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Abstract. Fuzzy set theory, soft set theory and rough set theory are mathematical tools for dealing with uncertainties and are closely related. In 1982, Pawlak initiated the rough set theory, Dubois and Prade combined fuzzy sets and rough sets all together. In 1999, Molodtsov introduced the concept of soft sets to solve complicated problems and various types of uncertainties. Maji et al. studied the (Zadeh's) fuzzification of the soft set theory. As a generalization, I define the notion of a soft set in **L**-set theory, introduce several operators for **L**-soft set theory, and investigate the rough operators on L^X induced by an **L**-soft set.

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Corresponding Author: Xueyou Chen (math-chen@qq.com)

1. Introduction

As a generalization of Zadeh's(classical) notion of a fuzzy set [24], the notion of an **L**-set was introduced in [10]. The basic ideas of **L**-set theory and its extensions, as well as many interesting applications, can be found in [1].

In 1982, Pawlak [19] initiated rough set theory to study incomplete and insufficient information. In rough set theory, the approximation of an arbitrary subset of a universe by two definable subsets are called lower and upper approximations, which correspond to two rough operators. The two rough operators were first defined by means of a given indiscernibility relation in [19]. Usually indiscernibility relations are supposed to be equivalences. Furthermore, as generalizations, they also were defined by an arbitrary binary relation in [23], a mapping in [4, 11], and other methods. Dubois and Prade [6] first investigated fuzzy rough set and rough fuzzy set.

Ganter and Wille [9] put forward formal concept analysis theory(FCA), which is an order-theoretical analysis of scientific data. Formal context, concept lattice are two main notions and tools. Bělohlávek [1] established **L**-order, **L**-context and **L**-concept lattice from the point of view of graded approach.

In 1999, Molodtsov [15] introduced the concept of soft sets to solve complicated problems and various types of uncertainties. Maji [13] and Ali et al. [16] introduced several operators for soft set theory. Moreover, Maji et al. [12] studied fuzzy soft set theory. Feng et al. [7, 8] investigated the problem of combining fuzzy sets, rough sets with soft sets. Recently, soft set theory has been developed rapidly by some scholars in theory and practice (See [14, 17, 18, 20, 21, 22]).

I defined the notion of a soft set in **L**-set theory and introduced several operators for **L**-soft set theory in [5], and investigated the rough operators on the set of all **L**-soft sets induced by the rough operators on L^X in [4]. As a continuation, in the paper, I define an **L**-order $\preceq$, an **L**-equality $\approx$, the union, the extended (restricted) intersection, AND, OR, the complement on **L**-soft sets, support them by examples, so a basic version of **L**-soft set theory is provided. On the other hand, I define the two rough operators on the set L^X of all **L**-sets induced by an **L**-soft set, discuss the lattice structure of **L**-rough sets, show that it includes the two rough operators on L^X in [4] as a special case.

The above contents are arranged into three parts, Section 3: **L**-soft sets, and Section 4: Rough operators on L^X induced by a soft set. In Section 2, I give an overview of **L**-sets, FCA, soft sets and fuzzy soft sets, rough sets, which surveys Preliminaries.

2. Preliminaries

The section is devoted to some main notions for each area, i.e., **L**-sets [1, 10], formal concept analysis[1, 9], soft sets [7, 13, 15, 18] and rough sets [2, 3, 4, 11, 19, 23].

2.1. **L-sets.** The seminal paper on fuzzy sets is [24]. As a generalization, the notion of an **L**-set was introduced in [10]. An overview of the theory of **L**-sets and **L**-relations (i.e., fuzzy sets and relations in the framework of complete residuated lattices) can be found in [1].

First, residuated lattice is one of the fundamental concepts.

Definition 2.1. A residuated lattice is an algebra **L**= $\langle L, \vee, \wedge, \otimes, \to, 0, 1\rangle$ such that

(i) $\langle L, \vee, \wedge, 0, 1\rangle$ is a lattice with the least element 0 and the greatest element 1,

(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e., $\otimes$ is associative, commutative, and it holds the identity $a \otimes 1 = a$,

(iii) $\otimes, \to$ form an adjoint pair, i.e., $a\otimes b \le c$ iff $a \le b \to c$ holds for all $a, b, c \in L$.

Residuated lattice **L** is called complete if $\langle L, \vee, \wedge\rangle$ is a complete lattice. In this paper, we assume that **L** is complete.

Next, an **L**-set is defined in the following manner.

For a universe set X, an **L**-set in X is a mapping $\tilde{A} : X \to L$. $\tilde{A}(x)$ indicates the truth degree of "x belongs to $\tilde{A}$". We use the symbol L^X to denote the set of all **L**-sets in X. For instance: $\tilde{1}_X : X \to L$, $\tilde{0}_X : X \to L$ are defined as: for all $x \in X$, $\tilde{1}_X(x) = 1, \tilde{0}_X(x) = 0$, respectively.

The negation operator is defined: for $\tilde{A} \in L^X$, $\tilde{A}^*(x) = \tilde{A}(x) \to 0$ for every $x \in X$. The classical order $\leq$ and equality $=$ are generalized in fuzzy setting. i.e., **L**-relation, **L**-equality.

$I \in L^{X\times X}$ is called an **L**-binary relation. The truth degree to which elements x and y are related by an **L**-relation I is denoted by $I(x, y)$ or (xIy).

A binary **L**-relation I on X is an **L**-equivalence if it satisfies: $\forall x, y, z \in X$, $I(x, x) = 1$ (reflexivity), $I(x, y) = I(y, x)$ (symmetry), $I(x, y) \otimes I(y, z) \leq I(x, z)$ (transitivity). An **L**-equivalence is an **L**-equality if it satisfies: $I(x, y) = 1$ implies $x = y$.

An **L**-order on X with an **L**-equality relation $\approx$ is a binary **L**-relation $\preceq$ which is compatible with respect to $\approx$ and satisfies: $\forall x, y, z \in X$, $(x \preceq x) = 1$ (reflexivity), $(x \preceq y) \wedge (y \preceq x) \leq (x \approx y)$ (antisymmetry), $(x \preceq y) \otimes (y \preceq z) \leq (x \preceq z)$ (transitivity). A set X equipped with an **L**-order $\preceq$ and an **L**-equality $\approx$ is called an **L**-ordered set $\langle\langle X, \approx\rangle, \preceq\rangle$.

