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A study of hesitant fuzzy soft multiset theory

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Abstract. In this paper, we recall the definition of soft set, fuzzy soft set, hesitant fuzzy set and hesitant fuzzy soft set and some of their examples. We define the concept of hesitant fuzzy soft multiset which combines hesitant fuzzy soft set and soft multiset theory. We also define basic terms in hesitant fuzzy soft multiset with relevant examples. Some basic operations such as restricted intersection, extended intersection, union, restricted union, AND–product and OR–product and their properties are given, supported with illustrative examples. We finally establish some important results, including De Morgan's inclusions and laws.

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1. Introduction

Most of the available classical mathematical tools used for modeling, reasoning and computing are crisp and deterministic in character. But in real life situation, crisp data is not always the case with the complicated problems encountered in engineering, economics, environment and medical sciences. Consequently, the traditional standard methods cannot be automatically employed due to their various kinds of uncertainties which are inherently associated with these problems. There are theories such as theory of probability, theory of fuzzy set [10, 12, 17, 19], theory of intuitionistic fuzzy sets [1, 2], theory of vague sets [5], theory of interval mathematics [1] theory of rough sets [11], which are well known classical mathematical tools for handling uncertainties. But difficulties present in all these theories have been pointed out by Molodtsov in [9]. The cause of these problems is possibly related to the inadequacy of the parameterization tools of the theories. As a result Molodtsov initiated the concept of soft set theory as a new mathematical tool for solving the uncertainties which is free from the above mention difficulties. Presence of vagueness demanded fuzzy soft set [6] to come in to picture. But satisfactory evaluation of membership values is not always possible because of the insufficiency in the available information (besides the presence of vagueness) situation. Evaluation of non-membership values is also not always possible for the same reason as a result there exists an indeterministic part upon which hesitations survives. Certainly fuzzy soft set theory is not suitable to solve such problems. In such situations intuitionistic fuzzy soft set theory [7, 8] may be more applicable.

Soft multiset theory was introduced as a generalization of Molodtsov's soft set [3]. Also, introduced is the notion of fuzzy soft multiset theory [4]. Hesitant fuzzy soft set and its application in multicriteria decision making has as well been investigated [15].

In order to tackle the difficulty in establishing the degree of membership of an element in a set, Torra and Narukawa [13] and Torra [14] proposed the concept of hesitant fuzzy set. In fact, this new extension of fuzzy set can handle the cases that the difficulty in establishing the membership degree does not arise from a margin of error (as in intuitionistic or interval-valued fuzzy sets) or a specified possibility distribution of the possible values (as in type-2 fuzzy sets), but rather arises from our hesitation among a few different values [18]. Therefore, a hesitant fuzzy set can more accurately reflect the people′s hesitancy in stating their preferences over objects, compared to the fuzzy set and its many classical extensions.

In this paper, we extend hesitant fuzzy soft set model to soft multiset and thus, we establish a new soft set model called hesitant fuzzy soft multiset.

2. Some preliminary concepts

2.1 Soft Set

We first recall some basic notions in soft set theory. Let U be an initial universe set, E be a set of parameters or attributes with respect to $U, P(U)$ be the power set of U and $\mathrm{A} \subseteq \mathrm{E}$.

Definition 2.1 ([9])**.** A pair $(\Gamma,\ A)$ is called a soft set over U, where Γ is a mapping given by $\Gamma : A \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For $x \in A$, $\Gamma(x)$ may be considered as the set of x-elements or as the set of x-approximate elements of the soft set $(\Gamma,\ A)$.

The soft set $(\Gamma,\ A)$ can be represented as a set of ordered pairs as follows:

$$(\Gamma,\ A) = \{(x,\ \Gamma(x)),\ x \in A,\ \Gamma(x) \in P(U)\}.$$

Definition 2.2 ([6])**.** Let $(\Gamma,\ A)$ and $(\mathrm{G},\ \mathrm{B})$ be two soft sets over U.

(i) $(\Gamma,\ A)$ is said to be a soft subset of $(\mathrm{G},\ \mathrm{B})$, denoted by $(\Gamma,\ A) \,\widetilde{\subseteq}\, (\mathrm{G},\ \mathrm{B})$, if

$$A \subseteq \mathrm{B} \text{ and } \Gamma(x) \subseteq \mathrm{G}(x),\ \forall x \in \mathrm{A}.$$

(ii) $(\Gamma,\ A)$ and $(G,\ B)$ are said to be soft equal, denoted by $(\Gamma, A) = (\mathrm{G}, \mathrm{B})$, if

$$(\Gamma, A) \,\widetilde{\subseteq}\, (\mathrm{G}, \mathrm{B}) \text{ and } (\mathrm{G}, \mathrm{B}) \,\widetilde{\subseteq}\, (\Gamma, A).$$

Definition 2.3 ([7])**.** Let (Γ, A) be a soft set over U. Then

(i) the support of (Γ, A), denoted by $supp(\Gamma, A)$, is the set defined as:

$$supp(\Gamma, A) = \{x \in A :\ \Gamma(x) \neq \phi\},$$

(ii) (Γ, A) is called a non-null soft set, if supp $(\Gamma,\ A) \neq \varnothing$,
(iii) (Γ, A) is called a relative null soft set, denoted by ϕ_A, if $\Gamma(x) = \phi$, $\forall x \in A$,
(iv) (Γ, A) is called a relative whole soft set, denoted by U_A, if $\Gamma(x) = U$, $\forall x \in A$.

Definition 2.4. Let (Γ, A) be a soft set over U. If $\Gamma(x) \neq \phi$ for all $x \in A$, then (Γ, A) is called a non-empty soft set.

Definition 2.5 ([7])**.** Let (Γ, A) and (G, B) be two soft sets over U. Then the union of (Γ, A) and (G, B), denoted by $(\Gamma, A)\widetilde{\cup}(G, B)$ is a soft set defined as:

$$(\Gamma, A)\widetilde{\cup}(G, B) = (H, C),$$

where $C = A \cup B$ and $\forall x \in C$,

$$\tilde{H}(x) = \begin{cases} \Gamma(x) & \text{if } x \in A - B \\ G(x) & \text{if } x \in B - A \\ \Gamma(x) & \text{if } x \in A \cap B. \end{cases}$$

Definition 2.6 ([7])**.** Let (Γ, A) and $(G,\ B)$ be two soft sets over U. The restricted union of (Γ, A) and (G, B), denoted by $(\Gamma, A)\widetilde{\cup}_R(G, B)$, is a soft set defined as:

$$(\Gamma, A)\widetilde{\cup}_R(G, B) = (H,\ C),$$

where $C = A \cap B\ \neq \phi$ and $\forall x \in C,\ H(x) = \Gamma(x) \cup G(x)$.

Definition 2.7 ([7])**.** Let (Γ, A) and (G, B) be two soft sets over U. The extended intersection of (Γ, A) and (G, B), denoted by $(\Gamma, A)\widetilde{\cap}_E(G, B)$, is a soft set defined as:

$$(\Gamma, A)\widetilde{\cap}_E(G, B) = (H, C),$$

where $C = A \cup B$ and $\forall x \in C$,

$$H(x) = \begin{cases} \Gamma(x) & \text{if } x \in A - B \\ G(x) & \text{if } x \in B - A \\ \Gamma(x)\ \cap\ G(x) & \text{if } x \in A \cap B. \end{cases}$$

Definition 2.8 ([7])**.** Let (Γ, A) and (G, B) be two soft sets over U. Then the restricted intersection of (Γ, A) and (G, B), denoted by $(\Gamma, A) \cap (G, B)$, is a soft set defined as:

$$(\Gamma, A) \cap (G, B) = (H, C),$$

where $C = A \cap B \neq \phi$ and $\forall x \in C, H(x) = \Gamma(x) \cap G(x)$.

Definition 2.9 ([6])**.** Let (Γ, A) and (G, B) be two soft sets over U. Then the AND-product or AND-intersection of (Γ, A) and (G, B) denoted by $(\Gamma, A)\widetilde{\bigwedge}(G, B)$ is a soft set defined as:

$$(\Gamma,\ A)\widetilde{\bigwedge}(G, B) = (H, C),$$

where $C = A \times B$ and $forall(x, y) \in A \times B,\ H(x, y) = \Gamma(x) \cap G(y)$.

Definition 2.10 ([6])**.** Let (Γ, A) and (G, B) be two soft sets over U. Then the OR-product or OR-union of (Γ, A) and (G, B), denoted by $(\Gamma, A)\widetilde{\bigvee}(G, B)$ is a soft set defined as:

$$(\Gamma, A)\widetilde{\bigvee}(G, B) = (H, C),$$

where $C = A \times B$ and $\forall\ (x, y) \in A \times B, H(x, y) = \Gamma(x) \cup G(y)$.

2.2 Fuzzy Soft Set

Let U be an initial universe set and E be a set of parameters (which are fuzzy words or sentences involving fuzzy words). Let $P(U)$ denotes the set of all fuzzy subsets of U and $A \subseteq E$.

Definition 2.11 ([6])**.** A pair $(\widetilde{\Gamma}, A)$ is called a fuzzy soft set over U, where $\widetilde{\Gamma}$ is a mapping given by $\widetilde{\Gamma} : A \longrightarrow P(U)$. $\widetilde{\Gamma}$ is called fuzzy approximation function of the fuzzy set $(\widetilde{\Gamma}, A)$ and the values $\widetilde{\Gamma}(x)$ are fuzzy subsets of U, $\forall\ x \in A$. Then, a fuzzy soft set $(\widetilde{\Gamma}, A)$ over U can be represented by the set of ordered pairs:

$$(\widetilde{\Gamma}, A) = \left\{ \left(x,\ \widetilde{\Gamma}\,(x)\right) : x \in A,\ \widetilde{\Gamma}(x) \in P(U) \right\}.$$

Example 2.12. Suppose that $U = \{h_1, h_2, h_3, h_4, h_5\}$ be a universe set and $E = \{x_1, x_2, x_3, x_4\}$ be a set of parameters. Let $A = \{x_1, x_2, x_3\} \subseteq E$, $\widetilde{\Gamma}\,(x_1) = \left\{\frac{h_2}{0.8}, \frac{h_4}{0.6}\right\}$, $\widetilde{\Gamma}\,(x_2) = U$ and $\widetilde{\Gamma}\,(x_3) = \left\{\frac{h_1}{0.3}, \frac{h_4}{0.4}, \frac{h_5}{0.9}\right\}$. Then the fuzzy soft set $\left(\widetilde{\Gamma},\ A\right)$ is written as:

$$\left(\widetilde{\Gamma},\ A\right) = \left\{ \left(x_1, \left\{\frac{h_2}{0.8}, \frac{h_4}{0.6}\right\}\right), (x_2,\ U), \left(x_3, \left\{\frac{h_1}{0.3}, \frac{h_4}{0.4}, \frac{h_5}{0.9}\right\}\right) \right\}.$$

2.3 Hesitant Fuzzy Set

Definition 2.13 ([14])**.** A hesitant fuzzy set on U is in terms of a function that when applied to U returns a subset of [0,1] which can be represented with the following mathematical symbol:

$$\tilde{A} = \ \{\langle u,\ h_{\tilde{A}}(u)\rangle : u \in U\},$$

where $h_{\tilde{A}}(u)$ is a set of values in [0, 1], denoting the possible membership degrees of the element $u \in U$ to the set $\tilde{A}$. For convenience, we call $h_{\tilde{A}}(u)$ a hesitant fuzzy element and H the set of all hesitant fuzzy elements.

Definition 2.14 ([16])**.** For a hesitant fuzzy element h, $S\,(h) = \left(\frac{1}{l(h)}\right) \sum_{\gamma \in h} \gamma$ is called the score function of h, where $l(h)$ is the number of values in h.

2.4 Hesitant Fuzzy Soft Set

Definition 2.15 ([15])**.** Let $\tilde{H}(U)$ be the set of all hesitant fuzzy sets in U. Then a pair $(\tilde{F}, A)$ is called a hesitant fuzzy soft set over U, where $\tilde{F}$ is a mapping given by

$$\tilde{F} : A \longrightarrow \tilde{H}(U).$$

A hesitant fuzzy soft set is a mapping from parameters to $\tilde{H}(U)$. It is a parameterized family of hesitant fuzzy subsets of U. For $e \in A$, $\tilde{F}(e)$ may be considered as the set of $e-$approximate elements of the hesitant fuzzy soft set $(\tilde{F},\ A)$.

Example 2.16. Suppose that $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ is a set of houses and $A = \{e_1, e_2, e_3, e_4, e_5\}$ is a set of parameters, which stands for the parameters "cheap," "beautiful," "size," "location" and "surrounding environment," respectively. Then a hesitant fuzzy soft set $(\tilde{F},\ A)$ can describe the characteristics of the houses under hesitant fuzzy information.

