary intuitionistic smooth topology on A by τ.

Proof. For each $B \in 2^A$, let $B = A \cap C$ and $C \in 2^X$. Since $\tau \in OIST(X)$, $\mu_\tau(C) \leq 1 - \nu_\tau(C)$. Thus

$$\bigvee_{C \in 2^X,\ B = A \cap C} \mu_\tau(C) \leq \bigvee_{C \in 2^X,\ B = A \cap C} (1 - \nu_\tau(C)) = 1 - \bigwedge_{C \in 2^X,\ B = A \cap C} \nu_\tau(C).$$

So $\mu_{\tau_A}(B) \leq 1 - \nu_{\tau_A}(B)$. Hence $\tau_A : 2^A \to I \oplus I$ is a mapping.

It is obvious that the condition (OIST1) holds, i.e., $\tau_A(\phi) = \tau_A(A) = (1, 0)$.

Let $B_1, B_2 \in 2^A$. Then, by proof of Proposition 5.1 in [27], $\mu_{\tau_A}(B_1 \cap B_2) \geq \mu_{\tau_A}(B_1) \wedge \mu_{\tau_A}(B_2)$.

Let us show that $\nu_{\tau_A}(B_1 \cap B_2) \leq \nu_{\tau_A}(B_1) \vee \nu_{\tau_A}(B_2)$. Then

$$\begin{aligned}
\nu_{\tau_A}(B_1) \vee \nu_{\tau_A}(B_2) &= (\textstyle\bigwedge_{C_1 \in 2^X,\ B_1 = A \cap C_1} \nu_\tau(C_1)) \vee (\bigwedge_{C_2 \in 2^X,\ B_2 = A \cap C_2} \nu_\tau(C_2)) \\
&= \textstyle\bigwedge_{C_1,\ C_1 \in 2^X,\ B_1 \cap B_2 = A \cap (C_1 \cap C_2)} [\nu_\tau(C_1) \vee \nu_\tau(C_2)] \\
&\geq \textstyle\bigwedge_{C_1,\ C_1 \in 2^X,\ B_1 \cap B_2 = A \cap (C_1 \cap C_2)} \nu_\tau(C_1 \cap C_2) \\
&= \nu_{\tau_A}(B_1 \cap B_2).
\end{aligned}$$

Thus the condition (OIST2) holds.

Now let $\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^A$. Then, by proof of Proposition 5.1 in [27], $\mu_{\tau_A}(\bigcup_{\alpha \in \Gamma} B_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \mu_{\tau_A}(B_\alpha)$.

On the other hand,

$$\begin{aligned}
\nu_{\tau_A}(\textstyle\bigcup_{\alpha \in \Gamma} B_\alpha) &= \textstyle\bigwedge_{C_\alpha \in 2^X,\ (\bigcup_{\alpha \in \Gamma} C_\alpha) \cap A = \bigcup_{\alpha \in \Gamma} B_\alpha} \nu_\tau(\bigcup_{\alpha \in \Gamma} C_\alpha) \\
&\leq \textstyle\bigwedge_{C_\alpha \in 2^X,\ (\bigcup_{\alpha \in \Gamma} C_\alpha) \cap A = \bigcup_{\alpha \in \Gamma} B_\alpha} [\bigwedge_{\alpha \in \Gamma} \nu_\tau(C_\alpha)] \\
&= \textstyle\bigwedge_{\alpha \in \Gamma} [\bigwedge_{C_\alpha \in 2^X,\ (\bigcup_{\alpha \in \Gamma} C_\alpha) \cap A = \bigcup_{\alpha \in \Gamma} B_\alpha} \nu_\tau(C_\alpha)] \\
&= \textstyle\bigwedge_{\alpha \in \Gamma} \nu_{\tau_A}(B_\alpha).
\end{aligned}$$

Thus the condition (OIST3) holds. So $\tau_A \in OIST(A)$.

Furthermore, we can easily see that $\mu_{\tau_A}(B) \geq \mu_\tau(B)$ and $\nu_{\tau_A}(B) \leq \nu_\tau(B)$, for each $B \in 2^A$. This completes the proof. □

The following is the immediate result of Proposition 3.24.

Corollary 3.25. *Let (A, τ_A) be an ordinary intuitionistic smooth subspace of (X, τ) and let $B \in 2^A$.*

(1) $\mathcal{C}_A(B) = (\bigvee_{C \in 2^X, B = C \cap A} \mu_{\mathcal{C}}(C), \bigwedge_{C \in 2^X, B = C \cap A} \nu_{\mathcal{C}}(C))$*, where* $\mathcal{C}_A(B) = \tau_A(A - B)$.

(2) *If $Z \subset Y \subset X$, then $\tau_Z = (\tau_Y)_Z$.*

4. Ordinary intuitionistic smooth neighborhood structures of a point

In this section, we define an ordinary intuitionistic smooth neighborhood system of a point and we show that it has the similar properties of a neighborhood system in a classical topological space.

Definition 4.1. Let (X, τ) be an *oists* and let $x \in X$. Then a mapping $\mathcal{N}_x : 2^X \to I \oplus I$ is called the ordinary intuitionistic smooth neighborhood system of x, if for each $A \in 2^X$,

$$A \in \mathcal{N}_x := \exists B(B \in \tau) \wedge (x \in B \subset A)),$$

i.e.,

$$[A \in \mathcal{N}_x] = \mathcal{N}_x(A) = (\mu_{\mathcal{N}_x}(A), \nu_{\mathcal{N}_x}(A)) = (\bigvee_{x \in B \subset A} \mu_\tau(B), \bigwedge_{x \in B \subset A} \nu_\tau(B)).$$

Lemma 4.2. *Let (X, τ) be an oists and let $A \in 2^X$. Then*

$$\bigwedge_{x \in A} \bigvee_{x \in B \subset A} \mu_\tau(B) = \mu_\tau(A)$$

and

$$\bigvee_{x \in A} \bigwedge_{x \in B \subset A} \nu_\tau(B) = \nu_\tau(A).$$

Proof. By Lemma 3.1 in [43], it is obvious that $\bigwedge_{x \in A} \bigvee_{x \in B \subset A} \mu_\tau(B) = \mu_\tau(A)$.