The subsethood degree $S(\tilde{A}, \tilde{B})$ is defined as: for $\tilde{A}, \tilde{B} \in L^X$,

$$S(\tilde{A}, \tilde{B}) = \bigwedge_{x\in X} \tilde{A}(x) \to \tilde{B}(x), \text{ and } (\tilde{A} \preceq \tilde{B}) = S(\tilde{A}, \tilde{B}),$$

$$(\tilde{A} \approx \tilde{B}) = S(\tilde{A}, \tilde{B}) \wedge S(\tilde{B}, \tilde{A}). \text{ We write } \tilde{A} \subseteq \tilde{B}, \text{ if } S(\tilde{A}, \tilde{B}) = 1.$$

Example 2.2. For $\varnothing \neq W \subseteq L^X$, we obtain that $\langle\langle W, \approx\rangle, S\rangle$ is an **L**-ordered set. In fact, reflexivity and antisymmetry are trivial, we have to prove transitivity and compatibility.

Transitivity: $S(\tilde{A}, \tilde{B}) \otimes S(\tilde{B}, \tilde{C}) \leq S(\tilde{A}, \tilde{C})$ holds if and only if $S(\tilde{A}, \tilde{B}) \otimes S(\tilde{B}, \tilde{C}) \leq \tilde{A}(x) \to \tilde{C}(x)$, i.e., $\forall x \in X$, $\tilde{A}(x) \otimes S(\tilde{A}, \tilde{B}) \otimes S(\tilde{B}, \tilde{C}) \leq \tilde{C}(x)$, and it is true since $\tilde{A}(x) \otimes S(\tilde{A}, \tilde{B}) \otimes S(\tilde{B}, \tilde{C}) \leq \tilde{A}(x) \otimes (\tilde{A}(x) \to \tilde{B}(x)) \otimes (\tilde{B}(x) \to \tilde{C}(x)) \leq \tilde{C}(x)$.

In the similarly way, we also prove Compatibility:

$$S(\tilde{A}, \tilde{B}) \otimes (\tilde{A} \approx \tilde{A}_1) \otimes (\tilde{B} \approx \tilde{B}_1) \leq S(\tilde{A}_1, \tilde{B}_1).$$

By Example 2.2, we know that $\langle\langle L^X, \approx\rangle, \preceq\rangle$ is an **L**-ordered set.

Corresponding the operations $\vee, \wedge, \otimes$ on **L**, three operations are defined on **L**-sets as follows.

Definition 2.3. (i) Suppose $\{\tilde{A}_i \mid i \in I\} \subseteq L^X$ is a system of **L**-sets. Then $\bigvee_{i\in I} \tilde{A}_i$ and $\bigwedge_{i\in I} \tilde{A}_i$ are two **L**-sets defined as follows, for every $x \in X$,

$$(\bigvee_{i\in I} \tilde{A}_i)(x) = \bigvee_{i\in I} \tilde{A}_i(x), \quad (\bigwedge_{i\in I} \tilde{A}_i)(x) = \bigwedge_{i\in I} \tilde{A}_i(x).$$

(ii) For $\tilde{A}, \tilde{B} \in L^X$, $\tilde{A} \otimes \tilde{B}$ is an **L**-set in X defined as: for every $x \in X$,

$$(\tilde{A} \otimes \tilde{B})(x) = \tilde{A}(x) \otimes \tilde{B}(x).$$

2.2. **Formal concept analysis.** FCA is an order-theoretic method for the mathematical analysis of scientific data, pioneered by Ganter and Wille [9], has attracted a growing number of researchers and practitioners. I introduce by formalizing the notion of (formal) context.

A context is a triple (G, M, I) consisting of two sets G and M and a relation I between them. The elements of G are called the objects and the elements of M are called the attributes. We write gIm or $(g, m) \in I$ to show that the object g has the attribute m.

For a set $A \subseteq G$ of objects define $A^{'} = \{m \in M \mid gIm \ for \ all \ g \in A\}$. Correspondingly, for a set $B \subseteq M$ of attributes define $B^{'} = \{g \in G \mid gIm \ for \ all \ m \in B\}$. A concept of the context (G, M, I) is a pair (A, B) where $A \subseteq G, B \subseteq M$, $A^{'} = B$ and $B^{'} = A$. We call A the extent and B the intent of the concept (A, B).

Suppose (A_1, B_1), (A_2, B_2) are two concepts of the context (G, M, I). Then we define an order $\leq$, where $(A_1, B_1) \leq (A_2, B_2)$, if $A_1 \subseteq A_2$ (which is equivalent to $B_1 \supseteq B_2$).

Let $\mathcal{C}(G, M, I, \leq)$ be the set of all concepts of the context (G, M, I) with the order $\leq$. It forms a complete lattice (Concept Lattice) in which join and meet are given by:

$$\bigvee_{j\in J}(A_j, B_j) = ((\bigcup_{j\in J} A_j)^{''}, \bigcap_{j\in J} B_j), \qquad \bigwedge_{j\in J}(A_j, B_j) = (\bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{''}).$$

In [1], from the point of view of graded approach, Bělohlávek investigated **L**-order, **L**-Galois connection and **L**-concept lattice.

2.3. **Soft sets and fuzzy soft sets.** In 1999, Molodtsov [15] proposed soft sets and established the fundamental results of the new theory, to solve complicated problems and various types of uncertainties. A soft set is an approximate description of an object precisely consisting of two parts, namely predicate and approximate value set.

Let X be an initial universe set and E_X (simple E) be a collection of all possible parameters with respect to X. Usually, parameters are attributes, characteristics, or properties of objects in X (The role of E is the same with M in FCA).

In [15], Molodtsov introduced the notion of a soft set as follows.

Definition 2.4. A pair (F, A) is called a soft set over X, if $A \subseteq E$, and $F : A \to 2^X$, where 2^X is the power set of X.

In other words, a soft set over X is a parameterized family of subsets of the universe X. For $t \in A$, $F(t)$ may be viewed as the set of t-approximate elements of the soft set (F, A) (See [16]).

In fact, a soft set can be seen as a formal context and vice verse. Then it is related to the theory of formal concept analysis (See[9]).

As a generalization, in [12], Maji et al. defined the notion of a fuzzy soft set.

Definition 2.5. A pair (F, A) is called a fuzzy soft set over X if $A \subseteq E$, and $F : A \to [0, 1]^X$, where $[0, 1]^X$ is the collection of all fuzzy sets on X.

Maji et al. [12], and Ali and Shabir [17] introduced several operators for fuzzy soft set theory: equality of two fuzzy soft sets, subset and superset of a fuzzy soft set, complement of a fuzzy soft set, null fuzzy soft set, and absolute fuzzy soft set, the union, the intersection, etc.. Some researchers have studied soft sets, rough sets, fuzzy sets and soft topology (See [7, 8, 13, 15, 18, 20]).

2.4. **Rough sets.** Pawlak [19] initiated the rough set theory. Let (X, R) be an approximation space, and $R \subseteq X \times X$ be an equivalence relation. Then for $A \subseteq X$, two subsets $\underline{R}(A)$ and $\overline{R}(A)$ of X are defined:

$$\underline{R}(A) = \{x \in X \mid [x]_R \subseteq A\}, \quad \overline{R}(A) = \{x \in X \mid [x]_R \cap A \neq \varnothing\},$$

where $[x]_R = \{y \in X \mid xRy\}$.

