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Abstract. We introduce the notion of intuitionistic fuzzy edge graph $\widehat{G} = (V, B)$, where V is a crisp vertex set and B is an intuitionistic fuzzy relation on V, and present some of its properties. Using α-level graphs and (α, β)-level graphs, we characterize intuitionistic fuzzy graph $G = (A, B)$, where A is an intuitionistic fuzzy set on V and B is an intuitionistic fuzzy relation on V.

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1. Introduction

Graph theory is useful tool for solving numerous problems in different areas including computer science, engineering, operations research and optimization. In many cases, some aspects of a graph-theoretic problem may be uncertain. In such cases, it is natural to deal with the uncertainty using the methods of fuzzy sets and fuzzy logic. Fuzzy graph theory is finding an increasing number of applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems.
Based on Zadeh's fuzzy relations [21] Kaufanm defined in [11] a fuzzy graph. Rosenfeld [16] described the structure of fuzzy graphs obtaining analogs of several graph theoretical concepts. Bhattacharya [8] gave some remarks on operations on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson and Peng [13]. Mordeson and Nair presented a valuable contribution on fuzzy graphs as well as fuzzy hypergraphs in [14]. Shannon and Atanassov [19] introduced the

concept of intuitionistic fuzzy graphs and investigated some of their properties. Parvathi *et al.* defined operations on intuitionistic fuzzy graphs in [15]. Akram *et al.* [1, 2, 3, 4, 5, 17, 18] introduced many new concepts, including strong intuitionistic fuzzy graphs, intuitionistic fuzzy trees, intuitionistic fuzzy hypergraphs, and operations on interval-valued fuzzy graphs. Nowaways intuitionistic fuzzy graphs are playing a substantial role in chemistry, economics, computer science, engineering, medicine and decision making problems. The advantages of intuitionistic fuzzy sets and graphs are that they give more accuracy into the problems and reduce the cost of implementation and improve efficiency. In this article, we introduce the notion of intuitionistic fuzzy edge graph $\widehat{G} = (V, B)$, where V is a crisp vertex set and B is an intuitionistic fuzzy relation on V, and present some its properties. Next, using α-level graphs and (α, β)-level graphs we characterize intuitionistic fuzzy graph $G = (A, B)$, where A is an intuitionistic fuzzy set on V and B is an intuitionistic fuzzy relation on V.
We have used standard definitions and terminologies in this paper. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [9, 10, 22].

2. Preliminaries

A graph is an ordered pair $G^* = (V, E)$, where V is the set of vertices of G^* and $E \subset \widetilde{V^2}$ is the set of edges of G^*. Two vertices x and y in a graph G^* are said to be adjacent in G^*, if $\{x, y\}$ is in an edge of G^*. (For simplicity, an edge $\{x, y\}$ will be denoted by xy.) A simple graph is a graph without loops and multiple edges. Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be two graphs and let $V = V_1 \times V_2$. The union of graphs G_1^* and G_2^* is the graph $(V_1 \cup V_2, E_1 \cup E_2)$. The graph $(V_1 \cup V_2, E_1 \cup E_2 \cup E')$, where E' is the set of edges joining vertices of $(V_1$ and V_2, is denoted by $G_1^* + G_2^*$ and is called the join of graphs G_1^* and G_2^*. The Cartesian product of graphs G_1^* and G_2^*, denoted by $G_1^* \times G_2^*$, is the graph (V, E) with $E = \{(x, x_2)(x, y_2) | x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) | z \in V_2, x_1y_1 \in E_1\}$. The cross product of graphs G_1^* and G_2^*, denoted by $G_1^* * G_2^*$, is the graph (V, E) such that $E = \{(x_1, x_2)(y_1, y_2) | x_1y_1 \in E_1, x_2y_2 \in E_2\}$. The lexicographic product of graphs G_1^* and G_2^*, denoted by $G_1^* \bullet G_2^*$, is the graph (V, E) such that $E = \{(x, x_2)(x, y_2) | x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, x_2)(y_1, y_2) | x_1y_1 \in E_1, x_2y_2 \in E_2\}$. The strong product of graphs G_1^* and G_2^*, denoted by $G_1^* \boxtimes G_2^*$, is the graph (V, E) such that $E = \{(x, x_2)(x, y_2) | x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) | z \in V_2, x_1y_1 \in E_1\} \cup \{(x_1, x_2)(y_1, y_2) | x_1y_1 \in E_1, x_2y_2 \in E_2\}$. The composition of graphs G_1^* and G_2^*, denoted by $G_1^*[G_2^*]$, is the graph (V, E) such that $E = \{(x, x_2)(x, y_2) | x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)(y_1, z) | z \in V_2, x_1y_1 \in E_1\} \cup \{(x_1, x_2)(y_1, y_2) | x_2, y_2 \in V_2, x_2 \neq y_2, x_1y_1 \in E_1\}$.

A fuzzy subset [20] μ on a set V is a map $\mu : V \to [0, 1]$. A fuzzy binary relation on V is a fuzzy subset μ on $V \times V$. By a fuzzy relation we mean a fuzzy binary relation given by $\mu : V \times V \to [0, 1]$. A fuzzy graph $G = (\mu, \nu)$ is a non-empty set V together with a pair of mapping $\mu : V \to [0, 1]$ and $\nu : V \times V \to [0, 1]$ such that for all $x, y \in V$, $\nu(x, y) \leq \mu(x) \wedge \mu(y)$ and ν is a symmetric fuzzy relation on μ. An intuitionistic fuzzy set [6, 7] is an object of the form $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in V\}$, where the mappings $\mu_A : V \to [0, 1]$ and $\nu_A : V \to [0, 1]$ denote, respectively, the degree of membership and non-membership of each element $x \in V$ such that $\mu_A(x)+\nu_A(x) \leq 1$ for all $x \in V$.

3. Intuitionistic fuzzy graphs by level graphs

We define here an intuitionistic fuzzy edge graph $\widehat{G} = (V, B)$ when V is a crisp vertex set and $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy relation on V.

Definition 3.1. An intuitionistic fuzzy edge graph on a nonempty set V is an ordered pair of the form $\widehat{G} = (V, B)$, where V is the crisp vertex set and B is an intuitionistic fuzzy relation on V such that $\mu_B(xy) \leq \min\{\mu_A(x), \mu_A(y)\}$, $\nu_B(xy) \geq \max\{\nu_A(x), \nu_A(y)\}$ and $0 \leq \mu_B(xy) + \nu_B(xy) \leq 1$, for all $xy \in E$. Note that $\mu_B(xy) = \nu_B(xy) = 0$, for all $xy \in V \times V - E$.

We consider intuitionistic fuzzy edge graphs with crisp vertex set, i.e., intuitionistic fuzzy graphs $\widehat{G} = (V, B)$ for which $\mu_A(x) = 1$, $\nu_A(x) = 0 \ \forall \ x \in V$, and edges with membership and non-membership degrees in $[0, 1]$.

Example 3.2. Consider a simple graph $G^* = (V, E)$ such that $V = \{a, b, c\}$ and $E = \{ab, bc, ac\}$. Let B be an intuitionistic fuzzy relation on V defined by $B = \{(ab, 0.3, 0.1), (bc, 0.4, 0.1), (ac, 0.2, 0.1)\}$. By routine calculations, it is easy to see from Figure 1 that $\widehat{G} = (V, B)$ is an intutionistic fuzzy edge graph with crisp vertex set and intuitionistic fuzzy set of edges.

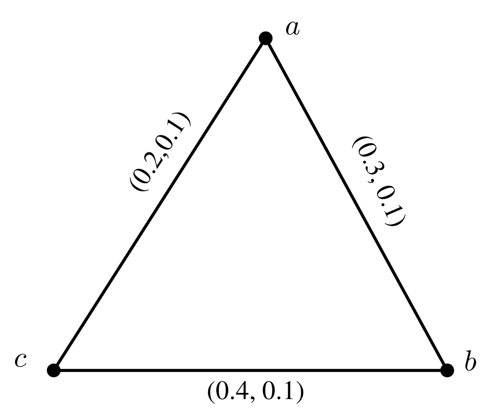

Figure 1. Intuitionistic fuzzy edge graph $\widehat{G} = (V, B)$

We state the following propositions without their proofs.