On the other hand, it is clear that $\bigvee_{x \in A} \bigwedge_{x \in B \subset A} \nu_\tau(B) \geq \nu_\tau(A)$. Now let $\mathcal{B}_x = \{B \in 2^X : x \in B \subset A\}$ and let $f \in \Pi_{x \in A} \mathcal{B}_x$. Then clearly, $\bigcup_{x \in A} f(x) = A$. Thus

$$\bigvee_{x \in A} \nu_\tau(f(x)) \leq \nu_\tau(\bigcup_{x \in A} f(x)) = \nu_\tau(A).$$

So

$$\bigvee_{x \in A} \bigwedge_{x \in B \subset A} \nu_\tau(B) = \bigwedge_{f \in \Pi_{x \in A}} \bigvee_{x \in A} \nu_\tau(f(x)) \leq \nu_\tau(A).$$

Hence $\bigvee_{x \in A} \bigwedge_{x \in B \subset A} \nu_\tau(B) = \nu_\tau(A)$. □

Theorem 4.3. *Let (X, τ) be an oists, let $A \in 2^X$ and let $x \in X$. Then*

$$\vDash (A \in \tau) \leftrightarrow \forall (x \in A \to \exists B(B \in \mathcal{N}_x) \wedge (B \subset A)),$$

i.e.,

$$[A \in \tau] = [\forall (x \in A \to \exists B(B \in \mathcal{N}_x) \wedge (B \subset A)],$$

i.e.,

$$[A \in \tau] = (\mu_\tau(A), \nu_\tau(A)) = (\bigwedge_{x \in A} \bigvee_{B \subset A} \mu_{\mathcal{N}_x}(B), \bigvee_{x \in A} \bigwedge_{B \subset A} \nu_{\mathcal{N}_x}(B)).$$

Proof. From Theorem 3.1 in [43], it is clear that $\mu_\tau(A) = \bigwedge_{x\in A}\bigvee_{B\subset A}\mu_{\mathcal{N}_x}(B)$.

On the other hand,

$\nu_\tau(A) = \bigvee_{x\in A}\bigwedge_{x\in C\subset A}\nu_\tau(C)$ [By Lemma 4.2]
$= \bigvee_{x\in A}\bigwedge_{B\subset A}\bigwedge_{x\in C\subset B}\nu_\tau(C)$
$= \bigvee_{x\in A}\bigwedge_{B\subset A}\nu_{\mathcal{N}_x}(B)$. [By Definition 4.1]

This completes the proof. □

Definition 4.4. $A \in (I\oplus I)^X$ is said to be normal, if there is $x \in X$ such that $A(x) = (1, 0)$.

We will denote the set of all normal intuitionistc fuzzy subsets of 2^X as $(I\oplus I)^{2^X}_N$.

From the following result, we can see that an ordinary intuitionistic smooth neighborhood system has the same properties in a classical neighborhood system.

Theorem 4.5. *Let (X,τ) be an oists and let $\mathcal{N} : X \to (I\oplus I)^{2^X}_N$ be the mapping given by $\mathcal{N}(x) = \mathcal{N}_x$, for each $x \in X$. Then $\mathcal{N}$ has the following properties:*

(1) *for any $x \in X$, $A \in 2^X$, $\models A \in \mathcal{N}_x \to x \in A$,*
(2) *for any $x \in X$, $A, B \in 2^X$, $\models (A \in \mathcal{N}_x) \wedge (B \in \mathcal{N}_x) \to A\cap B \in \mathcal{N}_x$,*
(3) *for any $x \in X$, $A, B \in 2^X$, $\models (A \subset B) \to (A \in \mathcal{N}_x \to B \in \mathcal{N}_x)$,*
(4) *for any $x \in X$, $\models (A \in \mathcal{N}_x) \to \exists C((C \in \mathcal{N}_x) \wedge (C \subset A) \wedge \forall y(y \in C \to C \in \mathcal{N}_y))$.*

Conversely, if a mapping $\mathcal{N} : X \to (I\oplus I)^{2^X}_N$ satisfies the above properties (2) and (3), then there is an ordinary intuitionistic smooth topology $\tau : 2^X \to (I\oplus I)$ on X defined as follows: for each $A \in 2^X$,

$$A \in \tau := \forall x(x \in A \to A \in \mathcal{N}_x),$$

i.e.,

$$[A \in \tau] = \tau(A) = (\mu_\tau(A), \nu_\tau(A)) = (\bigwedge_{x\in A}\mu_{\mathcal{N}_x}(A), \bigvee_{x\in A}\nu_{\mathcal{N}_x}(A)).$$

In particular, if $\mathcal{N}$ satisfies the above properties (1) and (4) also, then for each $x \in X$, $\mathcal{N}_x$ is an ordinary intuitionistic smooth neighborhood system of x with respect to τ.

Proof. (1) Since $A \in 2^X$, $A = (\chi_A, \chi_{A^c}) \in (I\oplus I)^X$. Then

$$[x \in A] = (\chi_A(x), \chi_{A^c}(x)) = (1, 0).$$

On the other hand, $[A \in \mathcal{N}_x] = (\bigvee_{x\in C\subset A}\mu_\tau(C), \bigwedge_{x\in C\subset A}\nu_\tau(C))$. Clearly, $\bigvee_{x\in C\subset A}\mu_\tau(C) > 0$ and $\bigwedge_{x\in C\subset A}\nu_\tau(C) < 1$. Thus $[A \in \mathcal{N}_x] \leq [x \in A]$.

(2) By the definition of $\mathcal{N}_x$, $[A\cap B \in \mathcal{N}_x] = (\bigvee_{x\in C\subset A\cap B}\mu_\tau(C), \bigwedge_{x\in C\subset A\cap B}\nu_\tau(C))$. From the proof of Theorem 3.2 (2) in [43], it is obvious that $\mu_{\mathcal{N}_x}(A\cap B) \geq \mu_{\mathcal{N}_x}(A) \wedge \mu_{\mathcal{N}_x}(B)$. Then it is sufficient to show that $\nu_{\mathcal{N}_x}(A\cap B) \leq \nu_{\mathcal{N}_x}(A) \vee \nu_{\mathcal{N}_x}(B)$. On the other hand,

$\nu_{\mathcal{N}_x}(A\cap B) = \bigwedge_{x\in C\subset A\cap B}\nu_\tau(C) = \bigwedge_{x\in C_1\subset A,\ x\in C_2\subset A}\nu_\tau(C_1\cap C_2)$
$\leq \bigwedge_{x\in C_1\subset A,\ x\in C_2\subset A}(\nu_\tau(C_1) \vee \nu_\tau(C_2))$
$= \bigwedge_{x\in C_1\subset A}\nu_\tau(C_1) \vee \bigwedge_{x\in C_2\subset A}\nu_\tau(C_2)$
$= \nu_{\mathcal{N}_x}(A) \vee \nu_{\mathcal{N}_x}(B)$.

Thus $[A\cap B \in \mathcal{N}_x] \geq [(A \in \mathcal{N}_x) \wedge (B \in \mathcal{N}_x)]$.

(3) The proof is immediate.

(4) It is clear that

$[\exists C((C \in \mathcal{N}_x) \wedge (C \subset A) \wedge \forall y(y \in C \to C \in \mathcal{N}_y))]$
$= (\bigvee_{C\subset A}[\mu_{\mathcal{N}_x}(C) \wedge \bigwedge_{y\in C}\mu_{\mathcal{N}_y}(C)], \bigwedge_{C\subset A}[\nu_{\mathcal{N}_x}(C) \vee \bigvee_{y\in C}\nu_{\mathcal{N}_y}(C)])$.