If $\underline{R}(A) = \overline{R}(A)$, A is called a definable set; if $\underline{R}(A) \neq \overline{R}(A)$, A is called an undefinable set, and $(\underline{R}(A), \overline{R}(A))$ is referred to as a pair of rough set. Therefore, $\underline{R}$ and $\overline{R}$ are called two rough operators.

In [23], Yao defined the two rough operators by an arbitrary binary relation. Furthermore, there are many generalizations of the theory of rough sets. Dubois and Prade investigated fuzzy rough set and rough fuzzy set in [6].

In [11], Järvinen introduced the two rough operators in a lattice-theoretical setting and studied their properties. Suppose $(P, 0, 1, \vee, \wedge, ')$ is an atomic Boolean lattice, Q is the set of all atoms. As a generalization, for an arbitrary mapping $\varphi : Q \to P$, Järinven defined two rough approximation operators as follows: for every $a \in P, x \in Q$,

$$N(a) = \vee\{x \mid \varphi(x) \leq a\}, \quad H(a) = \vee\{x \mid a \wedge \varphi(x) \neq 0\} \qquad (*)$$

We generalized the method in fuzzy setting (See[4]). First, the operation ρ was defined between **L**-sets.

Definition 2.6. Suppose $\tilde{A}, \tilde{B} \in L^X$, let $\rho(\tilde{A}, \tilde{B}) = \bigvee\limits_{x \in X} \tilde{A}(x) \otimes \tilde{B}(x)$, which is the related degree of $\tilde{A}$ and $\tilde{B}$ with respect to $\otimes$.

Then, we gave the following definition, which is a graded extension of [11].

Definition 2.7. Suppose X is a universe set, L^X is the set of all **L**-sets on X, $M = \{\{a/x\} \mid a \in L, a > 0, x \in X\}$ is the set of all singletons, $\varphi : M \to L^X$ is an arbitrary mapping, then we obtain two **L**-rough operators N_φ and H_φ as follows: for every $\tilde{A} \in L^X, x \in X$,

$$N_\varphi(\tilde{A})(x) = \bigvee_{\{a/x\} \in M} a \otimes S(\varphi(\{a/x\}), \tilde{A}),$$

$$H_\varphi(\tilde{A})(x) = \bigvee_{\{a/x\} \in M} a \otimes \rho(\varphi(\{a/x\}), \tilde{A}).$$

If $N_\varphi(\tilde{A}) = H_\varphi(\tilde{A})$, then $\tilde{A}$ is called a definable **L**-set; otherwise, $\tilde{A}$ is called an undefinable **L**-set. $(N_\varphi(\tilde{A}), H_\varphi(\tilde{A}))$ is referred to as a pair of **L**-rough set.

When L^X is Boolean, the above definitions coincide with the formula (*).

Example 2.8. Suppose $X = \{x_1, x_2, x_3\}$, and $L = [0, 1]$ with $a \otimes b = min(a, b)$, $a \to b = 1$, if $a \leq b$; $a \to b = b$, if $a > b$. (Gödel Structure)

Let $\varphi(\{a/x_i\}) = \{a \otimes 0.5/x_i\}$ for $i = 1, 2, 3$. Then we have

$$\begin{aligned} S(\varphi(\{a/x_i\}), \tilde{A}) &= \bigwedge_{y \in X} \varphi(\{a/x_i\})(y) \to \tilde{A}(y) \\ &= \bigwedge_{y \in X} (\{a \otimes 0.5/x_i\})(y) \to \tilde{A}(y) \end{aligned}$$

$$= a \otimes 0.5 \to \tilde{A}(x_i), \text{ for } i = 1,\ 2,\ 3$$

and

$$\begin{aligned}\rho(\varphi(\{a/x_i\}), \tilde{A}) &= \bigvee_{y \in X} \varphi(\{a/x_i\})(y) \otimes \tilde{A}(y) \\ &= \bigvee_{y \in X} (\{a \otimes 0.5/x_i\})(y) \otimes \tilde{A}(y) \\ &= [a \otimes 0.5] \otimes \tilde{A}(x_i), \text{ for } i = 1,\ 2,\ 3.\end{aligned}$$

For $\tilde{A} = \{0.6/x_1, 0.2/x_2, 0.7/x_3\}$, we obtain

$$\begin{aligned}N_\varphi(\tilde{A})(x_1) &= \bigvee_{\{a/x_1\} \in M} a \otimes S(\varphi(\{a/x_1\}), \tilde{A}) \\ &= \bigvee_{\{a/x_1\} \in M} a \otimes [a \otimes 0.5 \to \tilde{A}(x_1)] \\ &= \bigvee_{\{a/x_1\} \in M} a \otimes [a \otimes 0.5 \to 0.6] \\ &= \bigvee_{a \in L} a \otimes [a \otimes 0.5 \to 0.6] = 1.\end{aligned}$$

Similarly, we obtain

$$N_\varphi(\tilde{A}) = \{1/x_1, 0.2/x_2, 1/x_3\},\ H_\varphi(\tilde{A}) = \{0.5/x_1, 0.2/x_2, 0.5/x_3\}.$$

3. L-soft sets

In the section, I generalize the notion of a soft set in fuzzy setting, define **L**-order, **L**-equivalence relation, and several operators on the set of all **L**-soft sets over X. The definitions are accompanies with examples.

Suppose X is a universe set, L^X is the set of all **L**-sets in X. We know that $(L^X, \bigcup, \bigcap, *, \tilde{1}_X, \tilde{0}_X)$ is not a Boolean algebra. Let E be a collection of all possible parameters with respect to X.

First, I define the notion of a soft set in fuzzy setting.

Definition 3.1. A pair (F, A) is called an **L**-soft set over X, if $A \subseteq E$ and $F : A \to L^X$, denoted by $\theta = (F, A)$.

Clearly, when **L=2**, the above definition coincides with Definition 2.4, when **L**=[0, 1], the above definition coincides with Definition 2.5.

Example 3.2. Suppose $X = \{x_1, x_2, x_3\}$, and $L = [0, 1]$ equipped with Gödel Structure. Let $E = \{t_1, t_2, t_3, t_4\}$, $A_1 = \{t_1, t_2, t_3\}$ and let $F_1 : A_1 \to L^X$, where $F_1(t_1) = \{0.7/x_1\ \}$, $F_1(t_2) = \{1/x_1, 0.5/x_2\}$, $F_1(t_3) = \{0.6/x_1,\ 0.2/x_2,\ 0.7/x_3\}$. Then clearly, (F_1, A_1) is an **L**-soft set.