Proposition 3.3. *The Cartesian product* $\widehat{G}_1 \times \widehat{G}_2 = (V_1 \times V_2, B_1 \times B_2)$ *of two intuitionistic fuzzy edge graphs* $\widehat{G}_1$ *and* $\widehat{G}_2$ *is an intuitionistic fuzzy edge graph.*

Proposition 3.4. *The composition* $\widehat{G}_1[\widehat{G}_2]$ *of intuitionistic fuzzy edge graphs* $\widehat{G}_1$ *and* $\widehat{G}_2$ *is an intuitionistic fuzzy edge graph.*

Proposition 3.5. *The union* $\widehat{G}_1 \bigcup \widehat{G}_2$ *of two intuitionistic fuzzy edge graphs* $\widehat{G}_1$ *and* $\widehat{G}_2$ *is an intuitionistic fuzzy edge graph.*

Proposition 3.6. *The join $\widehat{G}_1 + \widehat{G}_2$ of two intuitionistic fuzzy edge graphs $\widehat{G}_1$ and $\widehat{G}_2$ is an intuitionistic fuzzy edge graph.*

Proposition 3.7. *The lexicographic product $\widehat{G}_1 \bullet \widehat{G}_2$ of two intuitionistic fuzzy edge graphs $\widehat{G}_1$ and $\widehat{G}_2$ is an intuitionistic fuzzy edge graph.*

Proposition 3.8. *The strong product $\widehat{G}_1 \boxtimes \widehat{G}_2$ of two intuitionistic fuzzy edge graphs $\widehat{G}_1$ and $\widehat{G}_2$ is an intuitionistic fuzzy edge graph.*

Proposition 3.9. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be crisp graphs with $V_1 \cap V_2 = \varnothing$. Let B_1 and B_2 be intuitionistic fuzzy relations on V_1 and V_2, respectively. Then $\widehat{G}_1 \cup \widehat{G}_2 = (V_1 \cup V_2, B_1 \cup B_2)$ is an intuitionistic fuzzy edge graph of $G_1^* \cup G_2^*$ if and only if $\widehat{G}_1 = (V_1, B_1)$ and $\widehat{G}_2 = (V_2, B_2)$ are intuitionistic fuzzy edge graphs of G_1^* and G_2^*, respectively.*

Proposition 3.10. *Let $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ be crisp graphs and let $V_1 \cap V_2 = \varnothing$. Let B_1 and B_2 be intuitionistic fuzzy relations of V_1 and V_2, respectively. Then $\widehat{G}_1 + \widehat{G}_2 = (V_1 + V_2, B_1 + B_2)$ is an intuitionistic fuzzy edge graph of G^* if and only if $\widehat{G}_1 = (V_1, B_1)$ and $\widehat{G}_2 = (V_2, B_2)$ are intuitionistic fuzzy edge graphs of G_1^* and G_2^*, respectively.*

Definition 3.11 ([2])**.** Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy sets on a nonempty set V. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy relation on a set V, then $A = (\mu_A, \nu_A)$ is called an intuitionistic fuzzy relation on $B = (\mu_B, \nu_B)$, if $\mu_A(x, y) \leq \min(\mu_B(x), \mu_B(y))$ and $\nu_A(xy) \geq \max(\nu_B(x), \nu_B(y))$, for all x, $y \in V$. An intuitionistic fuzzy relation A on V is called symmetric, if $\mu_A(x, y) = \mu_A(y, x)$ and $\nu_A(x, y) = \nu_A(y, x)$, for all x, $y \in V$.

Definition 3.12 ([2])**.** An intuitionistic fuzzy graph on a nonempty set V is an ordered pair of the form $G = (A, B)$, where $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy set on V and $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy relation on V such that

$$\mu_B(xy) \leq \min\{\mu_A(x), \mu_A(y)\},$$
$$\nu_B(xy) \geq \max\{\nu_A(x), \nu_A(y)\},$$

and

$$\mu_B(xy) + \nu_B(xy) \leq 1, \text{ for all x, y} \in \text{V}.$$

Definition 3.13. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set on V. Then the set

$$A_{(\alpha,\beta)} = \{x \in V \mid \mu_A(x) \geq \alpha,\ \nu_A(x) \leq \beta\},$$

where $(\alpha, \beta) \in [0, 1] \times [0, 1]$, $\alpha + \beta \leq 1$ is called the (α, β)-level set of A.

Let $B = (\mu_B, \nu_B)$ be an intuitionistic fuzzy relation on V. Then the set

$$B_{(\alpha,\beta)} = \{xy \in V \times V \mid \mu_B(xy) \geq \alpha,\ \nu_B(xy) \leq \beta\},$$

where $(\alpha, \beta) \in [0, 1] \times [0, 1]$, $\alpha + \beta \leq 1$ is called the (α, β)-level set of B.

$G_{(\alpha,\beta)} = (A_{(\alpha,\beta)}, B_{(\alpha,\beta)})$ is called (α, β)-level graph.

In case of $\alpha = \beta$, where $\alpha \leq 1$ we write level graph by G_α instead of $G_{(\alpha,\beta)}$. Note that

$$A_{(\alpha,\beta)} = \{x \in V \mid \mu_A(x) \geq \alpha\} \cap \{x \in V \mid \nu_A(x) \leq \beta\} = U(\mu; \alpha) \cap L(\nu; \beta),$$

$$B_{(\alpha,\beta)} = \{xy \in V \times V \mid \mu_B(xy) \geq \alpha\} \cap \{xy \in V \times V \mid \nu_B(xy) \leq \beta\}$$
$$= U(\mu;\alpha) \cap L(\nu;\beta).$$

Proposition 3.14. *The level graph $G_{(\alpha,\beta)} = (A_{(\alpha,\beta)}, B_{(\alpha,\beta)})$ is a subgraph of $G^* = (V,E)$.*

Example 3.15. Consider a simple graph $G^* = (V,E)$ such that $V = \{a,b,c,d\}$ and $E = \{ab, bc, cd, ad, ac, bd\}$. By routine calculations, it is easy to see from Figure 2 that $G = (A,B)$ is an intuitionistic fuzzy graph.

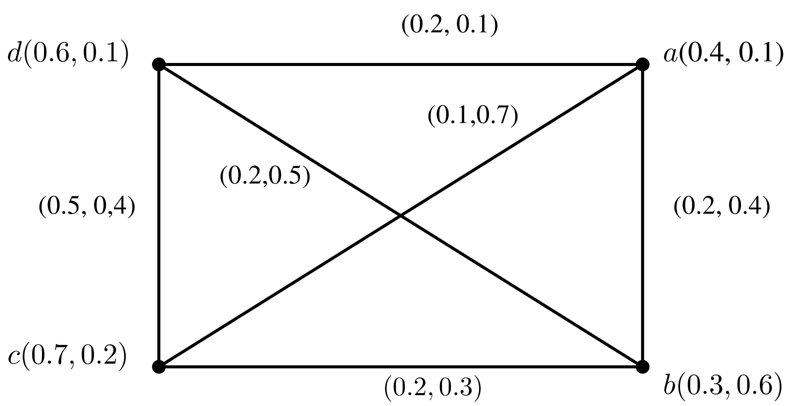

FIGURE 2. Intuitionistic fuzzy graph $G = (A,B)$

(i) Take $\alpha = 0.5$. We have $A_{0.5} = \{c,d\}$, $B_{0.5} = \{cd\}$. Clearly, the 0.5-level graph $G_{0.5}$ is a subgraph of G^*.

(ii) Take $\alpha = 0.2$, $\beta = 0.3$. Using Definition 3.13, we have $A_{(0.2,0.3)} = \{a,c,d\}$, $B_{(0.2,0.3)} = \{bc\}$. It is easy to see that $(0.2, 0.3)$-level graph $G_{(0.2,0.3)}$ is a subgraph of G^*.

We state the following Theorem without its proof.

Theorem 3.16. *Let $G = (A,B)$ be an intuitionistic fuzzy graph of G^*. Then $G_{(\alpha,\beta)} = (A_{(\alpha,\beta)}, B_{(\alpha,\beta)})$ is a subgraph of G^* for every $(\alpha,\beta) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\nu_A)$ with $\alpha + \beta \leq 1$.*

Corollary 3.17. *Let $G = (A,B)$ be an intuitionistic fuzzy graph of G^*. Then $G = (A,B)$ is intuitionistic fuzzy graph of G^* if and only if $U(\mu;\alpha)$ and $L(\nu;\beta)$ are subgraphs of G^* for every $\alpha \in [0, \mu(0)]$ and $\beta \in [\nu(0), 1]$ with $\alpha + \beta \leq 1$.*

The following Theorem is important in this paper. It is substantial solidification of the transfer principle for fuzzy sets described in [12].

Theorem 3.18. *$G = (A,B)$ is an intuitionistic fuzzy graph if and only if $G_{(\alpha,\beta)}$ is a crisp graph for each pair $(\alpha,\beta) \in [0,1] \times [0,1]$, $\alpha + \beta \leq 1$.*

Proof. Let $G = (A,B)$ be an intuitionistic fuzzy graph. For every $(\alpha,\beta) \in [0,1] \times [0,1]$. Take $xy \in B_{(\alpha,\beta)}$. Then $\mu_B(xy) \geq \alpha$ and $\nu_B(xy) \leq \beta$. Since G is an intuitionistic fuzzy graph, it follows that

$$\alpha \leq \mu_B(xy) \leq \min(\mu_A(x), \mu_A(y)),$$

$$\beta \geq \nu_B(xy) \geq \max(\nu_A(x), \nu_A(y)).$$

This shows that $\alpha \leq \mu_A(x)$, $\alpha \leq \mu_A(y)$, $\beta \geq \nu_A(x)$, $\beta \geq \nu_A(y)$, that is, $x, y \in A_{(\alpha,\beta)}$. Thus, $G_{(\alpha,\beta)}$ is a graph for each $(\alpha, \beta) \in [0, 1] \times [0, 1]$.