Consider

$$\begin{aligned}
\tilde{F}(e_1) &= \left\{ \tfrac{h_1}{\{0.2,0.3\}}, \tfrac{h_2}{\{0.5,0.6\}}, \tfrac{h_3}{\{0.3\}}, \tfrac{h_4}{\{0.3,0.5\}}, \tfrac{h_5}{\{0.4,0.5\}}, \tfrac{h_6}{\{0.6,0.7\}} \right\},\\
\tilde{F}(e_2) &= \left\{ \tfrac{h_1}{\{0.4,0.6,0.7\}}, \tfrac{h_2}{\{0.5,0.7,0.8\}}, \tfrac{h_3}{\{0.6,\ 0.8\}}, \tfrac{h_4}{\{0.7,0.9\}}, \tfrac{h_5}{\{0.3,0.4,0.5\}}, \tfrac{h_6}{\{0.3\}} \right\},\\
\tilde{F}(e_3) &= \left\{ \tfrac{h_1}{\{0.2,0.4\}}, \tfrac{h_2}{\{0.6,0.7\}}, \tfrac{h_3}{\{0.8,0.9\}}, \tfrac{h_4}{\{0.3,0.5\}}, \tfrac{h_5}{\{0.4,0.6\}}, \tfrac{h_6}{\{0.7\}} \right\},\\
\tilde{F}(e_4) &= \left\{ \tfrac{h_1}{\{0.3,0.5,0.6\}}, \tfrac{h_2}{\{0.2\}}, \tfrac{h_3}{\{0.5\}}, \tfrac{h_4}{\{0.6,0.7\}}, \tfrac{h_5}{\{0.5,0.6\}}, \tfrac{h_6}{\{0.8\}} \right\},\\
\tilde{F}(e_5) &= \left\{ \tfrac{h_1}{\{0.6\}}, \tfrac{h_2}{\{0.2,0.3,0.5\}}, \tfrac{h_3}{\{0.5,0.7\}}, \tfrac{h_4}{\{0.2,0.4\}}, \tfrac{h_5}{\{0.5,0.7\}}, \tfrac{h_6}{\{0.3,0.5\}} \right\}.
\end{aligned}$$

$$\left(\widetilde{F},\ A\right) = \left\{ \begin{array}{c}
\left(e_1, \left(\left\{ \tfrac{h_1}{\{0.2,0.3\}}, \tfrac{h_2}{\{0.5,0.6\}}, \tfrac{h_3}{\{0.3\}}, \tfrac{h_4}{\{0.3,0.5\}}, \tfrac{h_5}{\{0.4,0.5\}}, \tfrac{h_6}{\{0.6,0.7\}} \right\}\right)\right),\\
\left(e_2, \left(\left\{ \tfrac{h_1}{\{0.4,0.6,0.7\}}, \tfrac{h_2}{\{0.5,0.7,0.8\}}, \tfrac{h_3}{\{0.6,\ 0.8\}}, \tfrac{h_4}{\{0.7,0.9\}}, \tfrac{h_5}{\{0.3,0.4,0.5\}}, \tfrac{h_6}{\{0.3\}} \right\}\right)\right),\\
\left(e_3, \left(\left\{ \tfrac{h_1}{\{0.2,0.4\}}, \tfrac{h_2}{\{0.6,0.7\}}, \tfrac{h_3}{\{0.8,0.9\}}, \tfrac{h_4}{\{0.3,0.5\}}, \tfrac{h_5}{\{0.4,0.6\}}, \tfrac{h_6}{\{0.7\}} \right\}\right)\right),\\
\left(e_4, \left(\left\{ \tfrac{h_1}{\{0.3,0.5,0.6\}}, \tfrac{h_2}{\{0.2\}}, \tfrac{h_3}{\{0.5\}}, \tfrac{h_4}{\{0.6,0.7\}}, \tfrac{h_5}{\{0.5,0.6\}}, \tfrac{h_6}{\{0.8\}} \right\}\right)\right),\\
\left(e_5, \left(\left\{ \tfrac{h_1}{\{0.6\}}, \tfrac{h_2}{\{0.2,0.3,0.5\}}, \tfrac{h_3}{\{0.5,0.7\}}, \tfrac{h_4}{\{0.2,0.4\}}, \tfrac{h_5}{\{0.5,0.7\}}, \tfrac{h_6}{\{0.3,0.5\}} \right\}\right)\right)
\end{array} \right\}.$$

2.5 Soft Multiset

Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i\in I} U_i = \varnothing$ and let $\{E_{U_i} : i \in I\}$ be a collection of sets of parameters. Let $U = \biguplus_{\mathbf{i\in I}} P(U_{\mathbf{i}})$, where $P(U_i)$ denotes the power sets of $U_i's$, $E = \biguplus_{\mathbf{i\in I}} E_{U_i}$ and $A \subseteq E$.

Definition 2.17 ([3]). A pair (F, A) is called a soft multiset over U, where F is a mapping given by $F : A \longrightarrow U$.

In other words, a soft multiset over U is a parameterized family of subsets of U. For $\varepsilon \in A, F(\varepsilon)$ may be considered as the set of $\varepsilon-$approximate elements of the soft multiset (F, A). Based on the definition, any change in the order of the universes will produce a different soft multiset.

Example 2.18. Suppose that there are three universes U_1, U_2 and U_3. Let us consider a soft multiset (F, A) which describes the "attractiveness of houses", "cars" and "hotels" that Mr. X is considering for accommodation purchase, transportation purchase, and venue to hold a wedding celebration respectively.
Let $U_1 = \{h_1, h_2, h_3, h_4, h_5, h_6\}, U_2 = \{c_1, c_2, c_3, c_4, c_5\}$ and $U_3 = \{v_1, v_2, v_3, v_4\}$.
Let $E_U = \{E_{U_1}, E_{U_2}, E_{U_3}\}$ be a collection of sets of decision parameters related to the above universes, where

$$E_{U_1} = \left\{ \begin{array}{c} e_{U_1}, 1 = expensive,\ e_{U_1}, 2 = cheap,\ e_{U_1}, 3 = beautiful,\\ e_{U_1}, 4 = wooden,\ e_{U_1}, 5 = in\ green\ surroundings \end{array} \right\},$$

$$E_{U_2} = \left\{ \begin{array}{c} e_{U_2}, 1 = expensive,\ e_{U_2}, 2 = cheap,\ e_{U_2}, 3 = Model\ 2000\ and\ above,\\ e_{U_2}, 4 = Black,\ e_{U_2}, 5 = Made\ in\ Japan,\ e_{U_2}, 6 = Made\ in\ Malaysia \end{array} \right\},$$

$$E_{U_3} = \left\{ \begin{array}{c} e_{U_3}, 1 = expensive,\ e_{U_3}, 2 = cheap,\ e_{U_3}, 3 = majestic,\\ e_{U_3}, 4 = in\ Kuala\ Lumpur,\ e_{U_3}, 5 = in\ Kajang \end{array} \right\}.$$

Let $U = \biguplus_{i=1}^{3} P_{\mathbf{i}}(U_{\mathbf{i}}),\quad E = \biguplus_{i=1}^{3} E_{U_i}\ and\ A \subseteq E$ such that

$$A = \{a_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1)\,,\ a_2 = (e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1)\,,$$

$$a_3 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1)\,,\ a_4 = (e_{U_1}, 5, e_{U_2}, 4, e_{U_3}, 2)\,,$$

$$a_5 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3)\,,\ a_6 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 2)\,,$$

$$a_7 = (e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1)\,,\ a_8 = (e_{U_1}, 1, e_{U_2}, 3, e_{U_3}, 2)\}.$$

Suppose that

$$F(a_1) = (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})\,,$$

$$F(a_2) = (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2\})\,,$$

$$F(a_3) = (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \varnothing)\,,$$

$$F(a_4) = (\{h_1, h_4, h_6\}, \varnothing, \{v_1, v_4\})\,,$$

$$F(a_5) = (\{h_1, h_4\}, \{c_1, c_3\}, \{v_1\})\,,$$

$$F(a_6) = (\{h_1, h_4, h_5\}, \{c_1, c_3\}, U_3)\,,$$

$$F(a_7) = (\{h_1, h_4\}, \varnothing, \{v_3\})\,,$$

$$F(a_8) = (\{h_2, h_3, h_6\}, \{c_1, c_3\}, \{v_1, v_4\})\,.$$

Then we can view the soft multiset (F, A) as consisting of the following collection of approximations:

$$(F, A) = \{(a_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\}))\,,$$

$$(a_2,\ (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2\})),$$

$$(a_3, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \varnothing)),\ (a_4,\ (\{h_1, h_4, h_6\}, \varnothing, \{v_1, v_4\})),$$

$$(a_5,\ (\{h_1, h_4\}, \{c_1, c_3\}, \{v_1\})), (a_6, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, U_3)),$$

$$(a_7, (\{h_1, h_4\}, \varnothing, \{v_3\})),\ (a_8,\ (\{h_2, h_3, h_6\}, \{c_1, c_3\}, \{v_1, v_4\}))\}.$$

3. Hesitant fuzzy soft multiset

Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i \in I} U_i = \phi$ and let $\{E_{U_i} : i \in I\}$ be a collection of sets of parameters or attributes related to the universes. Let $U = \biguplus_{\mathbf{i \in I}} HFS(U_i)$, where $HFS\ (U_i)$ denotes the set of all hesitant fuzzy submultisets of the U_i's, $E = \biguplus_{\mathbf{i \in I}} E_{U_i}$ and $A \subseteq E$.

Definition 3.1. A pair $(\widetilde{\Gamma},\ A)$ is called a hesitant fuzzy soft multiset over U, where $\widetilde{\Gamma}$ is a mapping given by $\widetilde{\Gamma} : A \longrightarrow U$. In other words, a hesitant fuzzy soft multiset over U is a parameterized family of hesitant fuzzy submultisets of U. For $e \in A$, $\widetilde{\Gamma}(e)$ may be considered as the set of $e-$approximate elements of the hesitant fuzzy soft Multiset $(\widetilde{\Gamma}, A)$. Based on the above definition, any change in the order of universes will produce a different hesitant fuzzy soft Multiset.

Example 3.2. Suppose that there are three universes U_1, U_2 and U_3. Suppose that Mr. X has a budget to purchase a house, a car and rent a venue to hold a wedding celebration. Let us consider a hesitant fuzzy soft Multiset $(\widetilde{\Gamma},\ A)$ which describes the "houses," "Cars"and "hotels" that Mr. X with enough budget is considering for accommodation, transportation and venue to hold a wedding celebration with hesitant fuzzy elements respectively.

Let $U_1 = \{h_1, h_2,\ h_3, h_4\}$, $U_2 = \{c_1, c_2,\ c_3\}$ and $U_3 = \{v_1, v_2\}$. Let $\{E_{U_1},\ E_{U_2}, E_{U_3}\}$ be a collection of sets of decision parameters or attributes related to the above universes, where

$$E_{U_1} = \{e_{U_1},\ 1 = expensive,\ e_{U_1},\ 2 = cheap,\ e_{U_1},\ 3 = wooden,\ e_{U_1},\ 4 = in\ green\ surrounding\},$$
$$E_{U_2} = \{e_{U_2},\ 1 = expensive,\ e_{U_2},\ 2 = cheap,\ e_{U_2},\ 3 = sporty\},$$
$$E_{U_3} = \{e_{U_3},\ 1 = expensive,\ e_{U_3},\ 2 = cheap,\ e_{U_3},\ 3 = in\ Kuala\ Lumpur,\ e_{U_3},\ 4 = Majestic\}.$$

Let $U = \biguplus_{i=1}^{3} HFS(U_i)$, $E = \biguplus_{i=1}^{3} E_{U_i}$ and $A \subseteq E$ such that $A =$

$$\left\{ \begin{array}{l} a_1 = (e_{U_1}, 1, e_{U_2}, 1,\ e_{U_3}, 1),\ a_2 = (e_{U_1}, 1, e_{U_2}, 2,\ e_{U_3}, 1),\ a_3 = (e_{U_1}, 2, e_{U_2}, 2,\ e_{U_3}, 1), \\ a_4 = (e_{U_1}, 4, e_{U_2}, 3,\ e_{U_3}, 2),\ a_5 = (e_{U_1}, 4, e_{U_2}, 2,\ e_{U_3}, 2),\ a_6 = (e_{U_1}, 2, e_{U_2}, 2,\ e_{U_3}, 2) \end{array} \right\}$$