Let LS(X) be the set of all **L**-soft sets over X. On which, there exist two kinds of special elements: one is called a absolute soft set $(1_A, A)$, $\forall t \in A$, $1_A(t) = \tilde{1}_X$, denoted by $\Gamma_A = (1_A, A)$; the other is called a null soft set $(0_A, A)$, $\forall t \in A$, $0_A(t) = \tilde{0}_X$, denoted by $\Phi_A = (0_A, A)$.

Second, I introduce the relation **L**-order $\preceq$, and **L**-equivalence relation $\approx$ which correspond the relations $\tilde{\subseteq}, =$ in classical case (See [12, 13, 15]). For two **L**-soft sets $\theta_1 = (F, A),\ \theta_2 = (G, B)\ \in$LS(X),

$$(\theta_1 \preceq \theta_2) = S(\theta_1, \theta_2) = \bigwedge_{t \in A} S(F(t), G(t)),$$

$$(\theta_1 \approx \theta_2) = S(\theta_1, \theta_2) \wedge S(\theta_2, \theta_1).$$

Example 3.3. Follows Example 3.2, (F_1, A_1) is an **L**-soft set. Let $A_2 = \{t_1, t_2, t_3, t_4\}$ and let $F_2 : A_2 \to L^X$, where $F_2(t_1) = \{0.4/x_1\}$, $F_2(t_2) = \{0.9/x_1, 0.5/x_2, 0.3/x_3\}$, $F_2(t_3) = \{0.4/x_1, 0.2/x_2, 0.5/x_3\}$, $F_2(t_4) = \{1/x_1, 0.7/x_2, 0.6/x_3\}$. Then (F_2, A_2) is also an **L**-soft set. Thus we obtain

$$\begin{aligned} & S((F_1, A_1), (F_2, A_2)) \\ = & \bigwedge_{t \in A_1} S(F_1(t), F_2(t)) \\ = & S(F_1(t_1), F_2(t_1)) \wedge S(F_1(t_2), F_2(t_2)) \wedge S(F_1(t_3), F_2(t_3)) = 0.4, \\ & S((F_2, A_2), (F_1, A_1)) = \bigwedge_{t \in A_2} S((F_2(t), F_1(t)) = 0. \end{aligned}$$

Clearly, we have

$\theta_1 \tilde{\subseteq} \theta_2 \Leftrightarrow S(\theta_1, \theta_2) = 1 \Leftrightarrow A \subseteq B$, and $\forall t \in A, F(t) \subseteq G(t)$,

$\theta_1 = \theta_2 \Leftrightarrow S(\theta_1, \theta_2) = 1, S(\theta_2, \theta_1) = 1 \Leftrightarrow A = B$, and $\forall t \in A, F(t) = G(t)$.

So $\langle\langle$LS(X),$\approx$,$\rangle$, $\preceq\rangle$ is an **L**-order set (See[1]). When **L**=**2**, the above definitions coincide with [15], when **L**=[0, 1], the above definition coincides with [12].

Example 3.4. Follows Example 3.2, (F_1, A_1) is a **L**-soft set. Let $A_3 = A_1$ and let $F_3 : A_3 \to L^X$, where $F_3(t_1) = \{\ 0.4/x_1\ \}$, $F_3(t_2) = \{0.9/x_1, 0.5/x_2\}$, $F_3(t_3) = \{0.4/x_1,\ 0.2/x_2,\ 0.5/x_3\}$. Then (F_3, A_3) is also an **L**-soft set and $(F_3, A_3) \tilde{\subseteq} (F_1, A_1)$.

Third, I introduce the union and the extended (restricted) intersection of two **L**-soft sets. Maji et al. [12]defined the union of two fuzzy soft sets as follows.

Definition 3.5. Suppose $(F, A), (G, B)$ ∈LS(X) are two **L**-soft sets. Then the union of (F, A) and (G, B) is an **L**-soft set (H, C), where $C = A \cup B$, and for $t \in C$,

$$H(t) = \begin{cases} F(t) & \text{if } \ t \in A - B \\ G(t) & \text{if } \ t \in B - A \\ F(t) \vee G(t) & \text{if } \ t \in A \cap B \end{cases}$$

and written as $(F, A) \tilde{\bigcup} (G, B) = (H, C)$.

About some properties of the union, I combine Proposition 3.2 in [12] and Proposition 2 in [17] as follows.

Proposition 3.6. (1) $(F, A) \tilde{\bigcup} (F, A) = (F, A)$,

(2) $(F, A) \tilde{\bigcup} (G, B) = (G, B) \tilde{\bigcup} (F, A)$,

(3) $((F, A) \tilde{\bigcup} (G, B)) \tilde{\bigcup} (H, C) = (F, A) \tilde{\bigcup} ((G, B) \tilde{\bigcup} (H, C))$,

(4) $(F, A) \tilde{\subseteq} (F, A) \tilde{\bigcup} (G, B)$, *and* $(G, B) \tilde{\subseteq} (F, A) \tilde{\bigcup} (G, B)$

(5) $(F, A) \tilde{\subseteq} (G, B) \Rightarrow (F, A) \tilde{\bigcup} (G, B) = (G, B)$,

(6) $(F, A) \tilde{\bigcup} \Phi_A = (F, A)$,

(7) $(F, A) \tilde{\bigcup} \Gamma_A = \Gamma_A$.

In [12], Maji et al. also defined the intersection of two fuzzy soft sets, i.e., suppose $(F, A), (G, B)$ ∈LS(X) are two **L**-soft sets, then the intersection of (F, A) and (G, B) is also an **L**-soft set (K, D), where $D = A \cap B$, and for $t \in D$, $K(t) = F(t)$ or $G(t)$ (as both are the same **L**-set).

But generally $F(t) = G(t)$ does not hold, and $A \cap B$ may be a empty set. So Ali et al. introduced a new definition (See Definition 3.3 in [16] and Definition 10 in [17]). In [16, 17], it is called the restricted intersection($\cap$).

Definition 3.7. Suppose $(F, A), (G, B) \in$LS(X) are two **L**-soft sets such that $A \cap B \neq \varnothing$. Then the (restricted) intersection of (F, A) and (G, B) is also an **L**-soft set (K, D), where $D = A \cap B$, and for $t \in D$, $K(t) = F(t) \wedge G(t)$. It is denoted by $(F, A)\tilde{\bigcap}(G, B) = (K, D)$.

Example 3.8. Follows Example 3.3, we obtain $(H, C) = (F_1, A_1)\tilde{\bigcup}(F_2, A_2)$, where $C = A_1 \cup A_2 = \{t_1, t_2, t_3, t_4\}$, and $H(t_1) = F_1(t_1) \vee F_2(t_1) = \{0.7/x_1\}$, $H(t_2) = F_1(t_2) \vee F_2(t_2) = \{1/x_1, 0.5/x_2, 0.3/x_3\}$, $H(t_3) = F_1(t_3) \vee F_2(t_3) = \{0.6/x_1, 0.2/x_2, 0.7/x_3\}$, $H(t_4) = F_2(t_4) = \{1/x_1, 0.7/x_2, 0.6/x_3\}$.

