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An approach for decision making using intuitionistic trapezoidal fuzzy soft set

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Abstract. Introduction of soft sets by Molodtsov (1999) has evolved a revolution in the decision making paradigm. Researchers have used soft sets with different extensions of fuzzy sets to satisfy the various types of uncertainties involved with real life decision making problems. This paper introduces intuitionistic trapezoidal fuzzy soft set (ITrFSS) by combining intuitionistic trapezoidal fuzzy set (ITrFS) with soft set. Firstly, we generalize the adjustable approach applied to intuitionistic fuzzy soft set (IFSS) based decision making developed by Jiang et al. (2011) and then present an approach to ITrFSS based decision making using threshold ITrFSs and level soft sets. Moreover, we propose weighted ITrFSS and apply it in a decision making problem. Fuzzy analytic hierarchy process (AHP) has been used to derive the attribute weights. This paper has also validated the outcome of the adjustable approaches based on ITrFSS and weighted ITrFSS using closeness coefficient measure. Finally, two illustrative examples are provided to show the feasibility of the proposed approaches in real life decision making problems.

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Keywords: Intuitionistic trapezoidal fuzzy soft set, intuitionistic trapezoidal fuzzy set, level soft set, decision making.

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1. Introduction

Uncertainty is everywhere in our real life which are difficult to solve with the traditional mathematical tools. To deal with it, Molodtsov [28] introduced soft set as a generic mathematical tool. Soft set uses two important ideas which are parameterization and approximate description on the set of objects. Due to having these properties, applicability of soft sets to real life problems have been emerging rapidly. To deal with the inexact and incomplete uncertain realistic problems, Maji et al. [22] combined soft set with fuzzy set and introduced fuzzy soft set (FSS). Since its introduction, FSS theory has been successfully applied in many different fields such as decision making, data analysis, forecasting, simulation, optimization, texture classification, etc. Intuitionistic fuzzy soft set (IFSS) was introduced by Maji et al. [23, 24] as a generalization of FSS. Due to its inherent capability to represent human thought, IFSS has been widely used by many researches and practitioners to solve various real life decision making problems. The suitability of IFSS for the decision making applications has been explained in [21]. Various operations, properties and algebraic structure of IFSSs are studied in [45]. Some significant contributions on decision making based on IFSS are narrated below. Jiang et al. [20] investigated entropy and distance measurements on IFSSs. Zhang [39] proposed a new rough set model and applied it in IFSS based decision making problems. Das and Kar [7] proposed an algorithmic approach for decision making using IFSS, where the authors used cardinals of IFSS as a novel concept. Feng et al. [18] pointed out some limitations of the method proposed by Roy and Maji [31]. In order to overcome those limitations, they introduced an adjustable approach in [18] for FSSs based decision making using level soft sets. In the proposed adjustable approaches, the authors considered different types of thresholds and consequently different level soft sets. As a more generalized version of the method presented in [18], Jiang et al. [19] proposed an adjustable approach to IFSSs based decision making problem using the concept of level soft sets. The authors also extended the adjustable approach based on weighted IFSS. Qin et al. [30] further generalized the approaches introduced by Feng et al. [18] and Jiang et al. [19]. They developed the notion of reduct IFSS and presented an adjustable approach to interval-valued intuitionistic fuzzy soft set (IVIFSS) based decision making by using reduct IFSSs and level soft sets. Zhang et al. [40] also developed an adjustable approach for decision making based on IVIFSSs by using level soft sets. They further defined the concept of weighted IVIFSS and applied it to decision making environment. Das et al. [8, 9] proposed two decision making approaches using IVIFSS. Trapezoidal fuzzy soft set was introduced by Xiao et al [41].

During the last decade, the study on hybrid models combining FSS and other mathematical concepts have become much popular among the researchers and many researchers have contributed to this domain. Recently, Das et al. [13] have studied the evolutions of FSS and its extensions and extensively reviewed their applications in decision making problems. Tang [33] have proposed a novel FSS based approach using grey relational analysis (GRA) and DempsterÔÇôShafer theory of evidence, where GRA is used to determine the uncertain degree of various parameters and DempsterÔÇôShafer theory is used for aggregation purpose. Muthukumar et al. [29] proposed a new similarity measure on IFSSs which minimized the influence of imprecise parameters and discussed some of their basic properties. Alcantud [2] investigated the FSS based decision making approaches given by Roy and Maji [31] and Feng et al. [18]. The author further suggested that the methodology given in [18] can be explored to have more benefits. In their proposed approach, the author [31] aggregated the data coming from multi-sources into a resultant FSS using a more accurate operator. Wang et al. [34] used ambiguity measure and DempsterÔÇôShafer theory of evidence to propose an improved decision making approach based on FSS. The authors firstly used ambiguity measure to obtain the uncertainty degree of each parameter and then used DempsterÔÇÖs combination rule to rank the alternatives. Triangular fuzzy soft set (TFSS) was introduced in [11]. Dutta and Limboo [17] introduced Bell shaped FSS and applied it in medical diagnosis problem. Aiwu and Hongjun [1] proposed fuzzy linguistic soft set (FLSS) by combining fuzzy set, linguistic variable and soft set. Dey and Pal [15] introduced generalised multi-fuzzy soft set (GMFSS) and studied their various operations and properties. Finally they applied GMFSS in decision making problems. Intuitionistic fuzzy parameterized soft set theory was defined by Deli and Cagman [14]. The authors also presented as adjustable approach for decision making based on intuitionistic fuzzy parameterized soft set. In [3], Alcantud extensively studied the underlying relationships among fuzzy sets, soft sets, and their extensions. Das et al. [12] proposed correlation coefficient of hesitant fuzzy soft set (HFSS) and using it they developed correlation efficiency. The authors presented a decision making algorithm using the proposed concept. They also proposed correlation coefficient in the framework of interval-valued hesitant fuzzy soft set (IVHFSS). In [6], Das et al. proposed a novel approach to estimate the missing or unknown information in incomplete fuzzy soft sets (FSSs). Then they generalized their approach to find missing or unknown information in the context of interval valued fuzzy soft sets. In [10], Das et al. studied an algorithmic approach for group decision making (GDM) problems using neutrosophic soft matrix (NSM) and relative weights of experts. They considered NSM as the matrix representation of neutrosophic soft sets (NSSs), where NSS is the combination of neutrosophic set and soft set.

An object is described in fuzzy set using only the membership grade and fuzzy set considers that the sum of membership and non-membership grade is 1. But in real life, this sum may be less than 1. Intuitionistic fuzzy set (IFS) [4] solved this kind of problem by incorporating an additional parameter called hesitance margin. This hesitancy parameter makes IFS significant to solve the uncertain real life problems. Intuitionistic trapezoidal fuzzy set (ITrFS) was introduced by Wang and Zhang [35] as a generalization of IFS [4]. In ITrFS, the membership and non-membership functions are expressed using trapezoidal fuzzy numbers. The membership and non-membership functions of an intuitionistic trapezoidal fuzzy number are piecewise linear and trapezoidal, which can effectively transform the linguistic terms into numerical variables. So ITrFS is considered to be more useful to express the uncertain situations. Recently many researchers have contributed on ITrFSs based decision making. Ye [43, 44] investigated the similarity measures and distance based similarity measures between ITrFNs and applied to decision making problems. Ye [42] also studied expected value method for intuitionistic trapezoidal fuzzy set based decision making. Du and Liu [16] extended fuzzy VIKOR method with ITrFNs. Wei et al. [38] proposed grey relational analysis (GRA) model for selecting an enterprise resource planning (ERP) system in trapezoidal intuitionistic fuzzy setting. Yu [46] introduced generalized intuitionistic trapezoidal fuzzy weighted averaging operator (GITFWA) and studied its various properties. Saaty [32] introduced analytic hierarchy process (AHP), which has been successfully applied in decision making problems. To include imprecise judgement, researchers used fuzzy AHP [5], where priority vectors are derived from the fuzzy pair wise comparison matrices. To derive weights from fuzzy comparison matrices, Mikhailov [25, 26, 27] proposed fuzzy preference programming (FPP) method. Wang and Chin [37] proposed a priority method utilizing a linear goal programming (LGP) model to derive normalized fuzzy weights from the fuzzy pair wise comparison matrices.

