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A new proof of generalized Tychonoff theorem in (L, M)-fuzzy topological spaces

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Abstract. In this paper, using the structures of (L, M)-fuzzy topological product spaces which were introduced by Hu Zhao, Sheng-Gang Li and Gui-Xiu Chen, we directly give another version on the proof of generalized Tychonoff theorem in (L, M)-fuzzy topological spaces which was introduced by Hong-Yan Li and Fu-Gui Shi.

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Keywords: (L, M)-fuzzy topology, (L, M)-fuzzy neighborhood system, Product space, Fuzzy compactness degree, Generalized Tychonoff theorem.

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1. Introduction and Preliminaries

The notion of the measures (or degrees) of fuzzy compactness in (L, M)-fuzzy topological spaces was introduced by Hong-Yan Li and Fu-Gui Shi [4, 5] and a version on the proof of generalized Tychonoff theorem was obtained indirectly through using the subbase of (L, M)-fuzzy topology.

The relationship between (L, M)-fuzzy topology and (L, M)-fuzzy neighborhood system were further studied [8], and the initial structures of (L, M)-fuzzy neighborhood subspaces and (L, M)-fuzzy topological product spaces were given.

The construction of initial structures in the category of (L, M)-fuzzy topological spaces through those in the category of (L, M)-fuzzy neighborhood systems really looks rather interesting; the fact that the two categories are isomorphic [8], however, enables researchers to substitute one of them with the other, to find a solution of a complicated problem. A natural problem is: Can the proof of generalized Tychonoff theorem be given directly in an (L, M)-fuzzy topological space?

In this paper, using the structures of (L, M)-fuzzy topological product spaces [8], we directly give another version on the proof of generalized Tychonoff theorem in (L, M)-fuzzy topological spaces.

The following preliminaries will be used throughout this paper, which can be found in [1, 6].

A complete lattice L is called completely distributive, if one of the following conditions hold (the second then following as a consequence [1]):

(CD1)

$$\bigwedge_{i\in I}\left(\bigvee_{i\in J_i} a_{i,j}\right)=\bigvee_{f\in\prod J_i}\left(\bigwedge_{i\in I} a_{i,f(i)}\right),$$

(CD2)

$$\bigvee_{i\in I}\left(\bigwedge_{i\in J_i} a_{i,j}\right)=\bigwedge_{f\in\prod J_i}\left(\bigvee_{i\in I} a_{i,f(i)}\right),$$

where for each $i\in I$ and $j\in J_i, a_{i,j}\in L$ and $f\in\prod J_i$ means that f is a mapping $f: I\to\bigcup J_i$ such that $f(i)\in J_i$ for each $i\in I$.

An element $a\neq 0$ in a lattice is called coprime if $a\leq b\vee c$ implies $a\leq b$ or $a\leq c$ for all $b,c\in L$. Further, a is said to be join irreducible if $a=b\vee c$ implies $a=b$ or $a=c$ for all $b,c\in L$. The set of all coprime elements (resp. join irreducible elements) of L is denoted by $\mathrm{Copr}(L)$ (resp. $J(L)$). It can be verified that if L is distributive, then $a\in L$ is coprime iff it is join irreducible, which means $\mathrm{Copr}(L)=J(L)$. So, for convenience, we usually use $J(L)$ to stand for the set of all coprime elements of L if L is distributive. If L is a completely distributive lattice and $x\lhd\bigvee_{t\in T} y_t$, then there must be $t^\star\in T$ such that $x\lhd y_{t^\star}$ (here $x\lhd a$ means: $K\subset L, a\leq\bigvee K\Rightarrow\exists y\in K$ such that $x\leq y$). Some more properties of $\lhd$ can be found in [6].

Let L be a complete lattice, let $b\in L$, and let $A\subseteq L$. If (i) $\bigvee A=b$, (ii) if $C\subseteq L$ and $\bigvee C\geq b$, then $\forall x\in A$, there esists $y\in C$ such that $y\geq x$. Then A is said to be a minimal family of b. It can prove that the supremum of several minimal families of b is still a minimal family of b. Thus, if b has a minimal family, there must be a maximum minimum family, denoted as $\beta(b)$. It can be verified that if L is a completely distributive lattice iff each element b in L has a minimal family, and $\beta(b)(=\{a\in L\mid a\lhd b\})$ is the greatest minimal family of b, $\beta^*(b)=\beta(b)\cap J(L)$.