Conversely, let $G_{(\alpha,\beta)}$ be a graph for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$. For every $xy \in \widetilde{V^2}$, let $\nu_B(xy) = \beta$ and $\mu_B(xy) = \alpha$. Then $xy \in B_{(\alpha,\beta)}$. Since $G_{(\alpha,\beta)}$ is a graph, we have $x, y \in A_{(\alpha,\beta)}$; hence $\mu_A(x) \geq \alpha$, $\mu_A(y) \geq \alpha$, $\nu_A(x) \leq \beta$, $\nu_A(y) \leq \beta$. Thus,

$$\mu_B(xy) = \alpha \leq \min(\mu_A(x), \mu_A(y)),$$

$$\nu_B(xy) = \beta \geq \max(\nu_A(x), \nu_A(y)),$$

that is, $G = (A, B)$ is an intuitionistic fuzzy graph. □

Corollary 3.19. *$G = (A, B)$ is an intuitionistic fuzzy graph if and only if G_α is a crisp graph for each $\alpha \in [0, 1]$.*

Corollary 3.20. *$\widehat{G} = (V, B)$ is an intuitionistic fuzzy graph if and only if $\widetilde{G}_{(\alpha,\beta)}$ is a crisp graph for each $\alpha, \beta \in [0, 1]$.*

Definition 3.21 ([15])**.** Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The Cartesian product $G_1 \times G_2$ is the pair (A, B) of intuitionistic fuzzy sets defined on the Cartesian product $G_1^* \times G_2^*$ such that

(i) $\mu_A(x_1, x_2) = \min(\mu_{A_1}(x_1), \mu_{A_2}(x_2))$,
$\nu_A(x_1, x_2) = \max(\nu_{A_1}(x_1), \nu_{A_2}(x_2))$, for all $(x_1, x_2) \in V_1 \times V_2$,

(ii) $\mu_B((x, x_2)(x, y_2)) = \min(\mu_{A_1}(x), \mu_{B_2}(x_2y_2))$,
$\nu_B((x, x_2)(x, y_2)) = \max(\nu_{A_1}(x), \nu_{B_2}(x_2y_2))$, for all $x \in V_1$ and $x_2y_2 \in E_2$,

(iii) $\mu_B((x_1, z)(y_1, z)) = \min(\mu_{B_1}(x_1y_1), \mu_{A_2}(z))$,
$\nu_B((x_1, z)(y_1, z)) = \max(\nu_{B_1}(x_1y_1), \nu_{A_2}(z))$, for all $z \in V_2$ and $x_1y_1 \in E_1$.

Theorem 3.22. *$G = (A, B)$ is the Cartesian product of G_1 and G_2 if and only if for each pair $(\alpha, \beta) \in [0, 1] \times [0, 1]$, $\alpha + \beta \leq 1$ the (α, β)-level graph $G_{(\alpha,\beta)}$ is the Cartesian product of $(G_1)_{(\alpha,\beta)}$ and $(G_2)_{(\alpha,\beta)}$.*

Proof. Let $G = (A, B)$ be the Cartesian product of intuitionistic graphs G_1 and G_2. For every $(\alpha, \beta) \in [0, 1] \times [0, 1]$, if $(x, y) \in A_{(\alpha,\beta)}$, then

$$\min(\mu_{A_1}(x), \mu_{A_2}(y)) = \mu_A(x, y) \geq \beta$$

and

$$\max(\nu_{A_1}(x), \nu_{A_2}(y)) = \nu_A(x, y) \leq \alpha.$$

Thus $x \in (A_1)_{(\alpha,\beta)}$ and $y \in (A_2)_{(\alpha,\beta)}$, that is, $(x, y) \in (A_1)_{(\alpha,\beta)} \times (A_2)_{(\alpha,\beta)}$. So $A_{(\alpha,\beta)} \subseteq (A_1)_{(\alpha,\beta)} \times (A_2)_{(\alpha,\beta)}$.

Now if $(x, y) \in (A_1)_{(\alpha,\beta)} \times (A_2)_{(\alpha,\beta)}$, then $x \in (A_1)_{(\alpha,\beta)}$ and $y \in (A_2)_{(\alpha,\beta)}$. It follows that $\min(\mu_{A_1}(x), \mu_{A_2}(y)) \geq \beta$ and $\max(\nu_{A_1}(x), \nu_{A_2}(y)) \leq \alpha$. Since (A, B) is the Cartesian product of G_1 and G_2, $\mu_A(x, y) \geq \beta$ and $\nu_A(x, y) \leq \alpha$, that is, $(x, y) \in A_{(\alpha,\beta)}$. Thus $(A_1)_{(\alpha,\beta)} \times (A_2)_{(\alpha,\beta)} \subseteq A_{(\alpha,\beta)}$. So $(A_1)_{(\alpha,\beta)} \times (A_2)_{(\alpha,\beta)} = A_{(\alpha,\beta)}$.

We now prove $B_{(\alpha,\beta)} = E$, where E is the edge set of the Cartesian product $(G_1)_{(\alpha,\beta)} \times (G_2)_{(\alpha,\beta)}$, for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$. Let $(x_1, x_2)(y_1, y_2) \in B_{(\alpha,\beta)}$. Then, $\mu_B((x_1, x_2)(y_1, y_2)) \geq \beta$ and $\nu_B((x_1, x_2)(y_1, y_2)) \leq \alpha$. Since (A, B) is the Cartesian product of G_1 and G_2, one of the following cases hold:

(i) $x_1 = y_1$ and $x_2y_2 \in E_2$.
(ii) $x_2 = y_2$ and $x_1y_1 \in E_1$.

For the cases (i), we have

$$\mu_B((x_1, x_2)(y_1, y_2)) = \min(\mu_{A_1}(x_1), \mu_{B_2}(x_2y_2)) \geq \beta,$$

$$\nu_B((x_1, x_2)(y_1, y_2)) = \max(\nu_{A_1}(x_1), \nu_{B_2}(x_2y_2)) \leq \alpha.$$

Thus $\mu_{A_1}(x_1) \geq \beta,\ \nu_{A_1}(x_1) \leq a,\ \mu_{B_2}(x_2y_2) \geq \beta$ and $\nu_{B_2}(x_2y_2) \leq \alpha$. It follows that $x_1 = y_1 \in (A_1)_{(\alpha,\beta)},\ x_2y_2 \in (B_2)_{(\alpha,\beta)}$, that is, $(x_1, x_2)(y_1, y_2) \in E$.

Similarly, for the case (ii), we conclude that $(x_1, x_2)(y_1, y_2) \in E$. So, $B_{(\alpha,\beta)} \subseteq E$. For every $(x, x_2)(x, y_2) \in E,\ \mu_{A_1}(x) \geq \beta,\ \nu_{A_1}(x) \leq \alpha,\ \mu_{B_2}(x_2y_2) \geq b$ and $\nu_{B_2}(x_2y_2) \leq \alpha$. Since (A, B) is the Cartesian product of G_1 and G_2, we have

$$\mu_B((x, x_2)(x, y_2)) = \min(\mu_{A_1}(x), \mu_{B_2}(x_2y_2)) \geq \beta,$$

$$\nu_B((x, x_2)(x, y_2)) = \max(\nu_{A_1}(x), \nu_{B_2}(x_2y_2)) \leq \alpha.$$

Hence $(x, x_2)(x, y_2) \in B_{(\alpha,\beta)}$.

Similarly, for every $(x_1, z)(y_1, z) \in E$, we have $(x_1, z)(y_1, z) \in B_{(\alpha,\beta)}$. Therefore, $E \subseteq B_{(\alpha,\beta)}$, and so $B_{(\alpha,\beta)} = E$. Converse part is obvious.

□

Corollary 3.23. *$G = (A, B)$ is the Cartesian product of G_1 and G_2 if and only if for each $\alpha \in [0, 1]$, G_α is the Cartesian product of $(G_1)_\alpha$ and $(G_2)_\alpha$.*

Corollary 3.24. *$\widehat{G} = (V, B)$ is the Cartesian product of $\widehat{G}_1$ and $\widehat{G}_2$ if and only if for each pair $(\alpha, \beta) \in [0, 1] \times [0, 1]$, $\alpha + \beta \leq 1$ the (α, β)-level graph $\widehat{G}_{(\alpha,\beta)}$ is the Cartesian product of $(\widehat{G}_1)_{(\alpha,\beta)})$ and $(\widehat{G}_2)_{(\alpha,\beta)}$.*

Definition 3.25 ([15])**.** Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The composition $G_1[G_2]$ is the pair (A, B) of intuitionistic fuzzy sets defined on the composition $G_1^*[G_2^*]$ such that

(i) $\mu_A(x_1, x_2) = \min(\mu_{A_1}(x_1), \mu_{A_2}(x_2))$,
$\nu_A(x_1, x_2) = \max(\nu_{A_1}(x_1), \nu_{A_2}(x_2))$, for all $(x_1, x_2) \in V_1 \times V_2$,
(ii) $\mu_B((x, x_2)(x, y_2)) = \min(\mu_{A_1}(x), \mu_{B_2}(x_2y_2))$,
$\nu_B((x, x_2)(x, y_2)) = \max(\nu_{A_1}(x), \nu_{B_2}(x_2y_2))$, for all $x \in V_1$ and for all $x_2y_2 \in E_2$,
(iii) $\mu_B((x_1, z)(y_1, z)) = \min(\mu_{B_1}(x_1y_1), \mu_{A_2}(z))$,
$\nu_B((x_1, z)(y_1, z)) = \max(\nu_{B_1}(x_1y_1), \nu_{A_2}(z))$, for all $z \in V_2$ and for all $x_1y_1 \in E_1$,
(iv) $\mu_B((x_1, x_2)(y_1, y_2)) = \min(\mu_{A_2}(x_2), \mu_{A_2}(y_2), \mu_{B_1}(x_1y_1))$,
$\nu_B((x_1, x_2)(y_1, y_2)) = \max(\nu_{A_2}(x_2), \nu_{A_2}(y_2), \nu_{B_1}(x_1y_1))$, for all $x_2, y_2 \in V_2$, where $x_2 \neq y_2$ and for all $x_1y_1 \in E_1$.

