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Abstract. In this paper the concept of an intuitionistic L-fuzzy prime submodule of M is given, and some fundamental lemmas are proved. Also a characterization of an intuitionistic L-fuzzy prime submodule is given. Finally, we show that an intuitionistic L-fuzzy prime submodule is inherited by an R-module epimorphism.

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Corresponding Author: P. K. Sharma (pksharma@davjalandhar.com)

1. Introduction

Atanassov in [2, 4, 5] introduced the notion of an intuitionistic fuzzy subset A of a non-empty set X as a function from X to $[0,1] \times [0,1]$ as a generalization of fuzzy set given by Zadeh [21] which is a function from X to $[0,1]$. Atanassov and Stoeva in [3] generalized the notion of intuitionistic fuzzy subset of X to that of an intuitionistic L-fuzzy subset, namely a function from X to lattice $L \times L$ which is also a generalization of L-fuzzy set given by Goguen [9]. The development of Algebra in fuzzy setting are very much evident in the book of Kandasamy [13], Mordeson and Malik [15]. Acar in [1] gave the L-fuzzication of the notion of prime submodules.

In [8], Biswas considered the intuitionistic fuzzification of algebraic structures and introduced the notion of intuitionistic fuzzy subgroup of a group. Hur et al. [10], introduced and examined the notion of an intuitionistic fuzzy ideal of a ring. Rahman, Saikia in [17], Isaac, John in [11], and Sharma in [18] studied some aspects of intuitionistic fuzzy submodules. Since then several authors have obtained interesting results on intuitionistic L-fuzzy subgroup of the group G, intuitionistic L-fuzzy subring and ideal of the ring R and BP-Algebras, for example: see [16], [7] and [12].

In [6] the notion of intuitionistic fuzzy prime ideal of a ring over $[0,1]$ is given in terms of intuitionistic fuzzy singletons and the intuitionistic fuzzy prime spectrum of a ring is studied by Sharma and Kaur in [19]. The annihilator of intuitionistic fuzzy prime modules is discussed in [20]. In Section 3 of this paper, we generalize their definition to any complete lattice L when R is a commutative ring with identity. In Theorem (3.6) we give a characterization of intuitionistic L-fuzzy prime submodules which is one of the original results obtained in this paper. In Section 4, we investigate the behaviour of intuitionistic L-fuzzy prime submodules under R-module homomorphisms, which constitutes another original result of our work.

2. Preliminaries

Throughout the paper R is a commutative ring with identity, M a unitary R-module with zero element θ. Let $(L, \leq)$ be a lattice such that $(L, \vee, \wedge, ^{'}, 0, 1)$ be a complete lattice with least element 0 and greatest element 1, where $a \vee b = lub\{a, b\}$ and $a \wedge b = glb\{a, b\}$ for all $a, b \in L$ and "$'$" is the order-reversing involution on L.

Let $a, b \in L$. Then b is called a complement of a if $a \vee b = 1$ and $a \wedge b = 0$. We write $b = a^{'}$. Thus $1^{'} = 0$, $0^{'} = 1$ and $(a^{'})^{'} = a, \forall a \in L$. If $a \leq b$ then $b^{'} \leq a^{'}, \forall a, b \in L$.

An element $\alpha \in L, 1 \neq \alpha$, is called a prime element in L if for all $a; b \in L$ if $a \wedge b \leq \alpha$ implies $a \leq \alpha$ or $b \leq \alpha$.

If μ, ν are L-fuzzy submodules of an R-module M such that $\nu \subseteq \mu$. Then ν is called an L-fuzzy prime submodule of μ if r_t, x_s be any two L-fuzzy point of R and M respectively $(r \in R, x \in M, t, s \in L)$, $r_t x_s \in \nu$ implies that either $x_s \in \nu$ or $x_t \mu \subseteq \nu$. In particular if $\mu = \chi_M$, then ν is called an L-fuzzy prime submodule of M ([1]).

Given a nonempty set X, an intuitionistic L-fuzzy subset A of X is a function $A = (\mu_A, \nu_A) : X \to L \times L$ with the condition that $\nu_A(x) \leq (\mu_A(x))^{'}, \forall x \in X$, where "$'$" is the order-reversing involution on L. When $\nu_A(x) = (\mu_A(x))^{'}, \forall x \in X$, then A is called an L-fuzzy subset of X. We denote by $ILFS(X)$ the set of all intuitionistic L-fuzzy subsets of X. For $A, B \in ILFS(X)$ we say $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$. Also, $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$.