As stated above, adjustable approach has drawn the attention of many researchers due to its flexible nature on decision making paradigm. Initially adjustable approach was applied on FSS, then gradually it was extended to IFSS and IVIFSS. Similarly ITrFS has also many significant contributions to handle the uncertainties. However, none has concentrated on the extension of ITrFS with soft set to explore ITrFSS. The significance of ITrFSS on decision making problems is inevitable due to its ability of parameterization and representation of the linguistic information in more practical and realistic manner. ITrFSS based adjustable approaches can solve the problems where both the membership and non-membership functions are expressed using trapezoidal fuzzy numbers. As we have observed, researchers have considered simply membership and non-membership functions to propose adjustable approaches, but none has used trapezoidal membership and non-membership functions to implement adjustable approaches. Some complex situations in our real life may require trapezoidal representations of both the membership and non-membership functions, the concepts of level soft sets, as well as the concept of parameterizations to solve the problems. Hence the need of ITrFSS based adjustable approaches are significant. The purpose of this paper is to present an adjustable approach to solve real life decision making problems using ITrFSSs and weighted ITrFSSs. Firstly, we introduce the concept of ITrFSS by combining intuitionistic trapezoidal fuzzy set and soft set. Then some threshold intuitionistic trapezoidal fuzzy sets are introduced, which are used to compute the level soft sets of the ITrFSS. This study has presented an adjustable approach for decision making using ITrFSSs and level soft sets, which may be considered as a generalized approach of the methods introduced by Feng et al. [18] and Jiang et al. [19]. We have also introduced weighted ITrFSS, where weights of the attributes are derived using fuzzy AHP [5]. Then the weighted ITrFSS has been applied in decision making problems based on threshold intuitionistic trapezoidal fuzzy sets and level soft sets. Finally, we have used a closeness coefficient measure [41] to validate the adjustable approaches which are based on ITrFSS and weighted ITrFSS.

The rest of this paper is organized as follows. Section 2 briefly presents some relevant concepts including soft sets, fuzzy soft sets, intuitionistic fuzzy soft sets, and intuitionistic trapezoidal fuzzy numbers. In section 3, we introduce ITrFSS, some threshold intuitionistic trapezoidal fuzzy sets, and present an adjustable approach for decision making using ITrFSSs. Section 4 contains weighted ITrFSS based decision making followed by an illustrative example and validation of results using closeness coefficient. Comparative study is given in Section 5. Finally conclusions are drawn in section 6.

TABLE 1. Tabular representation of $(\bar{F}_{\{A\}}, E)$

U/E	s_1	s_2	s_3	s_4	s_5
d_1	0.2	0	0.6	0	0.6
d_2	0.4	0.8	0.2	0	0.7
d_3	0.9	0.1	0.8	0	0.5
d_4	0.7	0.7	0	0	0.8
d_5	0	0	0	0	0

2. Preliminaries

This section briefly presents some basic ideas relevant with this paper. Let U be the initial universe and E be the set of parameters.

Definition 2.1 ([28])**.** Let $P(U)$ denotes the power set of U and $A \subseteq E$. A pair $(F_{\{A\}}, E)$ is called a soft set over U, where $F_{\{A\}}$ is a mapping given by $F_{\{A\}} : E \to P(U)$ such that $F_{\{A\}}(e) = \varnothing$ if $e \notin A$. For any parameter $e \in A, F_{\{A\}}(e)$ may be considered as the set of e-approximate elements of the soft set $(F_{\{A\}}, E)$.

Example 2.2. Let U be the set of five diseases (Viral fever, Malaria, Typhoid, Gastric ulcer, Pneumonia) given by:

$$U = \{d_1, d_2, d_3, d_4, d_5\}$$

and E be the set of five symptoms given by:

$$E = \{Temperature, Headache, Stomachpain, Cough, Chestpain\} = \{s_1, s_2, s_3, s_4, s_5\}.$$

Let $A = \{s_1, s_2, s_3\} \subset E$. Now consider that $F_{\{A\}}$ is a mapping given by:

$$F_{\{A\}}(s_1) = \{d_1, d_2, d_3, d_4\}, F_{\{A\}}(s_2) = \{d_2, d_3, d_4\},$$
$$F_{\{A\}}(s_3) = \{d_1, d_2, d_3\},\ F_{\{A\}}(s_4) = \{d_1, d_2, d_3, d_4\}.$$

Then the soft set $(F_{\{A\}}, E)$ can be represented as follows:

$$(F_{\{A\}}, E) = \{(s_1, \{d_1, d_2, d_3, d_4\}), (s_2, \{d_2, d_3, d_4\}), (s_3, \{d_1, d_2, d_3\}), (s_4, \{d_1, d_2, d_3, d_4\}), (s_5, \{\varnothing\})\}.$$

Definition 2.3 ([22])**.** Let $FS(U)$ denotes the set of all fuzzy subsets of U and $A \subset E$. A pair $(\bar{F}_{\{A\}}, E)$ is called a fuzzy soft set (FSS) over U,
where $\bar{F}_{\{A\}}$ is a mapping $\bar{F}_{\{A\}} : E \to FS(U)$

Example 2.4. Consider the previous Example 2.2. If one considers that the association of symptom s_1 with the disease d_1 is more important, then this information cannot be expressed simply with 0 and 1. Such type of situation can be well expressed using a membership value in $[0, 1]$ instead of the crisp numbers 0 or 1. Assume that:

$$\bar{F}_{\{A\}}(s_1) = \{d_1/0.2, d_2/0.4, d_3/0.9, d_4/0.7\}, \bar{F}_{\{A\}}(s_2) = \{d_2/0.8, d_3/0.1, d_4/0.7\},$$
$$\bar{F}_{\{A\}}(s_3) = \{d_1/0.6, d_2/0.2, d_3/0.8\}, \bar{F}_{\{A\}}(s_5) = \{d_1/0.6, d_2/0.7, d_3/0.5, d_4/0.8\}.$$

Then the tabular representation of the FSS $(\bar{F}_{\{A\}}, E)$ is shown in Table 1.

Definition 2.5. [23] Let $IFS(U)$ denotes the set of all intuitionistic fuzzy subsets of U and $A \subset E$. A pair $(\tilde{F}_{\{A\}}, E)$ is called an intuitionistic fuzzy soft set (IFSS) over U, where $\tilde{F}_{\{A\}}$ is a mapping given by $\tilde{F}_{\{A\}} : E \to IFS(U)$.

Table 2. Tabular representation of $(\tilde{F}_{\{A\}}, E)$

U/E	s_1	s_2	s_3	s_4	s_5
d_1	$(0.2, 0.5)$	0	$(0.6, 0.3)$	0	$(0.6, 0.1)$
d_2	$(0.4, 0.3)$	$(0.8, 0.2)$	$(0.2, 0.6)$	0	$(0.7, 0.2)$
d_3	$(0.9, 0.1)$	$(0.1, 0.8)$	$(0.8, 0.1)$	0	$(0.5, 0.2)$
d_4	$(0.7, 0.2)$	$(0.7, 0.2)$	0	0	$(0.8, 0.2)$
d_5	0	0	0	0	0

Example 2.6. Consider Example 2.4. When one hesitates to find out the association of symptom s_1 with the disease then this information cannot be expressed using only the membership grade. Both membership and non membership values can be considered in this kind of situation. Let us take
$\tilde{F}_{\{A\}}(s_1) = \{d_1/(0.2, 0.5), d_2/(0.4, 0.3), d_3/(0.9, 0.1), d_4/(0.7, 0.2)\}$,
$\tilde{F}_{\{A\}}(s_2) = \{d_2/(0.8, 0.2), d_3/(0.1, 0.8), d_4/(0.7, 0.2)\}$,
$\tilde{F}_{\{A\}}(s_3) = \{d_1/(0.6, 0.3), d_2/(0.2, 0.6), d_3/(0.8, 0.1)\}$,
$\tilde{F}_{\{A\}}(s_5) = \{d_1/(0.6, 0.1), d_2/(0.7, 0.2), d_3/(0.5, 0.2), d_4/(0.8, 0.2)\}$.
Then the tabular representation of the IFSS $(\tilde{F}_{\{A\}}, E)$ is shown in Table 2.