An element $a\neq 1$ in a lattice is called prime if $a\geq b\wedge c$ implies $a\geq b$ or $a\geq c$ for all $b,c\in L$. The set of all primes of L is denoted by $P(L)$. If L is a completely distributive lattice, then for each $a\in L$, there exists $B_x\subseteq P(L)$ such that $\bigwedge B_x=x$. $\alpha(b)$ is the greatest maximal family of b, $\alpha^*(b)=\alpha(b)\cap P(L)$ (see [7]).

In the rest of the paper, L and M always denote Hutton algebras. A Hutton algebra L, is a completely distributive lattice with order-reversing involution with the least element 0 and the greatest element 1. Recall that an order-reversing involution $'$ on L is a map $(-)': L\longrightarrow L$ such that for any $a,b\in L$, the following conditions hold: (1) $a\leq b$ implies $b'\leq a'$. (2) $a''=a$. The following properties hold for any subset $\{b_i: i\in I\}\in L$: (1) $(\bigvee_{i\in I} b_i)'=\bigwedge_{i\in I} b_i'$; (2) $(\bigwedge_{i\in I} b_i)'=\bigvee_{i\in I} b_i'$. We notice that L^X, the set of all L-subsets of X, is also a Hutton algebra with pointwise order. Its smallest element and the largest element are denoted 0_X and 1_X, respectively. For each $A\in L^X$, the L-subset A' is defined $A'(x)=(A(x))'$ for each $x\in X$. Clearly, $J(L^X)=\{x_\lambda: x\in X,\lambda\in J(L)\}$, where x_λ is defined by $x_\lambda(y)=\lambda$ if $y=x$ and $x_\lambda(y)=0$ otherwise.

For a subfamily $\varphi\subseteq L^X$, $2^{(\varphi)}$ denotes the set of all finite subfamilies of φ.

Definition 1.1 ([2, 3])**.** An (L,M)-fuzzy topology on a set X is a map $\mathcal{T}: L^X \longrightarrow M$ such that

(LMFT1) $\mathcal{T}(1_X) = \mathcal{T}(0_X) = 1$,
(LMFT2) $\forall\, U, V \in L^X, \mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V)$,
(LMFT3) $\forall \{U_j : j \in J\} \subseteq L^X, \mathcal{T}\left(\bigvee_{j\in J} U_j\right) \geqslant \bigwedge_{j\in J} \mathcal{T}(U_j)$.

$\mathcal{T}(U)$ can be interpreted as the degree to which U is an open L-set, $\mathcal{T}^\star(U) = \mathcal{T}(U')$ will be called the degree of closedness. The pair $(X, \mathcal{T})$ is called (L,M)-fuzzy topological space. A mapping $f: X \longrightarrow Y$ from an (L,M)-fuzzy topological space $(X, \mathcal{T}_1)$ to another (L,M)-fuzzy topological space $(Y, \mathcal{T}_2)$ is said to be continuous if $\mathcal{T}_1(f^{\leftarrow}(B)) \geq \mathcal{T}_2(B)$ for each $B \in L^Y$. The category of all (L,M)-fuzzy topological spaces and their continuous mappings is denoted by (L,M)-**FTOP**.

The next Definition 1.2 and Lemma 1.3 were introduced by Shi [9] for an L-fuzzy topology, but could be easily reformulated for (L,M)-fuzzy topology as follows (See also, [8, 9]).

Definition 1.2. An (L,M)-fuzzy neighborhood system on a set X is a map $\mathcal{N}: L^X \longrightarrow M^{J(L^X)}$ satisfying the following conditions:

(LMFN1) $\mathcal{N}(1_X)(x_\lambda) = 1,\ \mathcal{N}(0_X)(x_\lambda) = 0\ \ (\forall\, x_\lambda \in J(L^X))$,
(LMFN2) $\mathcal{N}(U)(x_\lambda) = 0\ \ (\forall\, U \in L^X, \forall\, x_\lambda \in J(L^X), x_\lambda \not\leq U)$,
(LMFN3) $\mathcal{N}(U \wedge V)(x_\lambda) = \mathcal{N}(U)(x_\lambda) \wedge \mathcal{N}(V)(x_\lambda)\ \ (\forall\, U, V \in L^X, \forall\, x_\lambda \in J(L^X))$,
(LMFN4) $\mathcal{N}(U)(x_\lambda) = \bigvee_{x_\lambda \leq V \leq U} \bigwedge_{y_\mu \lhd V} \mathcal{N}(V)(y_\mu)\ (\forall U \in L^X, x_\lambda, y_\mu \in J(L^X))$.