Let $A \in ILFS(X)$ and $p, q \in L$. Then the set $A_{(p,q)} = \{x \in X : \mu_A(x) \geq p \text{ and } \nu_A(x) \leq q\}$ is called the (p, q)-cut subset of X with respect to A. By an intuitionistic L-fuzzy point $(ILFP)$ $x_{(p,q)}$ of $X, x \in X$ and $p, q \in L \backslash \{0\}$ such that $p \vee q \leq 1$, we mean $x_{(p,q)} \in ILFS(X)$ defined by

$$x_{(p,q)}(y) = \begin{cases} (p, q), & \text{if } y = x \\ (0, 1), & \text{if otherwise.} \end{cases}$$

If $x_{(p,q)}$ is an intuitionistic L-fuzzy point of X and $x_{(p,q)} \subseteq A \in ILFS(X)$, we write $x_{(p,q)} \in A$. Let $A = (\mu_A, \nu_A)$ be an ILFS of X and $Y \subseteq X$. Then the restriction of

A to the set Y is an ILFS $A_Y = (\mu_{A_Y}, \nu_{A_Y})$ of Y and is defined as:

$$\mu_{A_Y}(y) = \begin{cases} \mu_A(y), & \text{if } y \in Y \\ 0, & \text{if otherwise} \end{cases}; \quad \nu_{A_Y}(y) = \begin{cases} \nu_A(y), & \text{if } y \in Y \\ 1, & \text{otherwise.} \end{cases}$$

The following are two very basic definitions given in [14] and [19].

Definition 2.1. Let $A \in ILFS(R)$. Then A is called an intuitionistic L-fuzzy ideal ($ILFI$) of R, if for all $x, y \in R$, the followings are satisfied:
(i) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$,
(ii) $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$,
(iii) $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$,
(iv)$\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$.

Definition 2.2. Let $A \in ILFS(M)$. Then A is called an intuitionistic L-fuzzy module ($ILFM$) of M, if for all $x, y \in M, r \in R$, the followings are satisfied:
(i) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$,
(ii) $\mu_A(rx) \geq \mu_A(x)$,
(iii) $\mu_A(\theta) = 1$,
(iv) $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$,
(v)$\nu_A(rx) \leq \nu_A(x)$,
(vi)$\nu_A(\theta) = 0$.

Let $IF_L(M)$ denote the set of all intuitionistic L-fuzzy R-modules of M and $IF_L(R)$ denote the set of all intuitionistic L-fuzzy ideals of R. We note that when $R = M$, then $A \in IF_L(M)$ if and only if $\mu_A(\theta) = 1, \nu_A(\theta) = 0$ and $A \in IF_L(R)$.

Definition 2.3. Let $C \in ILFS(R)$ and $B \in ILFS(M)$. Define the composition $C \circ B$ and product CB respectively as follows: for all $w \in M$,

$$\mu_{C \circ B}(w) = \begin{cases} Sup[\mu_C(r) \wedge \mu_B(x)] & \text{if } w = rx, r \in R, x \in M \\ 0, & \text{if } w \text{ is not expressible as } w = rx \end{cases}$$

$$\nu_{C \circ B}(w) = \begin{cases} Inf[\nu_C(r) \vee \nu_B(x)] & \text{if } w = rx, r \in R, x \in M \\ 1, & \text{if } w \text{ is not expressible as } w = rx \end{cases}$$

and

$$\mu_{CB}(w) = \begin{cases} Sup[Inf_{i=1}^n\{\mu_C(r_i) \wedge \mu_B(x_i)\}] & \text{if } w = \Sigma_{i=1}^n r_i x_i, r_i \in R, x_i \in M, n \in N \\ 0, & \text{if } w \text{ is not expressible as } w = \Sigma_{i=1}^n r_i x_i \end{cases}$$

$$\nu_{CB}(w) = \begin{cases} Inf[Sup_{i=1}^n\{\nu_C(r_i) \vee \nu_B(x_i)\}] & \text{if } w = \Sigma_{i=1}^n r_i x_i, r_i \in R, x_i \in M, n \in N \\ 1, & \text{if } w \text{ is not expressible as } w = \Sigma_{i=1}^n r_i x_i, \end{cases}$$

where as usual supremum and infimum of an empty set are taken to be 0 and 1 respectively. Clearly, $C \circ B \subseteq CB$.

The following lemma can be found in [6, 14]. It gives the basic operations between intuitionistic L-fuzzy ideals and intuitionistic L-fuzzy modules where L is a complete lattice satisfying the infinite distributive law.