Then, by the proof of Theorem 3.2 (4) in [43], it is obvious that

$$\bigvee_{C\subset A}[\mu_{\mathcal{N}_x}(C)\wedge\bigwedge_{y\in C}\mu_{\mathcal{N}_y}(C)]\geq\mu_{\mathcal{N}_x}(A).$$

From Lemma 4.2, $\bigvee_{y\in C}\nu_{\mathcal{N}_y}(C)=\bigvee_{y\in C}\bigwedge_{y\in D\subset C}\nu_\tau(D)=\nu_\tau(C)$. Thus

$$\begin{aligned}\textstyle\bigwedge_{C\subset A}[\nu_{\mathcal{N}_x}(C)\vee\bigvee_{y\in C}\nu_{\mathcal{N}_y}(C)]&=\textstyle\bigwedge_{C\subset A}[\nu_{\mathcal{N}_x}(C)\vee\nu_\tau(C)]=\bigwedge_{C\subset A}\nu_\tau(C)\\&\leq\textstyle\bigwedge_{x\in C\subset A}\nu_\tau(C)=\nu_{\mathcal{N}_x}(A).\end{aligned}$$

So $[\exists C((C\in\mathcal{N}_x)\wedge(C\subset A)\wedge\forall y(y\in C\rightarrow C\in\mathcal{N}_y))]\geq[A\in\mathcal{N}_x]$.

Conversely suppose $\mathcal{N}$ satisfies the above properties (2) and (3) and let

$$\tau(A)=(\bigwedge_{x\in A}\mu_{\mathcal{N}_x}(A),\bigvee_{x\in A}\nu_{\mathcal{N}_x}(A)).$$

Then clearly, $\tau(\phi)=(1,0)$. Since $\mathcal{N}_x$ is intuitionistic fuzzy normal, there is $A_0\in 2^X$ such that $\mathcal{N}_x(A_0)=(1,0)$. Thus $\mathcal{N}_x(X)=(1,0)$. So

$$\tau(X)=(\bigwedge_{x\in X}\mu_{\mathcal{N}_x}(X),\bigvee_{x\in X}\nu_{\mathcal{N}_x}(X))=(1,0).$$

Hence τ satisfies the axiom (OIST1).

From the proof of 3.2 in [43], it is clear that $\mu_\tau(A\cap B)\geq\mu_\tau(A)\wedge\mu_\tau(B)$.

On the other hand,

$$\begin{aligned}\nu_\tau(A\cap B)&=\textstyle\bigvee_{x\in A\cap B}\nu_{\mathcal{N}_x}(A\cap B)\leq\bigvee_{x\in A\cap B}(\nu_{\mathcal{N}_x}(A)\vee\nu_{\mathcal{N}_x}(B))\\&=\textstyle\bigvee_{x\in A\cap B}\nu_{\mathcal{N}_x}(A)\vee\bigvee_{x\in A\cap B}\nu_{\mathcal{N}_x}(B)\\&\leq\textstyle\bigvee_{x\in A}\nu_{\mathcal{N}_x}(A)\vee\bigvee_{x\in B}\nu_{\mathcal{N}_x}(B)\\&=\nu_\tau(A)\vee\nu_\tau(B).\end{aligned}$$

Then τ satisfies the axiom (OIST2). Moreover, we can easily see that τ satisfies the axiom (OIST3). Thus $\tau\in OIST(X)$.

Now suppose $\mathcal{N}$ satisfies additionally the above properties (1) and (4). Then, from the proof of 3.2 in [43], $\mu_{\mathcal{N}_x}(A)=\bigvee_{x\in B\subset A}\mu_\tau(B)$, for each $x\in X$ and each $A\in 2^X$.

Let $x\in X$ and let $A\in 2^X$. Then, by the property (4),

$$\nu_{\mathcal{N}_x}(A)\geq\bigwedge_{C\subset A}[\nu_{\mathcal{N}_x}(C)\vee\bigvee_{y\in C}\nu_{\mathcal{N}_y}(C)].$$

From the property (1), $\nu_{\mathcal{N}_x}(C)=1$, for any $x\notin C$. Thus

$$\begin{aligned}\nu_{\mathcal{N}_x}(A)&\geq\textstyle\bigwedge_{x\in C\subset A}[\nu_{\mathcal{N}_x}(C)\vee\bigvee_{y\in C}\nu_{\mathcal{N}_y}(C)]\\&\geq\textstyle\bigwedge_{x\in C\subset A}\bigvee_{y\in C}\nu_{\mathcal{N}_y}(C)\\&=\textstyle\bigwedge_{x\in B\subset A}\nu_\tau(B).\end{aligned}$$

Now suppose $x\in C\subset A$. Then clearly, $\bigvee_{y\in C}\nu_{\mathcal{N}_y}(C)\geq\nu_{\mathcal{N}_x}(C)\geq\nu_{\mathcal{N}_x}(A)$. Thus

$$\bigwedge_{x\in B\subset A}\nu_\tau(B)=\bigwedge_{x\in C\subset A}\bigvee_{y\in C}\nu_{\mathcal{N}_y}(C)\geq\nu_{\mathcal{N}_x}(A).$$

So $\nu_{\mathcal{N}_x}(A)=\bigwedge_{x\in B\subset A}\nu_\tau(B)$. This completes the proof. □

5. Ordinary intuitionistic smooth bases and subbases

Definition 5.1. Let (X,τ) be an *oists* and let $\mathcal{B}:2^X\rightarrow I\oplus I$ be a mapping such that $\mu_{\mathcal{B}}\leq\mu_\tau$ and $\nu_{\mathcal{B}}\geq\nu_\tau$. Then $\mathcal{B}$ is called an ordinary intuitionistic smooth base for τ, if for each $A\in 2^X$,

$$\mu_\tau(A)=\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma}B_\alpha}\ \bigwedge_{\alpha\in\Gamma}\mu_{\mathcal{B}}(B_\alpha)$$

and

$$\nu_\tau(A) = \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_{\mathcal{B}}(B_\alpha).$$

Example 5.2. (1) Let X be a set and let $\mathcal{B} : 2^X \to I \oplus I$ be the mapping defined by:

$$\mathcal{B}(\{x\}) = (1, 0),\ \forall x \in X.$$

Then $\mathcal{B}$ is an ordinary intuitionistic smooth base for τ_X.

(2) Let $X = \{a, b, c\}$, let $(r, s) \in I_1 \oplus I_0$ be fixed and let $\mathcal{B} : 2^X \to I \oplus I$ be the mapping as follows: for each $A \in 2^X$,

$$\tau(A) = \begin{cases} (1,0) & \text{if either } A = \{a, b\} \text{ or } \{b, c\} \text{ or } X, \\ (r, s) & \text{otherwise.} \end{cases}$$

Then $\mathcal{B}$ is not an ordinary intuitionistic smooth base for an *oist* on X.