Theorem 3.26. *$G = (A, B)$ is the composition of G_1 and G_2 if and only if for each $(\alpha, \beta) \in [0, 1] \times [0, 1]$, $\alpha + \beta \leq 1$, the (α, β)-level graph $G_{(\alpha,\beta)}$ is the composition of $(G_1)_{(\alpha,\beta)}$ and $(G_2)_{(\alpha,\beta)}$.*

Proof. Let $G = (A, B)$ be the composition of intuitionistic fuzzy graphs G_1 and G_2. By the definition of $G_1[G_2]$ and the same argument as in the proof of Theorem 3.22, we have $A_{(\alpha,\beta)} = (A_1)_{(\alpha,\beta)} \times (A_2)_{(\alpha,\beta)}$. Now we prove $B_{(\alpha,\beta)} = E$, where E is the edge set of the composition $(G_1)_{(\alpha,\beta)}[(G_2)_{(\alpha,\beta)}]$, for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$. Let $(x_1, x_2)(y_1, y_2) \in B_{(\alpha,\beta)}$. Then $\mu_B((x_1, x_2)(y_1, y_2)) \geq \beta$ and $\nu_B((x_1, x_2)(y_1, y_2)) \leq \alpha$. Since $G = (A, B)$ is the composition $G_1[G_2]$, one of the following cases hold:

(i) $x_1 = y_1$ and $x_2y_2 \in E_2$.
(ii) $x_2 = y_2$ and $x_1y_1 \in E_1$.
(iii) $x_2 \neq y_2$ and $x_1y_1 \in E_1$.

For the cases (i) and (ii), similarly, as in the cases (i) and (ii) in the proof of Theorem 3.22, we obtain $(x_1, x_2)(y_1, y_2) \in E$.

For the case (iii), we have

$$\mu_B((x_1, x_2)(y_1, y_2)) = \min(\mu_{A_2}(x_2), \mu_{A_2}(y_2), \mu_{B_1}(x_1y_1)) \geq \beta,$$

$$\nu_B((x_1, x_2)(y_1, y_2)) = \max(\nu_{A_2}(x_2), \nu_{A_2}(y_2), \nu_{B_1}(x_1y_1)) \leq \alpha.$$

Thus, $\mu_{A_2}(x_2) \geq \beta$, $\mu_{A_2}(y_2) \geq b$, $\mu_{B_1}(x_1y_1)) \geq \beta$, $\nu_{A_2}(x_2) \leq a$, $\nu_{A_2}(y_2) \leq \alpha$ and $\nu_{B_1}(x_1y_1)) \leq \alpha$. It follows that $x_2y_2 \in (A_2)_{(\alpha,\beta)}$ and $x_1y_1 \in (B_1)_{(\alpha,\beta)}$, that is, $(x_1, x_2)(y_1, y_2) \in E$. So, $B_{(\alpha,\beta)} \subseteq E$.

For every $(x, x_2)(x, y_2) \in E$, $\mu_{A_1}(x) \geq \beta$, $\nu_{A_1}(x) \leq a$, $\mu_{B_2}(x_2y_2) \geq \beta$ and $\nu_{B_2}(x_2y_2) \leq \alpha$. Since $G = (A, B)$ is the composition $G_1[G_2]$, we have

$$\mu_B((x, x_2)(x, y_2)) = \min(\mu_{A_1}(x), \mu_{B_2}(x_2y_2)) \geq \beta,$$

$$\nu_B((x, x_2)(x, y_2)) = \max(\nu_{A_1}(x), \nu_{B_2}(x_2y_2)) \leq \alpha.$$

Hence, $(x, x_2)(x, y_2) \in B_{(\alpha,\beta)}$.

Similarly, for every $(x_1, z)(y_1, z) \in E$, we have $(x, x_2)(x, y_2) \in B_{(\alpha,\beta)}$. For every $(x_1, x_2)(y_1, y_2) \in E$, where $x_2 \neq y_2$, $x_1 \neq y_1$, $\mu_{B_1}(x_1y_1) \geq \beta$, $\nu_{B_1}(x_1y_1) \leq a$, $\mu_{A_2}(y_2) \geq \beta$, $\nu_{A_2}(y_2) \leq a$, $\mu_{A_2}(x_2) \geq \beta$ and $\nu_{A_2}(x_2) \leq \alpha$. Since $G = (A, B)$ is the composition $G_1[G_2]$, we have

$$\mu_B((x_1, x_2)(y_1, y_2)) = \min(\mu_{A_2}(x_2), \mu_{A_2}(y_2), \mu_{B_1}(x_1y_1)) \geq \beta,$$

$$\nu_B((x_1, x_2)(y_1, y_2)) = \max(\nu_{A_2}(x_2), \nu_{A_2}(y_2), \nu_{B_1}(x_1y_1)) \leq \alpha,$$

Then $(x_1, x_2)(y_1, y_2) \in B_{(\alpha,\beta)}$. Thus $E \subseteq B_{(\alpha,\beta)}$. So $E = B_{(\alpha,\beta)}$.

Conversely, suppose that$(A_{(\alpha,\beta)}, B_{(\alpha,\beta)})$, where $(\alpha, \beta) \in [0, 1] \times [0, 1]$, is the composition of $(G_1)_{(\alpha,\beta)}$ and $((A_2)_{(\alpha,\beta)}, (B_2)_{(\alpha,\beta)})$.

Similarly, by the same arguments as in the proof of Theorem 3.22, we obtain

$$\mu_B((x_1, x_2)(y_1, y_2)) = \min(\mu_{A_2}(x_2), \mu_{A_2}(y_2), \mu_{B_1}(x_1y_1)),$$

$$\nu_B((x_1, x_2)(y_1, y_2)) = \max(\nu_{A_2}(x_2), \nu_{A_2}(y_2), \nu_{B_1}(x_1y_1)),$$

for all $x_2, y_2 \in V_2(x_2 \neq y_2)$ and for all $x_1y_1 \in E_1$. This completes the proof. □

Corollary 3.27. *$G = (A, B)$ is the composition of G_1 and G_2 if and only if for $\alpha \in [0, 1]$ the α-level graph G_α is the composition of $(G_1)_\alpha$ and $(G_2)_\alpha$.*

Corollary 3.28. *$\widehat{G} = (V, B)$ is the composition of $\widehat{G}_1$ and $\widehat{G}_2$ if and only if for each $\alpha, \beta \in [0, 1]$, $\alpha + \beta \leq 1$ the (α, β)-level graph $\widehat{G}_{(\alpha,\beta)}$ is the composition of $(\widehat{G}_1)_{(\alpha,\beta)})$ and $(\widehat{G}_2)_{(\alpha,\beta)}$.*

Definition 3.29 ([15]). Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be intuitionistic fuzzy graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The union $G_1 \cup G_2$ is defined as the pair (A, B) of intuitionistic fuzzy sets determined on the union of graphs G_1^* and G_2^* such that

(i) $\mu_A(x) = \begin{cases} \mu_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \notin V_2 \\ \mu_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \notin V_1 \\ \max(\mu_{A_1}(x), \mu_{A_2}(x)) & \text{if } x \in V_1 \cap V_2 \end{cases}$

(ii) $\nu_A(x) = \begin{cases} \nu_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \notin V_2 \\ \nu_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \notin V_1 \\ \min(\nu_{A_1}(x), \nu_{A_2}(x)) & \text{if } x \in V_1 \cap V_2 \end{cases}$

(iii) $\mu_B(xy) = \begin{cases} \mu_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2 \\ \mu_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1 \\ \max(\mu_{B_1}(xy), \mu_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2 \end{cases}$

(iv) $\nu_B(xy) = \begin{cases} \nu_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2 \\ \nu_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1 \\ \min(\nu_{B_1}(xy), \nu_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2. \end{cases}$

Theorem 3.30. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, and $V_1 \cap V_2 = \varnothing$. Then $G = (A, B)$ is the union of G_1 and G_2 if and only if each (α, β)-level graph $G_{(\alpha,\beta)}$ is the union of $(G_1)_{(\alpha,\beta)}$ and $(G_2)_{(\alpha,\beta)}$.*

Proof. Let $G = (A, B)$ be the union of intuitionistic fuzzy graphs G_1 and G_2. We show that $A_{(\alpha,\beta)} = (A_1)_{(\alpha,\beta)} \cup (A_2)_{(\alpha,\beta)}$, for each $(\alpha, \beta) \in [0, 1] \times [0, 1]$. Let $x \in A_{(\alpha,\beta)}$. Then $x \in V_1 \setminus V_2$ or $x \in V_2 \setminus V_1$. If $x \in V_1 \setminus V_2$, then $\mu_{A_1}(x) = \mu_A(x) \geq \beta$ and $\nu_{A_1}(x) = \nu_A(x) \leq \alpha$, which implies $x \in (A_1)_{(\alpha,\beta)}$. Analogously, $x \in V_2 \setminus V_1$ implies $x \in (A_2)_{(\alpha,\beta)}$. Thus, $x \in (A_1)_{(\alpha,\beta)} \cup (A_2)_{(\alpha,\beta)}$. So $A_{(\alpha,\beta)} \subseteq (A_1)_{(\alpha,\beta)} \cup (A_2)_{(\alpha,\beta)}$.