Lemma 2.4. *Let $C \in IF_L(R), A, B \in IF_L(M)$ and let L be a complete lattice satisfying the infinite distributive law.*

(1) *$CB \subseteq A$ if and only if $C \circ B \subseteq A$.*

(2) *$r_{(s,t)} \in ILFS(R), x_{(p,q)} \in ILFS(M)$ be $ILFPs$. Then $r_{(s,t)} \circ x_{(p,q)} = (rx)_{(s \wedge p, t \vee q)}$.*

(3) *If $\mu_C(0) = 1, \nu_C(0) = 0$ then $CA \in IF_L(M)$.*

(4) *Let $r_{(s,t)} \in ILFS(R)$ be an $ILFP$. Then for all $w \in M$,*

$$\mu_{r_{(s,t)} \circ B}(w) = \begin{cases} Sup[s \wedge \mu_B(x)] & \text{if } w = rx, r \in R, x \in M \\ 0, & \text{if } w \text{ is not expressible as } w = rx \end{cases}$$

and

$$\nu_{r_{(s,t)} \circ B}(w) = \begin{cases} Inf[t \vee \nu_B(x)] & \text{if } w = rx, r \in R, x \in M \\ 1, & \text{if } w \text{ is not expressible as } w = rx. \end{cases}$$

The following theorem gives a relation between an intuitionistic L-fuzzy modules on M and submodules of M. It is a very practical method to construct an intuitionistic L-fuzzy module on M.

Theorem 2.5. *Let $A \in ILFS(M)$. Then A is an intuitionistic L-fuzzy module if and only if for all $\alpha, \beta \in L$ with $\alpha \vee \beta \leq 1$ such that $A_{(\alpha,\beta)}$ is an R-submodules of M.*

Proof. Simple proof □

Definition 2.6 ([6])**.** For a non-constant $C \in IF_L(R), C$ is called an intuitionistic L-fuzzy prime ideal of R, if for any intuitionistic L-fuzzy points $x_{(p,q)}, y_{(r,s)} \in ILFS(R)$, $x_{(p,q)} y_{(r,s)} \in C$ implies that either $x_{(p,q)} \in C$ or $y_{(r,s)} \in C$.

3. Intuitionistic L-fuzzy prime submodules

In this section, we will give a characterization of an intuitionistic L-fuzzy prime submodule of M.

Definition 3.1. For $A, B \in IF_L(M), B$ is called an intuitionistic L-fuzzy submodule of A, if $B \subseteq A$.

In particular, if $A = \chi_M$, then we say B is an intuitionistic L-fuzzy submodule of M.

Definition 3.2. Let B be an intuitionistic L-fuzzy submodule of A, B is called an intuitionistic L-fuzzy prime submodule of A, if $r_{(s,t)} \in ILFS(R), x_{(p,q)} \in ILFS(M)$ $(r \in R, x \in M, s, t, p, q \in L)$, $r_{(s,t)} x_{(p,q)} \in B$ implies that either $x_{(p,q)} \in B$ or $r_{(s,t)} A \subseteq B$.

In particular, taking $A = \chi_M$, if for $r_{(s,t)} \in ILFS(R), x_{(p,q)} \in ILFS(M)$ we have $r_{(s,t)} x_{(p,q)} \in B$ implies that either $x_{(p,q)} \in B$ or $r_{(s,t)} \chi_M \subseteq B$, then B is called an intuitionistic L-fuzzy prime submodule of M.

The following theorem says that intuitionistic L-fuzzy prime submodule and intuitionistic L-fuzzy prime ideals coincide when R is considered to be a module over itself.

Theorem 3.3. *If $M = R$, then $B \in ILFS(M)$, is an intuitionistic L-fuzzy prime submodule of M if and only if $B \in IF_L(R)$ is an intuitionistic L-fuzzy prime ideal.*

Proof. Let B be an intuitionistic L-fuzzy prime submodule of M. Since $B \in IF_L(M)$ and R is a commutative ring, $B \in IF_L(R)$.

For $a_{(p,q)}, b_{(s,t)} \in ILFS(R), a_{(p,q)}b_{(s,t)} \in B$ implies $a_{(p,q)} \in B$ or $b_{(s,t)}\chi_M \subseteq B$.

If $a_{(p,q)} \in B$, then B is an intuitionistic L-fuzzy prime ideal.

If $b_{(s,t)}\chi_M \subseteq B$, then for each $m \in M$,

$$\mu_{b_{(s,t)}}\chi_M(bm) \leq \mu_B(bm)$$

and

$$\nu_{b_{(s,t)}}\chi_M(bm) \geq \nu_B(bm).$$

Since R has identity, $b = b1$ and $\mu_{b_{(s,t)}}\chi_M(b1) = s \leq \mu_B(b)$ and $\nu_{b_{(s,t)}}\chi_M(b1) = t \geq \nu_B(b)$. Thus $s = \mu_{b_{(s,t)}}(b) \leq \mu_B(b)$ and $t = \nu_{b_{(s,t)}}(b) \geq \nu_B(b)$. So $b_{(s,t)} \in B$.