Assume that $\mathcal{B}$ is an ordinary intuitionistic smooth base for an *oist* τ on X. Then clearly, $\mathcal{B} \leq \tau$. Moreover, $\tau(\{a, b\}) = \tau(\{b, c\}) = (1, 0)$. Thus

$$\mu_\tau(\{b\}) = \mu_\tau(\{a, b\} \cap \tau(\{b, c\}) \geq \mu_\tau(\{a, b\} \wedge \mu_\tau(\{b, c\} = 1$$

and

$$\nu_\tau(\{b\}) = \nu_\tau(\{a, b\} \cap \tau(\{b, c\}) \leq \nu_\tau(\{a, b\} \wedge \nu_\tau(\{b, c\} = 0.$$

So $\tau(\{b\}) = (1, 0)$. On the other hand, by the definition of $\mathcal{B}$,

$$\mu_\tau(\{b\}) = \bigvee_{\{A_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ \{b\}=\bigcup_{\alpha\in\Gamma} A_\alpha} \bigwedge_{\alpha\in\Gamma} \mu_{\mathcal{B}}(A_\alpha) = r$$

and

$$\nu_\tau(\{b\}) = \bigwedge_{\{A_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ \{b\}=\bigcup_{\alpha\in\Gamma} A_\alpha} \bigvee_{\alpha\in\Gamma} \nu_{\mathcal{B}}(A_\alpha) = s.$$

This is a contradiction. Hence $\mathcal{B}$ is not an ordinary intuitionistic smooth base for an *oist* on X.

Theorem 5.3. *Let (X, τ) be an oists and let $\mathcal{B} : 2^X \to I \oplus I$ be a mapping such that $\mathcal{B} \leq \tau$. Then $\mathcal{B}$ is an ordinary intuitionistic smooth base for τ if and only if for each $x \in X$ and each $A \in 2^X$, $\mu_{\mathcal{N}_x}(A) \leq \bigvee_{x\in B\subset A} \mu_{\mathcal{B}}(B)$ and $\nu_{\mathcal{N}_x}(A) \geq \bigwedge_{x\in B\subset A} \nu_{\mathcal{B}}(B)$.*

Proof. (⇒): Suppose $\mathcal{B}$ is an ordinary intuitionistic smooth base for τ. Let $x \in X$ and let $A \in 2^X$. Then, by Theorem 4.4 in [27], it is obvious that $\mu_{\mathcal{N}_x}(A) \leq \bigvee_{x\in B\subset A} \mu_{\mathcal{B}}(B)$.

On the other hand,

$\nu_{\mathcal{N}_x}(A) = \bigwedge_{x\in B\subset A} \nu_\tau(B)$ [By Definition 4.1]

$= \bigwedge_{x\in B\subset A} \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ B=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_{\mathcal{B}}(B_\alpha)$. [By Definition 4.2]

If $x \in B \subset A$ and $B = \bigcup_{\alpha\in\Gamma} B_\alpha$, then there is $\alpha_0 \in \Gamma$ such that $x \in B_{\alpha_0}$. Thus

$$\bigvee_{\alpha\in\Gamma} \nu_{\mathcal{B}}(B_\alpha) \geq \nu_{\mathcal{B}}(B_{\alpha_0}) \geq \bigwedge_{x\in B\subset A} \nu_{\mathcal{B}}(B).$$

So $\nu_{\mathcal{N}_x}(A) \geq \bigwedge_{x\in B\subset A} \nu_{\mathcal{B}}(B)$. Hence the necessary condition holds.

(⇐): Suppose the necessary condition holds. Then, by Theorem 4.4 in [27], it is clear that

$$\mu_\tau(A) = \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mu_{\mathcal{B}}(B_\alpha).$$

Let $A \in 2^X$. Suppose $A = \bigcup_{\alpha\in\Gamma} B_\alpha$ and $\{B_\alpha\} \subset 2^X$. Then

$\nu_\tau(A) \leq \bigvee_{\alpha\in\Gamma} \nu_\tau(B_\alpha)$ [By the axiom (IOST3)]
$\leq \bigvee_{\alpha\in\Gamma} \nu_\mathcal{B}(B_\alpha)$. [Since $\mathcal{B} \leq \tau$]

Thus

$$\nu_\tau(A) \leq \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_\mathcal{B}(B_\alpha). \tag{5.3.1}$$

On the other hand,

$\nu_\tau(A) = \bigvee_{x\in A} \bigwedge_{x\in B\subset A} \nu_\tau(B)$ [By Lemma 4.2]
$= \bigvee_{x\in A} \nu_{\mathcal{N}_x}(A)$ [By Definition 4.1]
$= \bigvee_{x\in A} \bigwedge_{x\in B\subset A} \nu_\mathcal{B}(B)$ [By the hypothesis]
$= \bigwedge_{f\in\Pi_{x\in A}\mathcal{B}_x} \bigvee_{x\in A} \nu_\mathcal{B}(f(x))$,

where $\mathcal{B}_x = \{B \in 2^X : x \in B \subset A\}$. Furthermore, $A = \bigcup_{x\in A} f(x)$, for each $f \in \Pi_{x\in A}\mathcal{B}_x$. So

$$\bigwedge_{f\in\Pi_{x\in A}\mathcal{B}_x} \bigvee_{x\in A} \nu_\mathcal{B}(f(x)) = \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_\mathcal{B}(B_\alpha).$$

Hence

$$\nu_\tau(A) \geq \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_\mathcal{B}(B_\alpha). \tag{5.3.2}$$

By (5.3.1) and (5.3.2), $\nu_\tau(A) = \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_\mathcal{B}(B_\alpha)$. Therefore $\mathcal{B}$ is an ordinary intuitionistic smooth base for τ. □

Theorem 5.4. *Let $\mathcal{B} : 2^X \to I \oplus I$ be a mapping. Then $\mathcal{B}$ is an ordinary intuitionistic smooth base for some oist τ on X if and only if it has the following conditions:*

(1) $(\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mu_\mathcal{B}(B_\alpha), \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_\mathcal{B}(B_\alpha)) = (1, 0)$,
(2) *for any $A_1,\ A_2 \in 2^X$ and each $x \in A_1 \cap A_2$,*

$$\mu_\mathcal{B}(A_1) \wedge \mu_\mathcal{B}(A_2) \leq \bigvee_{x\in A\subset A_1\cap A_2} \mu_\mathcal{B}(A)$$

and

$$\nu_\mathcal{B}(A_1) \vee \nu_\mathcal{B}(A_2) \geq \bigwedge_{x\in A\subset A_1\cap A_2} \nu_\mathcal{B}(A).$$

In fact, $\tau : 2^X \to I \oplus I$ is the mapping defined as follows: for each $A \in 2^X$,

$$\mu_\tau(A) = \begin{cases} 1 & \text{if } A = \phi, \\ \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mu_\mathcal{B}(B_\alpha) & \text{otherwise} \end{cases}$$

and

$$\nu_\tau(A) = \begin{cases} 0 & \text{if } A = \phi, \\ \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_\mathcal{B}(B_\alpha) & \text{otherwise.} \end{cases}$$

In this case, τ is called the ordinary intuitionistic smooth topology on X induced by $\mathcal{B}$.