Definition 2.7 ([4])**.** The membership and non-membership functions of an intuitionistic trapezoidal fuzzy number (ITrFN) $\acute{a}$ are respectively defined as:

$$\mu_{\acute{a}}(x) = \begin{cases} \dfrac{x-a}{b-a}\mu_{\acute{a}} & a \leq x < b \\ \mu_{\acute{a}} & b \leq x \leq c \\ \dfrac{d-x}{d-c}\mu_{\acute{a}} & c < x \leq d \\ 0, & \text{otherwise} \end{cases}$$

and its non-membership function is defined as:

$$\nu_{\acute{a}}(x) = \begin{cases} \dfrac{b-x+\nu_{\acute{a}}(x-a_1)}{b-a_1}\mu_{\acute{a}} & a_1 \leq x < b \\ \nu_{\acute{a}} & b \leq x \leq c \\ \dfrac{x-c+\nu_{\acute{a}}(d_1-x)}{d_1-c}\mu_{\acute{a}}, & c < x \leq d_1 \\ 0 & \text{otherwise.} \end{cases}$$

Here $0 \leq \mu_{\acute{a}} \leq 1, 0 \leq \nu_{\acute{a}} \leq 1$ and $\mu_{\acute{a}} + \nu_{\acute{a}} \leq 1$, $a, b, c, d, a_1, d_1 \in \Re$. $\acute{a} = \langle([a, b, c, d] : \mu_{\acute{a}}), ([a_1, b, c, d_1] : \nu_{\acute{a}})\rangle$ is called an ITrFN. For convenience, ITrFN $\acute{a}$ is written as $\acute{a} = ([a, b, c, d]; \mu_{\acute{a}}, \nu_{\acute{a}})$. Let $\acute{\alpha}_1 = ([a_1, b_1, c_1, d_1]; \mu_{\acute{\alpha}_1}, \nu_{\acute{\alpha}_1})$ and $\acute{\alpha}_2 = ([a_2, b_2, c_2, d_2]; \mu_{\acute{\alpha}_2}, \nu_{\acute{\alpha}_2})$ be two ITrFNs and $\lambda \geq 0$. Then following operations are defined on $\acute{\alpha}_1$ and $\acute{\alpha}_2$:
(i) $\acute{\alpha}_1 \oplus \acute{\alpha}_2 = ([a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2]; \mu_{\acute{\alpha}_1} + \mu_{\acute{\alpha}_2} - \mu_{\acute{\alpha}_1}\mu_{\acute{\alpha}_2}, \nu_{\acute{\alpha}_1}\nu_{\acute{\alpha}_2})$,
(ii) $\acute{\alpha}_1 \otimes \acute{\alpha}_2 = ([a_1a_2, b_1b_2, c_1c_2, d_1d_2]; \mu_{\acute{\alpha}_1}\mu_{\acute{\alpha}_2}, \nu_{\acute{\alpha}_1} + \nu_{\acute{\alpha}_2} - \nu_{\acute{\alpha}_1}\nu_{\acute{\alpha}_2})$,
(iii) $\lambda\acute{\alpha}_1 = \big([\lambda a, \lambda b, \lambda c, \lambda d]; 1 - (1 - \mu_{\acute{\alpha}_1})^\lambda, \nu_{\acute{\alpha}_1}^\lambda\big)$,
(v) $\acute{\alpha}_1^\lambda = \big([a^\lambda, b^\lambda, c^\lambda, d^\lambda]; \mu_{\acute{\alpha}_1}^\lambda, 1 - (1 - \nu_{\acute{\alpha}_1})^\lambda\big)$.

In order to compare any two ITrFNs, Wang and Zhang [35] defined the score function, accuracy function, and expected value as given below.

Definition 2.8 ([35])**.** Let $\acute{\alpha}_i = ([a_i, b_i, c_i, d_i]; \mu_{\acute{\alpha}_i}, \nu_{\acute{\alpha}_i})$ be an ITrFN. Score and accuracy functions of $acute\alpha$ are respectively defined by:

$$S(\acute{\alpha}) = I(\acute{\alpha}) \times (\mu_{\acute{\alpha}} - \nu_{\acute{\alpha}})$$

and

$$H(\acute{\alpha}) = I(\acute{\alpha}) \times (\mu_{\acute{\alpha}} + \nu_{\acute{\alpha}}),$$

where

$$I(\acute{\alpha}) = \frac{1}{8} \times [(a + b + c + d) \times (1 + \mu_{\acute{\alpha}} - \nu_{\acute{\alpha}})]$$

is the expected value of the ITrFN $\acute{alpha}$. In order to compare two ITrFNs $\acute{\alpha}_1 = ([a_1, b_1, c_1, d_1]; \mu_{\acute{\alpha}_1}, \nu_{\acute{\alpha}_1})$ and $\acute{\alpha}_2 = ([a_2, b_2, c_2, d_2]; \mu_{\acute{\alpha}_2}, \nu_{\acute{\alpha}_2})$, the score and accuracy functions are used as follows:

(i) if $S(\acute{\alpha}_1) > S(\acute{\alpha}_2)$, then $\acute{\alpha}_1 > \acute{\alpha}_2$,

(ii) suppose $S(\acute{\alpha}_1) = S(\acute{\alpha}_2)$, then

(a) if $H(\acute{\alpha}_1) > H(\acute{\alpha}_2)$, then $\acute{\alpha}_1 > \acute{\alpha}_2$,

(b) if $H(\acute{\alpha}_1) = H(\acute{\alpha}_2)$, then $\acute{\alpha}_1 = \acute{\alpha}_2$.

Distance between two fuzzy sets is a useful measure for calculating the difference between them. The Hamming and Euclidean distances between two intuitionistic trapezoidal fuzzy numbers are given below.

Definition 2.9 ([36])**.** Normalized Hamming distance between two ITrFNs $\acute{\alpha}_1 = ([a_1, b_1, c_1, d_1]; \mu_{\acute{\alpha}_1}, \nu_{\acute{\alpha}_1})$ and $\acute{\alpha}_2 = ([a_2, b_2, c_2, d_2]; \mu_{\acute{\alpha}_2}, \nu_{\acute{\alpha}_2})$ is defined as:

$$D_H(\acute{\alpha}_1, \acute{\alpha}_2) = \frac{1}{8} \left(\begin{array}{l} |(1 + \mu_{\acute{\alpha}_1} - \nu_{\acute{\alpha}_1})\, a_1 - (1 + \mu_{\acute{\alpha}_2} - \nu_{\acute{\alpha}_2})\, a_2| \\ + |(1 + \mu_{\acute{\alpha}_1} - \nu_{\acute{\alpha}_1})\, b_1 - (1 + \mu_{\acute{\alpha}_2} - \nu_{\acute{\alpha}_2})\, b_2| \\ + |(1 + \mu_{\acute{\alpha}_1} - \nu_{\acute{\alpha}_1})\, c_1 - (1 + \mu_{\acute{\alpha}_2} - \nu_{\acute{\alpha}_2})\, c_2| \\ + |(1 + \mu_{\acute{\alpha}_1} - \nu_{\acute{\alpha}_1})\, d_1 - (1 + \mu_{\acute{\alpha}_2} - \nu_{\acute{\alpha}_2})\, d_2| \end{array} \right).$$

Definition 2.10 ([36])**.** Normalized Euclidean distance between two ITrFNs $\acute{\alpha}_1 = ([a_1, b_1, c_1, d_1]; \mu_{\acute{\alpha}_1}, \nu_{\acute{\alpha}_1})$ and $\acute{\alpha}_2 = ([a_2, b_2, c_2, d_2]; \mu_{\acute{\alpha}_2}, \nu_{\acute{\alpha}_2})$ is defined as:

$$D_E(\acute{\alpha}_1, \acute{\alpha}_2) = \sqrt{\frac{1}{8} \left(\begin{array}{l} \{(1 + \mu_{\acute{\alpha}_1} - \nu_{\acute{\alpha}_1})\, a_1 - (1 + \mu_{\acute{\alpha}_2} - \nu_{\acute{\alpha}_2})\, a_2\}^2 \\ + \{(1 + \mu_{\acute{\alpha}_1} - \nu_{\acute{\alpha}_1})\, b_1 - (1 + \mu_{\acute{\alpha}_2} - \nu_{\acute{\alpha}_2})\, b_2\}^2 \\ + \{(1 + \mu_{\acute{\alpha}_1} - \nu_{\acute{\alpha}_1})\, c_1 - (1 + \mu_{\acute{\alpha}_2} - \nu_{\acute{\alpha}_2})\, c_2\}^2 \\ + \{(1 + \mu_{\acute{\alpha}_1} - \nu_{\acute{\alpha}_1})\, d_1 - (1 + \mu_{\acute{\alpha}_2} - \nu_{\acute{\alpha}_2})\, d_2\}^2 \end{array} \right)}.$$