Similarly, $(K, D) = (F_1, A_1)\tilde{\bigcap}(F_2, A_2)$, where $D = A_1 \cap A_2 = \{t_1, t_2, t_3\}$, and $K(t_1) = F_1(t_1) \wedge F_2(t_1) = \{0.4/x_1\}$, $K(t_2) = F_1(t_2) \wedge F_2(t_2) = \{0.9/x_1, 0.5/x_2\}$, $K(t_3) = F_1(t_3) \wedge F_2(t_3) = \{0.4/x_1, 0.2/x_2, 0.5/x_3\}$.

In [16, 17],Ali et al. defined a new intersection, which is called the extended intersection.

Definition 3.9. Suppose $(F, A), (G, B) \in$LS(X) are two **L**-soft sets. Then the extended intersection of (F, A) and (G, B) is also an **L**-soft set (J, C), where $C = A \cup B$, and for $t \in C$,

$$J(t) = \begin{cases} F(t) & \text{if } \ t \in A - B \\ G(t) & \text{if } \ t \in B - A \\ F(t) \wedge G(t) & \text{if } \ t \in A \cap B \end{cases}$$

and written as $(F, A) \sqcap (G, B) = (J, C)$.

Example 3.10. Follows Example 3.8, we obtain $(J, C) = (F_1, A_1) \sqcap (F_2, A_2)$, where $C = A_1 \cup A_2 = \{t_1, t_2, t_3, t_4\}$, and $J(t_1) = F_1(t_1) \wedge F_2(t_1) = \{0.4/x_1\}$, $J(t_2) = F_1(t_2) \wedge F_2(t_2) = \{0.9/x_1, 0.5/x_2\}$, $J(t_3) = F_1(t_3) \wedge F_2(t_3) = \{0.4/x_1, 0.2/x_2, 0.5/x_3\}$, $J(t_4) = F_2(t_4) = \{1/x_1, 0.7/x_2, 0.6/x_3\}$.

Fourth, I consider the complement of an **L**-soft set (F, A). Maji et al. [12] introduced the notion of NOT SET OF A SET OF PARAMETERS, that is, Definition 3.11.

Definition 3.11. Let $E = \{t_1, t_2, \cdots, t_n\}$ be the set of parameters. The NOT set of E, denoted by $\rceil E$, is defined by $\rceil E = \{\neg t_1, \neg t_2, \cdots, \neg t_n\}$, where $\rceil t_i = not\ t_i$, $\forall i$. (It may be noted that $\rceil$ and $\neg$ are different operators).

About the NOT SET OF E, the following proposition holds.

Proposition 3.12 ([12]). (1) $\rceil(\rceil A) = A$,
(2) $\rceil(A \cup B) = \rceil A \cup \rceil B$,
(3) $\rceil(A \cap B) = \rceil A \cap \rceil B$.

Definition 3.13. The complement of an **L**-soft set (F, A) is denoted by $(F, A)^c$, and is defined by $(F, A)^c = (F^c, \rceil A)$, where $F^c : \rceil A \to L^X$, for every $\neg t \in \rceil A$,

$$F^c(\neg t) = F^*(t) = F(t) \to 0.$$

Example 3.14. Suppose $X = \{x_1, x_2, x_3\}$ and $L = [0, 1]$ with $a \otimes b = max(a + b - 1, 0)$, $a \to b = min(1 - a + b, 1)$ (Lukasiewicz Structure). $a^* = a \to 0$.

Consider (F_1, A_1) defined in Example 3.2. Then $(F_1, A_1)^c = (F_1^c, \rceil A_1)$, $\rceil A_1 = \{\neg t_1, \neg t_2, \neg t_3\}$, $F_1^c : \rceil A_1 \to L^X$, where
$F_1^c(\neg t_1) = \{0.7/x_1\}^* = \{0.3/x_1, 1/x_2, 1/x_3\}$,
$F_1^c(\neg t_2) = \{1/x_1, 0.5/x_2\}^* = \{0/x_1, 0.5/x_2, 1/x_3\}$,
$F_1^c(\neg t_3) = \{0.6/x_1,\ 0.2/x_2, 0.7/x_3\}^* = \{0.4/x_1, 0.8/x_2, 0.3/x_3\}$.
Thus Clearly, $\Gamma_A^c = \Phi_A$ and $\Phi_A^c = \Gamma_A$ hold.

Furthermore, **L** satisfies the law of double negation, if it satisfies $a = (a \to 0) \to 0$, for every $a \in L$, that is $a^{**} = a$ (See p. 32 in [1]).

In [17], the following proposition was proved.

Proposition 3.15 ([17], De Morgan Law)**.** *Suppose* **L** *satisfies the law of double negation. Then*
(1) $[(F, A)\tilde{\bigcup}(G, B)]^c = (F, A)^c \sqcap (G, B)^c$,
(2) $[(F, A) \sqcap (G, B)]^c = (F, A)^c\tilde{\bigcup}(G, B)^c$.

Finally, I introduce the operators OR, AND on LS(X).

Definition 3.16. Suppose $(F, A), (G, B) \in$LS(X), $(F, A)AND(G, B)$ is an **L**-soft set, denoted by $(F, A)\tilde{\wedge}(G, B) = (H, C)$, where $C = A \times B$, for every $t_1 \in A, t_2 \in B$, $H(t_1, t_2) = F(t_1) \wedge G(t_2)$.

$(F, A)OR(G, B)$ is an **L**-soft set, denoted by $(F, A)\tilde{\vee}(G, B) = (K, D)$, where $D = A \times B$, for every $t_1 \in A, t_2 \in B$, $K(t_1, t_2) = F(t_1) \vee G(t_2)$.