and suppose that

$$\widetilde{\Gamma}(a_1) = \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.5,0.8,0.3\}}, \frac{h_2}{\{0.4,0.5\}}, \frac{h_3}{\{0.9,0.7,0.5\}}, \frac{h_4}{\{0.6,0.4\}} \right\}, \\ \left\{ \frac{c_1}{\{0.5,0.3\}}, \frac{c_2}{\{0.7,0.6,0.8\}}, \frac{c_3}{\{0.9,0.2,0.3\}} \right\}, \left\{ \frac{v_1}{\{0.7,0.5,0.8\}}, \frac{v_2}{\{0.9,0.7,0.8\}} \right\} \end{array} \right),$$
$$\widetilde{\Gamma}(a_2) = \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.7,0.7,0.9\}}, \frac{h_2}{\{0.3,0.6,0.1\}}, \frac{h_3}{\{0.8,0.6,0.2\}}, \frac{h_4}{\{0.3,0.2,\ 0.2\}} \right\}, \\ \left\{ \frac{c_1}{\{0.7,0.6,0.2\}}, \frac{c_2}{\{0.2,0.4,0.8\}}, \frac{c_3}{\{0.7,0.9,0.8\}} \right\}, \left\{ \frac{v_1}{\{0.8,0.9,0.6\}}, \frac{v_2}{\{0.5,0.6,0.9\}} \right\} \end{array} \right),$$
$$\widetilde{\Gamma}(a_3) = \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.2,0.3,0.7\}}, \frac{h_2}{\{0.7,0.9,0.2\}}, \frac{h_3}{\{0.2,0.3,0.4\}}, \frac{h_4}{\{0.6,0.5,0.2\}} \right\}, \\ \left\{ \frac{c_1}{\{0.7,0.8,0.9\}}, \frac{c_2}{\{0.2,0.4,0.5\}}, \frac{c_3}{\{0.9,0.7,0.2\}} \right\}, \left\{ \frac{v_1}{\{0.9,0.8\}}, \frac{v_2}{\{0.4,0.2,0.5\}} \right\} \end{array} \right),$$
$$\widetilde{\Gamma}(a_4) = \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.5,0.4,0.6\}}, \frac{h_2}{\{0.4,0.8,0.9\}}, \frac{h_3}{\{0.9,0.8,0.1\}}, \frac{h_4}{\{0.5,0.2,0.3\}} \right\}, \\ \left\{ \frac{c_1}{\{0.2,0.8,0.1\}}, \frac{c_2}{\{0.6,0.2,0.8\}}, \frac{c_3}{\{0.1,0.2,0.7\}} \right\}, \left\{ \frac{v_1}{\{0.8,0.9}, \frac{v_2}{0.2,0.4\}} \right\} \end{array} \right),$$
$$\widetilde{\Gamma}(a_5) = \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.2,0.3,0.7\}}, \frac{h_2}{\{0.3,0.2,0.5\}}, \frac{h_3}{\{0.7,0.8\}}, \frac{h_4}{\{0.7,0.8,0.9\}} \right\}, \\ \left\{ \frac{c_1}{\{0.9,0.2\}}, \frac{c_2}{\{0.7,0.8,0.9\}}, \frac{c_3}{\{0.2,0.7\}} \right\}, \left\{ \frac{v_1}{\{0.5,0.7\}}, \frac{v_2}{\{0.7,0.8,0.9\}} \right\} \end{array} \right),$$
$$\widetilde{\Gamma}(a_6) = \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.2,0.4,0.5\}}, \frac{h_2}{\{0.9,0.2\}}, \frac{h_3}{\{0.4,0.6,0.7\}}, \frac{h_4}{\{0.9,0.8,0.3\}} \right\}, \\ \left\{ \frac{c_1}{\{0.4,0.5\}}, \frac{c_2}{\{0.9,0.2,0.3\}}, \frac{c_3}{\{0.4,0.6\}} \right\}, \left\{ \frac{v_1}{\{0.9,0.2\}}, \frac{v_2}{\{0.9,0.7,0.2\}} \right\} \end{array} \right).$$

Then, we can view the hesitant fuzzy soft multiset $(\widetilde{\Gamma}, A)$ as consisting of the following collection of approximations:

$$\left(\widetilde{\Gamma},\ A\right)=\left\{\begin{array}{c}
\left(a_1,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.5,0.8,0.3\}},\frac{h_2}{\{0.4,0.5\}},\frac{h_3}{\{0.9,0.7,0.5\}},\frac{h_4}{\{0.6,0.4\}}\right\},\\ \left\{\frac{c_1}{\{0.5,0.3\}},\frac{c_2}{\{0.7,0.6,0.8\}},\frac{c_3}{\{0.9,0.2,0.3\}}\right\},\left\{\frac{v_1}{\{0.7,0.5,0.8\}},\frac{v_2}{\{0.9,0.7,0.8\}}\right\}\end{array}\right)\right),\\
\left(a_2,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.7,0.7,0.9\}},\frac{h_2}{\{0.3,0.6,0.1\}},\frac{h_3}{\{0.8,0.6,0.2\}},\frac{h_4}{\{0.3,0.2,\ 0.2\}}\right\},\\ \left\{\frac{c_1}{0.7,0.6,0.2},\frac{c_2}{0.2,0.4,0.8},\frac{c_3}{0.7,0.9,0.8}\right\},\left\{\frac{v_1}{\{0.8,0.9,0.6\}},\frac{v_2}{\{0.5,0.6,0.9\}}\right\}\end{array}\right)\right),\\
\left(a_3,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.2,0.3,0.7\}},\frac{h_2}{\{0.7,0.9,0.2\}},\frac{h_3}{\{0.2,0.3,0.4\}},\frac{h_4}{\{0.6,0.5,0.2\}}\right\},\\ \left\{\frac{c_1}{0.7,0.8,0.9},\frac{c_2}{0.2,0.4,0.5},\frac{c_3}{0.9,0.7,0.2}\right\},\left\{\frac{v_1}{0.9,0.8},\frac{v_2}{0.4,0.2,0.5}\right\}\end{array}\right)\right),\\
\left(a_4,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.5,0.4,0.6\}},\frac{h_2}{\{0.4,0.8,0.9\}},\frac{h_3}{\{0.9,0.8,0.1\}},\frac{h_4}{\{0.5,0.2,0.3\}}\right\},\\ \left\{\frac{c_1}{\{0.2,0.8,0.1\}},\frac{c_2}{\{0.6,0.2,0.8\}},\frac{c_3}{\{0.1,0.2,0.7\}}\right\},\left\{\frac{v_1}{\{0.8,0.9},\frac{v_2}{0.2,0.4\}}\right\}\end{array}\right)\right),\\
\left(a_5,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.2,0.3,0.7\}},\frac{h_2}{\{0.3,0.2,0.5\}},\frac{h_3}{\{0.7,0.8\}},\frac{h_4}{\{0.7,0.8,0.9\}}\right\},\\ \left\{\frac{c_1}{\{0.9,0.2\}},\frac{c_2}{\{0.7,0.8,0.9\}},\frac{c_3}{\{0.2,0.7\}}\right\},\ \left\{\frac{v_1}{\{0.5,0.7\}},\frac{v_2}{\{0.7,0.8,0.9\}}\right\}\end{array}\right)\right),\\
\left(a_6,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.2,0.4,0.5\}},\frac{h_2}{\{0.9,0.2\}},\frac{h_3}{\{0.4,0.6,0.7\}},\frac{h_4}{\{0.9,0.8,0.3\}}\right\},\\ \left\{\frac{c_1}{\{0.4,0.5\}},\frac{c_2}{\{0.9,0.2,0.3\}},\frac{c_3}{\{0.4,0.6\}}\right\},\left\{\frac{v_1}{\{0.9,0.2\}},\frac{v_2}{\{0.9,0.7,0.2\}}\right\}\end{array}\right)\right)
\end{array}\right\}.$$

Each approximation has two parts, namely a predicate name and an approximate value set. We can logically explain the above example as follows: We know that $a_1=(e_{U_1},1,e_{U_2},1,\ e_{U_3},1)$, where $e_{U_1},\ 1=\textit{expensive houses}$, $e_{U_2},\ 1=\textit{expensive car}$, and $e_{U_3},\ 1=\textit{expensive venue}$. Then

$$\widetilde{\Gamma}\left(a_1\right)=\left(\begin{array}{c}\left\{\frac{h_1}{\{0.5,0.8,0.3\}},\frac{h_2}{\{0.4,0.5\}},\frac{h_3}{\{0.9,0.7,0.5\}},\frac{h_4}{\{0.6,0.4\}}\right\},\\ \left\{\frac{c_1}{\{0.5,0.3\}},\frac{c_2}{\{0.7,0.6,0.8\}},\frac{c_3}{\{0.9,0.2,0.3\}}\right\},\left\{\frac{v_1}{\{0.7,0.5,0.8\}},\frac{v_2}{\{0.9,0.7,0.8\}}\right\}\end{array}\right).$$

Thus we can see that the membership score functional value of h_1, h_2 and h_4 approximated to one decimal place is 0.5, so these houses are fairly expensive for Mr X; also we can see that the membership score functional value of h_3 is 0.7, this means that the house h_3 is expensive. Since the first set is concerning expensive houses, then we can explain the second set as follows: the membership score functional value for c_1 is 0.4, this means this car is not expensive for him. The membership score functional value of c_2 is 0.7, so this car is expensive (however, this car may not be expensive if the first set is concerning cheap houses), also we can see that the membership score functional value of c_3 is 0.4, this means that this car is not expensive for him. Now, since the first set is concerning expensive houses and the second set is concerning expensive cars, then we can also explain the third set as follows: since the membership score functional value for the v_1 is 0.6, then this venue is quite expensive, also we can see that, the membership score functional value of v_2 is 0.8, so this venue is expensive (this venue may not be expensive if the first set is concerning cheap houses or / and the second set is concerning cheap cars). So depending on the previous explanation we can say the following:

If $\left\{\frac{h_1}{\{0.5,0.8,0.3\}}, \frac{h_2}{\{0.4,0.5\}}, \frac{h_3}{\{0.9,0.7,0.5\}}, \frac{h_4}{\{0.6,0.4\}}\right\}$ is hesitant fuzzy set of expensive houses, then the hesitant fuzzy set of relatively expensive cars is

$$\left\{\frac{c_1}{\{0.5, 0.3\}}, \frac{c_2}{\{0.7, 0.6, 0.8\}}, \frac{c_3}{\{0.9, 0.2, 0.3\}}\right\}$$

and if $\left\{\frac{h_1}{\{0.5,0.8,0.3\}}, \frac{h_2}{\{0.4,0.5\}}, \frac{h_3}{\{0.9,0.7,0.5\}}, \frac{h_4}{\{0.6,0.4\}}\right\}$ is the hesitant fuzzy set of expensive houses and $\left\{\frac{c_1}{\{0.5,0.3\}}, \frac{c_2}{\{0.7,0.6,0.8\}}, \frac{c_3}{\{0.9,0.2,0.3\}}\right\}$ is the hesitant fuzzy set of relatively expensive cars, then the hesitant fuzzy set of relatively expensive venues is $\left\{\frac{v_1}{\{0.7,0.5,0.8\}}, \frac{v_2}{\{0.9,0.7,0.8\}}\right\}$. It is obvious that the relation in hesitant fuzzy soft multiset is a conditional relation.

Definition 3.3. For any hesitant fuzzy soft multiset $(\widetilde{\Gamma}, A)$, a pair $(e_{U_i}, j,\ \widetilde{\Gamma}e_{U_i}, j)$ is called a U_i-hesitant fuzzy soft multiset part $\forall\ e_{U_i}, j \in a_k$ and $\widetilde{\Gamma}e_{U_i}, j \subseteq\ \widetilde{\Gamma}(A)$ is a hesitant fuzzy approximate value set, where $a_k \in A,\ k = \{1, 2, \ldots, n\},\ i \in \{1, 2, \ldots, m\},\ j \in \{1, 2, \ldots, r\}$.

Example 3.4. Consider Example 3.2. Then

$$(e_{U_i}, j,\ \widetilde{\Gamma}e_{U_i}, j) = \left\{\begin{array}{c} \left(e_{U_1}, 1, \left\{\frac{h_1}{\{0.5,0.8,0.3\}}, \frac{h_2}{\{0.4,0.5\}}, \frac{h_3}{\{0.9,0.7,0.5\}}, \frac{h_4}{\{0.6,0.4\}}\right\}\right), \\ \left(e_{U_1}, 1, \left\{\frac{h_1}{\{0.7,0.7,0.9\}}, \frac{h_2}{\{0.3,0.6\}}, \frac{h_3}{\{0.8,0.6,0.2\}}, \frac{h_4}{\{0.3,0.2,0.2\}}\right\}\right), \\ \left(e_{U_1}, 2, \left\{\frac{h_1}{\{0.2,0.3,0.7\}}, \frac{h_2}{\{0.7,0.9,0.2\}}, \frac{h_3}{\{0.2,0.3,0.4\}}, \frac{h_4}{\{0.6,0.5,0.2\}}\right\}\right), \\ \left(e_{U_1}, 4, \left\{\frac{h_1}{\{0.5,0.4,0.6\}}, \frac{h_2}{\{0.4,0.8,0.9\}}, \frac{h_3}{\{0.9,0.8,0.1\}}, \frac{h_4}{\{0.5,0.2,0.3\}}\right\}\right), \\ \left(e_{U_1}, 4, \left\{\frac{h_1}{\{0.2,0.3,0.7\}}, \frac{h_2}{\{0.3,0.2,0.5\}}, \frac{h_3}{\{0.7,0.8\}}, \frac{h_4}{\{0.7,0.8,0.9\}}\right\}\right), \\ \left(e_{U_1}, 2, \left\{\frac{h_1}{\{0.2,0.4,0.5\}}, \frac{h_2}{\{0.9,0.2\}}, \frac{h_3}{\{0.4,0.6,0.7\}}, \frac{h_4}{\{0.9,0.8,0.3\}}\right\}\right) \end{array}\right\}$$

is a U_1-hesitant fuzzy soft multiset part of $(\widetilde{\Gamma}, A)$.

Definition 3.5. For any hesitant fuzzy soft multisets $(\widetilde{\Gamma}, A)$ and $(\tilde{G}, B)$ over U, $(\widetilde{\Gamma}, A)$ is called a hesitant fuzzy soft multisubset of $(\tilde{G}, B)$, if the following properties are satisfied:

(i) $A \subseteq B$,

(ii) $\forall\ e_{U_i}, j \in a_k$, $(e_{U_i}, j,\ \widetilde{\Gamma}e_{U_i}, j)$ is a hesitant fuzzy subset of $(e_{U_i}, j,\ \tilde{G}e_{U_i}, j)$, where $a_k \in A,\ k = \{1, 2, \ldots, n\}$, $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, r\}$.