Conversely, let B be an intuitionistic L-fuzzy prime ideal of R. Then $B \subset \chi_R$ and $B \in IF_L(M)$. Now, let $r_{(s,t)}x_{(p,q)} \in B$, for any $r_{(s,t)} \in ILFS(R), x_{(p,q)} \in ILFS(M)$.

If $x_{(p,q)} \in B$, then B is an intuitionistic L-fuzzy prime submodule of M.

If $x_{(p,q)} \notin B$, then $r_{(s,t)} \in B$. Thus by the definition of intuitionistic L-fuzzy ideal of R,

$$\mu_{r_{(s,t)}\chi_M}(rm) = s \leq \mu_B(r) \leq \mu_B(rm)$$

and

$$\nu_{r_{(s,t)}\chi_M}(rm) = t \geq \nu_B(r) \geq \nu_B(rm).$$

So $r_{(s,t)}\chi_M \subseteq B$. □

The following theorem, which relates intuitionistic fuzzy submodule to prime submodules of the module, will be needed in the proof of Theorem 3.6.

Theorem 3.4. *Let B be an intuitionistic L-fuzzy prime submodule of A. If $B_{(\alpha,\beta)} \neq A_{(\alpha,\beta)}, \alpha, \beta \in L$, then $B_{(\alpha,\beta)}$ is a prime submodule of $A_{(\alpha,\beta)}$.*

Proof. Let $B_{(\alpha,\beta)} \neq A_{(\alpha,\beta)}$ and $rx \in B_{(\alpha,\beta)}$, for some $r \in R, x \in M$. If $rx \in B_{(\alpha,\beta)}$, then $\mu_B(rx) \geq \alpha$ and $\nu_B(rx) \leq \beta$. Thus $(rx)_{(\alpha,\beta)} = r_{(\alpha,\beta)}x_{(\alpha,\beta)} \in B$. Since B is an intuitionistic L-fuzzy prime submodule of A, either $x_{(\alpha,\beta)} \in B$ or $r_{(\alpha,\beta)}A \subseteq B$.

Case(i): If $x_{(\alpha,\beta)} \in B$, then $\mu_B(x) \geq \alpha$ and $\nu_B(x) \leq \beta$. Thus $x \in B_{(\alpha,\beta)}$.

Case(ii): If $r_{(\alpha,\beta)}A \subseteq B$, then for any $w \in rA_{(\alpha,\beta)}, w = rz$, for some $z \in A_{(\alpha,\beta)}$. Thus $\mu_A(z) \geq \alpha$ and $\nu_A(z) \leq \beta$. On the other hand,

$$\alpha = \alpha \wedge \mu_A(z) \leq Sup\{\alpha \wedge \mu_A(x) : w = rx\} = \mu_{r_{(\alpha,\beta)}A}(w) \leq \mu_B(w).$$

Similarly, we have

$$\beta = \beta \vee \nu_A(z) \geq Inf\{\beta \vee \nu_A(x) : w = rx\} = \nu_{r_{(\alpha,\beta)}A}(w) \geq \nu_B(w).$$

So $w \in B_{(\alpha,\beta)}$. Hence $rA_{(\alpha,\beta)} \subseteq B_{(\alpha,\beta)}$. Therefore $B_{(\alpha,\beta)}$ is a prime submodule of $A_{(\alpha,\beta)}$. □

Corollary 3.5. *Let B be an intuitionistic L-fuzzy prime submodule of M. Then*

$$B_* = \{x \in M : \mu_B(x) = \mu_B(\theta) \text{ and } \nu_B(x) = \nu_B(\theta)\}$$

is a prime submodule of M.

Proof. Clear from Theorem 3.4 as $B_{(\alpha,\beta)} = B_*$, when $\alpha = \mu_B(\theta)$ and $\beta = \nu_B(\theta)$. □

The following theorem is the main result of section 3. It generalize the work of [10] from $[0, 1]$ to a complete lattice L.

Theorem 3.6. (1) *Let N be a prime submodule of M and α a prime element in L. If A is an ILFS of M defined by*

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha, & \text{if otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } y \in N \\ \alpha^{'}, & \text{otherwise,} \end{cases}$$

for all $x \in M$, where $\alpha^{'}$ is complement of α in L. Then A is an intuitionistic L-fuzzy prime submodule of M.