Now let $x \in (A_1)_{(\alpha,\beta)} \cup (A_2)_{(\alpha,\beta)}$. Then $x \in (A_1)_{(\alpha,\beta)}$, $x \notin (A_2)_{(\alpha,\beta)}$ or $x \in (A_2)_{(\alpha,\beta)}$, $x \notin (A_1)_{(\alpha,\beta)}$. For the first case, we have $\mu_{A_1}(x) = \mu_A(x) \geq \beta$ and $\nu_{A_1}(x) = \nu_A(x) \leq \alpha$, which implies $x \in A_{(\alpha,\beta)}$. For the second case, we have $\mu_{A_2}(x) = \mu_A(x) \geq \beta$ and $\nu_{A_2}(x) = \nu_A(x) \leq \alpha$. Hence $x \in A_{(\alpha,\beta)}$. Therefore, $(A_1)_{(\alpha,\beta)} \cup (A_2)_{(\alpha,\beta)} \subseteq A_{(\alpha,\beta)}$.

To prove that $B_{(\alpha,\beta)} = (B_1)_{(\alpha,\beta)} \cup (B_2)_{(\alpha,\beta)}$, for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$, consider $xy \in B_{(\alpha,\beta)}$. Then $xy \in E_1 \setminus E_2$ or $xy \in E_2 \setminus E_1$. For $xy \in E_1 \setminus E_2$, we have $\mu_{B_1}(xy) = \mu_B(xy) \geq \beta$ and $\nu_{B_1}(xy) = \nu_B(xy) \leq \alpha$. Thus $xy \in (B_1)_{(\alpha,\beta)}$.

Similarly, $xy \in E_2 \setminus E_1$ gives $xy \in (B_2)_{(\alpha,\beta)}$. So $B_{(\alpha,\beta)} \subseteq (B_1)_{(\alpha,\beta)} \cup (B_2)_{(\alpha,\beta)}$. If $xy \in (B_1)_{(\alpha,\beta)} \cup (B_2)_{(\alpha,\beta)}$, then $xy \in (B_1)_{(\alpha,\beta)} \setminus (B_2)_{(\alpha,\beta)}$ or $xy \in (B_2)_{(\alpha,\beta)} \setminus (B_1)_{(\alpha,\beta)}$. For the first case, $\mu_{B_1}(xy) = \mu_B(xy) \geq \beta$ and $\nu_{B_1}(xy) = \nu_B(xy) \leq \alpha$. Hence $xy \in B_{(\alpha,\beta)}$. In the second case, we obtain $xy \in B_{(\alpha,\beta)}$. Therefore, $(B_1)_{(\alpha,\beta)} \cup (B_2)_{(\alpha,\beta)} \subseteq B_{(\alpha,\beta)}$.

The converse part is obvious.

□

Corollary 3.31. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, and $V_1 \cap V_2 = \varnothing$. Then $G = (A, B)$ is the union of G_1 and G_2 if and only if each α-level graph G_α is the union of $(G_1)_\alpha$ and $(G_2)_\alpha$.*

Corollary 3.32. *Let $\widehat{G}_1 = (V_1, B_1)$ and $\widehat{G}_2 = (V_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, and $V_1 \cap V_2 = \varnothing$. Then $\widehat{G} = (V, B)$ is the union of $\widehat{G}_1$ and $\widehat{G}_2$ if and only if each (α, β)-level graph $\widehat{G}_{(\alpha,\beta)}$ is the union of $(\widetilde{G}_1)_{(\alpha,\beta)}$ and $(\widehat{G}_2)_{(\alpha,\beta)}$.*

Definition 3.33 ([15])**.** Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The join $G_1 + G_2$ is the pair (A, B) of intuitionistic fuzzy sets de fined on the join *join* $G_1^* + G_2^*$ such that

$$\text{(i)}\ \ \mu_A(x) = \begin{cases} \mu_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \notin V_2 \\ \mu_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \notin V_1 \\ \max(\mu_{A_1}(x), \mu_{A_2}(x)) & \text{if } x \in V_1 \cap V_2 \end{cases}$$

$$\text{(ii)}\ \ \nu_A(x) = \begin{cases} \nu_{A_1}(x) & \text{if } x \in V_1 \text{ and } x \notin V_2 \\ \nu_{A_2}(x) & \text{if } x \in V_2 \text{ and } x \notin V_1 \\ \min(\nu_{A_1}(x), \nu_{A_2}(x)) & \text{if } x \in V_1 \cap V_2 \end{cases}$$

$$\text{(iii)}\ \ \mu_B(xy) = \begin{cases} \mu_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2 \\ \mu_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1 \\ \max(\mu_{B_1}(xy), \mu_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2 \\ \min(\mu_{A_1}(x), \mu_{A_2}(y)) & \text{if } xy \in E' \end{cases}$$

$$\text{(iv)}\ \ \nu_B(xy) = \begin{cases} \nu_{B_1}(xy) & \text{if } xy \in E_1 \text{ and } xy \notin E_2 \\ \nu_{B_2}(xy) & \text{if } xy \in E_2 \text{ and } xy \notin E_1 \\ \min(\nu_{B_1}(xy), \nu_{B_2}(xy)) & \text{if } xy \in E_1 \cap E_2 \\ \max(\nu_{A_1}(x), \nu_{A_2}(y)) & \text{if } xy \in E'. \end{cases}$$

Theorem 3.34. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, and $V_1 \cap V_2 = \varnothing$. Then $G = (A, B)$ is the join of G_1 and G_2 if and only if each (α, β)-level graph $G_{(\alpha,\beta)}$ is the join of $(G_1)_{(\alpha,\beta)}$ and $(G_2)_{(\alpha,\beta)}$.*

Proof. Let $G = (A, B)$ be the join of intuitionistic fuzzy graphs G_1 and G_2. Then by the definition and the proof of Theorem 3.30, $A_{(\alpha,\beta)} = (A_1)_{(\alpha,\beta)} \cup (A_2)_{(\alpha,\beta)}$, for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$. We show that $B_{(\alpha,\beta)} = (B_1)_{(\alpha,\beta)} \cup (B_2)_{(\alpha,\beta)} \cup E'_{(\alpha,\beta)}$, for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$, where $E'_{(\alpha,\beta)}$ is the set of all edges joining the vertices of $(A_1)_{(\alpha,\beta)}$ and $(A_2)_{(\alpha,\beta)}$.

From the proof of theorem 3.30, it follows that $(B_1)_{(\alpha,\beta)} \cup (B_2)_{(\alpha,\beta)} \subseteq B_{(\alpha,\beta)}$. If $xy \in E'_{(\alpha,\beta)}$, then $\mu_{A_1}(x) \geq \beta,\ \nu_{A_1}(x) \leq a,\ \mu_{A_2}(y) \geq \beta$ and $\nu_{A_2}(y) \leq \alpha$. Thus

$$\mu_B(xy) = \min(\mu_{A_1}(x), \mu_{A_2}(y)) \geq \beta$$

and

$$\nu_B(xy) = \max(\nu_{A_1}(x), \nu_{A_2}(y)) \leq \alpha.$$

It follows that $xy \in B_{(\alpha,\beta)}$. So, $(B_1)_{(\alpha,\beta)} \cup (B_2)_{(\alpha,\beta)} \cup E'_{(\alpha,\beta)} \subseteq B_{(\alpha,\beta)}$. For every $xy \in B_{(\alpha,\beta)}$, if $xy \in E_1 \cup E_2$, then $xy \in (B_1)_{(\alpha,\beta)} \cup (B_2)_{(\alpha,\beta)}$, by the proof of Theorem 3.30. If $x \in V_1$ and $y \in V_2$, then

$$\min(\mu_{A_1}(x), \mu_{A_2}(y)) = \mu_B(xy) \geq \beta$$

and

$$\max(\nu_{A_1}(x), \nu_{A_2}(y)) = \nu_B(xy) \leq \alpha.$$

Hence $x \in (A_1)_{(\alpha,\beta)}$ and $y \in (A_2)_{(\alpha,\beta)}$. So $xy \in E'_{(\alpha,\beta)}$. Therefore, $B_{(\alpha,\beta)} \subseteq (B_1)_{(\alpha,\beta)} \cup (B_2)_{(\alpha,\beta)} \cup E'_{(\alpha,\beta)}$.