Proof. (⇒): Suppose $\mathcal{B}$ is an ordinary intuitionistic smooth base for some *oist* τ on X. Then by Definition 5.1 and the axiom (OIST1),

$$\Big(\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mu_\mathcal{B}(B_\alpha), \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_\mathcal{B}(B_\alpha)\Big) = \tau(X) = (1, 0).$$

Thus the condition (1) holds.

Let $A_1,\ A_2 \in 2^X$ and let $x \in A_1 \cap A_2$. Then, by the proof of Theorem 4.2 in [43], it is obvious that $\mu_{\mathcal{B}}(A_1) \wedge \mu_{\mathcal{B}}(A_2) \leq \bigvee_{x\in A\subset A_1\cap A_2} \mu_{\mathcal{B}}(A)$. On the other hand,

$$\nu_{\mathcal{B}}(A_1) \vee \nu_{\mathcal{B}}(A_2) \geq \nu_\tau(A_1) \vee \nu_\tau(A_2) \geq \nu_\tau(A_1 \cap A_2) \geq \nu_{\mathcal{N}_x}(A_1 \cap A_2) \geq \bigwedge_{x\in A\subset A_1\cap A_2} \nu_{\mathcal{B}}(A).$$

Thus $\nu_{\mathcal{B}}(A_1) \vee \nu_{\mathcal{B}}(A_2) \geq \bigwedge_{x\in A\subset A_1\cap A_2} \nu_{\mathcal{B}}(A)$. So the condition (2) holds.

($\Leftarrow$): Suppose the necessary conditions (1) and (2) are satisfied. Then, by the proof of Theorem 4.2 in [43], we can see that the followings hold:

$\mu_\tau(X) = \mu_\tau(\phi) = 1$,

$\mu_\tau(A \cap B) \geq \mu_\tau(A) \wedge \mu_\tau(B)$, for any $A,\ B \in 2^X$

and

$\mu_\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \mu_\tau(A_\alpha)$, for each $\{A_\alpha\}_{\alpha\in\Gamma} \subset 2^X$.

From the definition of τ, it is obvious that $\nu_\tau(X) = \nu_\tau(\phi) = 0$. Thus τ satisfies the axiom (OIST1).

Let $\{A_\alpha\}_{\alpha\in\Gamma} \subset 2^X$ and let $\mathcal{B}_\alpha = \{\{B_{\delta_\alpha} : \delta_\alpha \in \Gamma_\alpha\} : \bigcup_{\delta_\alpha\in\Gamma_\alpha} B_{\delta_\alpha} = A_\alpha\}$. Let $f \in \Pi_{\alpha\in\Gamma}\mathcal{B}_\alpha$. Then clearly, $\bigcup_{\alpha\in\Gamma}\bigcup_{B_{\delta_\alpha}\in f(\alpha)} B_{\delta_\alpha} = \bigcup_{\alpha\in\Gamma} A_\alpha$. Thus

$$\begin{aligned}\nu_\tau(\textstyle\bigcup_{\alpha\in\Gamma} A_\alpha) &= \textstyle\bigwedge_{\bigcup_{\delta\in\Gamma} B_\delta=\bigcup_{\alpha\in\Gamma} A_\alpha} \bigvee_{\delta\in\Gamma} \nu_{\mathcal{B}}(B_\delta)\\ &\leq \textstyle\bigwedge_{f\in\Pi_{\alpha\in\Gamma}\mathcal{B}_\alpha} \bigvee_{\alpha\in\Gamma} \bigvee_{B_{\delta_\alpha}\in f(\alpha)} \nu_{\mathcal{B}}(B_{\delta_\alpha})\\ &= \textstyle\bigvee_{\alpha\in\Gamma} \bigwedge_{\{B_{\delta_\alpha}:\delta_\alpha\in\Gamma_\alpha\}\in\mathcal{B}_\alpha} \bigvee_{\delta_\alpha\in\Gamma_\alpha} \nu_{\mathcal{B}}(B_{\delta_\alpha})\\ &= \textstyle\bigvee_{\alpha\in\Gamma} \nu_\tau(A_\alpha).\end{aligned}$$

So τ satisfies the axiom (OIST3).

Now let $A,\ B \in 2^X$ and suppose $\nu_\tau(A) < t$ and $\nu_\tau(B) < t$. Then there are $\{A_{\alpha_1} : \alpha_1 \in \Gamma_1\}$ and $\{B_{\alpha_2} : \alpha_2 \in \Gamma_2\}$ such that $\bigcup_{\alpha_1\in\Gamma_1} A_{\alpha_1} = A$, $\bigcup_{\alpha_2\in\Gamma_2} B_{\alpha_2} = B$ and $\nu_{\mathcal{B}}(A_{\alpha_1}) < t$ for each $\alpha_1 \in \Gamma_1$, $\nu_{\mathcal{B}}(B_{\alpha_2}) < t$ for each $\alpha_2 \in \Gamma_2$. Let $x \in A \cap B$. Then there are $\alpha_{1x} \in \Gamma_1$ and $\alpha_{2x} \in \Gamma_2$ such that $x \in A_{\alpha_{1x}} \cap B_{\alpha_{2x}}$. Thus, from the assumption,

$$t > \nu_{\mathcal{B}}(A_{\alpha_{1x}}) \vee \nu_{\mathcal{B}}(B_{\alpha_{2x}}) \geq \bigwedge_{x\in C\subset A_{\alpha_{1x}}\cap B_{\alpha_{2x}}} \nu_{\mathcal{B}}(C).$$

Moreover, there is C_x such that $x \in C_x \subset A_{\alpha_{1x}} \cap B_{\alpha_{2x}} \subset A \cap B$ and $\nu_{\mathcal{B}}(C_x) < t$. Since $\bigcup_{x\in A\cap B} C_x = A \cap B$, we obtain

$$t \geq \bigvee_{x\in A\cap B} \nu_{\mathcal{B}}(C_x) \geq \bigwedge_{\bigcup_{\alpha\in\Gamma} B_\alpha=A\cap B} \bigvee_{\alpha\in\Gamma} \nu_{\mathcal{B}}(B_\alpha) = \nu_\tau(A \cap B).$$

Now let $k = \nu_\tau(A) \vee \nu_\tau(B)$ and let n be any natural number. Then $\nu_\tau(A) < k + 1/n$ and $\nu_\tau(B) < k + 1/n$. Thus $\nu_\tau(A \cap B) \leq k + 1/n$. So $\nu_\tau(A \cap B) \leq k = \nu_\tau(A) \vee \nu_\tau(B)$. Hence τ satisfies the axiom (OIST2). This completes the proof. □

Example 5.5. (1) Let $X = \{a, b, c\}$ and let $(r, s) \in I_1 \oplus I_0$ be fixed. We define the mapping $\mathcal{B} : 2^X \to I \oplus I$ as follows: for each $A \in 2^X$,

$$\mu_{\mathcal{B}}(A) = \begin{cases} 1 & \text{if } A = \{b\} \text{ or } \{a, b\} \text{ or } \{b, c\} \\ r & \text{otherwise} \end{cases}$$

and

$$\nu_{\mathcal{B}}(A) = \begin{cases} 0 & \text{if } A = \{b\} \text{ or } \{a, b\} \text{ or } \{b, c\} \\ s & \text{otherwise.} \end{cases}$$