Example 2.11. Let $\acute{\alpha_1} = ([0.5, 0.6, 0.7, 0.9]; 0, 6, 0.3)$ and $\acute{\alpha_2} = ([0.4, 0.5, 0.6, 0.7]; 0.5, 0.3)$ be two ITrFNs. The normalized Hamming distance $D_H(\acute{\alpha}_1, \acute{\alpha}_2)$ and Euclidean distance $D_E(\acute{\alpha}_1, \acute{\alpha}_2)$ measurements between two ITrFNs $\acute{\alpha}_1$ and $\acute{\alpha}_2$ are given below:

$$D_H(\acute{\alpha}_1, \acute{\alpha}_2) = \frac{1}{8} \left(\begin{array}{l} |(1 + 0.6 - 0.3) \times 0.5 - (1 + 0.5 - 0.3) \times 0.4| \\ + |(1 + 0.6 - 0.3) \times 0.6 - (1 + 0.5 - 0.3) \times 0.5| \\ + |(1 + 0.6 - 0.3) \times 0.7 - (1 + 0.5 - 0.3) \times 0.6| \\ + |(1 + 0.6 - 0.3) \times 0.9 - (1 + 0.5 - 0.3) \times 0.7| \end{array} \right) = 0.11,$$

$$D_E(\acute{\alpha}_1, \acute{\alpha}_2) = \sqrt{\frac{1}{8}\begin{pmatrix} \{(1+0.6-0.3)\times 0.5-(1+0.5-0.3)\times 0.4\}^2 \\ +\{(1+0.6-0.3)\times 0.6-(1+0.5-0.3)\times 0.5\}^2 \\ +\{(1+0.6-0.3)\times 0.7-\{(1+0.5-0.3)\times 0.6\}^2 \\ +\{(1+0.6-0.3)\times 0.9-(1+0.5-0.3)\times 0.7\}^2 \end{pmatrix}} = 0.16.$$

3. Adjustable approach to ITrFSS based decision making

In this section, we present an adjustable approach for decision making based on ITrFSS, which extends the methods introduced by Feng et al. [18] and Jiang et al. [19]. Firstly, we introduce ITrFSS, which is defined as a combination of intuitionistic trapezoidal fuzzy set (ITrFS) and soft set. Then we propose various threshold ITrFSs, such as mid-level threshold ITrFS, top-level threshold ITrFS, and bottom-level threshold ITrFS. These threshold ITrFSs are necessary to adjust the decision makers judgements based on the decision environment and have significant contribution in decision making under various circumstances. This section also presents positive ideal and negative ideal ITrFNs, and closeness coefficient of alternatives.

Definition 3.1. Let $\acute{P}(U)$ be the set of all intuitionistic trapezoidal fuzzy subsets of U, E be the set of parameters and $A \subset E$. A pair $(\acute{F}_{\{A\}}, E)$ is called intuitionistic trapezoidal fuzzy soft set (ITrFSS) over U, where $\acute{F}_{\{A\}}$ is a mapping given by $\tilde{F}_{\{A\}}$: $E \to P(U)$.

In other words, ITrFSS is a parameterized family of intuitionistic trapezoidal fuzzy subsets of U. For any parameter $e \in A, \acute{F}_{\{A\}}(e)$ is considered as the intuitionistic trapezoidal fuzzy value set for the parameter e.

Example 3.2. Let us consider Example 2.6 mentioned earlier. ITrFN supports piecewise linear and trapezoidal representation of the intuitionistic fuzzy values. When a decision maker expresses his/her opinion in linguistic terms, ITrFN is considered to be suitable for numerical conversion of those linguistic terms. For simplicity, here we take numerical values to present the association of the diseases with each of the symptoms. ITrFN $\acute{a} = ([a,b,c,d];\mu_{\acute{a}},\nu_{\acute{a}})\,, a,b,c,d,\mu_{\acute{a}},\nu_{\acute{a}} \in \Re$ is used to present the association. Let us take

$$\acute{F}_{\{A\}}(s_1) = \{d_1/(0.4,0.5,0.6,0.7)(0.2,0.5), d_2/(0.1,0.2,0.3,0.4)(0.4,0.3), d_3/(0.7,0.8,0.8,0.9)(0.9,0.1), d_4/(0.5,0.6,0.7,0.8)(0.7,0.2)\}$$

$$\acute{F}_{\{A\}}(s_2) = \{d_2/(0.4,0.5,0.5,0.6)(0.8,0.2), d_3/(0.3,0.4,0.5,0.6)(0.1,0.8), d_4/(0.1,0.2,0.2,0.4)(0.7,0.2)\}$$

$$\acute{F}_{\{A\}}(s_3) = \{d_1/(0.1,0.2,0.2,0.3)(0.6,0.3), d_2/(0.3,0.4,0.5,0.6)(0.2,0.6), d_3/(0.7,0.8,0.8,0.9)(0.8,0.1)\}$$

$$\acute{F}_{\{A\}}(s_5) = \{d_1/(0.5,0.6,0.6,0.7)(0.6,0.1), d_2/(0.4,0.5,0.6,0.7)(0.7,0.2), d_3/(0.1,0.2,0.3,0.4)(0.5,0.2), d_4/(0.2,0.3,0.4,0.5)(0.8,0.2)\}.$$

Table 3. Tabular representation of $(\acute{F}_{\{A\}}, E)$

U/E	s_1	s_2	s_3	s_4	s_5
d_1	$(0.4, 0.5, 0.6, 0.7)(0.2, 0.5)$	0	$(0.1, 0.2, 0.2, 0.3)(0.6, 0.3)$	0	$(0.5, 0.6, 0.6, 0.7)(0.6, 0.1)$
d_2	$(0.1, 0.2, 0.3, 0.4)(0.4, 0.3)$	$(0.4, 0.5, 0.5, 0.6)(0.8, 0.2)$	$(0.3, 0.4, 0.5, 0.6)(0.2, 0.6)$	0	$(0.4, 0.5, 0.6, 0.7)(0.7, 0.2)$
d_3	$(0.7, 0.8, 0.8, 0.9)(0.9, 0.1)$	$(0.3, 0.4, 0.5, 0.6)(0.1, 0.8)$	$(0.7, 0.8, 0.8, 0.9)(0.8, 0.1)$	0	$(0.1, 0.2, 0.3, 0.4)(0.5, 0.2)$
d_4	$(0.5, 0.6, 0.7, 0.8)(0.7, 0.2)$	$(0.1, 0.2, 0.2, 0.4)(0.7, 0.2)$	0	0	$(0.2, 0.3, 0.4, 0.5)(0.8, 0.2)$
d_5	0	0	0	0	0

Then the tabular representation of the ITrFSS $(\acute{F}_{\{A\}}, E)$ is shown in Table 3.

Let $z = (\acute{F}_{\{A\}}, E)$ be an ITrFSS over U and $A \subseteq E$, Let $\acute{z} = ([a, b, c, d]; \mu_{\acute{z}}, \nu_{\acute{z}})$ be an ITrFN for the ITrFSS $(\acute{F}_{\{A\}}, E)$. These notations are used in all of the following definitions.

Definition 3.3. For any ITrFN $\acute{c} = ([c_1, c_2, c_3, c_4]; \mu_{\acute{c}}, \nu_{\acute{c}})$ the $\acute{c}$-level soft set of ITrFSS $z = (\acute{F}_{\{A\}}, E)$ is a crisp soft set $L(z, \acute{c}) = \left(F_{\{A\}}(\epsilon), \acute{c}\right)$ which is defined below.

$$L\left(F_{\{A\}}(\epsilon); \acute{c}\right) = \{x \in U | \acute{z} > \acute{c}\}, \forall \epsilon \in A.$$

The ITrFN $\acute{c} = ([c_1, c_2, c_3, c_4]; \mu_{\acute{c}}, \nu_{\acute{c}})$ is considered as threshold ITrFN, which is chosen by the decision maker in advance for the real life decision making problems based on ITrFSS.