Example 3.17. Follows Example 3.3, (F_1, A_1) and (F_2, A_2) are two **L**-soft sets. Then we have, $(F_1, A_1)\tilde{\wedge}(F_2, A_2) = (H, C)$, where $C = A \times B$,

C	H
(t_1, t_1)	$\{0.4/x_1\}$
(t_1, t_2)	$\{0.7/x_1\}$
(t_1, t_3)	$\{0.4/x_1\}$
(t_2, t_1)	$\{0.4/x_1\}$
(t_2, t_2)	$\{0.9/x_1, 0.5/x_2\}$
(t_2, t_3)	$\{0.4/x_1, 0.2/x_2\}$
(t_3, t_1)	$\{0.4/x_1\}$
(t_3, t_2)	$\{0.6/x_1, 0.2/x_2, 0.3/x_3\}$
(t_3, t_3)	$\{0.4/x_1, 0.2/x_2, 0.5/x_3\}$

$(F_1, A_1)\tilde{\vee}(F_2, A_2) = (K, D)$, where $D = A \times B$,

D	K
(t_1, t_1)	$\{0.7/x_1\}$
(t_1, t_2)	$\{0.9/x_1, 0.5/x_2, 0.3/x_3\}$
(t_1, t_3)	$\{0.7/x_1, 0.2/x_2, 0.5/x_3\}$
(t_2, t_1)	$\{1/x_1, 0.5/x_2\}$
(t_2, t_2)	$\{1/x_1, 0.5/x_2, 0.3/x_3\}$
(t_2, t_3)	$\{1/x_1, 0.5/x_2, 0.5/x_3\}$
(t_3, t_1)	$\{0.6/x_1, 0.2/x_2, 0.7/x_3\}$
(t_3, t_2)	$\{0.9/x_1, 0.5/x_2, 0.7/x_3\}$
(t_3, t_3)	$\{0.6/x_1, 0.2/x_2, 0.7/x_3\}$

About the operators OR, AND on LS(X), the following De Morgan's types of results hold.

Proposition 3.18. *Suppose* **L** *satisfies the law of double negation. Then*
(1) $((F, A)\tilde{\vee}(G, B))^c = (F, A)^c\tilde{\wedge}(G, B)^c$,
(2) $((F, A)\tilde{\wedge}(G, B))^c = (F, A)^c\tilde{\vee}(G, B)^c$.

Remark 3.19. In [5], by means of N_φ, H_φ on L^X, we defined two rough operators N, H on LS(X) in following manner: for every $\theta = (F, A)$ and for every $t \in A$,

$$F_*(t) : A \to L^X, \quad F_*(t) = N_\varphi(F(t)),$$

$$F^*(t) : A \to L^X, \quad F^*(t) = H_\varphi(F(t)).$$

Then we obtain two **L**-soft sets $N(\theta) = (F_*, A)$, $H(\theta) = (F^*, A)$. The operators N, H are called the lower and upper rough approximations of **L**-soft sets. If $N(\theta) = H(\theta)$, the **L**-soft set θ is said to be definable, otherwise $(N(\theta), H(\theta))$ is called a pair of rough **L**-soft set.

I present the following example.

Example 3.20. Let $(F, A) = (F_1, A_1)$ defined in Example 3.2. Then we may obtain,
$F_*(t_1) = N_\varphi(F(t_1)) = \{1/x_1\}$,
$F_*(t_2) = N_\varphi(F(t_2)) = \{1/x_1, 1/x_2\}$,
$F_*(t_3) = N_\varphi(F_1(t_3)) = \{1/x_1, 0.2/x_2, 1/x_2\}$; and
$F^*(t_1) = H_\varphi(F_1(t_1)) = \{0.5/x_1\}$,
$F^*(t_2) = H_\varphi(F_1(t_2)) = \{0.5/x_1, 0.5/x_2\}$,
$F^*(t_3) = H_\varphi(F_1(t_3)) = \{0.5/x_1, 0.2/x_2, 0.5/x_2\}$.
Then $N(\theta) = (F_*, A)$, $H(\theta) = (F^*, A)$ is the lower and upper approximations of $\theta = (F, A)$. For more details, see [5].

4. Rough operators on L^X defined by a soft set

In the section, suppose X is a universe set, L^X is the set of all **L**-sets in X, E is a collection of all possible parameters with respect to X. I wish investigate two rough operators on L^X induced by an **L**-soft set.

Definition 4.1. Suppose $\theta = (F, A)$ is an **L**-soft set, two **L**-rough operators N and H are defined as follows: for every $\tilde{A} \in L^X, x \in X$,

$$N(\tilde{A})(x) = \bigvee_{t\in A} F(t)(x) \otimes S(F(t), \tilde{A}),$$

$$H(\tilde{A})(x) = \bigvee_{t\in A} F(t)(x) \otimes \rho(F(t), \tilde{A}). \quad (**)$$

If $N(\tilde{A}) = H(\tilde{A})$, then $\tilde{A}$ is called a definable **L**-set, otherwise, $\tilde{A}$ is called an undefinable **L**-set. $(N(\tilde{A}), H(\tilde{A}))$ is referred to as a pair of **L**-rough set.

Next, I introduce some examples.

Example 4.2. In [4], for L^X, let M be the set of all singletons. Then we define two **L**-rough approximation operators N_φ and H_φ induced by an arbitrary mapping $\varphi : M \to L^X$, for every $A \in L^X, x \in X$,

$$N_\varphi(A)(x) = \bigvee_{\{a/x\}\in M} a \otimes S(\varphi(\{a/x\}), A),$$

$$H_\varphi(A)(x) = \bigvee_{\{a/x\}\in M} a \otimes \rho(\varphi(\{a/x\}), A).$$

Obviously, if we choose M is the set of all possible parameters with respect to X, then (φ, M) is an **L**-soft set, and according to the formula (**). Thus we obtain $N = N_\varphi, H = H_\varphi$.

Furthermore, if $(L^X, \bigcup, \bigcap, *, \tilde{1}_X, \tilde{0}_X)$ is a Boolean lattice, and E is the set of all atoms. When $A = E$, the above definition coincides with the case in [11].

Example 4.3. Suppose $X = \{x_1, x_2, x_3\}$ and $L = [0, 1]$ equipped with Gödel Structure. Let $E = \{t_1, t_2, t_3, t_4\}$, $A = \{t_1, t_2, t_3\}$ and let $F : A \to L^X$ is defined as:

$F(t_1) = \{0.5/x_1\}$, $F(t_2) = \{0.4/x_1, 0.7/x_2, 0.7/x_3\}$, $F(t_3) = \{0.7/x_2, 0.4/x_3\}$. Then (F, A) is an **L**-soft set.