Conversely, let each level graph $G_{(\alpha,\beta)}$ be the join of $(G_1)_{(\alpha,\beta)}$ and $((A_2)_{(\alpha,\beta)}, (B_2)_{(\alpha,\beta)})$. From the proof of the Theorem 3.30, we have

(i) $\begin{cases} \mu_A(x) = \mu_{A_1}(x) & \text{if } x \in V_1 \\ \mu_A(x) = \mu_{A_2}(x) & \text{if } x \in V_2 \end{cases}$

(ii) $\begin{cases} \nu_A(x) = \nu_{A_1}(x) & \text{if } x \in V_1 \\ \nu_A(x) = \nu_{A_2}(x) & \text{if } x \in V_2. \end{cases}$

(iii) $\begin{cases} \mu_B(xy) = \mu_{B_1}(xy) & \text{if } xy \in E_1 \\ \mu_B(xy) = \mu_{B_2}(xy) & \text{if } xy \in E_2 \end{cases}$

(iv) $\begin{cases} \nu_B(xy) = \nu_{B_1}(xy) & \text{if } xy \in E_1 \\ \nu_B(xy) = \nu_{B_2}(xy) & \text{if } xy \in E_2. \end{cases}$

Let $x \in V_1, y \in V_2,\ \min(\mu_{A_1}(x), \mu_{A_2}(y)) = b,\ \max(\nu_{A_1}(x), \nu_{A_2}(y)) = a,\ \mu_B(xy) = d$ and $\nu_B(xy) = c$. Then $x \in (A_1)_{(\alpha,\beta)}, y \in (A_2)_{(\alpha,\beta)}$ and $xy \in B_{(c,d)}$. It follows that $xy \in B_{(\alpha,\beta)}, x \in (A_1)_{(c,d)}$ and $y \in (A_2)_{(c,d)}$. Thus, $\mu_B(xy) \geq b,\ \nu_B(xy) \leq \alpha,\ \mu_{A_1}(x) \geq d,\ \nu_{A_1}(x) \leq c,\ \mu_{A_2}(y) \geq d$ and $\nu_{A_2}(y) leqc$. So,

$$\mu_B(xy) \geq \beta = \min(\mu_{A_1}(x), \mu_{A_2}(y)) \geq d = \mu_B(xy)$$

$$\nu_B(xy) \leq \alpha = \max(\nu_{A_1}(x), \nu_{A_2}(y)) \leq c = \nu_B(xy).$$

Hence

$$\mu_B(xy) = \min(\mu_{A_1}(x), \mu_{A_2}(y)), \qquad \nu_B(xy) = \max(\nu_{A_1}(x), \nu_{A_2}(y))$$

as desired. □

Corollary 3.35. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, and $V_1 \cap V_2 = \varnothing$. Then $G = (A, B)$ is the join of G_1 and G_2 if and only if each α-level graph G_α is the join of $(G_1)_\alpha$ and $(G_2)_\alpha$.*

Corollary 3.36. *Let $\widehat{G}_1 = (V_1, B_1)$ and $\widehat{G}_2 = (V_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, and $V_1 \cap V_2 = \varnothing$. Then $\widehat{G} = (V, B)$ is the join of $\widehat{G}_1$ and $\widehat{G}_2$ if and only if each (α, β)-level graph $\widehat{G}_{(\alpha,\beta)}$ is the join of $(\widehat{G}_1)_{(\alpha,\beta)}$ and $(\widehat{G}_2)_{(\alpha,\beta)}$.*

Definition 3.37. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The Cross product $G_1 * G_2$ is the pair (A, B) of intuitionistic fuzzy sets defined on the cross product $G_1^* * G_2^*$ such that

(i) $\mu_A(x_1, x_2) = \min(\mu_{A_1}(x_1), \mu_{A_2}(x_2))$,
$\nu_A(x_1, x_2) = \max(\nu_{A_1}(x_1), \nu_{A_2}(x_2))$, for all $(x_1, x_2) \in V_1 \times V_2$,

(ii) $\mu_B((x_1, x_2)(y_1, y_2)) = \min(\mu_{B_1}(x_1y_1), \mu_{B_2}(x_2y_2))$,
$\nu_B((x_1, x_2)(y_1, y_2)) = \max(\nu_{B_1}(x_1y_1), \nu_{B_2}(x_2y_2))$, for all $x_1y_1 \in E_1$ and for all $x_2y_2 \in E_2$.

Theorem 3.38. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A, B)$ is the cross* *product of G_1 and G_2 if and only if each level graph $G_{(\alpha,\beta)}$ is the cross product of $(G_1)_{(\alpha,\beta)}$ and $(G_2)_{(\alpha,\beta)}$.*

Proof. Let $G = (A, B)$ be the cross product of G_1 and G_2. Then by the definition of the Cartesian product and the proof of Theorem 3.22, we have $A_{(\alpha,\beta)} = (A_1)_{(\alpha,\beta)} \times (A_2)_{(\alpha,\beta)}$, for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$. We show that

$$B_{(\alpha,\beta)} = \{(x_1, x_2)(y_1, y_2)|x_1y_1 \in (B_1)_{(\alpha,\beta)}, x_2y_2 \in (B_2)_{(\alpha,\beta)}\},$$

for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$. Indeed, if $(x_1, x_2)(y_1, y_2) \in B_{(\alpha,\beta)}$, then

$$\mu_B((x_1, x_2)(y_1, y_2)) = \min(\mu_{B_1}(x_1y_1), \mu_{B_2}(x_2y_2)) \geq \beta,$$

$$\nu_B((x_1, x_2)(y_1, y_2)) = \max(\nu_{B_1}(x_1y_1), \mu_{B_2}(x_2y_2)) \leq \alpha.$$

Thus $\mu_{B_1}(x_1y_1) \geq \beta$, $\mu_{B_2}(x_2y_2) \geq \beta$, $\nu_{B_1}(x_1y_1) \leq \alpha$ and $\mu_{B_2}(x_2y_2) \leq \alpha$. So, $x_1y_1 \in (B_1)_{(\alpha,\beta)}$ and $x_2y_2 \in (B_2)_{(\alpha,\beta)}$.

Now if $x_1y_1 \in (B_1)_{(\alpha,\beta)}$ and $x_2y_2 \in (B_2)_{(\alpha,\beta)}$, then $\mu_{B_1}(x_1y_1) \geq \beta$, $\nu_{B_1}(x_1y_1) \leq a$, $\mu_{B_2}(x_2y_2) \geq \beta$ and $\nu_{B_2}(x_2y_2) \leq \alpha$. It follows that

$$\mu_B((x_1, x_2)(y_1, y_2)) = \min(\mu_{B_1}(x_1y_1), \mu_{B_2}(x_2y_2)) \geq \beta,$$

$$\nu_B((x_1, x_2)(y_1, y_2)) = \max(\nu_{B_1}(x_1y_1), \nu_{B_2}(x_2y_2)) \leq \alpha,$$

because $G = (A, B)$ is the cross product of $G_1 * G_2$. Hence, $(x_1, x_2)(y_1, y_2) \in B_{(\alpha,\beta)}$.

The converse part is obvious.

□

Corollary 3.39. *Let $\widehat{G}_1 = (V_1, B_1)$ and $\widetilde{G}_2 = (V_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $\widehat{G} = (V, B)$ is the cross product of $\widehat{G}_1$ and $\widehat{G}_2$ if and only if each $\widehat{G}_{(\alpha,\beta)}$ is the cross product of $\widehat{G_1}_{(\alpha,\beta)}$ and $\widehat{G_2}_{(\alpha,\beta)}$.*

Definition 3.40. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The lexicographic product $G_1 \bullet G_2$ is the pair (A, B) of intuitionistic fuzzy sets defined on the lexicographic product $G_1^* \bullet G_2^*$ such that

(i) $\mu_A(x_1, x_2) = \min(\mu_{A_1}(x_1), \mu_{A_2}(x_2))$,
$\nu_A(x_1, x_2) = \max(\nu_{A_1}(x_1), \nu_{A_2}(x_2))$ for all $(x_1, x_2) \in V_1 \times V_2$,

(ii) $\mu_B((x, x_2)(x, y_2)) = \min(\mu_{A_1}(x), \mu_{B_2}(x_2y_2))$,
$\nu_B((x, x_2)(x, y_2)) = \max(\nu_{A_1}(x), \nu_{B_2}(x_2y_2))$ for all $x \in V_1$ and for all $x_2y_2 \in E_2$,

(iii) $\mu_B((x_1, x_2)(y_1, y_2)) = \min(\mu_{B_1}(x_1y_1), \mu_{B_2}(x_2y_2))$,
$\nu_B((x_1, x_2)(y_1, y_2)) = \max(\nu_{B_1}(x_1y_1), \nu_{B_2}(x_2y_2))$ for all $x_1y_1 \in E_1$ and for all $x_2y_2 \in E_2$.