Definition 3.4. The mid-level intuitionistic trapezoidal fuzzy set (ITrFS) $mid\acute{z}$ for the ITrFSS $z = (\acute{F}_{\{A\}}, E)$ is defined as: for each $\epsilon \in A$,

$$[a_{mid\acute{z}}(\epsilon) = \frac{1}{|U|}\sum_{x\in U} a_{\acute{F}_{\{A\}}(\epsilon)}(x), b_{mid\acute{z}}(\epsilon) = \frac{1}{|U|}\sum_{x\in U} b_{\acute{F}_{\{A\}}(\epsilon)}(x),$$
$$c_{mid\acute{z}}(\epsilon) = \frac{1}{|U|}\sum_{x\in U} c_{\acute{F}_{\{A\}}(\epsilon)}(x), d_{mid\acute{z}}(\epsilon) = \frac{1}{|U|}\sum_{x\in U} d_{\acute{F}_{\{A\}}(\epsilon)}(x)],$$
$$[\mu_{mid\acute{z}}(\epsilon) = \frac{1}{|U|}\sum_{x\in U} \mu_{\acute{F}_{\{A\}}(\epsilon)}(x), \nu_{mid\acute{z}}(\epsilon) = \frac{1}{|U|}\sum_{x\in U} \nu_{\acute{F}_{\{A\}}(\epsilon)}(x)].$$

The mid-level ITrFS $mid\acute{z}$ is denoted as mid-level threshold for the ITrFSS $z = (\acute{F}_{\{A\}}, E)$. The level soft set of ITrFSS $\acute{z}$ concerned with the mid-level threshold $mid\acute{z}$ is called the mid-level soft set of $\acute{z}$ denoted by $L(\acute{z}; mid\acute{z})$ The mid-level threshold ITrFS is useful in those uncertain decision making problems, where the expert/decision maker takes a decision giving more focus on the average opinions.

Definition 3.5. Top-level threshold intuitionistic trapezoidal fuzzy set $top\acute{z}$ of the ITrFSS $z = (\acute{F}_{\{A\}}, E)$ is defined by: for each $\epsilon \in A$,

$$[a_{toptop\acute{z}}(\epsilon) = \max_{x\in U} a_{\acute{F}_{\{A\}}(\epsilon)}(x), b_{toptop\acute{z}}(\epsilon) = \max_{x\in U} b_{\acute{F}_{\{A\}}(\epsilon)}(x),$$
$$c_{toptop\acute{z}}(\epsilon) = \max_{x\in U} c_{\acute{F}_{\{A\}}(\epsilon)}(x), d_{toptop\acute{z}}(\epsilon) = \max_{x\in U} d_{\acute{F}_{\{A\}}(\epsilon)}(x)],$$
$$[\mu_{toptop\acute{z}}(\epsilon) = \max_{x\in U} \mu_{\acute{F}_{\{A\}}(\epsilon)}(x), \nu_{toptop\acute{z}}(\epsilon) = \min_{x\in U} \nu_{\acute{F}_{\{A\}}(\epsilon)}(x)].$$

The level soft set of ITrFSS $\acute{z}$ regarding the top-level threshold $top\acute{z}$ is called the top-level soft set of $\acute{z}$ and denoted by $L(\acute{z}; top\acute{z})$. This top-level soft set can be considered in those decision making problems, where the decision maker mostly considers the higher ITrFNs, i.e., the better opinions.

Definition 3.6. Bottom level threshold intuitionistic trapezoidal fuzzy set *bottomź* of the ITrFSS $z = (\acute{F}_{\{A\}}, E)$ is defined by: for each $\epsilon \in A$,

$$[a_{bottom\acute{z}}(\epsilon) = \min_{x\in U} a_{\acute{F}_{\{A\}}(\epsilon)}(x), b_{bottom\acute{z}}(\epsilon) = \min_{x\in U} b_{\acute{F}_{\{A\}}(\epsilon)}(x),$$
$$c_{bottom\acute{z}}(\epsilon) = \min_{x\in U} c_{\acute{F}_{\{A\}}(\epsilon)}(x), d_{bottom\acute{z}}(\epsilon) = \min_{x\in U} d_{\acute{F}_{\{A\}}(\epsilon)}(x)],$$
$$[\mu_{bottom\acute{z}}(\epsilon) = \min_{x\in U} \mu_{\acute{F}_{\{A\}}(\epsilon)}(x), \nu_{bottom\acute{z}}(\epsilon) = \max_{x\in U} \nu_{\acute{F}_{\{A\}}(\epsilon)}(x)].$$

The level soft set of ITrFSS $\acute{z}$ regarding the bottom-level threshold *bottomź* is called the bottom-level soft set of and denoted by $L(\acute{z}; bottom\acute{z})$ This bottom-level soft set can be considered in those decision making problems, where the decision maker want to consider almost all opinions for decision making purpose.

Definition 3.7. Positive ideal ITrFN $I^+ = (a_{I^+}, b_{I^+}, c_{I^+}, d_{I^+}); (\mu_{I^+}, \nu_{I^+})$ is defined by: for each $\epsilon \in A$,

$$a_{I^+} = \max_{x\in U} d_{\acute{F}_{\{A\}}(\epsilon)}(x),\ b_{I^+} = \max_{x\in U} b_{\acute{F}_{\{A\}}(\epsilon)}(x),$$
$$c_{I^+} = \max_{x\in U} c_{\acute{F}_{\{A\}}(\epsilon)}(x),\ d_{I^+} = \max_{x\in U} d_{\acute{F}_{\{A\}}(\epsilon)}(x),$$
$$\mu_{I^+} = \max_{x\in U} \mu_{\acute{F}_{\{A\}}(\epsilon)}(x),\ \nu_{I^+} = \min_{x\in U} \nu_{\acute{F}_{\{A\}}(\epsilon)}(x).$$

Negetive ideal ITrFN $I^- = (a_{I^-}, b_{I^-}, c_{I^-}, d_{I^-}); (\mu_{I^-}, \nu_{I^-})$ is defined by: for each $\epsilon \in A$,

$$a_{I^-} = \min_{x\in U} d_{\acute{F}_{\{A\}}(\epsilon)}(x),\ b_{I^-} = \min_{x\in U} b_{\acute{F}_{\{A\}}(\epsilon)}(x),$$
$$c_{I^-} = \min_{x\in U} c_{\acute{F}_{\{A\}}(\epsilon)}(x),\ d_{I^-} = \min_{x\in U} d_{\acute{F}_{\{A\}}(\epsilon)}(x),$$
$$\mu_{I^-} = \min_{x\in U} \mu_{\acute{F}_{\{A\}}(\epsilon)}(x),\ \nu_{I^-} = \max_{x\in U} \nu_{\acute{F}_{\{A\}}(\epsilon)}(x).$$

The distances of each object in an ITrFSS with the positive/negative ideal ITrFNs are computed by $D_i^+ = D_H(x_i, I^+), i = 1, 2, ..., m$ and $D_i^- = D_H(x_i, I^-), i = 1, 2, ..., m$. Closeness coefficient CC_i of each alternative $i = 1, 2, ..., m$ is calculated as

$$CC_i = \tfrac{D_i^-}{D_i^+ + D_i^-}$$

When the closeness coefficient of an alternative is more, the difference between it and the negative ideal ITrFN is more, i.e., the alternative is more similar to the positive ideal ITrFN. In other words, the closeness coefficient of an object is closer to 1 as it is closer to the positive ideal ITrFN and more distant from the negative ideal ITrFN.

Below we present the adjustable approach to ITrFSS based decision making using threshold ITrFSs and level soft sets. Let $\acute{z} = (\acute{F}_{\{A\}}, E)$ be an ITrFSS with alternatives $d_i, i = 1, ..., m$ and attributes/parameters $s_j, j = 1, 2, ...n$. Following steps explores the optimal alternatives(s) based on the proposed ideas.

Algorithm 1

Step 1. ITrFSS $\acute{z} = (\acute{F}_{\{A\}}, E)$ is taken as input.