This relationship is written as $(\widetilde{\Gamma},\ A) \widetilde{\subseteq} (\tilde{G},\ B)$. In this case $(\tilde{G},\ B)$ is called a hesitant fuzzy soft multisuperset of $(\widetilde{\Gamma},\ A)$.

Example 3.6. Let $A = \left\{\begin{array}{c} a_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), a_2 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1), \\ a_3 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3), a_4 = (e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1) \end{array}\right\}$

and let $B = \left\{\begin{array}{l} b_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), b_2 = (e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1), \\ b_3 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1), b_4 = (e_{U_1}, 5, e_{U_2}, 1, e_{U_3}, 2), \\ b_5 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3),\ b_6 = (e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1) \end{array}\right\}$.

Then clearly, $A \subseteq B$. Let $(\widetilde{\Gamma}, A)$ and $(\tilde{G}, B)$ be two hesitant fuzzy soft multisets over U such that

$$
\left(\widetilde{\Gamma},\ A\right) =
$$

$$
\left\{\begin{array}{l}
\left(a_1, \left(\begin{array}{c}
\left\{\frac{h_1}{\{0.5,0.8,0.3\}}, \frac{h_2}{\{0.4,0.5\}}, \frac{h_3}{\{0.9,0.7,0.5\}}, \frac{h_4}{\{0.6,0.4\}}\right\}, \\
\left\{\frac{c_1}{\{0.5,0.3\}}, \frac{c_2}{\{0.7,0.6,0.8\}}, \frac{c_3}{\{0.9,0.2,0.3\}}\right\}, \left\{\frac{v_1}{\{0.7,0.5,0.8\}}, \frac{v_2}{\{0.9,0.7,0.8\}}\right\}
\end{array}\right)\right), \\
\left(a_2, \left(\begin{array}{c}
\left\{\frac{h_1}{\{0.7,0.7,0.9\}}, \frac{h_2}{\{0.3,0.6,0.1\}}, \frac{h_3}{\{0.8,0.6,0.2\}}, \frac{h_4}{\{0.3,0.2,0.2\}}\right\}, \\
\left\{\frac{c_1}{\{0.7,0.6,0.2\}}, \frac{c_2}{\{0.2,0.4,0.8\}}, \frac{c_3}{\{0.7,0.9,0.8\}}\right\}, \left\{\frac{v_1}{\{0.8,0.9,0.6\}}, \frac{v_2}{\{0.5,0.6,0.9\}}\right\}
\end{array}\right)\right), \\
\left(a_3, \left(\begin{array}{c}
\left\{\frac{h_1}{\{0.2,0.3,0.7\}}, \frac{h_2}{\{0.7,0.9,0.2\}}, \frac{h_3}{\{0.2,0.3,0.4\}}, \frac{h_4}{\{0.6,0.5,0.2\}}\right\}, \\
\left\{\frac{c_1}{\{0.7,0.8,0.9\}}, \frac{c_2}{\{0.2,0.4,0.5\}}, \frac{c_3}{\{0.9,0.7,0.2\}}\right\}, \left\{\frac{v_1}{\{0.9,0.8\}}, \frac{v_2}{\{0.4,0.2,0.5\}}\right\}
\end{array}\right)\right), \\
\left(a_4, \left(\begin{array}{c}
\left\{\frac{h_1}{\{0.5,0.4,0.6\}}, \frac{h_2}{\{0.4,0.8,0.9\}}, \frac{h_3}{\{0.9,0.8,0.1\}}, \frac{h_4}{\{0.5,0.2,0.3\}}\right\}, \\
\left\{\frac{c_1}{\{0.2,0.8,0.1\}}, \frac{c_2}{\{0.6,0.2,0.8\}}, \frac{c_3}{\{0.1,0.2,0.7\}}\right\}, \left\{\frac{v_1}{\{0.8,0.9\}}, \frac{v_2}{\{0.2,0.4\}}\right\}
\end{array}\right)\right)
\end{array}\right\}
$$

$$
\left(\tilde{G},\ B\right) =
$$

$$
\left\{\begin{array}{l}
\left(b_1, \left(\begin{array}{c}
\left\{\frac{h_1}{\{0.5,0.9,0.4\}}, \frac{h_2}{\{0.4,0.6\}}, \frac{h_3}{\{0.9,0.8,0.5\}}, \frac{h_4}{\{0.6,0.5\}}\right\}, \\
\left\{\frac{c_1}{\{0.5,0.6\}}, \frac{c_2}{\{0.7,0.6,0.8\}}, \frac{c_3}{\{0.9,0.4,0.5\}}\right\}, \left\{\frac{v_1}{\{0.8,0.5,0.8\}}, \frac{v_2}{\{0.9,0.8,0.8\}}\right\}
\end{array}\right)\right), \\
\left(b_2, \left(\begin{array}{c}
\left\{\frac{h_1}{\{0.7,0.8,0.9\}}, \frac{h_2}{\{0.3,0.7,0.2\}}, \frac{h_3}{\{0.8,0.7,0.3\}}, \frac{h_4}{\{0.3,0.2,0.3\}}\right\}, \\
\left\{\frac{c_1}{\{0.7,0.7,0.4\}}, \frac{c_2}{\{0.2,0.4,0.8\}}, \frac{c_3}{\{0.7,0.9,0.9\}}\right\}, \left\{\frac{v_1}{\{0.8,0.9,0.7\}}, \frac{v_2}{\{0.6,0.7,0.9\}}\right\}
\end{array}\right)\right), \\
\left(b_3, \left(\begin{array}{c}
\left\{\frac{h_1}{\{0.2,0.3,0.8\}}, \frac{h_2}{\{0.7,0.9,0.3\}}, \frac{h_3}{\{0.2,0.3,0.4\}}, \frac{h_4}{\{0.6,0.5,0.2\}}\right\}, \\
\left\{\frac{c_1}{\{0.7,0.8,0.9\}}, \frac{c_2}{\{0.2,0.5,0.5\}}, \frac{c_3}{\{0.9,0.8,0.2\}}\right\}, \left\{\frac{v_1}{\{0.9,0.8\}}, \frac{v_2}{\{0.5,0.3,0.7\}}\right\}
\end{array}\right)\right), \\
\left(b_4, \left(\begin{array}{c}
\left\{\frac{h_1}{\{0.5,0.5,0.6\}}, \frac{h_2}{\{0.4,0.8,0.9\}}, \frac{h_3}{\{0.9,0.8,0.2\}}, \frac{h_4}{\{0.5,0.4,0.6\}}\right\}, \\
\left\{\frac{c_1}{\{0.4,0.8,0.5\}}, \frac{c_2}{\{0.6,0.4,0.8\}}, \frac{c_3}{\{0.2,0.4,0.7\}}\right\}, \left\{\frac{v_1}{\{0.8,0.9\}}, \frac{v_2}{\{0.4,0.6\}}\right\}
\end{array}\right)\right), \\
\left(b_5, \left(\begin{array}{c}
\left\{\frac{h_1}{\{0.7,0.8,0.9\}}, \frac{h_2}{\{0.5,0.1,0.7\}}, \frac{h_3}{\{0.2,0.5\}}, \frac{h_4}{\{0.7,0.8,0.2\}}\right\}, \\
\left\{\frac{c_1}{\{0.2,0.3,0.1\}}, \frac{c_2}{\{0.5,0.3,0.7\}}, \frac{c_3}{\{0.4,0.1,0.3\}}\right\}, \left\{\frac{v_1}{\{0.8,0.9\}}, \frac{v_2}{\{0.9,0.7,0.2\}}\right\}
\end{array}\right)\right), \\
\left(b_6, \left(\begin{array}{c}
\left\{\frac{h_1}{\{0.4,0.5,0.2\}}, \frac{h_2}{\{0.2,0.4,0.7\}}, \frac{h_3}{\{0.9,0.4,0.2\}}, \frac{h_4}{\{0.1,0.2,0.3\}}\right\}, \\
\left\{\frac{c_1}{\{0.1,0.2,0.5\}}, \frac{c_2}{\{0.8,0.9,0.1\}}, \frac{c_3}{\{0.7,0.4,0.8\}}\right\}, \left\{\frac{v_1}{\{0.7,0.8,0.4\}}, \frac{v_2}{\{0.2,0.7,0.1\}}\right\}
\end{array}\right)\right)
\end{array}\right\}.
$$

Then, $(\widetilde{\Gamma},\ A)\ \widetilde{\subseteq}\ (\tilde{G},\ B)$.

Definition 3.7. Two hesitant fuzzy soft multisets $(\widetilde{\Gamma}, A)$ and $(\tilde{G}, B)$ over U are said to be equal, if $\left(\widetilde{\Gamma},\ A\right)$ is a hesitant fuzzy soft multisubset of $(\tilde{G},\ B)$ and $(\tilde{G},\ B)$ is a hesitant fuzzy soft multisubset of $\left(\widetilde{\Gamma},\ A\right)$. Mathematically, written as $\left(\widetilde{\Gamma},\ A\right) = (\tilde{G},\ B)$ if and only if $(\widetilde{\Gamma},\ A)\widetilde{\subseteq}\ (\tilde{G},\ B)$ and $(\tilde{G},\ B)\widetilde{\subseteq}(\widetilde{\Gamma},\ A)$.

Definition 3.8. The complement of a hesitant fuzzy soft multiset $\left(\widetilde{\Gamma}, A\right)$ is denoted by $\left(\widetilde{\Gamma}, A\right)^c$ and is defined as:

$$
\left(\widetilde{\Gamma},\ A\right)^c = \left(\widetilde{\Gamma}^c,\ A\right),
$$

where $\widetilde{\Gamma}^c : A \longrightarrow U$ is a mapping given by $\widetilde{\Gamma}^c(a) = (\widetilde{\Gamma}(a))^c,\ \forall a \in A$.
Clearly, $(\widetilde{\Gamma}^c)^c$ is the same as $\widetilde{\Gamma}$ and $\left(\left(\widetilde{\Gamma}, A\right)^c\right)^c = (\widetilde{\Gamma}, A)$.

It is important to note that, in the above definition of complement, the parameter set of the complement $\left(\widetilde{\Gamma}, A\right)^c$ is still the original parameter set A instead of $\neg A$.

Example 3.9. Consider Example 3.2. The $\left(\widetilde{\Gamma}, A\right)^c$ can be calculated as follows:

$$\widetilde{\Gamma}^c(a_1) = \left(\begin{array}{c} \left\{\frac{h_1}{\{0.7,0.2,0.5\}}, \frac{h_2}{\{0.5,0.6\}}, \frac{h_3}{\{0.5,0.3,0.1\}}, \frac{h_4}{\{0.6,0.4\}}\right\}, \\ \left\{\frac{c_1}{\{0.7,0.5\}}, \frac{c_2}{\{0.2,0.4,0.3\}}, \frac{c_3}{\{0.7,0.8,0.1\}}\right\}, \left\{\frac{v_1}{\{0.2,0.5,0.3\}}, \frac{v_2}{\{0.2,0.3,0.1\}}\right\} \end{array}\right),$$

$$\widetilde{\Gamma}^c(a_2) = \left(\begin{array}{c} \left\{\frac{h_1}{\{0.1,0.3,0.3\}}, \frac{h_2}{\{0.9,0.4,0.7\}}, \frac{h_3}{\{0.8,0.4,0.2\}}, \frac{h_4}{\{0.8,0.8,\ 0.7\}}\right\}, \\ \left\{\frac{c_1}{\{0.8,0.4,0.3\}}, \frac{c_2}{\{0.2,0.6,0.3\}}, \frac{c_3}{\{0.2,0.1,0.3\}}\right\}, \left\{\frac{v_1}{\{0.4,0.1,0.2\}}, \frac{v_2}{\{0.1,0.4,0.5\}}\right\} \end{array}\right),$$

$$\widetilde{\Gamma}^c(a_3) = \left(\begin{array}{c} \left\{\frac{h_1}{\{0.3,0.7,0.8\}}, \frac{h_2}{\{0.8,0.1,0.3\}}, \frac{h_3}{\{0.6,0.7,0.8\}}, \frac{h_4}{\{0.8,0.5,0.4\}}\right\}, \\ \left\{\frac{c_1}{\{0.1,0.2,0.3\}}, \frac{c_2}{\{0.5,0.6,0.8\}}, \frac{c_3}{\{0.8,0.3,0.1\}}\right\}, \left\{\frac{v_1}{\{0.2,0.1\}}, \frac{v_2}{\{0.5,0.8,0.6\}}\right\} \end{array}\right),$$

$$\widetilde{\Gamma}^c(a_4) = \left(\begin{array}{c} \left\{\frac{h_1}{\{0.4,0.6,0.5\}}, \frac{h_2}{\{0.1,0.2,0.6\}}, \frac{h_3}{\{0.9,0.2,0.1\}}, \frac{h_4}{\{0.7,0.8,0.5\}}\right\}, \\ \left\{\frac{c_1}{\{0.9,0.2,0.8\}}, \frac{c_2}{\{0.2,0.8,0.4\}}, \frac{c_3}{\{0.3,0.8,0.9\}}\right\}, \left\{\frac{v_1}{\{0.1,0.2\}}, \frac{v_2}{\{0.6,0.8\}}\right\} \end{array}\right),$$

$$\widetilde{\Gamma}^c(a_5) = \left(\begin{array}{c} \left\{\frac{h_1}{\{0.3,0.7,0.8\}}, \frac{h_2}{\{0.5,0.8,0.7\}}, \frac{h_3}{\{0.2,0.3\}}, \frac{h_4}{\{0.1,0.2,0.3\}}\right\}, \\ \left\{\frac{c_1}{\{0.8,0.1\}}, \frac{c_2}{\{0.1,0.2,0.3\}}, \frac{c_3}{\{0.3,0.8\}}\right\}, \left\{\frac{v_1}{\{0.3,0.5\}}, \frac{v_2}{\{0.1,0.2,0.3\}}\right\} \end{array}\right),$$

$$\widetilde{\Gamma}^c(a_6) = \left(\begin{array}{c} \left\{\frac{h_1}{\{0.5,0.6,0.8\}}, \frac{h_2}{\{0.8,0.1\}}, \frac{h_3}{\{0.3,0.4,0.6\}}, \frac{h_4}{\{0.7,0.2,0.1\}}\right\}, \\ \left\{\frac{c_1}{\{0.5,0.6\}}, \frac{c_2}{\{0.7,0.8,0.1\}}, \frac{c_3}{\{0.4,0.6\}}\right\}, \left\{\frac{v_1}{\{0.8,0.1\}}, \frac{v_2}{\{0.8,0.3,0.1\}}\right\} \end{array}\right).$$

Definition 3.10. A hesitant fuzzy soft multiset $(\widetilde{\Gamma}, A)$ over U is called a seminull hesitant fuzzy soft multiset, denoted by $(\widetilde{\Gamma}, A)_{\approx\phi}$, if at least one of a hesitant fuzzy soft multiset parts of $\left(\widetilde{\Gamma}, A\right) = \phi$.