Theorem 3.41. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A, B)$ is the lexicographic product of G_1 and G_2 if and only if $G_{(\alpha,\beta)} = (G_1)_{(\alpha,\beta)} \bullet (G_2)_{(\alpha,\beta)}$, for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$, $\alpha + \beta \leq 1$.*

Proof. Let $G = (A, B) = G_1 \bullet G_2$. By the definition of Cartesian product $G_1 \times G_2$ and the proof of Theorem 3.22, we have $A_{(\alpha,\beta)} = (A_1)_{(\alpha,\beta)} \times (A_2)_{(\alpha,\beta)}$, for all $(\alpha, \beta) \in [0,1] \times [0,1]$. We show that $B_{(\alpha,\beta)} = E_{(\alpha,\beta)} \cup E'_{(\alpha,\beta)}$ for all $(\alpha, \beta) \in [0,1] \times [0,1]$, where $E_{(\alpha,\beta)} = \{(x, x_2)(x, y_2) | x \in V_1,\ x_2y_2 \in (B_2)_{(\alpha,\beta)}\}$ is the subset of the edge set of the direct product $G_{(\alpha,\beta)} = (G_1)_{(\alpha,\beta)} \times (G_2)_{(\alpha,\beta)}$, and $E'_{(\alpha,\beta)} = \{(x_1, x_2)(y_1, y_2) | x_1y_1 \in (B_1)_{(\alpha,\beta)}, x_2y_2 \in (B_2)_{(\alpha,\beta)}\}$ is the edge set of the cross product $(G_1)_{(\alpha,\beta)} * (G_2)_{(\alpha,\beta)}$. For every $(x_1, x_2)(y_1, y_2) \in B_{(\alpha,\beta)}$. $x_1 = y_1,\ x_2y_2 \in E_2$ or $x_1y_1 \in E_1,\ x_2y_2 \in E_2$. If $x_1 = y_1,\ x_2y_2 \in E_2$, then $(x_1, x_2)(y_1, y_2) \in E_{(\alpha,\beta)}$, by the definition of the Cartesian product and the proof of Theorem 3.22. If $x_1y_1 \in E_1,\ x_2y_2 \in E_2$, then $(x_1, x_2)(y_1, y_2) \in E'_{(\alpha,\beta)}$, by the definition of cross product and the proof Theorem 3.38. Thus, $B_{(\alpha,\beta)} \subseteq E_{(\alpha,\beta)} \cup E'_{(\alpha,\beta)}$. From the definition of the Cartesian product and the proof of Theorem 3.22, we conclude that $E_{(\alpha,\beta)} \subseteq B_{(\alpha,\beta)}$, and also from the definition of cross product and the proof Theorem 3.38, we obtain $E'_{(\alpha,\beta)} \subseteq B_{(\alpha,\beta)}$. So, $E_{(\alpha,\beta)} \cup E'_{(\alpha,\beta)} \subseteq B_{(\alpha,\beta)}$.

Conversely, let $G_{(\alpha,\beta)} = (A_{(\alpha,\beta)}, B_{(\alpha,\beta)}) = (G_1)_{(\alpha,\beta)} \bullet (G_2)_{(\alpha,\beta)}$, for all $(\alpha, \beta) \in [0,1] \times [0,1]$. We know that $(G_1)_{(\alpha,\beta)} \bullet (G_2)_{(\alpha,\beta)}$ has the same vertex set as the Cartesian product $(G_1)_{(\alpha,\beta)} \times (G_2)_{(\alpha,\beta)}$. Now by the proof of Theorem 3.22, we have

$$\mu_A((x_1, x_2)) = \min(\mu_{A_1}(x_1), \mu_{A_2}(x_2)),$$

$$\nu_A((x_1, x_2)) = \max(\nu_{A_1}(x_1), \nu_{A_2}(x_2)),$$

for all $(x_1, x_2) \in V_1 \times V_2$.

For $x \in V_1$ and $x_2y_2 \in E_2$, let $\min(\mu_{A_1}(x), \mu_{B_2}(x_2y_2)) = \beta$, $\max(\nu_{A_1}(x), \nu_{B_2}(x_2y_2)) = \alpha$, $\mu_B((x, x_2)(x, y_2)) = \beta_1$ and $\nu_B((x, x_2)(x, y_2)) = \alpha_1$. Then, in view of the definition of the Cartesian product and lexicographic products, we have

$$(x, x_2)(x, y_2) \in (B_1)_{(\alpha,\beta)} \bullet (B_2)_{(\alpha,\beta)} \iff (x, x_2)(x, y_2) \in (B_1)_{(\alpha,\beta)} \times (B_2)_{(\alpha,\beta)},$$

$$(x, x_2)(x, y_2) \in (B_1)_{(\alpha,\beta)} \bullet (B_2)_{(\alpha,\beta)} \iff (x, x_2)(x, y_2) \in (B_1)_{(\alpha,\beta)} \times (B_2)_{(\alpha,\beta)}.$$

From this, by the same argument as in the proof of Theorem 3.22, we conclude

$$\mu_B((x, x_2)(x, y_2)) = \min(\mu_A(x), \mu_{B_2}(x_2y_2)),$$

$$\nu_B((x, x_2)(x, y_2)) = \max(\nu_A(x), \nu_{B_2}(x_2y_2)).$$

Now let $\mu_B((x_1, x_2)(y_1, y_2)) = \beta_1$ $\nu_B((x_1, x_2)(y_1, y_2)) = \alpha_1$, $\min(\mu_{B_1}(x_1y_1), \mu_{B_2}(x_2y_2)) = \beta$ and $\max(\nu_{B_1}(x_1y_1), \nu_{B_2}(x_2y_2)) = \alpha$, for $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$. Then in view of the definitions of cross product and the lexicographic product, we have

$$(x_1, x_2)(y_1, y_2) \in (B_1)_{(\alpha,\beta)} \bullet (B_2)_{(\alpha,\beta)} \iff (x_1, x_2)(y_1, y_2) \in (B_1)_{(\alpha,\beta)} * (B_2)_{(\alpha,\beta)},$$

$$(x_1, x_2)(y_1, y_2) \in (B_1)_{(\alpha,\beta)} \bullet (B_2)_{(\alpha,\beta)} \iff (x_1, x_2)(y_1, y_2) \in (B_1)_{(\alpha,\beta)} * (B_2)_{(\alpha,\beta)}.$$

By the same argument as in the proof of Theorem 3.38, we can conclude

$$\mu_B((x_1, x_2)(y_1, y_2)) = \min(\mu_{B_1}(x_1y_1), \mu_{B_2}(x_2y_2)),$$

$$\nu_B((x_1, x_2)(y_1, y_2)) = \max(\nu_{B_1}(x_1y_1), \nu_{B_2}(x_2y_2)),$$

which completes the proof. □

Corollary 3.42. *Let $\widehat{G}_1 = (V_1, B_1)$ and $\widehat{G}_2 = (V_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $\widehat{G} = (V, B)$ is the lexicographic product of $\widehat{G}_1$ and $\widehat{G}_2$ if and only if $\widehat{G}_{(\alpha,\beta)} = (\widehat{G}_1)_{(\alpha,\beta)} \bullet (\widehat{G}_2)_{(\alpha,\beta)}$, for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$, $\alpha + \beta \leq 1$.*

Lemma 3.43. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, such that $V_1 = V_2$, $A_1 = A_2$ and $E_1 \cap E_2 = \varnothing$. Then $G = (A, B)$ is the union of G_1 and G_2 if and only if $G_{(\alpha,\beta)}$ is the union of $(G_1)_{(\alpha,\beta)}$ and $(G_2)_{(\alpha,\beta)}$, for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$.*

Proof. Let $G = (A, B)$ be the union of intuitionistic fuzzy graphs G_1 and G_2. Then by the definition of the union and the fact that $V_1 = V_2$, $A_1 = A_2$, we have $A = A_1 = A_2$. Then $A_{(\alpha,\beta)} = (A_1)_{(\alpha,\beta)} \cup (A_2)_{(\alpha,\beta)}$. We now show that $B_{(\alpha,\beta)} = (B_1)_{(\alpha,\beta)} \cup (B_2)_{(\alpha,\beta)}$, for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$. For every $xy \in (B_1)_{(\alpha,\beta)}$, we have $\mu_B(xy) = \mu_{B_1}(xy) \geq \beta$ and $\nu_B(xy) = \nu_{B_1}(xy) \leq \alpha$. Thus $xy \in B_{(\alpha,\beta)}$. So, $(B_1)_{(\alpha,\beta)} \subseteq B_{(\alpha,\beta)}$.

Similarly, we obtain $(B_2)_{(\alpha,\beta)} \subseteq B_{(\alpha,\beta)}$. Then, $((B_1)_{(\alpha,\beta)} \cup (B_2)_{(\alpha,\beta)}) \subseteq B_{(\alpha,\beta)}$. For every $xy \in B_{(\alpha,\beta)}$, $xy \in E_1$ or $xy \in E_2$. If $xy \in E_1$, then $\nu_{B_1}(xy) = \nu_B(xy) \leq \alpha$. Thus $xy \in (B_1)_{(\alpha,\beta)}$. If $xy \in E_2$, then we have $xy \in (B_2)_{(\alpha,\beta)}$. Thus, $B_{(\alpha,\beta)} \subseteq (B_1)_{(\alpha,\beta)} \cup (B_2)_{(\alpha,\beta)}$.

Conversely, suppose that the (α, β)-level graph $G_{(\alpha,\beta)}$ be the union of $(G_1)_{(\alpha,\beta)}$ and $((A_2)_{(\alpha,\beta)}, (B_2)_{(\alpha,\beta)})$. Let $\mu_A(x) = \beta$, $\nu_A(x) = \alpha$, $\mu_{A_1}(x) = \beta_1$ and $\nu_{A_1}(x) = \alpha_1$, for some $x \in V_1 = V_2$. Then $x \in A_{(\alpha,\beta)}$ and $x \in (A_1)_{(\alpha,\beta)}$. Thus $x \in (A_1)_{(\alpha,\beta)}$ and $x \in A_{(\alpha,\beta)}$, because $A_{(\alpha,\beta)} = (A_1)_{(\alpha,\beta)}$ and $A_{(\alpha,\beta)} = (A_1)_{(\alpha,\beta)}$. It follows that $\mu_{A_1}(x) \geq b$, $\nu_{A_1}(x) \leq \alpha$, $\mu_A(x) \geq d$ and $\nu_A(x) \leq c$. So, $\mu_{A_1}(x) \geq \mu_A(x)$, $\nu_{A_1}(x) \leq \nu_A(x)$, $\mu_A(x) \geq \mu_{A_1}(x)$ and $\nu_A(x) \leq \nu_{A_1}(x)$. Hence, $\mu_A(x) = \mu_{A_1}(x)$ and $\nu_A(x) = \nu_{A_1}(x)$. Since $A_1 = A_2$, $V_1 = V_2$, $A = A_1 = A_1 \cup A_2$.