Then we can easily see that $\mathcal{B}$ satisfies the conditions (1) and (2) in Theorem 5.4. Thus $\mathcal{B}$ is an ordinary intuitionistic smooth base for an *oist* τ on X. In fact, $\tau : 2^X \to I \oplus I$ is defined as follows: for each $A \in 2^X$,

$$\mu_\tau(A) = \begin{cases} 1 & \text{if } A \in \{\phi, \{b\}, \{a,b\}, \{b,c\}, X\} \\ r & \text{otherwise} \end{cases}$$

and

$$\nu_\tau(A) = \begin{cases} 0 & \text{if } A \in \{\phi, \{b\}, \{a,b\}, \{b,c\}, X\} \\ s & \text{otherwise.} \end{cases}$$

(2) Let $(r,s) \in I_1 \oplus I_0$ be fixed. We define the mapping $\mathcal{B} : 2^{\mathbb{R}} \to I \oplus I$ as follows: for each $A \in 2^{\mathbb{R}}$,

$$\mu_{\mathcal{B}}(A) = \begin{cases} 1 & \text{if } A = (a,b), \text{ for } a,\ b \in \mathbb{R} \text{ with } a \leq b \\ r & \text{otherwise} \end{cases}$$

and

$$\nu_{\mathcal{B}}(A) = \begin{cases} 0 & \text{if } A = (a,b), \text{ for } a,\ b \in \mathbb{R} \text{ with } a \leq b \\ s & \text{otherwise.} \end{cases}$$

Then it can be easily seen that $\mathcal{B}$ satisfies the conditions (1) and (2) in Theorem 5.4. Thus $\mathcal{B}$ is an ordinary intuitionistic smooth base for an *oist* $\tau_{(r,s)}$ on $\mathbb{R}$.

In this case, $\tau_{(r,s)}$ will be called the (r,s)-ordinary intuitionistic smooth usual topology on $\mathbb{R}$.

(3) Let $(r,s) \in I_1 \oplus I_0$ be fixed. We define the mapping $\mathcal{B} : 2^{\mathbb{R}} \to I \oplus I$ as follows: for each $A \in 2^{\mathbb{R}}$,

$$\mu_{\mathcal{B}}(A) = \begin{cases} 1 & \text{if } A = [a,b), \text{ for } a,\ b \in \mathbb{R} \text{ with } a \leq b \\ r & \text{otherwise} \end{cases}$$

and

$$\nu_{\mathcal{B}}(A) = \begin{cases} 0 & \text{if } A = [a,b), \text{ for } a,\ b \in \mathbb{R} \text{ with } a \leq b \\ s & \text{otherwise.} \end{cases}$$

Then we can easily see that $\mathcal{B}$ satisfies the conditions (1) and (2) in Theorem 5.4. Thus $\mathcal{B}$ is an ordinary intuitionistic smooth base for an *oist* τ_l on $\mathbb{R}$.

In this case, τ_l will be called the (r,s)-ordinary intuitionistic smooth lower-limit topology on $\mathbb{R}$.

Definition 5.6. Let $\tau_1,\ \tau_2 \in OIST(X)$, and let $\mathcal{B}_1$ and $\mathcal{B}_1$ be ordinary intuitionistic smooth bases for τ_1 and τ_2, respectively. Then $\mathcal{B}_1$ and $\mathcal{B}_1$ are equivalent, if $\tau_1 = \tau_2$.

Theorem 5.7. *Let $\tau_1,\ \tau_2 \in OIST(X)$, and let $\mathcal{B}_1$ and $\mathcal{B}_1$ be ordinary intuitionistic smooth bases for τ_1 and τ_2, respectively. Then τ_1 is coarser than τ_2, i.e., $\mu_{\tau_1} \leq \mu_{\tau_2}$ and $\nu_{\tau_1} \geq \nu_{\tau_2}$ if and only if for each $x \in X$ and each $A \in 2^X$, if $x \in A$, then $\mu_{\mathcal{B}_1}(A) \leq \bigvee_{x \in B \subset A} \mu_{\mathcal{B}_2}(B)$ and $\nu_{\mathcal{B}_1}(A) \geq \bigwedge_{x \in B \subset A} \nu_{\mathcal{B}_2}(B)$.*

Proof. $(\Rightarrow)$: Suppose τ_1 is finer than τ_2. For each $x \in X$, let $x \in A \in 2^X$. Then, by Theorem 4.8 in [27], $\mu_{\mathcal{B}_1} \leq \bigvee_{x \in B \subset A} \mu_{\mathcal{B}_2}(B)$. On the other hand,

$$\begin{aligned} \nu_{\mathcal{B}_1}(A) &\geq \nu_{\tau_1}(A) \text{ [Since } \mathcal{B}_1 \text{ is an ordinary intuitionistic smooth base for } \tau_1] \\ &\geq \nu_{\tau_2}(A) \text{ [By the hypothesis]} \\ &= \bigwedge_{\{A_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ A = \bigcup_{\alpha \in \Gamma} A_\alpha} \bigvee_{\alpha \in \Gamma} \nu_{\mathcal{B}_2}(A_\alpha). \end{aligned}$$

[Since $\mathcal{B}_2$ is an ordinary intuitionistic smooth base for τ_2]
Since $x \in A$ and $A = \bigcup_{\alpha\in\Gamma} A_\alpha$, there is $\alpha_0 \in \Gamma$ such that $x \in A_{\alpha_0}$. Thus

$$\bigwedge_{\{A_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ A=\bigcup_{\alpha\in\Gamma} A_\alpha} \bigvee_{\alpha\in\Gamma} \nu_{\mathcal{B}_2}(A_\alpha) \geq \nu_{\mathcal{B}_2}(A_{\alpha_0}) \geq \bigwedge_{x\in B\subset A} \nu_{\mathcal{B}_2}(B).$$

So $\nu_{\mathcal{B}_1}(A) \geq \bigwedge_{x\in B\subset A} \nu_{\mathcal{B}_2}(B)$.

($\Leftarrow$): Suppose the necessary conditions hold. Then, by Theorem 4.8 in [27], $\mu_{\tau_1} \leq \mu_{\tau_2}$. Let $A \in 2^X$. Then

$$\begin{aligned}\nu_{\tau_1}(A) &= \bigvee_{x\in A} \bigwedge_{x\in B\subset A} \nu_{\mathcal{B}_1}(B) \text{ [By Lemma 4.2]}\\ &\geq \bigvee_{x\in A} \bigwedge_{x\in B\subset A} \bigwedge_{x\in C\subset B} \nu_{\mathcal{B}_2}(C) \text{ [By the hypothesis]}\\ &= \bigwedge_{x\in C\subset A} \bigvee_{x\in A} \nu_{\mathcal{B}_2}(C)\\ &= \bigwedge_{\{C_x\}_{x\in A}\subset 2^X,\ A=\bigcup_{x\in A} C_x} \bigvee_{x\in A} \nu_{\mathcal{B}_2}(C_x)\\ &= \nu_{\tau_2}(A).\end{aligned}$$

Thus $\nu_{\tau_1} \geq \nu_{\tau_2}$. So τ_1 is coarser than τ_2. This completes the proof. □

The following is the immediate result of Definition 5.6 and Theorem 5.7.