Example 3.11. Consider Example 3.2. Let us consider a hesitant fuzzy soft multiset $(\widetilde{\Gamma},\ A)$ which describes "stone houses" "cars" and "hotels" with

$$A = \{a_1 = (e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1),\ a_2 = (e_{U_1}, 3, e_{U_2}, 3, e_{U_3}, 1), a_3 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3)\}.$$

Then a seminull hesitant fuzzy soft multiset denoted $(\widetilde{\Gamma},\ A)_{\approx\phi_1}$ is given by

$$\left\{\begin{array}{c} \left(a_1, \left(\begin{array}{c} \left\{\frac{h_1}{0}, \frac{h_2}{0}, \frac{h_3}{0}, \frac{h_4}{0}\right\}, \left\{\frac{c_1}{\{0.5,0.3\}}, \frac{c_3}{\{0.7,0.6,0.8\}}, \frac{c_4}{\{0.9,0.2,0.3\}}\right\}, \\ \left\{\frac{v_1}{\{0.7,0.5,0.8\}}, \frac{v_2}{\{0.9,0.7,0.8\}}\right\} \end{array}\right)\right), \\ \left(a_2, \left(\begin{array}{c} \left\{\frac{h_1}{0}, \frac{h_2}{0}, \frac{h_3}{0}, \frac{h_4}{0}\right\}, \left\{\frac{c_1}{\{0.7,0.6,0.2\}}, \frac{c_3}{\{0.2,0.4,0.8\}}, \frac{c_4}{\{0.7,0.9,0.8\}}\right\}, \\ \left\{\frac{v_1}{\{0.8,0.9,0.6\}}, \frac{v_2}{\{0.5,0.6,0.9\}}\right\} \end{array}\right)\right), \\ \left(a_3, \left(\begin{array}{c} \left\{\frac{h_1}{0}, \frac{h_2}{0}, \frac{h_3}{0}, \frac{h_4}{0}\right\}, \left\{\frac{c_1}{\{0.7,0.8,0.9\}}, \frac{c_3}{\{0.2,0.4,0.5\}}, \frac{c_4}{\{0.9,0.7,0.2\}}\right\}, \\ \left\{\frac{v_1}{\{0.9,0.8\}}, \frac{v_2}{\{0.4,0.2,0.5\}}\right\} \end{array}\right)\right) \end{array}\right\}.$$

Definition 3.12. A hesitant fuzzy soft multiset $(\widetilde{\Gamma},\ A)$ over U is a null hesitant fuzzy soft multiset, denoted by $(\widetilde{\Gamma}, A)_{\phi}$, if all the hesitant fuzzy soft multiset parts of $\left(\widetilde{\Gamma},\ A\right) = \phi$.

Example 3.13. Consider 3.2. Let us consider a hesitant fuzzy soft multiset $(\widetilde{\Gamma},\ A)$ which describes "stone houses", "cheap cars" and "hotels in Kuala Lumpur" with

$$A = \{a_1 = (e_{U_1}, 3, e_{U_2}, 2,\ e_{U_3}, 3)\,,\ a_2 = (e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 3)\}$$

$$(\widetilde{\Gamma},\ A)_{\varnothing} = \left\{ \begin{array}{c} \left(a_1, \left(\left\{\frac{h_1}{0}, \frac{h_2}{0}, \frac{h_3}{0}, \frac{h_4}{0}\right\}, \left\{\frac{c_1}{0},\ \frac{c_3}{0}, \frac{c_4}{0.}\right\}, \left\{\frac{v_1}{0}, \frac{v_2}{0}\right\}\right)\right), \\ \left(a_2, \left(\left\{\frac{h_1}{0}, \frac{h_2}{0}, \frac{h_3}{0}, \frac{h_4}{0}\right\}, \left\{\frac{c_1}{0},\ \frac{c_3}{0}, \frac{c_4}{0}\right\}, \left\{\frac{v_1}{0}, \frac{v_2}{0}\right\}\right)\right) \end{array} \right\}.$$

Definition 3.14. A hesitant fuzzy soft multiset $(\widetilde{\Gamma}, A)$ over U is called a semi-absolute hesitant fuzzy soft multiset, denoted by $(\widetilde{\Gamma}, A)_{\approx U_i}$, if $\left(e_{U_i,}j,\ \ \widetilde{\Gamma} e_{U_i}, j\right) = U_i$, for at least one i, $a_k \in A$, $k = \{1, 2, \ldots, n\}$, $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, r\}$.

Example 3.15. Consider 3.2. Let us consider a hesitant fuzzy soft multiset $(\widetilde{\Gamma},\ A)$ which describes "wooden houses," "cars" and "hotels"

$A = \{a_1 = (e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1)\,,\ a_2 = (e_{U_1}, 3, e_{U_2}, 3, e_{U_3}, 1)\,, a_3 = (e_{U_1}, 3, e_{U_2}, 3, e_{U_3}, 3)\}$.

Then a semi-absolute hesitant fuzzy soft multiset denoted $(\widetilde{\Gamma},\ A)_{\approx U_1}$ is given by

$$\left\{ \begin{array}{c} \left(a_1, \left(\begin{array}{c} \left\{\frac{h_1}{1}, \frac{h_2}{1}, \frac{h_3}{1}, \frac{h_4}{1}\right\}, \left\{\frac{c_1}{\{0.5,0.3\}},\ \frac{c_3}{\{0.7,0.6,0.8\}}, \frac{c_4}{\{0.9,0.2,0.3\}}\right\}, \\ \left\{\frac{v_1}{\{0.7,0.5,0.8\}}, \frac{v_2}{\{0.9,0.7,0.8\}}\right\} \end{array} \right)\right), \\ \left(a_2, \left(\begin{array}{c} \left\{\frac{h_1}{1}, \frac{h_2}{1}, \frac{h_3}{1}, \frac{h_4}{1}\right\}, \left\{\frac{c_1}{\{0.7,0.6,0.2\}},\ \frac{c_3}{\{0.2,0.4,0.8\}}, \frac{c_4}{\{0.7,0.9,0.8\}}\right\}, \\ \left\{\frac{v_1}{\{0.8,0.9,0.6\}}, \frac{v_2}{\{0.5,0.6,0.9\}}\right\} \end{array} \right)\right), \\ \left(a_3, \left(\begin{array}{c} \left\{\frac{h_1}{1}, \frac{h_2}{1}, \frac{h_3}{1}, \frac{h_4}{1}\right\}, \left\{\frac{c_1}{\{0.7,0.8,0.9\}},\ \frac{c_3}{\{0.2,0.4,0.5\}}, \frac{c_4}{\{0.9,0.7,0.2\}}\right\}, \\ \left\{\frac{v_1}{\{0.9,0.8\}}, \frac{v_2}{\{0.4,0.2,0.5\}}\right\} \end{array} \right)\right) \end{array} \right\}.$$

Definition 3.16. A hesitant fuzzy soft multiset $(\widetilde{\Gamma}, A)$ over U is called an absolute hesitant fuzzy soft multiset, denoted by $(\widetilde{\Gamma},\ A)_U$, if $\left(e_{U_i}, j,\ \ \widetilde{\Gamma} e_{U_i}, j\right) =\ U_i,\ \forall\ i$.

Example 3.17. Consider Example 3.2. Let us consider a hesitant fuzzy soft multiset $(\widetilde{\Gamma},\ A)$ which describes "wooden houses", "expensive classic cars" and "hotels in Kuala Lumpur" with

$$A = \{a_1 = (e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 3)\,,\ a_2 = (e_{U_1}, 3, e_{U_2}, 3, e_{U_3}, 3)\}.$$

Then an absolute hesitant fuzzy soft multiset $(\widetilde{\Gamma},\ A)_U$ is given as

$$(\widetilde{\Gamma},\ A)_U = \left\{ \begin{array}{c} \left(a_1, \left(\left\{\frac{h_1}{1}, \frac{h_2}{1}, \frac{h_3}{1}, \frac{h_4}{1}\right\}, \left\{\frac{c_1}{1},\ \frac{c_3}{1}, \frac{c_4}{1}\right\}, \left\{\frac{v_1}{1}, \frac{v_2}{1}\right\}\right)\right), \\ \left(a_2, \left(\left\{\frac{h_1}{1}, \frac{h_2}{1}, \frac{h_3}{1}, \frac{h_4}{1}\right\}, \left\{\frac{c_1}{1},\ \frac{c_3}{1}, \frac{c_4}{1}\right\}, \left\{\frac{v_1}{1}, \frac{v_2}{1}\right\}\right)\right) \end{array} \right\}.$$

Theorem 3.18. *If $(\widetilde{\Gamma},\ A)$ is a hesitant fuzzy soft multiset over U, Then*

(1) $\left((\widetilde{\Gamma}, A)^c\right)^c = (\widetilde{\Gamma},\ A)$,

(2) $(\widetilde{\Gamma},\ A)_{\approx\phi_i}{}^{c} = (\widetilde{\Gamma},\ A)_{\approx U_i}$,

(3) $(\widetilde{\Gamma},\ A)_{\phi}{}^{c} = (\widetilde{\Gamma},\ A)_{U}$,

(4) $(\widetilde{\Gamma},\ A)_{\approx U_i}{}^{c} = (\widetilde{\Gamma},\ A)_{\approx\phi_i}$,

(5) $(\widetilde{\Gamma},\ A)_{U}{}^{c} = (\widetilde{\Gamma},\ A)_{\phi}$.

Proof. The proof is straight forward. □

Definition 3.19. The union of two hesitant fuzzy soft multisets $(\widetilde{\Gamma}, A)$ and $(\tilde{G}, B)$ over U, denoted by $(\widetilde{\Gamma}, A)\widetilde{\cup}(\tilde{G}, B)$, is a hesitant fuzzy soft multiset $(\tilde{H}, C)$, where $C = A \cup B$ and for all $e \in C$,

$$\tilde{H}(e) = \begin{cases} \widetilde{\Gamma}(e) & \text{if } e \in A/B \\ \tilde{G}(e) & \text{if } e \in B/A \\ \widetilde{\Gamma}(e) \cup\ \tilde{G}(e) & \text{if } e \in A \cap B. \end{cases}$$

Example 3.20. Consider Example 3.2. Let

$$A = \left\{ \begin{array}{c} a_1 = (e_{U_1}, 1, e_{U_2}, 1, eu_3, 1), a_2 = (e_{U_1}, 2, e_{U_2}, 3, eu_3, 1), \\ a_3 = (e_{U_1}, 4, e_{U_2}, 3, eu_3, 3), a_4 = (e_{U_1}, 3, e_{U_2}, 1, eu_3, 1) \end{array} \right\}$$

and

$$B = \left\{ \begin{array}{c} b_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), b_2 = (e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1), \\ b_3 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1), b_4 = (e_{U_1}, 5, e_{U_2}, 4, e_{U_3}, 2), \\ b_5 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 2),\ b_6 = (e_{U_1}, 1, e_{U_2}, 3, e_{U_3}, 2) \end{array} \right\}.$$