(2) *Conversely, any intuitionistic L-fuzzy prime submodule can be obtained as in (1).*

Proof. (1) Since N is a prime submodule of M, $N \neq M$, we have that A is non-constant intuitionistic L-fuzzy submodule of M. We show that A is an intuitionistic L-fuzzy prime submodule of M.

Suppose $r_{(s,t)} \in ILFS(R), x_{(p,q)} \in ILFS(M)$ are such that $r_{(s,t)}x_{(p,q)} \in A$ and $x_{(p,q)} \notin A$.

If $x_{(p,q)} \notin A$, then $\mu_A(x) = \alpha$ and $\nu_A(x) = \alpha^{'}$. Thus $x \notin N$.

If $r_{(s,t)}x_{(p,q)} \in A$, then $\mu_{(rx)_{(s\wedge p, t\vee q)}}(rx) \leq \mu_A(rx)$ and $\nu_{(rx)_{(s\wedge p, t\vee q)}}(rx) \geq \nu_A(rx)$. Thus $s \wedge p \leq \mu_A(rx)$ and $t \vee q \geq \nu_A(rx)$.

If $\mu_A(rx) = 1$ and $\nu_A(rx) = 0$, then $rx \in N$. Since $x \notin N$ and N is a prime submodule of M, we have $rM \subseteq N$. Thus $\mu_A(rm) = 1$ and $\nu_A(rm) = 0$, for all $m \in M$. So $\mu_{r_{(s,t)}\chi_M}(rm) = s \leq \mu_A(rm)$ and $\nu_{r_{(s,t)}\chi_M}(rm) = t \geq \nu_A(rm)$.

If $\mu_A(rx) = \alpha$ and $\nu_A(x) = \alpha^{'}$, then $s \wedge p \leq \alpha$ and $t \vee q \geq \alpha^{'}$. As α is prime element of L, we have $s \wedge p \leq \alpha$ and $p \nleq \alpha$ implies $s \leq \alpha$ and $t \vee q \geq \alpha^{'}$ implies $t^{'} \vee q^{'} \geq \alpha$ and $q^{'} \ngeq \alpha$ implies $t^{'} \leq \alpha$, i.e., $t \geq \alpha^{'}$. Thus for all $w \in M$,

$\mu_{r_{(s,t)}\chi_M}(w) = s \leq \alpha \leq \mu_A(w)$ and $\nu_{r_{(s,t)}\chi_M}(w) = t \geq \alpha^{'} \geq \nu_A(w)$.

So $r_{(s,t)}\chi_M \subseteq A$. Hence A is an intuitionistic L-fuzzy prime submodule of M.

(2) Let A be an intuitionistic L-fuzzy prime submodule of M. We show that A is of the form

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha, & \text{if otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } y \in N \\ \alpha^{'}, & \text{otherwise,} \end{cases}$$

for all $x \in M$, where $\alpha^{'}$ is complement of the prime element α in L.

Claim (1): $A_* = \{x \in M : \mu_A(x) = \mu_A(\theta) \text{ and } \nu_A(x) = \nu_A(\theta)\}$ is a prime submodule of M.

Since A is a non-constant intuitionistic L-fuzzy prime submodule of M, $A_* \neq M$. For all $r \in R, m \in M$, if $rm \in A_*$ implies $\mu_A(rm) = \mu_A(\theta)$ and $\nu_A(rm) = \nu_A(\theta)$ so that $(rm)_{(\mu_A(\theta),\nu_A(\theta))} = r_{(\mu_A(\theta),\nu_A(\theta))}m_{(\mu_A(\theta),\nu_A(\theta))} \in A$, then $m_{(\mu_A(\theta),\nu_A(\theta))} \in A$ or $r_{(\mu_A(\theta),\nu_A(\theta))}\chi_M \subseteq A$.

Case(i): If $m_{(\mu_A(\theta),\nu_A(\theta))} \in A$, then $\mu_A(\theta) \leq \mu_A(m)$ and $\nu_A(\theta) \geq \nu_A(m)$ but $\mu_A(\theta) \geq \mu_A(m)$ and $\nu_A(\theta) \leq \nu_A(m)$ [by definition of $ILFSM$]. Thus $\mu_A(m) = \mu_A(\theta)$ and $\nu_A(m) = \nu_A(\theta)$. So $m \in A_*$.