$\mathcal{N}(U)(x_\lambda)$ is called the degree to which x_λ belongs to the neighborhood of U. The pair $(X, \mathcal{N})$ is called an (L,M)-fuzzy neighborhood space. A mapping $f: X \longrightarrow Y$ from an (L,M)-fuzzy neighborhood space $(X, \mathcal{N}_1)$ to another (L,M)-fuzzy neighborhood space $(Y, \mathcal{N}_2)$ is said to be continuous if $\mathcal{N}_2(U)(f^{\rightarrow}(x_\lambda)) \leq \mathcal{N}_1(f^{\leftarrow}(U))(x_\lambda)$ for each $U \in L^Y$ and each $x_\lambda \in J(L^X)$. The category of all (L,M)-fuzzy neighborhood spaces and their continuous mappings is denoted by (L,M)-**FNS**.

Lemma 1.3. (1) *Define* $\mathcal{N}_{\mathcal{T}}: L^X \longrightarrow M^{J(L^X)}$ *by*

$$\mathcal{N}_{\mathcal{T}}(U)(x_\lambda) = \bigvee_{x_\lambda \leq V \leq U} \mathcal{T}(V)\ \ (\forall U \in L^X, \forall x_\lambda \in J(L^X)).$$

Then $\mathcal{N}_{\mathcal{T}}$ *is an* (L,M)*-fuzzy neighborhood system induced by* $\mathcal{T}$*.*

(2) *Define* $\mathcal{T}_{\mathcal{N}}: L^X \longrightarrow M$ *by*

$$\mathcal{T}_{\mathcal{N}}(U) = \bigwedge_{x_\lambda \lhd U} \mathcal{N}(U)(x_\lambda)\ (\forall U \in L^X).$$

Then $\mathcal{T}_{\mathcal{N}}$ *is an* (L,M)*-fuzzy topology induced by* $\mathcal{N}$*.*

(3) $\mathcal{N}_{\mathcal{T}_{\mathcal{N}}} = \mathcal{N}$ *and* $\mathcal{T}_{\mathcal{N}_{\mathcal{T}}} = \mathcal{T}$*.*

(4) (L,M)-**FTOP** *is isomorphic to* (L,M)-**FNS***.*

Definition 1.4 ([8, 9])**.** For any set X, let $\{(X_j, \mathcal{T}_j)\}_{j\in I}$ be a family of (L,M)-**FTOP**-objects, let $X = \prod_{j\in I} X_j$, and let $p_j: X \longrightarrow X_j$ be the j-th projection. The product (L, M)-fuzzy topology on X, denoted by $\prod_{j\in I}\mathcal{T}_j$, is the weakest (L, M)-fuzzy topology on X such that p_j is continuous for each $j \in I$. The pair $(X, \prod_{j\in I}\mathcal{T}_j)$ is called the product space of $\{(X_j, \mathcal{T}_j)\}_{j\in I}$.

Theorem 1.5 ([8, 9]). (1) *If* $\mathcal{T} = \prod_{j\in I}\mathcal{T}_j$, *then* $\mathcal{T} = \bigvee_{j\in I} p_j^{\leftarrow}(\mathcal{T}_j)$.