Corollary 5.8. *Let $\mathcal{B}_1$ and $\mathcal{B}_1$ be ordinary intuitionistic smooth bases for two ordinary intuitionistic smooth topologies on a set X, respectively. Then*

$\mathcal{B}_1$ and $\mathcal{B}_1$ are equivalent if and only if

(1) for each $B_1 \in 2^X$ and each $x \in B_1$, $\mu_{\mathcal{B}_1}(B_1) \leq \bigvee_{x\in B_2\subset B_1} \mu_{\mathcal{B}_2}(B_2)$ and $\nu_{\mathcal{B}_1}(B_1) \geq \bigwedge_{x\in B_2\subset B_1} \nu_{\mathcal{B}_2}(B_2)$,

(2) for each $B_2 \in 2^X$ and each $x \in B_2$, $\mu_{\mathcal{B}_2}(B_2) \leq \bigvee_{x\in B_1\subset B_2} \mu_{\mathcal{B}_1}(B_1)$ and $\nu_{\mathcal{B}_2}(B_2) \geq \bigwedge_{x\in B_1\subset B_2} \nu_{\mathcal{B}_1}(B_1)$.

It is obvious that every ordinary intuitionistic smooth topology itself forms an ordinary intuitionistic smooth base. Then the following provides a sufficient condition for one to see if a mapping $\mathcal{B} : 2^X \to I \oplus I$ such that $\mu_\mathcal{B} \leq \mu_\tau$ and $\nu_\mathcal{B} \geq \nu_\tau$ is an ordinary intuitionistic smooth base for τ, where $\tau \in OIST(X)$.

Proposition 5.9. *Let (X, τ) be an oists, let $\mathcal{B} : 2^X \to I \oplus I$ be a mapping such that $\mu_\mathcal{B} \leq \mu_\tau$ and $\nu_\mathcal{B} \geq \nu_\tau$ and for each $x \in X$ and each $A \in 2^X$ such that $x \in A$, let $\mu_\tau \leq \bigvee_{x\in B\subset A} \mu_\mathcal{B}(B)$ and $\nu_\tau \geq \bigwedge_{x\in B\subset A} \nu_\mathcal{B}(B)$. Then $\mathcal{B}$ is an ordinary intuitionistic smooth base for τ.*

Proof. From the proof of Proposition 4.10 in [27], it is clear that the first part of the condition (1) of Theorem 5.4 holds, i.e., $\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mu_\mathcal{B}(B_\alpha) = 1$.

On the other hand,

$$\begin{aligned}&\bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_\mathcal{B}(B_\alpha)\\ &\geq \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_\tau(B_\alpha) \text{ [Since } \nu_\mathcal{B} \geq \nu_\tau]\\ &\geq \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \nu_\tau(\textstyle\bigcup_{\alpha\in\Gamma} B_\alpha) \text{ [By the axiom (OIST3)]}\\ &= \nu_\tau(X)\\ &= \bigvee_{x\in X} \bigwedge_{x\in B\subset X} \nu_\tau(B) \text{ [By Lemma 4.2]}\\ &\geq \bigvee_{x\in X} \bigwedge_{x\in B\subset X} \bigwedge_{x\in C\subset B} \nu_\mathcal{B}(C) \text{ [By the hypothesis]}\\ &= \bigwedge_{x\in C\subset X} \bigvee_{x\in X} \nu_\mathcal{B}(C)\\ &= \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_\mathcal{B}(B_\alpha).\end{aligned}$$

Since $\tau \in OIST(X)$, $\nu_\tau(X) = 0$. Thus $\bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_\mathcal{B}(B_\alpha) = 0$. So the condition (1) of Theorem 5.4 holds.

Now let $A_1,\ A_2 \in 2^X$ and let $x \in A_1 \cap A_2$. Then, by the proof of Proposition 4.10 in [27], it is obvious that $\mu_{\mathcal{B}}(A_1) \wedge \mu_{\mathcal{B}}(A_2) \leq \bigvee_{x \in A \subset A_1 \cap A_2} \mu_{\mathcal{B}}(A)$. On the other hand,

$$\begin{aligned}\nu_{\mathcal{B}}(A_1) \vee \nu_{\mathcal{B}}(A_2) &\geq \nu_\tau(A_1) \vee \nu_\tau(A_2) \text{ [Since } \nu_{\mathcal{B}} \geq \nu_\tau] \\ &\geq \nu_\tau(A_1 \cap A_2) \text{ [By the axiom (OIST2)]} \\ &\geq \textstyle\bigwedge_{x \in A \subset A_1 \cap A_2} \nu_{\mathcal{B}}(A). \text{ [By the hypothesis]}\end{aligned}$$

Thus the condition (2) of Theorem 5.4 holds. So, by Theorem 5.4, $\mathcal{B}$ is an ordinary intuitionistic smooth base for τ. This completes the proof. □

Definition 5.10. Let (X, τ) be an *oists* and let $\varphi : 2^X \to I \oplus I$ be a mapping. Then φ is called an ordinary intuitionistic smooth subbase for τ, if $\varphi^\sqcap$ is an ordinary intuitionistic smooth base for τ, where $\varphi^\sqcap : 2^X \to I \oplus I$ is the mapping defined as follows: for each $A \in 2^X$,

$$\varphi^\sqcap(A) = (\bigvee_{\{B_\alpha\} \sqsubset 2^X,\ A = \bigcap_{\alpha \in \Gamma} B_\alpha} \bigwedge_{\alpha \in \Gamma} \mu_\varphi(B_\alpha), \bigwedge_{\{B_\alpha\} \sqsubset 2^X,\ A = \bigcap_{\alpha \in \Gamma} B_\alpha} \bigvee_{\alpha \in \Gamma} \nu_\varphi(B_\alpha)),$$

where $\sqsubset$ stands for "a finite subset of".

Example 5.11. Let $(r, s) \in I_1 \oplus I_0$ be fixed. We define the mapping $\varphi : 2^{\mathbb{R}} \to I \oplus I$ as follows: for each $A \in 2^{\mathbb{R}}$,

$$\mu_\varphi(A) = \begin{cases} 1 & \text{if } A = (a, \infty) \text{ or } (-\infty, b) \text{ or } (a, b) \\ r & \text{otherwise} \end{cases}$$

and

$$\nu_\varphi(A) = \begin{cases} 0 & \text{if } A = (a, \infty) \text{ or } (-\infty, b) \text{ or } (a, b) \\ s & \text{otherwise,} \end{cases}$$

where $a,\ b \in \mathbb{R}$ such that $a < b$. Then we can easily see that φ is an ordinary intuitionistic smooth subbase for the (r, s)-ordinary intuitionistic smooth usual topology $\mathcal{U}_{(r,s)}$ on $\mathbb{R}$.