Case(ii): If $r_{(\mu_A(\theta),\nu_A(\theta))}\chi_M \subseteq A$, then $\mu_A(\theta) \leq \mu_A(rm)$ and $\nu_A(\theta) \geq \nu_A(rm)$. Thus $rm \in A_*$, for all $m \in M$. On the other hand,

$$\theta \in N \text{ and } \mu_A(\theta) = 1, \nu_A(\theta) = 0.$$

So for all $x \in A_*, \mu_A(\theta) = \mu_A(x) = 1$ and $\nu_A(\theta) = \nu_A(x) = 0$. Hence $A_* = N$.

Claim (2): A has two values.

Since A_* is a prime submodule of M, $A_* \neq M$. Then there exists $z \in M\backslash A_*$.

We will show that for all $y \in M$ such that $y \in A_*$,

$$\mu_A(y) = \mu_A(z) < \mu_A(\theta) \text{ and } \nu_A(y) = \nu_A(z) > \nu_A(\theta).$$

Then $z \in A_*$. Thus $\mu_A(z) < 1 = \mu_A(\theta)$ and $\nu_A(z) > 0 = \nu_A(\theta)$. so $z_{(1,0)} \notin A$ and $z_{(\mu_A(z),\nu_A(z))} = z_{(1,0)}1_{(\mu_A(z),\nu_A(z))} \in A$. Hence $1_{(\mu_A(z),\nu_A(z))}\chi_M \subseteq A$. Since $w = 1.w$, for all $w \in M$, we have $\mu_A(z) \leq \mu_A(w)$ and $\nu_A(z) \geq \nu_A(w)$.

Let $w = y$. Then $\mu_A(z) \leq \mu_A(y)$ and $\nu_A(z) \geq \nu_A(y)$. Similarly, $\mu_A(y) \leq \mu_A(z)$ and $\nu_A(y) \geq \nu_A(z)$. Thus $\mu_A(z) = \mu_A(y)$ and $\nu_A(z) = \nu_A(y)$.

Claim (3): Let $\mu_A(z) = \alpha$ and $\nu_A(z) = \alpha^{'}$, where α is prime element in L and $\alpha^{'}$ be its complement in L. First, let $s \wedge p \leq \alpha$ and $t \vee q \geq \alpha^{'}$, i.e., $t^{'} \wedge q^{'} \leq \alpha$ and let $p \nleq \alpha$ and $q^{'} \nleq \alpha$.

Suppose $x \in M\backslash A_*$. Then $x_{(p,q)} \notin A$. Thus $1_{(s,t)}x_{(p,q)} = x_{(s\wedge p,t\vee q)} \in A$. So $1_{(s,t)}\chi_M \subseteq A$, and for all $w \in M$, $\mu_{1_{(s,t)}\chi_M}(w) \leq \mu_A(w)$ and $\nu_{1_{(s,t)}\chi_M}(w) \geq \nu_A(w)$.

Let $w = x$. Then $s = \mu_{1_{(s,t)}\chi_M}(w) \leq \mu_A(x) = \alpha$ and $t = \nu_{1_{(s,t)}\chi_M}(w) \geq \nu_A(x) = \alpha^{'}$. Thus $s \leq \alpha$ and $t^{'} \leq \alpha$. Thus, every intuitionistic L-fuzzy prime submodule of M is of the form

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha, & \text{if otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in N \\ \alpha^{'}, & \text{otherwise,} \end{cases}$$

for all $x \in M$, where $\alpha^{'}$ is complement of the prime element α in L and N is a prime submodule of M. □

This theorem is particularly useful in deciding whether of not an intuitionistic fuzzy submodule is prime. The following example illustrate this.

Example 3.7. Let $M = Z$ be a module over $R = Z$. Then

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in 3Z \\ 0.25, & \text{if otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in 3Z \\ 0.75, & \text{otherwise} \end{cases}$$

is an intuitionistic L-fuzzy prime submodule of Z, since $3Z$ is prime submodule of Z and 0.25 is a prime element in $[0, 1]$.

4. Intuitionistic L-fuzzy prime submodules of homomorphic modules

In this section, we investigate the behaviour of intuitionistic L-fuzzy prime submodules under an R-module epimorphism. Firstly, we recall the definition of image and inverse image of an intuitionistic L-fuzzy subset under a R-module homomorphism. From now on, M and M_1 are R-modules.

Definition 4.1. Let f be a R-module homomorphism from M to M_1, $A \in ILFS(M)$ and $B \in ILFS(M_1)$. Then $f(A) \in ILFS(M_1)$ and $f^{-1}(B) \in ILFS(M)$ are defined by: $\forall w \in M_1$ and $\forall m \in M$,

$$f(A)(w) = \begin{cases} (Sup\{\mu_A(m) : m \in f^{-1}(w)\}, Inf\{\nu_A(m) : m \in f^{-1}(w)\}), & \text{if } f^{-1}(w) \neq \phi \\ (0,1), & \text{otherwise} \end{cases}.$$

and $f^{-1}(B)(m) = (\mu_B(f(m)), \nu_B(f(m)))$.