(2) *If* $(Y, \mathcal{T}_Y)$ *is an* (L, M)*-fuzzy topological space, then a mapping* $g : Y \longrightarrow X$ *is continuous if and only if* $p_j \circ g$ $(\forall j \in I)$ *is continuous.*

(3) $\forall x_\lambda \in J(L^X)$, $\forall A \in L^X$ *and every index set* I, *we have*

$$\mathcal{N}_{\mathcal{T}}(A)(x_\lambda) = \bigvee_{J\subseteq I finite}\left\{\bigwedge_{j\in J}\mathcal{N}_{\mathcal{T}_j}(A_j)(p_j^{\rightarrow}(x_\lambda)) \mid \bigwedge_{j\in J} p_j^{\leftarrow}(A_j) \leq A\right\}$$

and

$$(\prod_{j\in I}\mathcal{T}_j)(A) = \bigwedge_{x_\lambda\lhd A}\ \bigvee_{J\subseteq I finite}\left\{\bigwedge_{j\in J}\mathcal{N}_{\mathcal{T}_j}(A_j)(p_j^{\rightarrow}(x_\lambda)) \mid \bigwedge_{j\in J} p_j^{\leftarrow}(A_j) \leq A\right\}.$$

Definition 1.6 ([4, 5]). Let $\mathcal{T} : L^X \longrightarrow M$ be a map. $\forall A \in L^X$, let

$$\mathbb{S}_{\mathcal{T}}(A) = \left\{\mathcal{U} \subseteq L^X \mid \bigwedge_{x\in X}\left(A'(x) \vee \bigvee_{B\in\mathcal{U}} B(x)\right) \not\leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\ \bigwedge_{x\in X}\left(A'(x) \vee \bigvee_{B\in\mathcal{V}} B(x)\right)\right\},$$

$$FCD_{\mathcal{T}}(A) = \bigwedge_{\mathcal{U}\in\mathbb{S}_{\mathcal{T}}(A)}\ \bigvee_{B\in\mathcal{U}}\mathcal{T}'(B).$$

If $(X, \mathcal{T})$ is an (L, M)-fuzzy topological space, then $FCD_{\mathcal{T}}(A)$ is called the degree of fuzzy compactness of A with respect to $\mathcal{T}$. A is called fuzzy compact with respet to (L, M)-fuzzy topology $\mathcal{T}$, if $FCD_{\mathcal{T}}(A) = 1$.

Lemma 1.7 ([5]). *Let* $f : X \longrightarrow Y$ *be a set map.* $\mathcal{T}_1$ *be an* (L, M)*-fuzzy topology on* X, $\mathcal{T}_2$ *be an* (L, M)*-fuzzy topology on* Y, *and* $f : (X, \mathcal{T}_1) \longrightarrow (Y, \mathcal{T}_2)$ *be continuous. Then* $\forall A \in L^X$,

$$FCD_{\mathcal{T}_2}(f^{\rightarrow}(A)) \geq FCD_{\mathcal{T}_1}(A).$$

The main results are as follows:

Theorem 1.8 ([4, 5]). (1) *Let* $(X, \mathcal{T})$ *be the product* (L, M)*-fuzzy topological space of* $\{(X_j, \mathcal{T}_j)\}_{j\in I}$. *Then* $\forall A = \prod_{j\in I} A_j \in L^{\prod_{j\in I} X_j}$,

$$FCD_{\mathcal{T}}(A) \geq \bigwedge_{j\in I} FCD_{\mathcal{T}_j}(A_j),$$

where $A_j \in L^{X_j}$ *for any* $j \in I$.

(2) *Let* $(X, \mathcal{T})$ *be the product* (L, M)*-fuzzy topological space of* $\{(X_j, \mathcal{T}_j)\}_{j\in I}$. *Then*

$$FCD_{\mathcal{T}}(1_X) = \bigwedge_{j\in I} FCD_{\mathcal{T}_j}(1_{X_j}).$$

2. A new proof of the main results

The Proof of Theorem 1.8

Proof. (1) Suppose that $b \in M$ and $\bigwedge_{j\in I} FCD_{\mathcal{T}_j}(A_j) \not\leq b$. Then there exists $a \in \alpha^*(b)$ such that $\bigwedge_{j\in I} FCD_{\mathcal{T}_j}(A_j) \not\leq a$. Thus $FCD_{\mathcal{T}_j}(A_j) \not\leq a$ for any $j \in I$. Notice that

$$FCD_{\mathcal{T}_j}(A_j) = \bigwedge_{\mathcal{U}_j \in \mathbb{S}_{\mathcal{T}}(A_j)} \bigvee_{B\in\mathcal{U}_j} \mathcal{T}_j{}'(B),$$

we have

$\forall j \in I, \forall \mathcal{U}_j \in \mathbb{S}_{\mathcal{T}}(A_j)$, there exists $B \in \mathcal{U}_j$ such that $\mathcal{T}_j'(B) \not\leq a$, i.e.,

$\forall j \in I, \forall \mathcal{U}_j \subseteq L^{X_j}$, if $\forall B \in \mathcal{U}_j$, $\mathcal{T}_j(B) \geq a'$, then $\mathcal{U}_j \notin \mathbb{S}_{\mathcal{T}}(A_j)$.