Let $\tilde{A} = \{0.1/x_1, 0.5/x_2, 0.7/x_3\}$. Then we have

$$S(F(t_1), \tilde{A}) = \bigwedge_{y\in X} F(t_1)(y) \to \tilde{A}(y) = 0.1,$$

$$S(F(t_2), \tilde{A}) = \bigwedge_{y\in X} F(t_2)(y) \to \tilde{A}(y) = 0.1,$$

$$S(F(t_3), \tilde{A}) = \bigwedge_{y\in X} F(t_3)(y) \to \tilde{A}(y) = 0.5;$$

$$\rho(F(t_1), \tilde{A}) = \bigvee_{y\in X} F(t_1)(y) \otimes \tilde{A}(y) = 0.1,$$

$$\rho(F(t_2), \tilde{A}) = \bigvee_{y\in X} F(t_2)(y) \otimes \tilde{A}(y) = 0.7,$$

$$\rho(F(t_3), \tilde{A}) = \bigvee_{y\in X} F(t_3)(y) \otimes \tilde{A}(y) = 0.5.$$

Thus

$N(\tilde{A})(x_1)$

$= \bigvee_{t\in A} F(t)(x_1) \otimes S(F(t), A)$
$= [F(t_1)(x_1) \otimes S(F(t_1), A)] \vee [F(t_2)(x_1) \otimes S(F(t_2), A)] \vee [F(t_3)(x_1) \otimes S(F(t_3), A)]$
$= 0.1,$

$$N(\tilde{A})(x_2) = \bigvee_{t\in A} F(t)(x_2) \otimes S(F(t), A) = 0.5,$$

$$N(\tilde{A})(x_3) = \bigvee_{t\in A} F(t)(x_3) \otimes S(F(t), A) = 0.4.$$

So $N(\tilde{A}) = \{0.1/x_1, 0.5/x_2, 0.4/x_3\}$.

Similarly, we obtain $H(\tilde{A}) = \{0.4/x_1, 0.7/x_2, 0.7/x_3\}$.

Let $P = \{N(\tilde{A}) \mid \tilde{A} \in L^X\}$ and $Q = \{H(\tilde{A}) \mid \tilde{A} \in L^X\}$, which are the set of all lower approximations, and all upper approximations, respectively. We define the relation **L**-order $\preceq$ and **L**-equivalence relation $\approx$. For example, for $N(\tilde{A}), N(\tilde{B}) \in P$,

$$S(N(\tilde{A}), N(\tilde{B})) = \bigwedge_{x\in X} N(\tilde{A})(x) \to N(\tilde{B})(x),$$

$$(N(\tilde{A}) \preceq N(\tilde{B})) = S(N(\tilde{A}), N(\tilde{B})),$$

$$(N(\tilde{A}) \approx N(\tilde{B})) = S(N(\tilde{A}), N(\tilde{B})) \wedge S(N(\tilde{B}), N(\tilde{A})).$$

Then $\langle\langle P, \approx\rangle, \preceq\rangle$ and $\langle\langle Q, \approx\rangle, \preceq\rangle$ are two **L**-ordered sets.

Certainly, there is a natural bivalent order relation on the set of all **L**-rough sets $\{(N(\tilde{A}), H(\tilde{A})) \mid \tilde{A} \in L^X\}$ defined by: $(N(\tilde{A}), H(\tilde{A})) \leq (N(\tilde{B}), H(\tilde{B}))$ iff $N(\tilde{A}) \subseteq N(\tilde{B})$, and $H(\tilde{A}) \subseteq H(\tilde{B})$.

Furthermore, we may define the relation **L**-order $\preceq$, and **L**-equivalence relation $\approx$ on $\{(N(\tilde{A}), H(\tilde{A})) \mid \tilde{A} \in L^X\}$, for any $\tilde{A}, \tilde{B} \in L^X$,

$$S((N(\tilde{A}), H(\tilde{A})), (N(\tilde{B}), H(\tilde{B}))) = S(N(\tilde{A}), N(\tilde{B})) \wedge S(H(\tilde{A}), H(\tilde{B})),$$
$$((N(\tilde{A}), H(\tilde{A})) \preceq (N(\tilde{B}), H(\tilde{B}))) = S((N(\tilde{A}), H(\tilde{A})), (N(\tilde{B}), H(\tilde{B}))),$$

and

$$((N(\tilde{A}), H(\tilde{A})) \approx (N(\tilde{B}), H(\tilde{B}))) = S((N(\tilde{A}), H(\tilde{A})), (N(\tilde{B}), H(\tilde{B}))) \wedge S((N(\tilde{B}), H(\tilde{B})), (N(\tilde{A}), H(\tilde{A}))).$$

Thus $\{(N(\tilde{A}), H(\tilde{A})) \mid \tilde{A} \in L^X\}$ is also an **L**-ordered set.

Third, I investigate some properties of the two rough operators.

In the classical case, N and H are monotone increasing, i.e., if $\tilde{A} \subseteq \tilde{B}$, $N(\tilde{A}) \subseteq N(\tilde{B})$ and $H(\tilde{A}) \subseteq H(\tilde{B})$ hold. For L^X, from the point of view of graded approach, we will prove the two **L**-rough operators are monotone increasing for the subsethood degrees (See (2) and (3) in Proposition 4.4).

Proposition 4.4. (1) $H(\tilde{0}_X) = \tilde{0}_X$,
(2) $S(\tilde{A}, \tilde{B}) \leq S(N(\tilde{A}), N(\tilde{B}))$,
(3) $S(\tilde{A}, \tilde{B}) \leq S(H(\tilde{A}), H(\tilde{B}))$.

Proof. (1) For every $x \in X$, we have

$$H(\tilde{0}_X)(x) = \bigvee_{t\in A} F(t)(x) \otimes \rho(F(t), \tilde{0}_X) = \bigvee_{t\in A} F(t)(x) \otimes 0 = 0.$$

(2) For every $x \in X$, we have

$$\begin{aligned} S(\tilde{A},\tilde{B})\otimes N(\tilde{A})(x) &= S(\tilde{A},\tilde{B})\otimes \bigvee_{t\in A} F(t)(x)\otimes S(F(t),\tilde{A}) \\ &= \bigvee_{t\in A} S(\tilde{A},\tilde{B})\otimes F(t)(x)\otimes S(F(t),\tilde{A}) \\ &\leq \bigvee_{t\in A} F(t)(x)\otimes S(F(t),\tilde{B}) \\ &= N(\tilde{B})(x). \end{aligned}$$

Then we obtain $S(\tilde{A},\tilde{B}) \leq N(\tilde{A})(x) \to N(\tilde{B})(x)$. Thus $S(\tilde{A},\tilde{B}) \leq S(N(\tilde{A}),N(\tilde{B}))$ holds.

(3) For every $x \in X$, we have

$$\begin{aligned} S(\tilde{A},\tilde{B})\otimes H(\tilde{A})(x) &= S(\tilde{A},\tilde{B})\otimes \bigvee_{t\in A} F(t)(x)\otimes \rho(F(t),\tilde{A}) \\ &= \bigvee_{t\in A} S(\tilde{A},\tilde{B})\otimes F(t)(x)\otimes \rho(F(t),\tilde{A}) \\ &\leq \bigvee_{t\in A} F(t)(x)\otimes \rho(F(t),\tilde{B}) \\ &= H(\tilde{B})(x). \end{aligned}$$

Then we obtain $S(\tilde{A},\tilde{B}) \leq H(\tilde{A})(x) \to H(\tilde{B})(x)$. Thus $S(\tilde{A},\tilde{B}) \leq S(H(\tilde{A}),H(\tilde{B}))$ holds.