In the next two theorems we show that, both the image and inverse image of an intuitionistic L-fuzzy prime submodules under a R-module epimorphism are again intuitionistic L-fuzzy prime submodules. Here we need to assume that the complete lattice L is distributive.

Theorem 4.2. *Let f be an R-modules epimorphism from M to M_1, and suppose that L is distributive. If A is an intuitionistic L-fuzzy prime submodule of M such that $\chi_{kerf} \subseteq A$, then $f(A)$ is an intuitionistic L-fuzzy prime submodule of M_1.*

Proof. Let $w_1, w_2 \in M_1$. Then

$$\begin{aligned} & \mu_{f(A)}(w_1) \wedge \mu_{f(A)}(w_2) \\ &= [Sup\{\mu_A(m_1) : f(m_1) = w_1)\}] \wedge [Sup\{\mu_A(m_2) : f(m_2) = w_2\}] \\ &= Sup\{\mu_A(m_1) \wedge \mu_A(m_2) : f(m_1) = w_1, f(m_2) = w_2\} \\ &\leq Sup\{\mu_A(m_1 - m_2) : f(m_1) = w_1, f(m_2) = w_2\} \\ &\leq Sup\{\mu_A(m_1 - m_2) : f(m_1 - m_2) = w_1 - w_2\} \\ &= \mu_{f(A)}(w_1 - w_2). \end{aligned}$$

Thus $\mu_{f(A)}(w_1 - w_2) \geq \mu_{f(A)}(w_1) \wedge \mu_{f(A)}(w_2)$. Similarly, we can show that

$$\nu_{f(A)}(w_1 - w_2) \leq \nu_{f(A)}(w_1) \vee \nu_{f(A)}(w_2).$$

Furthermore, for all $w_1 \in M_1$ and $r \in R$, we have

$$\begin{aligned} \mu_{f(A)}(w_1) &= Sup\{\mu_A(m) : f(m) = w_1\}] \leq Sup\{\mu_A(rm) : f(m) = w_1\} \\ &= Sup\{\mu_A(rm) : f(rm) = rw_1)\} \\ &= \mu_{f(A)}(rw_1). \end{aligned}$$

Thus, $\mu_{f(A)}(rw_1) \geq \mu_{f(A)}(w_1)$. Similarly, we can show that $\nu_{f(A)}(rw_1) \leq \nu_{f(A)}(w_1)$.

Also, it is clear that $\mu_{f(A)}(\theta_1) = 1$ and $\nu_{f(A)}(\theta_1) = 0$. So, $f(A)$ is an intuitionistic L-fuzzy submodule of M_1.

Next, we show that $f(A)$ is an intuitionistic L-fuzzy prime submodule of M_1. Since A is an intuitionistic L-fuzzy prime submodule of M, so A is of the form

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha, & \text{if otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in N \\ \alpha', & \text{otherwise,} \end{cases}$$

for all $x \in M$, where α' is complement of the prime element α in L and $N = A_*$ is a prime submodule of M.

We first claim that if A_* is a prime submodule of M and $\chi_{kerf} \subseteq A$, then $f(A_*)$ is prime submodule of M_1.

Let $x \in \chi_{kerf}$. Then $\mu_{\chi_{kerf}}(x) = 1 \leq \mu_A(x)$ and $\nu_{\chi_{kerf}}(x) = 0 \geq \nu_A(x)$. Thus $\mu_A(x) = \mu_A(\theta)$ and $\nu_A(x) = \nu_A(\theta)$. So $x \in A_*$. Hence $kerf \subseteq A_*$.

For all $r \in R, w \in M_1, rw \in f(A_*)$, there exists $z \in A_*$ such that $rw = f(z)$. Since f is an epimorphism, there exists $m \in M$ such that $rw = rf(m) = f(rm) = f(z)$. Now, $rm \in A_*$ and A_* is a prime submodule of M. Then either $m \in A_*$ or $rM \subseteq A_*$.