We can prove that $FCD_{\mathcal{T}}(A) \not\leq b$. If not,

$$FCD'_{\mathcal{T}}(A) = \bigvee_{\mathcal{U}\in\mathbb{S}_{\mathcal{T}}(A)} \bigwedge_{C\in\mathcal{U}} \mathcal{T}(C) \geq b' \geq a',$$

then there exists $\mathcal{U}_0 \in \mathbb{S}_{\mathcal{T}}(A)$ and $\forall C \in \mathcal{U}_0$, $\mathcal{T}(C) \geq a'$. Notice that

$$\mathcal{T}(C) = (\prod_{j\in I} \mathcal{T}_j)(C) = \bigwedge_{x_\lambda \lhd C} \bigvee_{J\subseteq I \text{finite}} \left\{ \bigwedge_{j\in J} \mathcal{N}_{\mathcal{T}_j}(C_j)(p_j^{\rightarrow}(x_\lambda)) \mid \bigwedge_{j\in J} p_j^{\leftarrow}(C_j) \leq C \right\},$$

for any $C \in \mathcal{U}_0$.

Thus $\forall x_\lambda \lhd C$, there exists a finite J of I and $C_j \in L^{X_j}$ $(\forall j \in J)$ such that $\bigwedge_{j\in J} p_j^{\leftarrow}(C_j) \leq C$ and $a' \leq \mathcal{N}_{\mathcal{T}_j}(C_j)(p_j^{\rightarrow}(x_\lambda))$, for any $j \in J$. Further, there exists

$$V_j \in L^{X_j} \text{ such that } p_j^{\rightarrow}(x_\lambda) \leq V_j \leq C_j \text{ and } a' \leq \mathcal{T}_j(V_j),$$

since

$$\mathcal{N}_{\mathcal{T}_j}(C_j)(p_j^{\rightarrow}(x_\lambda)) = \bigvee_{p_j^{\rightarrow}(x_\lambda)\leq V_j\leq C_j} \mathcal{T}_j(V_j).$$

From the above proved, we can obtain the following result:

If there exists $\mathcal{U}_0 \in \mathbb{S}_{\mathcal{T}}(A)$, and $\forall C \in \mathcal{U}_0$, $\mathcal{T}(C) \geq a'$, then $\forall C \in \mathcal{U}_0$, there exists a finite J of I and $V_j \in L^{X_j}$ $(\forall j \in J)$ such that $\bigwedge_{j\in J} p_j^{\leftarrow}(V_j) = C$ and $a' \leq \mathcal{T}_j(V_j)$. Notice that, $\bigwedge_{j\in J} p_j^{\leftarrow}(V_j) = C$ implies $p_j^{\leftarrow}(V_j) = C$ $(\forall j \in J)$. In fact, $C \leq p_j^{\leftarrow}(V_j)$ is obvious.

On the other hand, $\forall j \in J$, let $b \in M$, $p_j^{\leftarrow}(V_j) \not\leq b$. Then there exists $a \in \alpha(b)$ such that $p_j^{\leftarrow}(V_j) \not\leq a$. thus $C = \bigwedge_{j\in J} p_j^{\leftarrow}(V_j) \not\leq b$. If not, then $\bigwedge_{j\in J} p_j^{\leftarrow}(V_j) \leq b$. By the definition of $\alpha(b)$, $\forall x \in \alpha(b)$, there exists $j_0 \in J$ such that $p_{j_0}^{\leftarrow}(V_{j_0}) \leq x$. This yields a contradiction. So, $p_j^{\leftarrow}(V_j) \leq C$.