By a similar method, we conclude that

$$\text{(i)} \begin{cases} \mu_B(xy) = \mu_{B_1}(xy) & \text{if } xy \in E_1 \\ \mu_B(xy) = \mu_{B_2}(xy) & \text{if } xy \in E_2 \end{cases}$$

$$\text{(ii)} \begin{cases} \nu_B(xy) = \nu_{B_1}(xy) & \text{if } xy \in E_1 \\ \nu_B(xy) = \nu_{B_2}(xy) & \text{if } xy \in E_2. \end{cases}$$

□

Definition 3.44. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be intuitionistic fuzzy pair of graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The strong product $G_1 \boxtimes G_2$ is the pair (A, B) of intuitionistic fuzzy sets defined on the strong product $G_1^* \boxtimes G_2^*$ such that

(i) $\mu_A(x_1, x_2) = \min(\mu_{A_1}(x_1), \mu_{A_2}(x_2))$,
$\nu_A(x_1, x_2) = \max(\nu_{A_1}(x_1), \nu_{A_2}(x_2))$, for all $(x_1, x_2) \in V_1 \times V_2$,

(ii) $\mu_B((x, x_2)(x, y_2)) = \min(\mu_{A_1}(x), \mu_{B_2}(x_2y_2))$,
$\nu_B((x, x_2)(x, y_2)) = \max(\nu_{A_1}(x), \nu_{B_2}(x_2y_2))$, for all $x \in V_1$ and for all $x_2y_2 \in E_2$,

(iii) $\mu_B((x_1, z)(y_1, z)) = \min(\mu_{B_1}(x_1y_1), \mu_{A_2}(z))$,
$\nu_B((x_1, z)(y_1, z)) = \max(\nu_{B_1}(x_1y_1), \nu_{A_2}(z))$, for all $z \in V_2$ and for all $x_1y_1 \in E_1$,

(iv) $\mu_B((x_1,x_2)(y_1,y_2)) = \min(\mu_{B_1}(x_1y_1), \mu_{B_2}(x_2y_2))$,
$\nu_B((x_1,x_2)(y_1,y_2)) = \max(\nu_{B_1}(x_1y_1), \nu_{B_2}(x_2y_2))$, for all $x_1y_1 \in E_1$ and for all $x_2y_2 \in E_2$.

Theorem 3.45. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be intuitionistic fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then $G = (A, B)$ is the strong product of G_1 and G_2 if and only if $G_{(\alpha,\beta)}$, where $(\alpha,\beta) \in [0,1] \times [0,1]$, $\alpha + \beta \leq 1$ is the strong product of $(G_1)_{(\alpha,\beta)}$ and $(G_2)_{(\alpha,\beta)}$.*

Proof. According to the definitions of the strong product, the cross product and the Cartesian product, we obtain $G_1 \boxtimes G_2 = (G_1 \times G_2) \cup (G_1 * G_2)$ and

$$(G_1)_{(\alpha,\beta)} \boxtimes (G_2)_{(\alpha,\beta)} = ((G_1)_{(\alpha,\beta)} \times (G_2)_{(\alpha,\beta)}) \cup ((G_1)_{(\alpha,\beta)} * (G_2)_{(\alpha,\beta)}),$$

for all $(\alpha,\beta) \in [0,1] \times [0,1]$.

Now by Theorem 3.38, Theorem 3.22 and Lemma 3.43, we see that

$$\begin{aligned} & G = G_1 \boxtimes G_2 \\ \iff & G = (G_1 \times G_2) \cup (G_1 * G_2) \\ \iff & G_{(\alpha,\beta)} = (G_1 \times G_2)_{(\alpha,\beta)} \cup (G_1 * G_2)_{(\alpha,\beta)} \\ \iff & G_{(\alpha,\beta)} = ((G_1)_{(\alpha,\beta)} \times (G_2)_{(\alpha,\beta)}) \cup ((G_1)_{(\alpha,\beta)} * (G_2)_{(\alpha,\beta)}) \\ \iff & G_{(\alpha,\beta)} = (G_1)_{(\alpha,\beta)} \boxtimes (G_2)_{(\alpha,\beta)}, \end{aligned}$$

for all $(\alpha,\beta) \in [0,1] \times [0,1]$. □

4. Conclusions

A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. When there is vagueness in the description of the objects or in its relationships or in both, it is natural that we need to design a fuzzy graph model. An intuitionistic fuzzy graph model is a generalization of fuzzy graph model. Intuitionistic fuzzy models give more precision, flexibility and compatibility to the system as compared to the classic and fuzzy models. Thus in this connection we have presented properties of intuitionistic fuzzy graphs by level graphs. Currently, we are extending our research work to (1) Pythagorean fuzzy(Intuitionistic fuzzy of second type) incidence graphs; (2) Rough Pythagorean fuzzy soft graphs; (3) Rough Pythagorean fuzzy soft hypergraphs; (4) Rough Pythagorean fuzzy bipolar soft graphs; (5) Rough Pythagorean fuzzy graphs; (6) Rough Pythagorean fuzzy incidence graphs.

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References

[1] M. Akram and W. A. Dudek, Interval-valued fuzzy graphs, Computer Math. Appl. 61 (2011) 289–299.
[2] M. Akram and G. Shahzadi, Certain characterization of m-polar fuzzy graphs by level graphs, Punjab Univ. J. Math. 49 (2017) 1–12.
[3] M. Akram and W. A. Dudek, Intuitionistic fuzzy hypergraphs with applications, Inform. Sci. 218 (2013) 182–193.
[4] M. Akram and N. O. Alshehri, Intuitionistic fuzzy cycles and intuitionistic fuzzy trees, Scientific World Journal 2014 (2014), Article ID 305836, 11 pages.

[5] M. Akram and B. Davvaz, Strong intuitionistic fuzzy graphs, Filomat. 26 (2012) 177–196.
[6] K. T. Atanassov, Intuitionistic fuzzy sets, VII ITKR's Session, Deposed in Central for Science-Technical Library of Bulgarian Academy of Sciences, 1697/84, Sofia, Bulgaria, June 1983, (Bulgarian).
[7] K. T. Atanassov, Intuitionistic fuzzy sets, Theory and Applications, Studies in Fuzziness and Soft Computing, Physica, Heidelberg, Germany 2012.
[8] P. Bhattacharya, Some remarks on fuzzy graphs, Pattern Recognition Letter 6 (1987) 297–302.
[9] Ch. Chaitany and T. V. P. Kumar, A note on uniform vertex and uniform edge fuzzy graphs, Int. J. Innovative Research in Sci. Eng. and Tech. 4 (7) (2015) 57–95.
[10] W. A. Dudek and A. A. Talebi, Operations on level graphs of bipolar fuzzy graphs, Buletinul Acad. Stiinte Republ. Moldova. Matematica. 81 (2016) 107–124.
[11] A. Kauffman, Introduction a la Theorie des Sous-emsembles Flous, Masson et Cie. l (1973).
[12] M. Kondo and W. A. Dudek, On the transfer principle in fuzzy theory, Mathware and Soft Computing 12 (2005) 41–55.
[13] J. N. Mordeson and P. Chang-Shyh, Operations on fuzzy graphs, Inform. Sci. 79 (1994) 159–170.
[14] J. N. Mordeson and P. S. Nair, Fuzzy graphs and fuzzy hypergraphs, Physica Verlag, Heidelberg 2001.
[15] R. Parvathi, M. G. Karunambigai and K. T. Atanassov, Operations on intuitionistic fuzzy graphs, Fuzzy Systems, 2009. FUZZ-IEEE 2009. IEEE International Conference, 1396–1401.
[16] A. Rosenfeld, Fuzzy graphs, Fuzzy Sets and their Applications(L. A. Zadeh, K. S. Fu, M. Shimura, Eds.), Academic Press, New York (1975) 77–95.
[17] M. Sarwar and M. Akram, An algorithm for computing certain metrics in intuitionistic fuzzy graphs, Journal of Intelligent and Fuzzy Systems (30) (2016) 2405–2416.
[18] S. Shahzadi and M. Akram, Graphs in an intuitionistic fuzzy soft environment, Axioms 2018 7 (2) 20, https://doi.org/10.3390/axioms7020020.
[19] A. Shannon and K. T. Atanassov, Intuitionistic fuzzy graphs from $\alpha-$, $\beta-$, (α, β)- levels, Notes on Intuitionistic Fuzzy Sets 1 (1995) 32–35.
[20] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[21] L. A. Zadeh, Similarity relations and fuzzy orderings, Inform. Sci. 3 (1971) 177–200.
[22] J. Zhan, H. Masood Malik and M. Akram, Novel decision-making algorithms based on intuitionistic fuzzy rough environment, Int. J. Mach. Learn. & Cyber. (2018). https://doi.org/10.1007/s13042-018-0827-4.

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