If $m \in A_*$, then $w = f(m) \in f(A_*)$ and if $rM \subseteq A_*$, then $rM_1 = f(rM) \subseteq f(A_*)$. Thus $f(A_*)$ is a prime submodule of M_1. Since α is a prime element in L, by Theorem 3.6, for all $w \in M_1$,

$$\mu_{f(A)}(w) = \begin{cases} 1, & \text{if } w \in f(A_*) \\ \alpha, & \text{if otherwise} \end{cases} ; \quad \nu_{f(A)}(w) = \begin{cases} 0, & \text{if } w \in f(A_*) \\ \alpha^{'}, & \text{otherwise.} \end{cases}$$

So $f(A)$ is an intuitionistic L-fuzzy prime submodule of M_1. □

Example 4.3. Let f be a homomorphism from Z to Z defined by $f(x) = 2x$, and let

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in 3Z \\ 0.25, & \text{if otherwise} \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in 3Z \\ 0.75, & \text{otherwise} \end{cases}$$

be an intuitionistic L-fuzzy prime submodule of Z. Then

$$\begin{aligned} f(A)(0) &= (Sup\{\mu_A(x) : f(n) = 0\}, Inf\{\nu_A(x) : f(n) = 0\}) \\ &= (\mu_A(0), \nu_A(0)) = (1, 0) \end{aligned}$$

and

$$\begin{aligned} f(A)(1) &= (Sup\{\mu_A(x) : f(n) = 1\}, Inf\{\nu_A(x) : f(n) = 1\}) \\ &= (0, 1) \ [\text{ As } f^{-1}(1) = \varnothing]. \end{aligned}$$

Similarly, we can find that $f(A)(3) = f(A)(5) = (0, 1)$ and $f(A)(2) = f(A)(4) = (0.25, 0.75)$ and so on. Thus we get

$$\mu_{f(A)}(x) = \begin{cases} 1, & \text{if } x \in 6Z \\ 0.25, & \text{if } x \in 2Z - 6Z \\ 0, & \text{if } x \in Z - 2Z \end{cases} ; \quad \nu_{f(A)}(x) = \begin{cases} 0, & \text{if } x \in 6Z \\ 0.75, & \text{if } x \in 2Z - 6Z \\ 1, & \text{if } x \in Z - 2Z \end{cases}$$

is not an intuitionistic L-prime fuzzy submodule of Z. This shows that the assumption that f be an epimorphism in Theorem 4.2(cannot be dropped.

Theorem 4.4. *Let f be a R-module epimorphism from M to M_1. If B is an intuitionistic L-fuzzy prime submodule of M_1, then $f^{-1}(B)$ is an intuitionistic L-fuzzy prime submodule of M.*

Proof. Let B be an intuitionistic L-fuzzy prime submodule of M_1. Then

$$\mu_B(x) = \begin{cases} 1, & \text{if } x \in B_* \\ \alpha, & \text{if otherwise} \end{cases} ; \quad \nu_B(x) = \begin{cases} 0, & \text{if } x \in B_* \\ \alpha^{'}, & \text{otherwise,} \end{cases}$$

for all $x \in M_1$, where $\alpha^{'}$ is complement of the prime element α in L and B_* is a prime submodule of M_1.

We first show that $f^{-1}(B_*)$ is a prime submodule of M.

For all $r \in R, m \in M$, if $rm \in f^{-1}(B_*)$, then $f(rm) \in B_*$, i.e., $rf(m) \in B_*$. As B_* is prime submodule of M_1, either $f(m) \in B_*$ or $rM_1 \subseteq B_*$.

If $f(m) \in B_*$, then $m \in f^{-1}(B_*)$ and if $rM_1 \subseteq B_*$, then $rf(M) = f(rM) \subseteq B_*$ Thus $rM \subseteq f^{-1}(B_*)$. So

$$\mu_{f^{-1}(B)}(x) = \begin{cases} 1, & \text{if } x \in f^{-1}(B_*) \\ \alpha, & \text{if otherwise} \end{cases}; \quad \nu_{f^{-1}(B)}(x) = \begin{cases} 0, & \text{if } x \in f^{-1}(B_*) \\ \alpha', & \text{otherwise.} \end{cases}$$

Hence $f^{-1}(B)$ is an intuitionistic L-fuzzy prime submodule of M. □

5. Conclusion

As the study of modules over a ring R provides us with an insight into the structure of R. In the same way the study of intuitionistic L-fuzzy modules provides us with an insight into the structure of lattice L. In this paper, we have given a characterization of intuitionistic L-fuzzy prime submodules and also investigate the behaviour of intuitionistic L-fuzzy prime submodules under R-homomorphisms. This is useful for the further study of intuitionistic L-fuzzy modules.

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P. K. Sharma (pksharma@davjalandhar.com)
Post-Graduate Department of mathematics, D.A.V.College, Jalandhar, Punjab, India

Kanchan (kanchan4usoh@gmail.com)
Research Scholar, IKGPT, University, Jalandhar, Punjab, India