Clearly, for $\tilde{A},\tilde{B} \in L^X$, if $\tilde{A} \subseteq \tilde{B}$, we also have $N(\tilde{A}) \subseteq N(\tilde{B}), H(\tilde{A}) \subseteq H(\tilde{B})$. □

Remark 4.5. (F,A) is called a full soft set, if $\bigvee_{t\in A} F(t) = \tilde{1}_X$. i.e., $\{F(t) \mid t \in A\}$ is a cover of L^X. In the case, we have $N(\tilde{1}_X) = \tilde{1}_X$.

In [11], the set of all lower approximations and the set of all upper approximations form complete lattices. In fuzzy setting, we also obtain the following propositions, which show the algebraic structure of P and Q.

Proposition 4.6. *Suppose* $\{\tilde{A}_i \mid i \in I\} \subseteq L^X$. *Then*

(1) $\bigvee_{i\in I} N(\tilde{A}_i) \subseteq N(\bigvee_{i\in I} \tilde{A}_i)$,

(2) $N(\bigwedge_{i\in I} \tilde{A}_i) \subseteq \bigwedge_{i\in I} N(\tilde{A}_i)$,

(3) $H(\bigwedge_{i\in I} \tilde{A}_i) \subseteq \bigwedge_{i\in I} H(\tilde{A}_i)$.

L satisfies idempotency, if it satisfies $a \otimes a = a$, for every $a \in L$ (See [1]).

Proposition 4.7. *Suppose* **L** *satisfies idempotency. Then for* $\tilde{A},\tilde{B} \in L^X$,

$$H(\tilde{A})\otimes H(\tilde{B}) = H(\tilde{A}\otimes\tilde{B}).$$

Proof. Suppose $\tilde{A},\tilde{B} \in L^X$. Then for every $x \in X$, we have

$$\begin{aligned} H(\tilde{A}\otimes\tilde{B})(x) &= \bigvee_{t\in A} F(t)(x)\otimes \rho(F(t),\tilde{A}\otimes\tilde{B}) \\ &= \bigvee_{t\in A} F(t)(x)\otimes \bigvee_{y\in X} F(t)(y)\otimes \tilde{A}(y)\otimes\tilde{B}(y) \\ &= \bigvee_{t\in A} F(t)(x)\otimes [\bigvee_{y\in X} F(t)(y)\otimes \tilde{A}(y)] \otimes [\bigvee_{y\in X} F(t)(y)\otimes \tilde{B}(y)] \\ &= \bigvee_{t\in A} F(t)(x)\otimes \rho(F(t),\tilde{A})\otimes \rho(F(t),\tilde{B}) \\ &= \bigvee_{t\in A} F(t)(x)\otimes \rho(F(t),\tilde{A})\otimes \bigvee_{t\in A} F(t)(x)\otimes \rho(F(t),\tilde{B}) \end{aligned}$$

$$= H(\tilde{A})(x) \otimes H(\tilde{B})(x).$$

Thus the above proposition holds. □

Proposition 4.8. *Suppose* $Q = \{H(\tilde{A}) \mid \tilde{A} \in L^X\}$ *is closed for* $\bigcup$, *that is,* $\mathcal{U}$ *is an* **L**-*set in* Q. *Then* $\bigcup \mathcal{U} \in Q$.

Proof. Suppose $\mathcal{U}$ is an **L**-set in Q. Then for every $x \in X$, we have

$$\bigcup \mathcal{U}(x) = \bigvee_{\tilde{A} \in L^X} \mathcal{U}(H(\tilde{A})) \otimes H(\tilde{A})(x).$$

For an **L**-set $\mathcal{U}$ in Q, there exists an **L**-set $\mathcal{U}^*$ in L^X such that for every $\tilde{A} \in L^X$, we have $\mathcal{U}^*(\tilde{A}) = \mathcal{U}(H(\tilde{A}))$. Let $\tilde{D} = \bigcup \mathcal{U}^*$, i.e.,

$$\tilde{D}(x) = \bigvee_{\tilde{A} \in L^X} \mathcal{U}^*(\tilde{A}) \otimes \tilde{A}(x) = \bigvee_{\tilde{A} \in L^X} \mathcal{U}(H(\tilde{A})) \otimes \tilde{A}(x).$$

Then

$$\begin{aligned}
\bigcup \mathcal{U}(x) &= \bigvee_{\tilde{A} \in L^X} \mathcal{U}(H(\tilde{A})) \otimes H(\tilde{A})(x) \\
&= \bigvee_{\tilde{A} \in L^X} \mathcal{U}(H(\tilde{A})) \otimes \bigvee_{t \in A} F(t)(x) \otimes \rho(F(t), \tilde{A}) \\
&= \bigvee_{t \in A} F(t)(x) \otimes \bigvee_{\tilde{A} \in L^X} \mathcal{U}(H(\tilde{A})) \otimes \rho(F(t), \tilde{A}) \\
&= \bigvee_{t \in A} F(t)(x) \otimes \bigvee_{\tilde{A} \in L^X} \mathcal{U}(H(\tilde{A})) \otimes \bigvee_{y \in X} F(t)(y) \otimes \tilde{A}(y) \\
&= \bigvee_{t \in A} F(t)(x) \otimes \bigvee_{y \in X} F(t)(y) \otimes \bigvee_{\tilde{A} \in L^X} \mathcal{U}(H(\tilde{A})) \otimes \tilde{A}(y) \\
&= \bigvee_{t \in A} F(t)(x) \otimes \bigvee_{y \in X} F(t)(y) \otimes \tilde{D}(y) \\
&= \bigvee_{t \in A} F(t)(x) \otimes \rho(F(t), \tilde{D}) \\
&= H(\tilde{D})(x).
\end{aligned}$$

Suppose $\{\tilde{A}_i \mid i \in I\} \subseteq Q$. Then Clearly, we also have

$$H(\bigvee_{i \in I} \tilde{A}_i) = \bigvee_{i \in I} H(\tilde{A}_i).$$

□

Remark 4.9. By the above propositions, we know that if **L** satisfies idempotency, Q is an **L**-open topology on X (See [1]). Unfortunately, P does not form an **L**-topology on X.

In fact, Q is a semilattice with respect to $\bigcup$, and the minimal element is 0_X.

Conclusion

In the paper, I investigated two problems. One is generalized the notion of soft set in fuzzy setting, and introduced several operators for **L**-soft set theory: the complement of an **L**-soft set; **L**-order, **L**-equivalence relation, the union, the intersection, OR, AND of two **L**-soft sets. The other is defined two rough operators on L^X by an **L**-soft set, and discussed some of their properties.

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Chen Xueyou (math-chen@qq.com)
School of Mathematics, Shandong University of Technology, Zibo, Shandong 255049, P. R. China