Let

$$\mathcal{V}_j = \{V_j \in L^{X_j} \mid p_j^{\leftarrow}(V_j) = C, a' \leq \mathcal{T}_j(V_j), C \in \mathcal{U}_0\},$$

and

$$\mathcal{R}_j = \{p_j^{\leftarrow}(V_j) \in L^X \mid V_j \in L^{X_j}, p_j^{\leftarrow}(V_j) = C, a' \leq \mathcal{T}_j(V_j), C \in \mathcal{U}_0\},$$

where $\forall j \in J \subseteq I$. Then $\mathcal{V}_j \notin \mathbb{S}_{\mathcal{T}}(A_j)$, for any $j \in J$, i.e., $\forall j \in J$. Then we can obtain

$$\bigwedge_{x_j \in X_j} \left(A'_j(x_j) \vee \bigvee_{V_j \in \mathcal{V}_j} V_j(x_j) \right) \leq \bigvee_{\mathcal{W}_j \in 2^{(\mathcal{V}_j)}} \bigwedge_{x_j \in X_j} \left(A'_j(x_j) \vee \bigvee_{V_j \in \mathcal{W}_j} V_j(x_j) \right).$$

Meanwhile, we can obtain

$$\bigvee_{C \in \mathcal{U}_0} C \leq \bigvee_{j \in J} \bigvee_{V_j \in \mathcal{V}_j} p_j^{\leftarrow}(V_j).$$

Let $r = \bigwedge_{x \in X} \left(A'(x) \vee \bigvee_{C \in \mathcal{U}_0} C(x) \right)$. Then

$$\begin{aligned} r &\leq \bigwedge_{x \in X} \left(\bigvee_{j \in I} A'_j(p_j^{\rightarrow}(x)) \vee \bigvee_{j \in J} \bigvee_{V_j \in \mathcal{V}_j} p_j^{\leftarrow}(V_j)(x) \right) \\ &= \bigwedge_{x \in X} \left(\bigvee_{j \notin J} A'_j(p_j^{\rightarrow}(x)) \vee \bigvee_{j \in J} \left(A'_j(p_j^{\rightarrow}(x)) \vee \bigvee_{V_j \in \mathcal{V}_j} V_j(p_j^{\rightarrow}(x) \right) \right). \end{aligned}$$

Taking any $d \in \beta^*(r)$.

Case1: If $d \leq \bigwedge_{x \in X} \bigvee_{j \in I} A'_j(p_j^{\rightarrow}(x))$, then

$$d \leq \bigwedge_{x \in X} \bigvee_{j \in I} A'_j(p_j^{\rightarrow}(x)) = \bigwedge_{x \in X} A'(x) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U}_0)}} \bigwedge_{x \in X} \left(A'(x) \vee \bigvee_{C \in \mathcal{V}} C(x) \right).$$

In this case, we have that

$$\bigwedge_{x \in X} \left(A'(x) \vee \bigvee_{C \in \mathcal{U}_0} C(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U}_0)}} \bigwedge_{x \in X} \left(A'(x) \vee \bigvee_{C \in \mathcal{V}} C(x) \right).$$

Case2: If $d \not\leq \bigwedge_{x \in X} \bigvee_{j \in I} A'_j(p_j^{\rightarrow}(x))$ $(= \bigvee_{j \in I} \bigwedge_{x_j \in X_j} A'_j(x_j))$, then there exists $e \in \beta^*(\bigwedge_{j \in I} \bigvee_{x_j \in X_j} A_j(x_j))$ such that $e \not\leq d'$. Thus $\forall j \in I$, there exists $x_j \in X_j$ such that $e \lhd A_j(x_j)$.

Next, we prove that

$$d \leq \bigvee_{j \in J} \bigwedge_{x_j \in X_j} \left(A'_j(x_j) \vee \bigvee_{V_j \in \mathcal{V}_j} V_j(x_j) \right).$$

If not, there exists $h \in \beta^* \left(\bigwedge_{j \in J} \bigvee_{x_j \in X_j} \left(A_j(x_j) \wedge \bigwedge_{V_j \in \mathcal{V}_j} V'_j(x_j) \right) \right)$ such that $h \not\leq d'$. Thus $\forall j \in J$,there exists $y_j \in X_j$ such that $h \lhd A_j(x_j) \wedge \bigwedge_{V_j \in \mathcal{V}_j} V'_j(x_j)$.