Suppose that $(\widetilde{\Gamma},\ A)$ and $(\tilde{G},\ B)$ are two hesitant fuzzy soft multisets over the same U such that

$$\left(\widetilde{\Gamma},\ A\right) =$$

$$\left\{ \begin{array}{c} \left(a_1, \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.4,0.8,0.4\}}, \frac{h_2}{\{0.5,0.7\}}, \frac{h_3}{\{0.8,0.7,0.6\}}, \frac{h_4}{\{0.5,0.3\}} \right\}, \\ \left\{ \frac{c_1}{\{0.5,0.3\}}, \frac{c_2}{\{0.6,0.6\}}, \frac{c_3}{\{0.6,0.2,0.2\}} \right\}, \left\{ \frac{v_1}{\{0.6,0.5,0.7\}}, \frac{v_2}{\{0.7,0.6,0.8\}} \right\} \end{array} \right)\right), \\ \left(a_2, \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.4,0.3,0.6\}}, \frac{h_2}{\{0.5,0.6\}}, \frac{h_3}{\{0.4,0.2,0.6\}}, \frac{h_4}{\{0.4,0.2,0.3\}} \right\}, \\ \left\{ \frac{c_1}{\{0.6,0.4,0.3\}}, \frac{c_2}{\{0.5,0.8,0.2\}}, \frac{c_3}{\{0.4,0.3,0.2\}} \right\}, \left\{ \frac{v_1}{\{0.8,0.4,0.4\}}, \frac{v_2}{\{0.5,0.7,0.1\}} \right\} \end{array} \right)\right), \\ \left(a_3, \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.2,0.3,0.7\}}, \frac{h_2}{\{0.7,0.9,0.2\}}, \frac{h_3}{\{0.1,0.2,0.3\}}, \frac{h_4}{\{0.5,0.5,0.1\}} \right\}, \\ \left\{ \frac{c_1}{\{0.6,0.7,0.8\}}, \frac{c_2}{\{0.1,0.3,0.4\}}, \frac{c_3}{\{0.8,0.6,0.2\}} \right\}, \left\{ \frac{v_1}{\{0.8,0.7\}}, \frac{v_2}{\{0.3,0.1,0.4\}} \right\} \end{array} \right)\right), \\ \left(a_4, \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.5,0.2,0.1\}}, \frac{h_2}{\{0.7,0.4,0.6\}}, \frac{h_3}{\{0.8,0.9,0.1\}}, \frac{h_4}{\{0.6,0.3,0.2\}} \right\}, \\ \left\{ \frac{c_1}{\{0.3,0.4,0.6\}}, \frac{c_2}{\{0.4,0.6\}}, \frac{c_3}{\{0.9,0.3,0.4\}} \right\}, \left\{ \frac{v_1}{\{0.9,0.4\}}, \frac{v_2}{\{0.9,0.8,0.4\}} \right\} \end{array} \right)\right) \end{array} \right\},$$

and $\left(\tilde{G},\ B\right) =$

$$
\left\{\begin{array}{c}
\left(b_1, \left(\begin{array}{c} \left\{\frac{h_1}{\{0.5,0.8,0.4\}}, \frac{h_2}{\{0.5,0.8\}}, \frac{h_3}{\{0.9,0.7,0.8\}}, \frac{h_4}{\{0.6,0.4\}}\right\}, \\ \left\{\frac{c_1}{\{0.5,0.3\}}, \frac{c_2}{\{0.7,0.6\}}, \frac{c_3}{\{0.9,0.2,0.3\}}\right\}, \left\{\frac{v_1}{\{0.7,0.5,0.8\}}, \frac{v_2}{\{0.9,0.7,0.8\}}\right\} \end{array}\right)\right), \\
\left(b_2, \left(\begin{array}{c} \left\{\frac{h_1}{\{0.7,0.7,0.9\}}, \frac{h_2}{\{0.3,0.6,0.1\}}, \frac{h_3}{\{0.8,0.6,0.2\}}, \frac{h_4}{\{0.3,0.2,0.2\}}\right\}, \\ \left\{\frac{c_1}{\{0.7,0.6,0.2\}}, \frac{c_2}{\{0.2,0.4,0.8\}}, \frac{c_3}{\{0.7,0.9,0.8\}}\right\}, \left\{\frac{v_1}{\{0.8,0.9,0.6\}}, \frac{v_2}{\{0.5,0.6,0.9\}}\right\} \end{array}\right)\right), \\
\left(b_3, \left(\begin{array}{c} \left\{\frac{h_1}{\{0.4,0.3,0.7\}}, \frac{h_2}{\{0.7,0.9,0.2\}}, \frac{h_3}{\{0.4,0.3,0.7\}}, \frac{h_4}{\{0.6,0.5,0.2\}}\right\}, \\ \left\{\frac{c_1}{\{0.7,0.8,0.9\}}, \frac{c_2}{\{0.5,0.9,0.5\}}, \frac{c_3}{\{0.9,0.7,0.2\}}\right\}, \left\{\frac{v_1}{\{0.9,0.8\}}, \frac{v_2}{\{0.4,0.3,0.6\}}\right\} \end{array}\right)\right), \\
\left(b_4, \left(\begin{array}{c} \left\{\frac{h_1}{\{0.5,0.4,0.6\}}, \frac{h_2}{\{0.4,0.8,0.9\}}, \frac{h_3}{\{0.9,0.8,0.1\}}, \frac{h_4}{\{0.5,0.2,0.3\}}\right\}, \\ \left\{\frac{c_1}{\{0.2,0.8,0.1\}}, \frac{c_2}{\{0.6,0.2,0.8\}}, \frac{c_3}{\{0.1,0.2,0.7\}}\right\}, \left\{\frac{v_1}{\{0.8,0.9\}}, \frac{v_2}{\{0.2,0.4\}}\right\} \end{array}\right)\right), \\
\left(b_5, \left(\begin{array}{c} \left\{\frac{h_1}{\{0.2,0.3,0.7\}}, \frac{h_2}{\{0.3,0.2,0.5\}}, \frac{h_3}{\{0.7,0.8\}}, \frac{h_4}{\{0.7,0.8,0.9\}}\right\}, \\ \left\{\frac{c_1}{\{0.9,0.2\}}, \frac{c_2}{\{0.7,0.8,0.9\}}, \frac{c_3}{\{0.2,0.7\}}\right\}, \left\{\frac{v_1}{\{0.5,0.7\}}, \frac{v_2}{\{0.7,0.8,0.9\}}\right\} \end{array}\right)\right), \\
\left(b_6, \left(\begin{array}{c} \left\{\frac{h_1}{\{0.2,0.4,0.5\}}, \frac{h_2}{\{0.9,0.2\}}, \frac{h_3}{\{0.4,0.6,0.7\}}, \frac{h_4}{\{0.9,0.8,0.3\}}\right\}, \\ \left\{\frac{c_1}{\{0.4,0.5\}}, \frac{c_2}{\{0.9,0.2,0.3\}}, \frac{c_3}{\{0.4,0.6\}}\right\}, \left\{\frac{v_1}{\{0.9,0.2\}}, \frac{v_2}{\{0.9,0.7,0.2\}}\right\} \end{array}\right)\right)
\end{array}\right\}.
$$

Then using the basic fuzzy union, we have $\left(\widetilde{\Gamma},\ A\right) \widetilde{\cup} \left(\tilde{G},\ B\right) = \left(\tilde{H}, D\right) =$

$$
\left\{\begin{array}{c}
\left(d_1, \left(\begin{array}{c} \left\{\frac{h_1}{\{0.5,0.8,0.4\}}, \frac{h_2}{\{0.5,0.8\}}, \frac{h_3}{\{0.9,0.7,0.8\}}, \frac{h_4}{\{0.6,0.4\}}\right\}, \\ \left\{\frac{c_1}{\{0.5,0.3\}}, \frac{c_2}{\{0.7,0.6\}}, \frac{c_3}{\{0.9,0.2,0.3\}}\right\}, \left\{\frac{v_1}{\{0.7,0.5,0.8\}}, \frac{v_2}{\{0.9,0.7,0.8\}}\right\} \end{array}\right)\right), \\
\left(d_2, \left(\begin{array}{c} \left\{\frac{h_1}{\{0.4,0.3,0.7\}}, \frac{h_2}{\{0.7,0.9,0.2\}}, \frac{h_3}{\{0.4,0.3,0.7\}}, \frac{h_4}{\{0.6,0.5,0.2\}}\right\}, \\ \left\{\frac{c_1}{\{0.7,0.8,0.9\}}, \frac{c_2}{\{0.5,0.9,0.5\}}, \frac{c_3}{\{0.4,0.7,0.2\}}\right\}, \left\{\frac{v_1}{\{0.9,0.8\}}, \frac{v_2}{\{0.4,0.3,0.5\}}\right\} \end{array}\right)\right), \\
\left(d_3, \left(\begin{array}{c} \left\{\frac{h_1}{\{0.2,0.3,0.7\}}, \frac{h_2}{\{0.7,0.9,0.2\}}, \frac{h_3}{\{0.1,0.2,0.3\}}, \frac{h_4}{\{0.5,0.5,0.1\}}\right\}, \\ \left\{\frac{c_1}{\{0.6,0.7,0.8\}}, \frac{c_2}{\{0.1,0.3,0.4\}}, \frac{c_3}{\{0.8,0.6,0.2\}}\right\}, \left\{\frac{v_1}{\{0.8,0.7\}}, \frac{v_2}{\{0.3,0.1,0.4\}}\right\} \end{array}\right)\right), \\
\left(d_4, \left(\begin{array}{c} \left\{\frac{h_1}{\{0.5,0.2,0.1\}}, \frac{h_2}{\{0.7,0.4,0.6\}}, \frac{h_3}{\{0.8,0.9,0.1\}}, \frac{h_4}{\{0.6,0.3,0.2\}}\right\}, \\ \left\{\frac{c_1}{\{0.3,0.4,0.6\}}, \frac{c_2}{\{0.4,0.6\}}, \frac{c_3}{\{0.9,0.3,0.4\}}\right\}, \left\{\frac{v_1}{\{0.9,0.4\}}, \frac{v_2}{\{0.9,0.8,0.4\}}\right\} \end{array}\right)\right) \\
\left(d_5, \left(\begin{array}{c} \left\{\frac{h_1}{\{0.7,0.7,0.9\}}, \frac{h_2}{\{0.3,0.6,0.1\}}, \frac{h_3}{\{0.8,0.6,0.2\}}, \frac{h_4}{\{0.3,0.2,0.2\}}\right\}, \\ \left\{\frac{c_1}{\{0.7,0.6,0.2\}}, \frac{c_2}{\{0.2,0.4,0.8\}}, \frac{c_3}{\{0.7,0.9,0.8\}}\right\}, \left\{\frac{v_1}{\{0.8,0.9,0.6\}}, \frac{v_2}{\{0.5,0.6,0.9\}}\right\} \end{array}\right)\right), \\
\left(d_6, \left(\begin{array}{c} \left\{\frac{h_1}{\{0.5,0.4,0.6\}}, \frac{h_2}{\{0.4,0.8,0.9\}}, \frac{h_3}{\{0.9,0.8,0.1\}}, \frac{h_4}{\{0.5,0.2,0.3\}}\right\}, \\ \left\{\frac{c_1}{\{0.2,0.8,0.1\}}, \frac{c_2}{\{0.6,0.2,0.8\}}, \frac{c_3}{\{0.1,0.2,0.7\}}\right\}, \left\{\frac{v_1}{\{0.8,0.9\}}, \frac{v_2}{\{0.2,0.4\}}\right\} \end{array}\right)\right), \\
\left(d_7, \left(\begin{array}{c} \left\{\frac{h_1}{\{0.2,0.3,0.7\}}, \frac{h_2}{\{0.3,0.2,0.5\}}, \frac{h_3}{\{0.7,0.8\}}, \frac{h_4}{\{0.7,0.8,0.9\}}\right\}, \\ \left\{\frac{c_1}{\{0.9,0.2\}}, \frac{c_2}{\{0.7,0.8,0.9\}}, \frac{c_3}{\{0.2,0.7\}}\right\}, \left\{\frac{v_1}{\{0.5,0.7\}}, \frac{v_2}{\{0.7,0.8,0.9\}}\right\} \end{array}\right)\right), \\
\left(d_8, \left(\begin{array}{c} \left\{\frac{h_1}{\{0.2,0.4,0.5\}}, \frac{h_2}{\{0.9,0.2\}}, \frac{h_3}{\{0.4,0.6,0.7\}}, \frac{h_4}{\{0.9,0.8,0.3\}}\right\}, \\ \left\{\frac{c_1}{\{0.4,0.5\}}, \frac{c_2}{\{0.9,0.2,0.3\}}, \frac{c_3}{\{0.4,0.6\}}\right\}, \left\{\frac{v_1}{\{0.9,0.2\}}, \frac{v_2}{\{0.9,0.7,0.2\}}\right\} \end{array}\right)\right)
\end{array}\right\},
$$

where $D = \{d_1 = a_1 = b_1,\ d_2 = a_2 = b_3,\ d_3 = a_3,\ d_4 = a_4,\ d_5 = b_2,\ d_6 = b_4, d_7 = b_5, d_8 = b_6\}$.