Table 4. Tabular representation of ITrFSS $(\acute{F}_{\{A\}}, E)$

U/E	s_1	s_2	s_3	s_4
u_1	$(0.3, 0.4, 0.4, 0.6)(0.3, 0.5)$	$(0.4, 0.5, 0.5, 0.6)(0.7, 0.3)$	$(0.3, 0.4, 0.5, 0.6)(0.6, 0.4)$	$(0.7, 0.8, 0.8, 0.9)(0.6, 0.3)$
u_2	$(0.5, 0.6, 0.6, 0.8)(0.6, 0.2)$	$(0.1, 0.3, 0.5, 0.6.)(0.5, 0.4)$	$(0.5, 0.6, 0.6, 0.7)(0.7, 0.1)$	$(0.6, 0.7, 0.8, 0.9)(0.3, 0.4)$
u_3	$(0.4, 0.5, 0.5, 0.7)(0.7, 0.1)$	$(0.7, 0.8, 0.8, 0.9)(0.3, 0.6)$	$(0.6, 0.8, 0.8, 0.9)(0.3, 0.5)$	$(0.4, 0.5, 0.5, 0.7)(0.4, 0.5)$
u_4	$(0.4, 0.5, 0.5, 0.6)(0.6, 0.4)$	$(0.5, 0.6, 0.6, 0.8)(0.4, 0.6)$	$(0.7, 0.8, 0.9, 1)(0.5, 0.3)$	$(0.3, 0.4, 0.4, 0.6)(0.4, 0.5)$
u_5	$(0.7, 0.8, 0.8, 1)(0.4, 0.5)$	$(0.6, 0.7, 0.7, 0.9)(0.7, 0.1)$	$(0.1, 0.2, 0.2, 0.3)(0.6, 0.4)$	$(0.4, 0.5, 0.5, 0.6)(0.5, 0.4)$

Step 2. The threshold intuitionistic trapezoidal fuzzy set of the ITrFSS $\acute{z} = (\acute{F}_{\{A\}}, E)$ is computed. As per requirements, one may compute mid-level threshold ITrFS $mid\acute{z}$ or top-level threshold ITrFS $top\acute{z}$ or bottom-level threshold ITrFS $bottom\acute{z}$ which are respectively defined in Definitions 3.4, 3.5, and 3.6.

Step 3. The level soft set is computed from the ITrFSS $\acute{z} = (\acute{F}_{\{A\}}, E)$ using $mid\acute{z}$ or $top\acute{z}$ or $bottom\acute{z}$ to produce respectively mid-level soft set $L(\acute{z}; mid\acute{z})$ or top-level soft set $L(\acute{z}; top\acute{z})$ or bottom-level soft set $L(\acute{z}; bottom\acute{z})$. Score and accuracy function of ITrFN, given in Definition 2.8, are used to compute the level soft sets.

Step 4. The level soft set $L(\acute{z}; mid\acute{z})$ or $L(\acute{z}; top\acute{z})$ or $L(\acute{z}; bottom\acute{z})$ is presented as a crisp relation R, where $R = [r_{ij}]_{mn}$. Here m be the number of alternatives and n be the number of parameters/attributes.

Step 5. The choice value c_i for each alternative $d_i, i = 1, 2, ..., m$ is determined form the relation R, $c_i = \sum_{j=1}^{n} r_{ij}$, where $r_{ij} \in \{0, 1\}$.

Step 6. Alternative $d_k, k \in 1, 2, ..., m$ is selected as optimal decision if $c_k = max_i c_i \forall i$

Step 7. If k has more than one value, then any one of d_k may be chosen.

Following steps 8, 9 and 10 are used to validate the results of Algorithm 1.

Step 8. The difference of each alternative $d_i, i = 1, 2, ..., m$ from the positive ideal ITrFN $I^+ = (a_{I^+}, b_{I^+}, c_{I^+}, d_{I^+}); (\mu_{I^+} \nu_{I^+},)$ and the negative ideal ITrFN $I^- = (a_{I^-}, b_{I^-}, c_{I^-}, d_{I^-}); (\mu_{I^-} \nu_{I^-},)$ is calculated. Normalized Hamming distance, given in Definition 2.9, is used to find the differences.

Step 9. The closeness coefficient CC_i of each alternative $d_i, i = 1, 2, ..., m$ is computed using the difference of the alternative from positive ideal ITrFN and negative ideal ITrFN.

Step 10. Alternative with maximum closeness coefficient is selected.

Algorithm 1 has been illustrated using the following example.

Example 3.8. Suppose Mr. X wants to buy a car among a set of five cars (alternatives) $U = u_1, u_2, u_3, u_4, u_5$={Scorpio, Innova, Bolero, Xylo, Xuv5} which have a set of four attributes given by $E = s_1, s_2, s_3, s_4$={Fuel economy, Price, Comfort, Design}. Information about the attributes of different cars is represented using an ITrFSS $z = (\acute{F}_{\{A\}}, E)$ which is given below in tabular form (Table 4).

Table 5. Tabular representation of the mid-level, top-level, and bottom-level soft sets and the corresponding choice values

	$L(\acute{z}; mid\acute{z})$					$L(\acute{z}; top\acute{z})$					$L(\acute{z}; bottom\acute{z})$				
	s_1	s_2	s_3	s_4	c_i	s_1	s_2	s_3	s_4	c_i	s_1	s_2	s_3	s_4	c_i
u_1	0	1	0	1	2	0	1	0	0	1	0	1	1	1	3
u_2	1	1	1	0	3	0	0	0	0	0	1	1	1	1	4
u_3	1	0	0	0	1	0	0	0	0	0	1	0	0	1	2
u_4	0	0	1	0	1	0	0	0	0	0	1	0	1	1	3
u_5	0	1	0	1	2	0	1	0	0	1	0	1	1	1	3

Mid-level threshold intuitionistic trapezoidal fuzzy set $mid\acute{z}$, top-level threshold intuitionistic trapezoidal fuzzy set $top\acute{z}$, and bottom-level threshold intuitionistic trapezoidal fuzzy set $bottom\acute{z}$ corresponding to the ITrFSS $\acute{z} = (\acute{F}_{\{A\}}, E)$ which is shown in Table 4, are given below. Computation procedures of these threshold intuitionistic trapezoidal fuzzy sets ($mid\acute{z}$, $top\acute{z}$ and $bottom\acute{z}$) are respectively given in Definitions 3.4, 3.5, and 3.6.

$$\begin{aligned} mid\acute{z} = \{&s_1, (0.46, 0.56, 0.56, 0.74)(0.52, 0.34)\}, \{s_2, (0.46, 0.58, 0.62, 0.76)(0.52, 0.4)\},\\ &\{s_3, (0.44, 0.56, 0.6, 0.7)(0.54, 0.34)\}, \{s_4, (0.48, 0.58, 0.6, 0.74)(0.44, 0.42)\} \end{aligned}$$

$$\begin{aligned} top\acute{z} = \{&s_1, (0.7, 0.8, 0.8, 0.1)(0.7, 0.1)\}, \{s_2, (0.7, 0.8, 0.8, 0.9)(0.7, 0.1)\},\\ &\{s_3, (0.7, 0.8, 0.9, 0.1)(0.7, 0.1)\}, \{s_4, (0.7, 0.8, 0.8, 0.9)(0.6, 0.3)\} \end{aligned}$$

$$\begin{aligned} bottom\acute{z} = \{&s_1, (0.3, 0.4, 0.4, 0.6)(0.3, 0.5)\}, \{s_2, (0.1, 0.3, 0.5, 0.6)(0.3, 0.6)\},\\ &\{s_3, (0.1, 0.2, 0.2, 0.3)(0.3, 0.5)\}, \{s_4, (0.3, 0.4, 0.4, 0.6)(0.3, 0.5)\}. \end{aligned}$$

Level soft sets such as mid-level soft set $L(\acute{z}; mid\acute{z})$ top-level soft set $L(\acute{z}; top\acute{z})$ and bottom-level soft set $L(\acute{z}; bottom\acute{z})$ and their corresponding choice values are displayed in Table 5.

Table 5 shows that the optimal decision is to select the car u_2 i.e., Innova if one uses mid-level threshold. Car Scorpio (u_1) or Xuv5 (u_5) will be selected if one uses top-level threshold. When one uses bottom-level threshold, car Innova (u_2) will be the optimal decision.

4. Adjustable approach to weighted ITrFSS based decision making

Definition 4.1. Let $\acute{P}(U)$ be the set of all intuitionistic trapezoidal fuzzy subsets of U, E be the set of parameters, and $A \subseteq E$. A weighted ITrFSS is a triple $\xi = (\acute{F}_{\{A\}}, E, \omega)$, where $(\acute{F}_{\{A\}}, E)$ is called ITrFSS over U and $\omega : A \to [0, 1]$ is an weight function specifying the weight $w_j = \omega(\epsilon_j)$ for each attribute $\epsilon_j \in A$. Weighted ITrFSS is considered as a special case of ITrFSS, where importance of different parameters are different and the parameters are categorized by their respective weights.

Below we present the adjustable approach to weighted ITrFSS based decision making using threshold ITrFSs and level soft sets.

Algorithm 2 The algorithm is explained below stepwise.

Step 1. Weighted ITrFSS $\xi = (\acute{F}_{\{A\}}, E, \omega)$ is taken as input. Weights of the attributes are computed using fuzzy AHP.