Theorem 5.12. *Let $\varphi : 2^X \to I \oplus I$ be a mapping. Then φ is an ordinary intuitionistic smooth subbase for some oist if and only if $\bigvee_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigwedge_{\alpha \in \Gamma} \mu_\varphi(B_\alpha) = 1$ and $\bigwedge_{\{B_\alpha\}_{\alpha \in \Gamma} \subset 2^X,\ X = \bigcup_{\alpha \in \Gamma} B_\alpha} \bigvee_{\alpha \in \Gamma} \nu_\varphi(B_\alpha) = 0$.*

Proof. ($\Rightarrow$): Suppose φ is an ordinary intuitionistic smooth subbase for some *oist*. Then, by Definition 5.10, it is clear that the necessary condition holds.

($\Leftarrow$): Suppose the necessary condition holds. We only show that $\varphi^\sqcap$ satisfies the condition (2) in Theorem 5.4. Let $A,\ B \in 2^X$ and $x \in A \cap B$, for each $x \in X$. Then, by the proof of Theorem 4.3 in [43], it is obvious that $\mu_{\varphi^\sqcap}(A) \wedge \mu_{\varphi^\sqcap}(B) \leq \bigvee_{x \in C \subset A \cap B} \mu_{\varphi^\sqcap}(C)$.

On the other hand,

$$\begin{aligned}&\nu_{\varphi^\sqcap}(A) \vee \nu_{\varphi^\sqcap}(B) \\ &= (\textstyle\bigwedge_{\bigcap_{\alpha_1 \in \Gamma_1} B_{\alpha_1} = A} \bigvee_{\alpha_1 \in \Gamma_1} \nu_\varphi(B_{\alpha_1})) \vee (\bigwedge_{\bigcap_{\alpha_2 \in \Gamma_2} B_{\alpha_2} = B} \bigvee_{\alpha_2 \in \Gamma_2} \nu_\varphi(B_{\alpha_2})) \\ &= \textstyle\bigwedge_{\bigcap_{\alpha_1 \in \Gamma_1} B_{\alpha_1} = A} \bigwedge_{\bigcap_{\alpha_2 \in \Gamma_2} B_{\alpha_2} = B} (\bigvee_{\alpha_1 \in \Gamma_1} \nu_\varphi(B_{\alpha_1}) \vee \bigvee_{\alpha_2 \in \Gamma_2} \nu_\varphi(B_{\alpha_2})) \\ &\geq \textstyle\bigwedge_{\bigcap_{\alpha \in \Gamma} B_\alpha = A \cap B} \bigvee_{\alpha \in \Gamma} \nu_\varphi(B_\alpha) \\ &= \nu_{\varphi^\sqcap}(A \cap B).\end{aligned}$$

Since $x \in A \cap B$, $\nu_{\varphi^\sqcap}(A) \vee \nu_{\varphi^\sqcap}(B) \geq \nu_{\varphi^\sqcap}(A \cap B) \geq \bigwedge_{x \in C \subset A \cap B} \nu_{\varphi^\sqcap}(C)$. Thus $\varphi^\sqcap$ satisfies the condition (2) in Theorem 5.4. This completes the proof. □

Example 5.13. Let $X = \{a, b, c, d, e\}$ and let $(r, s) \in I_1 \oplus I_0$ be fixed. We define the mapping $\varphi : 2^X \to I \oplus I$ as follows: for each $A \in 2^X$,

$$\mu_\varphi(A) = \begin{cases} 1 & \text{if } A \in \{\{a\}, \{a, b, c\}, \{b, c, d\}, \{c, e\}\} \\ r & \text{otherwise} \end{cases}$$

and

$$\nu_\varphi(A) = \begin{cases} 0 & \text{if } A \in \{\{a\}, \{a, b, c\}, \{b, c, d\}, \{c, e\}\} \\ s & \text{otherwise,} \end{cases}$$

Then $X = \{a\} \cup \{b, c, d\} \cup \{c, e\}$, $\mu_{\varphi^\sqcap}(\{a\}) = \mu_{\varphi^\sqcap}(\{b, c, d\}) = \mu_{\varphi^\sqcap}(\{c, e\}) = 1$ and $\nu_{\varphi^\sqcap}(\{a\}) = \nu_{\varphi^\sqcap}(\{b, c, d\}) = \nu_{\varphi^\sqcap}(\{c, e\}) = 0$. Thus

$$\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mu_\varphi(B_\alpha) = 1$$

and

$$\bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_\varphi(B_\alpha) = 0.$$

So, by Theorem 5.12, φ is an ordinary intuitionistic smooth subbase for some *oist*.

The following is the immediate result of Corollary 5.8 and Theorem 5.12.

Proposition 5.14. *$\varphi_1,\ \varphi_2 : 2^X \to I \oplus I$ be two mappings such that*

$$\Big(\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mu_{\varphi_1}(B_\alpha),\ \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_{\varphi_1}(B_\alpha)\Big) = (1, 0)$$

and

$$\Big(\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mu_{\varphi_2}(B_\alpha),\ \bigwedge_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\ X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigvee_{\alpha\in\Gamma} \nu_{\varphi_2}(B_\alpha)\Big) = (1, 0).$$

Suppose the two conditions hold:

(1) *for each $S_1 \in 2^X$ and each $x \in S_1$, $\mu_{\varphi_1}(S_1) \leq \bigvee_{x\in S_2\subset S_1} \mu_{\varphi_2}(S_2)$ and $\nu_{\varphi_1}(S_1) \geq \bigwedge_{x\in S_2\subset S_1} \nu_{\varphi_2}(S_2)$,*

(2) *for each $S_2 \in 2^X$ and each $x \in S_2$, $\mu_{\varphi_2}(S_2) \leq \bigvee_{x\in S_1\subset S_2} \mu_{\varphi_1}(S_1)$ and $\nu_{\varphi_2}(S_2) \geq \bigwedge_{x\in S_1\subset S_2} \nu_{\varphi_1}(S_1)$.*

Then φ_1 and φ_2 are ordinary intuitionistic smooth subbases for the same ordinary intuitionistic smooth topology on X.

6. Conclusions

We defined an ordinary intuitionistic smooth topology and level set of an *oist*, and obtain some their basic properties and gave some examples. Also we defined an ordinary intuitionistic smooth subspace. Next, we introduced the concept of an ordinary intuitionistic smooth neighborhood system and and we proved that an ordinary intuitionistic smooth neighborhood system has the same properties in a classical neighborhood system (See Theorem 4.5). Finally, we defined an ordinary intuitionistic smooth base and an ordinary intuitionistic smooth subbase, and obtain two characterization of an ordinary intuitionistic smooth base (See Theorems 5.3 and 5.4) and one characterization of an ordinary intuitionistic smooth subbase (See Theorem 5.12), and gave some their examples.

Acknowledgements. We are thankful to referees for the comments and suggestions they have made in finishing the paper.

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