Theorem 3.21. *If* $(\widetilde{\Gamma},\ A)$, $(\tilde{G},\ B)$ *and* $(\tilde{H},\ C)$ *are three hesitant fuzzy soft multisets over* U, *then*

(1) $\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}\left(\widetilde{\Gamma},\ A\right)=(\widetilde{\Gamma},\ A)$,

(2) $\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}\left(\left(\tilde{G},B\right)\widetilde{\cup}\left(\tilde{H},\ C\right)\right)=\left(\left(\widetilde{\Gamma},A\right)\widetilde{\cup}\left(\tilde{G},\ B\right)\right)\widetilde{\cup}\left(\tilde{H},\ C\right)$,

(3) $\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}\left(\tilde{G},\ A\right)_{\phi}=\left(\widetilde{\Gamma},\ A\right)$,

(4) $\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}\left(\tilde{G},\ A\right)_{\approx\phi_i}=\left(\tilde{R},\ A\right)$,

where $\tilde{R}$ *is as in Definition 3.19*,

(5) $\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}\left(\tilde{G},\ B\right)_{\approx\phi_i}=\left(\tilde{R},\ D\right)$,

where $D=A\cup B$ *and* $\tilde{R}$ *is as in Definition 3.19*,

(6) $\left(\widetilde{\Gamma},A\right)\widetilde{\cup}\left(\tilde{G},\ B\right)_{\phi}=\begin{cases}\left(\widetilde{\Gamma},A\right) & \text{if } A=B\\ \left(\tilde{R},D\right) & \text{if otherwise,}\end{cases}$

where $D=A\cup B$,

(7) $\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}\left(\tilde{G},\ A\right)_{\approx U_i}=\left(\tilde{R},\ A\right)_{\approx U_i}$,

(8) $\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}\left(\tilde{G},\ A\right)_{U}=\left(\tilde{G},A\right)_{U}$,

(9) $\left(\widetilde{\Gamma},A\right)\widetilde{\cup}\left(\tilde{G},\ B\right)_{\approx U_i}=\begin{cases}\left(\tilde{R},\ A\right)_{\approx U_i} & \text{if } A=B,\\ \left(\tilde{R},\ D\right) & \text{otherwise,}\end{cases}$

where $D=A\cup B$,

(10) $\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}\left(\tilde{G},\ B\right)_{U}=\begin{cases}\left(\tilde{G},B\right)_{U}, & \text{if } A=B,\\ \left(\tilde{R},\ D\right) & \text{otherwise,}\end{cases}$

where $D=A\cup B$.

Proof. The proof is straight forward. □

Definition 3.22 ([7])**.** The extended intersection of two hesitant fuzzy soft multisets $\left(\widetilde{\Gamma},A\right)$ and $\left(\tilde{G},B\right)$ over U, denoted by $\left(\widetilde{\Gamma},A\right)\widetilde{\cap}_E\left(\tilde{G},B\right)$, is the hesitant fuzzy soft multiset $(\tilde{H},C)$, where $C=A\cup B$ and for all $e\in C$,

$$\tilde{H}\left(e\right)=\begin{cases}\widetilde{\Gamma}\left(e\right) & \text{if } e\in A/B\\ \tilde{G}\left(e\right) & \text{if } e\in B/A\\ \widetilde{\Gamma}\left(e\right)\cap\tilde{G}\left(e\right) & \text{if } e\in A\cap B.\end{cases}$$

Example 3.23. Consider Example 3.2. By using the basic fuzzy intersection, we have

$$\left(\widetilde{\Gamma},\ A\right)\widetilde{\cap}_E\left(\tilde{G},\ B\right)=\left(\tilde{H},\ C\right)=$$

$$\left\{\begin{array}{c}
\left(c_1,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.4,0.8,0.4\}},\frac{h_2}{\{0.5,0.7\}},\frac{h_3}{\{0.8,0.7,0.6\}},\frac{h_4}{\{0.5,0.3\}}\right\},\\ \left\{\frac{c_1}{\{0.5,0.3\}},\frac{c_2}{\{0.6,0.6\}},\frac{c_3}{\{0.6,0.2,0.2\}}\right\},\left\{\frac{v_1}{\{0.6,0.5,0.7\}},\frac{v_2}{\{0.7,0.6,0.8\}}\right\}\end{array}\right)\right),\\
\left(c_2,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.4,0.3,0.6\}},\frac{h_2}{\{0.5,0.6\}},\frac{h_3}{\{0.4,0.2,0.6\}},\frac{h_4}{\{0.4,0.2,0.3\}}\right\},\\ \left\{\frac{c_1}{\{0.6,0.4,0.3\}},\frac{c_2}{\{0.5,0.8,0.2\}},\frac{c_3}{\{0.4,0.3,0.2\}}\right\},\left\{\frac{v_1}{\{0.8,0.4,0.4\}},\frac{v_2}{\{0.5,0.7,0.1\}}\right\}\end{array}\right)\right),\\
\left(c_3,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.2,0.3,0.7\}},\frac{h_2}{\{0.7,0.9,0.2\}},\frac{h_3}{\{0.1,0.2,0.3\}},\frac{h_4}{\{0.5,0.5,0.1\}}\right\},\\ \left\{\frac{c_1}{\{0.6,0.7,0.8\}},\frac{c_2}{\{0.1,0.3,0.4\}},\frac{c_3}{\{0.8,0.6,0.2\}}\right\},\left\{\frac{v_1}{\{0.8,0.7\}},\frac{v_2}{\{0.3,0.1,0.4\}}\right\}\end{array}\right)\right),\\
\left(c_4,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.5,0.2,0.1\}},\frac{h_2}{\{0.7,0.4,0.6\}},\frac{h_3}{\{0.8,0.9,0.1\}},\frac{h_4}{\{0.6,0.3,0.2\}}\right\},\\ \left\{\frac{c_1}{\{0.3,0.4,0.6\}},\frac{c_2}{\{0.4,0.6\}},\frac{c_3}{\{0.9,0.3,0.4\}}\right\},\left\{\frac{v_1}{\{0.9,0.4\}},\frac{v_2}{\{0.9,0.8,0.4\}}\right\}\end{array}\right)\right)\\
\left(c_5,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.7,0.7,0.9\}},\frac{h_2}{\{0.3,0.6,0.1\}},\frac{h_3}{\{0.8,0.6,0.2\}},\frac{h_4}{\{0.3,0.2,0.2\}}\right\},\\ \left\{\frac{c_1}{\{0.7,0.6,0.2\}},\frac{c_2}{\{0.2,0.4,0.8\}},\frac{c_3}{\{0.7,0.9,0.8\}}\right\},\left\{\frac{v_1}{\{0.8,0.9,0.6\}},\frac{v_2}{\{0.5,0.6,0.9\}}\right\}\end{array}\right)\right),\\
\left(c_6,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.5,0.4,0.6\}},\frac{h_2}{\{0.4,0.8,0.9\}},\frac{h_3}{\{0.9,0.8,0.1\}},\frac{h_4}{\{0.5,0.2,0.3\}}\right\},\\ \left\{\frac{c_1}{\{0.2,0.8,0.1\}},\frac{c_2}{\{0.6,0.2,0.8\}},\frac{c_3}{\{0.1,0.2,0.7\}}\right\},\left\{\frac{v_1}{\{0.8,0.9\}},\frac{v_2}{\{0.2,0.4\}}\right\}\end{array}\right)\right),\\
\left(c_7,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.2,0.3,0.7\}},\frac{h_2}{\{0.3,0.2,0.5\}},\frac{h_3}{\{0.7,0.8\}},\frac{h_4}{\{0.7,0.8,0.9\}}\right\},\\ \left\{\frac{c_1}{\{0.9,0.2\}},\frac{c_2}{\{0.7,0.8,0.9\}},\frac{c_3}{\{0.2,0.7\}}\right\},\left\{\frac{v_1}{\{0.5,0.7\}},\frac{v_2}{\{0.7,0.8,0.9\}}\right\}\end{array}\right)\right),\\
\left(c_8,\left(\begin{array}{c}\left\{\frac{h_1}{\{0.2,0.4,0.5\}},\frac{h_2}{\{0.9,0.2\}},\frac{h_3}{\{0.4,0.6,0.7\}},\frac{h_4}{\{0.9,0.8,0.3\}}\right\},\\ \left\{\frac{c_1}{\{0.4,0.5\}},\frac{c_2}{\{0.9,0.2,0.3\}},\frac{c_3}{\{0.4,0.6\}}\right\},\left\{\frac{v_1}{\{0.9,0.2\}},\frac{v_2}{\{0.9,0.7,0.2\}}\right\}\end{array}\right)\right)
\end{array}\right\}.$$

Where $C=\{c_1=a_1=b_1,\ c_2=a_2=b_3,\ c_3=a_3,\ c_4=\ a_4,\ c_5=b_2,\ c_6=b_4, c_7=b_5, c_8=b_6\}$.

Theorem 3.24. *If $(\widetilde{\Gamma},A)$, $(\tilde{G},B)$ and $(\tilde{H},C)$ are three hesitant fuzzy soft multisets over U, then*

(1) $\left(\widetilde{\Gamma},A\right)\widetilde{\cap}_E\left(\widetilde{\Gamma},A\right)=(\widetilde{\Gamma},A)$,

(2) $\left(\widetilde{\Gamma},A\right)\widetilde{\cap}_E\left(\left(\tilde{G},B\right)\widetilde{\cap}_E\left(\tilde{H},C\right)\right)=\left(\left(\widetilde{\Gamma},A\right)\widetilde{\cap}_E\left(\tilde{G},B\right)\right)\widetilde{\cap}_E\left(\tilde{H},C\right)$,

(3) $\left(\widetilde{\Gamma},A\right)\widetilde{\cap}_E\ \left(\tilde{G},A\right)_U=\left(\widetilde{\Gamma},A\right)$,

(4) $\left(\widetilde{\Gamma},A\right)\widetilde{\cap}_E\ \left(\tilde{G},A\right)_{\approx\phi_i}=\ \left(\tilde{R},A\right)_{\approx\phi_i}$,

where $\tilde{R}$ is as in Definition 3.22,

(5) $\left(\widetilde{\Gamma},A\right)\widetilde{\cap}_E\left(\tilde{G},A\right)_{\phi_i}=\ \left(\tilde{R},A\right)_{\phi_i}$,

where $\tilde{R}$ is as in Definition 3.22,

(6) $\left(\widetilde{\Gamma},A\right)\widetilde{\cap}_E\ \left(\tilde{G},B\right)_{\approx\phi_i}=\begin{cases}\left(\tilde{R},D\right)_{\approx\phi_i} & \textit{if } A\subseteq B\\ \left(\tilde{R},\ D\right) & \textit{if otherwise,}\end{cases}$

where $D=A\ \cup B$,

(7) $\left(\widetilde{\Gamma}, A\right) \widetilde{\cap}_E \left(\tilde{G}, B\right)_{\approx U_i} = \left(\tilde{R}, D\right)$,
where $D = A \cup B$ *and* $\widetilde{R}$ *is as in Definition 3.22,*
(8) $\left(\widetilde{\Gamma}, A\right) \widetilde{\cap}_E \left(\tilde{G}, A\right)_U = \left(\widetilde{\Gamma}, A\right)$,

(9) $\left(\widetilde{\Gamma}, A\right) \widetilde{\cap}_E \left(\tilde{G}, B\right)_\phi = \begin{cases} \left(\tilde{R},\ D\right)_\varnothing & \text{if } A \subseteq B \\ \left(\tilde{R},\ D\right) & \text{if otherwise,} \end{cases}$

where $D = A \cup B$,

(10) $\left(\widetilde{\Gamma}, A\right) \widetilde{\cap}_E \left(\tilde{G}, B\right)_U = \begin{cases} \left(\widetilde{\Gamma}, A\right) & \text{if } B \subseteq A \\ \left(\tilde{R}, D\right) & \text{if otherwise,} \end{cases}$

where $D = A \cup B$ *and* $\tilde{R}$ *is as in Definition 3.22.*

Proof. The proof is straight forward. □

Definition 3.25. The restricted intersection of two hesitant fuzzy soft multisets $\left(\widetilde{\Gamma}, A\right)$ and $(\tilde{G}, B)$ over U, denoted by $\left(\widetilde{\Gamma}, A\right) \cap (\tilde{G}, B)$ is the hesitant fuzzy soft multiset $(\tilde{H}, C)$, where $C = A \cap B \neq \phi$ and for all $e \in C$, $\tilde{H}(e) = \widetilde{\Gamma}(e) \cap \tilde{G}(e)$.

Example 3.26. Consider Example 3.2. Then we have $\left(\widetilde{\Gamma}, A\right) \Cap \left(\tilde{G},\ B\right) = \left(\tilde{H},\ C\right)$, where $C = A \cap B \neq \phi$, and for all $e \in C$, $\tilde{H}(e) = \widetilde{\Gamma}(e) \cap \tilde{G}(e) =$

$$\left\{ \begin{array}{l} \left(c_1, \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.4,0.8,0.4\}}, \frac{h_2}{\{0.5,0.7\}}, \frac{h_3}{\{0.8,0.7,0.6\}}, \frac{h_4}{\{0.5,0.3\}} \right\}, \\ \left\{ \frac{c_1}{\{0.5,0.3\}}, \frac{c_2}{\{0.6,0.6\}}, \frac{c_3}{\{0.6,0.2,0.2\}} \right\}, \left\{ \frac{v_1}{\{0.6,0.5,0.7\}}, \frac{v_2}{\{0.7,0.6,0.8\}} \right\} \end{array} \right) \right), \\ c_2, \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.4,0.3,0.6\}}, \frac{h_2}{\{0.5,0.6\}}, \frac{h_3}{\{0.4,0.2,0.6\}}, \frac{h_4}{\{0.4,0.2,0.3\}} \right\}, \\ \left\{ \frac{c_1}{\{0.6,0.4,0.3\}}, \frac{c_2}{\{0.5,0.8,0.2\}}, \frac{c_3}{\{0.4,0.3,0.2\}} \right\}, \left\{ \frac{v_1}{\{0.8,0.4,0.4\}}, \frac{v_2}{\{0.5,0.7,0.1\}} \right\} \end{array} \right) \end{array} \right\},$$

where $C = \{c_1 = a_1 = b_1, c_2 = a_2 = b_3\}$.