Taking $z = \{z_j\}_{j\in I}$ such that $z_j = y_j$, when $j \in J$ and $z_j = x_j$ otherwise. Then

$$d \lhd r \leq \bigwedge_{x\in X} \left(\bigvee_{j\notin J} A'_j(p_j^{\rightarrow}(x)) \vee \bigvee_{j\in J} \left(A'_j(p_j^{\rightarrow}(x)) \vee \bigvee_{V_j\in\mathcal{V}_j} V_j(p_j^{\rightarrow}(x)) \right) \right)$$

$$\leq \bigvee_{j\notin J} A'_j(p_j^{\rightarrow}(z)) \vee \bigvee_{j\in J} \left(A'_j(p_j^{\rightarrow}(z)) \vee \bigvee_{V_j\in\mathcal{V}_j} V_j(p_j^{\rightarrow}(z)) \right)$$

$$\leq \bigvee_{j\notin J} A'_j(x_j) \vee \bigvee_{j\in J} \left(A'_j(y_j) \vee \bigvee_{V_j\in\mathcal{V}_j} V_j(y_j) \right).$$

Thus $d' \geq \bigwedge_{j\notin J} A_j(x_j) \wedge \bigwedge_{j\in J} \left(A_j(y_j) \wedge \bigwedge_{V_j\in\mathcal{V}_j} V'_j(y_j) \right) \geq e \wedge h$. This implies $e \leq d'$ or $h \leq d'$. This yields a contradiction. So

$$d \leq \bigvee_{j\in J} \bigwedge_{x_j\in X_j} \left(A'_j(x_j) \vee \bigvee_{V_j\in\mathcal{V}_j} V_j(x_j) \right)$$

$$\leq \bigvee_{j\in J} \bigvee_{\mathcal{W}_j\in 2^{(\mathcal{V}_j)}} \bigwedge_{x_j\in X_j} \left(A'_j(x_j) \vee \bigvee_{V_j\in\mathcal{W}_j} V_j(x_j) \right)$$

$$= \bigvee_{j\in J} \bigvee_{\mathcal{W}_j\in 2^{(\mathcal{V}_j)}} \bigwedge_{x_j\in X_j} \left(A'_j \vee \bigvee \mathcal{W}_j \right)(x_j)$$

$$\leq \bigvee_{j\in J} \bigvee_{\mathcal{W}_j\in 2^{(\mathcal{V}_j)}} \bigwedge_{x\in X} \left(p_j^{\leftarrow}(A'_j) \vee \bigvee_{V_j\in\mathcal{W}_j} p_j^{\leftarrow}(V_j) \right)(x)$$

$$\leq \bigvee_{j\in J} \bigvee_{\mathcal{W}_j\in 2^{(\mathcal{V}_j)}} \bigwedge_{x\in X} \left(A' \vee \bigvee_{V_j\in\mathcal{W}_j} p_j^{\leftarrow}(V_j) \right)(x)$$

$$\leq \bigvee_{j\in J} \bigvee_{\mathcal{D}_j\in 2^{(\mathcal{R}_j)}} \bigwedge_{x\in X} \left(A' \vee \bigvee \mathcal{D}_j \right)(x)$$

$$\leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U}_0)}} \bigwedge_{x\in X} \left(A'(x) \vee \bigvee_{C\in\mathcal{V}} C(x) \right).$$

In this case, we also have that

$$\bigwedge_{x\in X} \left(A'(x) \vee \bigvee_{C\in\mathcal{U}_0} C(x) \right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U}_0)}} \bigwedge_{x\in X} \left(A'(x) \vee \bigvee_{C\in\mathcal{V}} C(x) \right).$$

Both cases 1 and 2, we know that $\mathcal{U}_0 \notin \mathbb{S}_{\mathcal{T}}(A)$. However, $\mathcal{U}_0 \in \mathbb{S}_{\mathcal{T}}(A)$, which is a contradiction. Hence

$$FCD_{\mathcal{T}}(A) \not\leq b.$$

Therefore,

$$FCD_{\mathcal{T}}(A) \geq \bigwedge_{j\in I} FCD_{\mathcal{T}_j}(A_j).$$

(2) By (1) and Lemma 1.7, we can easily obtained the result. Then we omit it. □

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