Step 2. The threshold intuitionistic trapezoidal fuzzy set of the ITrFSS $\acute{z} = (\acute{F}_{\{A\}}, E)$ is computed. As per requirements, one may compute mid-level threshold ITrFS $mid\acute{z}$ or top-level threshold ITrFS $top\acute{z}$ or bottom-level threshold ITrFS $bottom\acute{z}$ which are respectively defined in Definitions 3.4, 3.5, and 3.6.

Step 3. The level soft set is computed from the ITrFSS $\acute{z} = (\acute{F}_{\{A\}}, E)$ using $mid\acute{z}$ or $top\acute{z}$ or $bottom\acute{z}$ to produce respectively mid-level soft set $L(\acute{z}; mid\acute{z})$ or top-level soft set $L(\acute{z}; top\acute{z})$ or bottom-level soft set $L(\acute{z}; bottom\acute{z})$. Score and accuracy function of ITrFN, given in Definition 2.8, are used to compute the level soft sets.

Step 4. The level soft set $L(\acute{z}; mid\acute{z})$ or $L(\acute{z}; top\acute{z})$ or $L(\acute{z}; bottom\acute{z})$ is presented as a crisp relation R, where $R = [r_{ij}]_{mn}$. Here m be the number of alternatives and n be the number of parameters/attributes.

Step 5. The weighted choice value w_i for each alternative $d_i, i = 1, 2, ..., m$ is determined form the relation R, $w_i = \sum_{j=1}^{n} \omega_j r_{ij}$, where $r_{ij} \in \{0, 1\}$.

Step 6. Alternative $d_k, k \in 1, 2, ..., m$ is selected as optimal decision if $w_k = max_i w_i \forall i$

Step 7. If k has more than one value, then any one of d_k may be chosen.

Steps 8, 9 and 10 of algorithm 1, described earlier, are also used here to validate the results. Algorithm 2 has been illustrated using the following example.

Example 4.2. Let us consider the trapezoidal fuzzy comparison matrix [37] for the attributes $E = \{s_1, s_2, s_3, s_4\}$ as considered in Example 3.8, is given below:

$$\begin{bmatrix} 1 & (1,2,3,4) & (2,3,4,5) & (2,3,4,5) \\ (0.25,0.33,0.5,1) & 1 & (1,2,3,4) & (1,2,3,4) \\ (0.02,0.25,0.33,0.5) & (0.25,0.33,0.5,1) & 1 & (1,2,3,4) \\ (0.02,0.25,0.33,0.5) & (0.25,0.33,0.5,1) & (0.33,0.5,0.5,1) & 1 \end{bmatrix}.$$

Using the linear goal programming model [37], the normalized trapezoidal fuzzy weight vector is computed as:

$$W = \{(0.35, 0.45, 0.53, 0.53), (0.16, 0.22, 0.31, 0.38), (0.12, 0.15, 0.15, 0.19), (0.08, 0.09, 0.10, 0.15)^T\}.$$

Using weight vector and expected values, the attributes weights are computed as:

$$\omega_1 = 0.47, \omega_2 = 0.27, \omega_3 = 0.15, \omega_4 = 0.10.$$

Then the tabular representation of the ITrFSS $\acute{z} = (\acute{F}_{\{A\}}, E)$ along with the corresponding mid-level ($mid\acute{z}$), top-level ($top\acute{z}$), and bottom-level ($bottom\acute{z}$) threshold

Table 6. Tabular representation of the mid-level, top-level, and bottom-level soft sets and their weighted choice values

					$L(\acute{z}; mid\acute{z})$
	$s_1, \omega_1 = 0.47$	$s_2, \omega_2 = 0.27$	$s_3, \omega_3 = 0.15$	$s_4, \omega_4 = 0.10$	$\omega_i c_i$
u_1	0	1	0	1	0.37
u_2	1	1	1	0	**0.89**
u_3	1	0	0	0	0.47
u_4	0	0	1	0	0.15
u_5	0	1	0	1	0.37
					$L(\acute{z}; top\acute{z})$
	$s_1, \omega_1 = 0.47$	$s_2, \omega_2 = 0.27$	$s_3, \omega_3 = 0.15$	$s_4, \omega_4 = 0.10$	$\omega_i c_i$
u_1	0	1	0	0	0.27
u_2	0	0	0	0	0
u_3	0	0	0	0	0
u_4	0	0	0	0	0
u_5	0	1	0	0	0.27
					$L(\acute{z}; bottom\acute{z})$
	$s_1, \omega_1 = 0.47$	$s_2, \omega_2 = 0.27$	$s_3, \omega_3 = 0.15$	$s_4, \omega_4 = 0.10$	$\omega_i c_i$
u_1	0	1	1	1	0.52
u_2	1	1	1	1	**0.99**
u_3	1	0	0	1	0.57
u_4	1	0	1	1	0.72
u_5	0	1	1	1	0.52

intuitionistic trapezoidal fuzzy sets remain similar as shown in example 3.2. Level soft sets such as mid-level soft set $L(\acute{z}; mid\acute{z})$, top-level soft set $L(\acute{z}; top\acute{z})$, and bottom-level soft set $L(\acute{z}; bottom\acute{z})$ and their corresponding weighted choice values are given in Table 6.

Table 6 shows that the optimal decision is to select the car u_2 i.e., Innova if one uses mid-level threshold. Car Scorpio (u_1) or Xuv5 (u_5) will be selected if one uses top-level threshold. When one uses bottom-level threshold, car Innova (u_2) will be the optimal decision. Although the decisions are similar with table 5, but these may be changed with some changes in the trapezoidal fuzzy comparison matrix.

Validation of results using closeness coefficient

Table 7. Closeness coefficient of the alternatives

Closeness coefficient				
u_1	u_2	u_3	u_4	u_5
0.310701	**0.382799**	0.299685	0.30036	0.340284

The closeness coefficients of various alternatives are shown in Table 7, which reflects that the alternative u_2, i.e., car Innova has the highest closeness coefficient. In the proposed adjustable approaches based on ITrFSS and weighted ITrFSS, car Innova (u_2) has been selected as the optimal decision, when the middle-level or bottom-level thresholds are used. But the result of using top-level threshold is found to be different which is either u_1 or u_5. This is due to the fact that a few numbers of alternatives can participate in case of top-level threshold. In contrast, more numbers of alternatives are allowed to participate in mid-level and bottom-level thresholds. Due to the participation of more number of alternatives in the cases of mid-level and bottom-level thresholds, the final outcome is same as found as using closeness coefficient.

5. Comparative study

Decision makers often prefer to express their opinions using linguistic terms rather than exact numeric values since linguistic terms can capture the uncertainty more closely. For the purpose of computation, the linguistic terms must be converted to some suitable numerical values. Among some other options, ITrFNs are considered to be more suitable for the said task as ITrFNs represent both of the membership and non-membership functions using trapezoidal fuzzy numbers where the trapezoidal fuzzy numbers are piecewise linear and trapezoidal. Since ITrFSS based approaches use ITrFNs to represent the necessary information, thus ITrFSS based decision making approaches are more appropriate for many real life problems. In the existing adjustable approaches [18, 30, 19, 40], the authors have used FSS, IV-IFSS, and IFSS respectively to investigate the decision making problem, however, in this paper, we apply ITrFSS to solve the decision making problem. Since ITrFSS is a generalization of FSS and IFSS, hence the proposed approaches are more general compared to [18] and [19]. Moreover, the proposed approach has performed a validation testing using closeness coefficient whereas none of the relevant existing approaches performed validation testing.

6. Conclusions

In this paper, we generalize the approaches introduced by Feng et al. [18] and Jiang et al. [19] in the framework of ITrFSS. We present as adjustable approach to decision making using the proposed ITrFSS, thresholds and level soft sets. This approach is known as adjustable as it can adjust the decision makers judgements with respect to their thoughts whenever necessary using different thresholds. The subjectivity in the decision making activities and inexact human thoughts act as a basis of the adjustable approach. We have also introduced weighted ITrFSS and applied it in adjustable approach. The weights of various attributes of the weighted ITrFSS are derived using fuzzy AHP. Closeness coefficient has been used to validate the results of the adjustable approaches. In future, researchers may study the applications of level soft sets and thresholds in different extensions of fuzzy soft sets, vague sets and rough sets theory.

References

[1] Z. Aiwu and G. Hongjun, Fuzzy-valued linguistic soft set theory and multi-attribute decision-making application, Chaos, Solitons and Fractals 89 (2016) 2–7.

[2] J. C. R. Alcantud, A novel algorithm for fuzzy soft set based decision making from multiobserver input parameter data set, Information Fusion 29 (2016) 142–148.

[3] J. C. R. Alcantud, Some formal relationships among soft sets, fuzzy sets, and their extensions, International Journal of Approximate Reasoning 68 (2016) 45–53.

[4] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.

[5] M. Celik, I. Deha Er and A. F. Ozok, Application of fuzzy extended AHP methodology on shipping registry selection: the case of Turkish maritime industry, Expert Systems with Applications (2007) doi:10.1016/j.eswa.2007.09.004.

[6] S. Das, S. Ghosh, S. Kar and T. Pal, An algorithmic approach for predicting unknown information in incomplete fuzzy soft set, Arabian journal for science and engineering 42 (8) (2017) 3563–3571.

[7] S. Das and S. Kar, Group decision making in medical system: An intuitionistic fuzzy soft set approach, Applied Soft Computing 24 (2014) 196–211.

[8] S. Das, S. Kar and T. Pal, Group decision making using interval-valued intuitionistic fuzzy soft matrix and confident weight of experts, Journal of Artificial Intelligence and Soft Computing Research 4 (1) (2014) 57–77.

[9] S. Das, M. B. Kar, T. Pal and S. Kar, Multiple attribute group decision making using interval-valued intuitionistic fuzzy soft matrix, Proc. of IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), Beijing, July 6–11 (2014) 2222–2229, doi: 10.1109/FUZZ-IEEE.2014.6891687.

[10] S. Das, S. Kumar, S. Kar and T. Pal, Group decision making using neutrosophic soft matrix: An algorithmic approach, Journal of King Saud University - Computer and Information Sciences (2017), doi http://dx.doi.org/10.1016/j.jksuci.2017.05.001.

[11] S. Das, P. Kumari and A. K. Verma, Triangular fuzzy soft set and its application in MADM, International Journal of Computational Systems Engineering 2 (2) (2015) 85–93.

[12] S. Das, D. Malakar, S. Kar and T. Pal, Correlation measure of hesitant fuzzy soft sets and their application in decision making, Neural Computing and Applications 2017 DOI 10.1007/s00521-017-3135-0.

[13] S. Das. D. Malakar, S. Kar and T. Pal, A Brief Review and Future Outline on Decision Making Using Fuzzy Soft Set, International Journal of Fuzzy System Applications 7 (2) (2018) 1–43.

[14] I. Deli and N. Cagman, Intuitionistic fuzzy parameterized soft set theory and its decision making, Applied Soft Computing 28 (2015) 109–113.

[15] A. Dey and M. Pal, Generalised multi-fuzzy soft set and its application in decision making, Pacific Science Review A: Natural Science and Engineering 17 (2015) 23–28.

[16] Y. Du and P. D. Liu, Extended fuzzy VIKOR method with intuitionistic trapezoidal fuzzy numbers, Information 14 (2011) 2575–2583.

[17] P. Dutta and B. Limboo, Bell-shaped fuzzy soft sets and their application in medical diagnosis, Fuzzy Information and Engineering 9 (2017) 67–91.

[18] F. Feng, Y. B. Jun, X. Liu and L. Li, An adjustable approach to fuzzy soft set based decision making, J. Comput. Appl. Math. 234 (1) (2010) 10–20.

[19] Y. Jiang, Y. Tang and Q. Chen, An adjustable approach to intuitionistic fuzzy soft sets based decision making, Applied Mathematical Modelling 35 (2011) 824–836.

[20] Y. Jiang, Y. Tang, H. Liu and Z. Chen, Entropy on intuitionistic fuzzy soft sets and on interval-valued fuzzy soft sets, Inform. Sci.240 (2013) 95–114.

[21] P. K. Maji, An application of intuitionistic fuzzy soft sets in a decision making problem, in: IEEE International Conference on Progress in Informatics and Computing (PIC), vol. 1, December 10-12 (2010) 349–351.
[22] P. K. Maji and R. Biswas, A. R. Roy, Fuzzy soft sets, J. Fuzzy Math. 9 (3) (2001) 589–602.
[23] P. K. Maji and R. Biswas, A. R. Roy, Intuitionistic fuzzy soft set, J. Fuzzy Math. 9 (3) (2001) 677–692.
[24] P. K. Maji, A. R. Roy and R. Biswas, On intuitionistic fuzzy soft sets, J. Fuzzy Math. 12 (3) (2004) 669–683.
[25] L. Mikhailov, Fuzzy analytical approach to partnership selection in formation of virtual enterprises, Omega 30 (2002) 393–401.
[26] L. Mikhailov, Deriving priorities from fuzzy pairwise comparison judgments, Fuzzy Sets and Systems 134 (2003) 365–385.
[27] L. Mikhailov and P. Tsvetinov, Evaluation of services using a fuzzy analytic hierarchy process, Applied Soft Computing 5 (2004) 23–33.
[28] D. Molodtsov, Soft set theory-first results, Computers and Mathematics with Applications 37 (1999) 19–31.
[29] P. Muthukumar and G. Sai Sundara Krishnan, A similarity measure of intuitionistic fuzzy soft sets and its application in medical diagnosis, Applied Soft Computing 41 (2016) 148–156.
[30] H. Qin, X. Ma, T. Herawan and J. M. Zain, An adjustable approach to interval-valued intuitionistic fuzzy soft sets based decision making, N. T. Nguyen, C.-G. Kim, and A. Janiak (Eds.): ACIIDS 2011, LNAI 6592, pp. 80–89, 2011. Springer-Verlag Berlin Heidelberg 2011.
[31] A. R. Roy and P. K. Maji, A fuzzy soft set theoretic approach to decision making problems, J. Comput. Appl. Math. 203 (2) (2007) 412–418.
[32] T. L. Saaty, The Analytic Hierarchy Process, McGraw-Hill, New York 1980.
[33] H. Tang, A novel fuzzy soft set approach in decision making based on grey relational analysis and Dempster-Shafer theory of evidence, Applied Soft Computing 31 (2015) 317–325.
[34] J. Wang, Y. Hu, F. Xiao, X. Deng and Y. Deng, A novel method to use fuzzy soft sets in decision making based on ambiguity measure and Dempster-Shafer theory of evidence: An application in medical diagnosis, Artificial Intelligence in Medicine 69 (2016) 1–11.
[35] J. Q. Wang and Z. Zhang, Aggregation operators on intuitionistic trapezoidal fuzzy number and its application to multi-criteria decision making problems, Journal of Systems Engineering and Electronics 20 (2) (2009) 321–326.
[36] J. Q. Wang and Z. Zhang, Multi-criteria decision-making method with incomplete certain information based on intuitionistic fuzzy number, Control Decision 24 (2) (2009) 226–230.
[37] Y. M. Wang and K-S. Chin, A linear goal programming priority method for fuzzy analytic hierarchy process and its applications in new product screening, International Journal of Approximate Reasoning 49 (2008) 451–465.
[38] G. W. Wei, X. F. Zhao and H. J. Wang, GRA model for selecting an ERP system in trapezoidal intuitionistic fuzzy setting, Information 13 (2010) 1143–1148.
[39] Z. Zhang, A rough set approach to intuitionistic fuzzy soft set based decision making, Applied Mathematical Modelling 36 (2012) 4605–4633.
[40] Z. Zhang, C. Wang, D. Tian and K. Li, A novel approach to interval-valued intuitionistic fuzzy soft set based decision making, Applied Mathematical Modelling 38 (2014) 1255–1270.
[41] Z. Xiao, S. Xia, K. Gong and D. Li, The trapezoidal fuzzy soft set and its application in MCDM, Applied Mathematical Modelling 36 (2012) 5844–5855.
[42] J. Ye, Expected value method for intuitionistic trapezoidal fuzzy multi criteria decision-making problems, International Journal of General Systems 38 (2011) 11730–11734.
[43] J. Ye, Multicriteria group decision-making method using the distances-based similarity measures between intuitionistic trapezoidal fuzzy numbers, International Journal of General Systems 41 (2012) 729–739.
[44] J. Ye, Multi-criteria group decision-making method using vector similarity measures for trapezoidal intuitionistic fuzzy numbers, Group Decision and Negotiation 21 (2012) 519–530.
[45] Y. Yin, H. Li and Y. B. Jun, On algebraic structure of intuitionistic fuzzy soft sets, Computers and Mathematics with Applications 64 (2012) 2896–2911.

[46] D. Yu, Intuitionistic trapezoidal fuzzy information aggregation methods and their applications to teaching quality evaluation, Journal of Information & Computational Science 10 (6) (2013) 1861–1869.

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