Definition 3.27. Let $\left(\widetilde{\Gamma}, A\right)$ and $(\tilde{G}, B)$ be two hesitant soft multisets over U. Then the restricted union of $\left(\widetilde{\Gamma}, A\right)$ and $(\tilde{G},\ B)$, denoted by $\left(\widetilde{\Gamma}, A\right) \widetilde{\cup}_R (\tilde{G}, B)$ is the hesitant fuzzy soft multiset defined as:

$$\left(\widetilde{\Gamma},\ A\right) \widetilde{\cup}_R \left(\tilde{G},\ B\right) = \left(\tilde{H},\ C\right),$$

where $C = A \cap B \neq \phi$ and for all $x \in C$, $\tilde{H}(x) = \widetilde{\Gamma}(x) \cup \tilde{G}(x)$.

Example 3.28. Consider Example 3.2. Let $\left(\widetilde{\Gamma}, A\right) \widetilde{\cup}_R \left(\tilde{G}, B\right) = \left(\tilde{H}, C\right) =$

$$\left\{ \begin{array}{l} \left(c_1, \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.5,0.8,0.4\}}, \frac{h_2}{\{0.5,0.8\}}, \frac{h_3}{\{0.9,0.7,0.8\}}, \frac{h_4}{\{0.6,0.4\}} \right\}, \\ \left\{ \frac{c_1}{\{0.5,0.3\}}, \frac{c_2}{\{0.7,0.6\}}, \frac{c_3}{\{0.9,0.2,0.3\}} \right\}, \left\{ \frac{v_1}{\{0.7,0.5,0.8\}}, \frac{v_2}{\{0.9,0.7,0.8\}} \right\} \end{array} \right) \right), \\ c_2, \left(\begin{array}{c} \left\{ \frac{h_1}{\{0.4,0.3,0.7\}}, \frac{h_2}{\{0.7,0.9,0.2\}}, \frac{h_3}{\{0.4,0.3,0.7\}}, \frac{h_4}{\{0.6,0.5,0.2\}} \right\}, \\ \left\{ \frac{c_1}{\{0.7,0.8,0.9\}}, \frac{c_2}{\{0.5,0.9,0.5\}}, \frac{c_3}{\{0.4,0.7,0.2\}} \right\}, \left\{ \frac{v_1}{\{0.9,0.8\}}, \frac{v_2}{\{0.4,0.3,0.5\}} \right\} \end{array} \right) \end{array} \right\},$$

where $C = \{c_1 = a_1 = b_1, c_2 = a_2 = b_3\}$.

Theorem 3.29. *If* $(\widetilde{\Gamma},\ A)$, $(\tilde{G},\ B)$ *and* $(\tilde{H},\ C)$ *are three hesitant fuzzy soft multisets over* U, *then*

(1) $\left(\widetilde{\Gamma},\ A\right) \Cap \left(\widetilde{\Gamma},\ A\right) = \left(\widetilde{\Gamma},\ A\right)$,

(2) $\left(\widetilde{\Gamma},\ A\right) \widetilde{\cup}_R \left(\widetilde{\Gamma},\ A\right) = \left(\widetilde{\Gamma},\ A\right)$,

(3) $\left(\widetilde{\Gamma},\ A\right) \Cap \left(\tilde{G},\ B\right) = \left(\tilde{G},\ B\right) \Cap \left(\widetilde{\Gamma},\ A\right)$,

(4) $\left(\widetilde{\Gamma},\ A\right) \widetilde{\cup}_R (G,\ B) = \left(\tilde{G},\ B\right) \widetilde{\cup}_R \left(\widetilde{\Gamma},\ A\right)$,

(5) $\left(\left(\widetilde{\Gamma},\ A\right) \Cap \left(\tilde{G},\ B\right)\right) \Cap \left(\tilde{H},\ C\right) = \left(\widetilde{\Gamma},\ A\right) \Cap \left(\left(\tilde{G},\ B\right) \Cap \left(\tilde{H},\ C\right)\right)$,

(6) $\left(\left(\widetilde{\Gamma},\ A\right) \widetilde{\cup}_R \left(\tilde{G},\ B\right)\right) \widetilde{\cup}_R \left(\tilde{H},\ C\right) = \left(\widetilde{\Gamma},\ A\right) \widetilde{\cup}_R \left(\left(\tilde{G},\ B\right) \widetilde{\cup}_R \left(\tilde{H},\ C\right)\right)$.

Proof. The proof is quite straight forward. □

Theorem 3.30. *Suppose that* $(\widetilde{\Gamma}, A)$, $(\tilde{G}, B)$ *and* $(\tilde{H}, C)$ *are three hesitant fuzzy soft multisets over* U. *Then*

(1) $$\left(\widetilde{\Gamma},\ A\right) \widetilde{\cup} \left(\left(\tilde{G},\ B\right) \cap_{\mathrm{E}} \left(\tilde{H},\ C\right)\right) = \left(\left(\widetilde{\Gamma},\ A\right) \widetilde{\cup} \left(\tilde{G},\ B\right)\right) \cap_{\mathrm{E}} \left(\left(\widetilde{\Gamma},\ A\right) \widetilde{\cup} \left(\tilde{G},\ B\right)\right),$$

(2) $$\left(\widetilde{\Gamma},\ A\right) \cap_{\mathrm{E}} \left(\left(\tilde{G},\ B\right) \widetilde{\cup} \left(\tilde{H},\ C\right)\right) = \left(\left(\widetilde{\Gamma},\ A\right) \cap_{\mathrm{E}} \left(\tilde{G},\ B\right)\right) \widetilde{\cup} \left(\left(\widetilde{\Gamma},\ A\right) \cap_{\mathrm{E}} \left(\tilde{G},\ B\right)\right).$$

Proof. The proof is straight forward. □

Definition 3.31. The AND Operation on two hesitant fuzzy soft multisets $\left(\widetilde{\Gamma}, A\right)$ and $(\tilde{G}, B)$, denoted by $\left(\widetilde{\Gamma},\ A\right) \widetilde{\bigwedge} \left(\tilde{G},\ B\right)$, is defined as:

$$\left(\widetilde{\Gamma},\ A\right) \widetilde{\bigwedge} \left(\tilde{G},\ B\right) = (\tilde{M},\quad A \times B),$$

where $\tilde{M}\,(a,b) = \widetilde{\Gamma}(a) \cap \tilde{G}(b)$, for all $(a,\ b) \in A \times B$.

Example 3.32. Consider Example 3.6. Let

$$A = \{a_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1)\,, a_2 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1)\}$$

and

$$B = \{b_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), b_2 = (e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1), b_3 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1)\}.$$

Then

$$\tilde{M}\,(a_1,\ b_1) = \left\{ \begin{array}{c} \left\{ \frac{h_1}{\{0.5,0.9,0.4\}}, \frac{h_2}{\{0.4,0.6\}}, \frac{h_3}{\{0.9,0.8,0.5\}}, \frac{h_4}{\{0.6,0.4,0.3\}} \right\}, \\ \left\{ \frac{c_1}{\{0.5,0.6\}}, \frac{c_2}{\{0.7,0.6,0.8\}}, \frac{c_3}{\{0.9,0.4,0.5\}} \right\}, \left\{ \frac{v_1}{\{0.8,0.5,0.8\}}, \frac{v_2}{\{0.9,0.8,0.8\}} \right\} \end{array} \right\},$$

$$\tilde{M}\,(a_1,\ b_2) = \left\{ \begin{array}{c} \left\{ \frac{h_1}{\{0.7,0.8,0.9\}}, \frac{h_2}{\{0.4,0.7,0.2\}}, \frac{h_3}{\{0.9,0.7,0.5\}}, \frac{h_4}{\{0.3,0.2,0.3\}} \right\}, \\ \left\{ \frac{c_1}{\{0.7,0.7,0.4\}}, \frac{c_2}{\{0.7,0.6,,0.8\}}, \frac{c_3}{\{0.9,0.4,0.5\}} \right\}, \left\{ \frac{v_1}{\{0.8,0.9,0.7\}}, \frac{v_2}{\{0.9,0.7,0.9\}} \right\} \end{array} \right\},$$

$$\tilde{M}\,(a_1,\ b_3) = \left\{ \begin{array}{c} \left\{ \frac{h_1}{\{0.5,0.8,0.8\}}, \frac{h_2}{\{0.7,0.9,0.3\}}, \frac{h_3}{\{0.9,0.7,0.5\}}, \frac{h_4}{\{0.6,0.5,0.2\}} \right\}, \\ \left\{ \frac{c_1}{\{0.7,0.8,0.9\}}, \frac{c_2}{\{0.7,0.6,0.5\}}, \frac{c_3}{\{0.9,0.8,0.3\}} \right\}, \left\{ \frac{v_1}{\{0.9,0.8,0.8\}}, \frac{v_2}{\{0.9,0.7,0.8\}} \right\} \end{array} \right\},$$

$$\tilde{M}\left(a_2,\ b_1\right)=\left\{\begin{array}{c}\left\{\frac{h_1}{\{0.7,0.9,0.9\}},\frac{h_2}{\{0.4,0.6,0.1\}},\frac{h_3}{\{0.9,0.8,0.5\}},\frac{h_4}{\{0.6,0.5,0.2\}}\right\},\\ \left\{\frac{c_1}{\{0.7,0.6,0.2\}},\frac{c_2}{\{0.7,0.6,0.8\}},\frac{c_3}{\{0.9,0.8,0.8\}}\right\},\left\{\frac{v_1}{\{0.8,0.9,0.8\}},\frac{v_2}{\{0.9,0.8,0.9\}}\right\}\end{array}\right\},$$

$$\tilde{M}\left(a_2,\ b_2\right)=\left\{\begin{array}{c}\left\{\frac{h_1}{\{0.7,0.8,0.9\}},\frac{h_2}{\{0.3,0.7,0.2\}},\frac{h_3}{\{0.8,0.7,0.3\}},\frac{h_4}{\{0.3,0.2,0.3\}}\right\},\\ \left\{\frac{c_1}{\{0.7,0.7,0.4\}},\frac{c_2}{\{0.2,0.4,0.8\}},\frac{c_3}{\{0.7,0.9,0.9\}}\right\},\left\{\frac{v_1}{\{0.8,0.9,0.7\}},\frac{v_2}{\{0.6,0.7,0.9\}}\right\}\end{array}\right\},$$

$$\tilde{M}\left(a_2,\ b_3\right)=\left\{\begin{array}{c}\left\{\frac{h_1}{\{0.7,0.7,0.9\}},\frac{h_2}{\{0.7,0.9,0.3\}},\frac{h_3}{\{0.8,0.6,0.4\}},\frac{h_4}{\{0.6,0.5,0.2\}}\right\},\\ \left\{\frac{c_1}{\{0.7,0.8,0.9\}},\frac{c_2}{\{0.2,0.5,0.8\}},\frac{c_3}{\{0.9,0.9,0.8\}}\right\},\left\{\frac{v_1}{\{0.9,0.9,0.6\}},\frac{v_2}{\{0.5,0.6,0.9\}}\right\}\end{array}\right\}.$$

Definition 3.33. The OR Operation on two hesitant fuzzy soft multisets $\left(\widetilde{\Gamma},\ A\right)$ and $(\tilde{G},\ B)$, denoted by $\left(\widetilde{\Gamma},\ A\right)\widetilde{\bigvee}\left(\tilde{G},\ B\right)$ is defined as:

$$\left(\widetilde{\Gamma},\ A\right)\widetilde{\bigvee}\left(\tilde{G},\ B\right)=(\tilde{N},\quad A\times B),$$

where $\tilde{N}\left(a,b\right)=\widetilde{\Gamma}(a)\cup\tilde{G}(b)$, for all $(a,\ b)\in A\times B$.

Example 3.34. Consider Example 3.6. Let

$$A=\left\{a_1=\left(e_{U_1},1,e_{U_2},1,e_{U_3},1\right),a_2=\left(e_{U_1},2,e_{U_2},3,e_{U_3},1\right)\right\}$$

and

$$B=\{b_1=(e_{U_1},1,e_{U_2},1,e_{U_3},1),b_2=(e_{U_1},1,e_{U_2},2,e_{U_3},1),\\ b_3=(e_{U_1},2,e_{U_2},3,e_{U_3},1)\}.$$

Then,

$$\tilde{N}\left(a_1,\ b_1\right)=\left\{\begin{array}{c}\left\{\frac{h_1}{\{0.5,0.9,0.4\}},\frac{h_2}{\{0.4,0.6\}},\frac{h_3}{\{0.9,0.8,0.5\}},\frac{h_4}{\{0.6,0.4,0.3\}}\right\},\\ \left\{\frac{c_1}{\{0.5,0.6\}},\frac{c_2}{\{0.7,0.6,0.8\}},\frac{c_3}{\{0.9,0.4,0.5\}}\right\},\left\{\frac{v_1}{\{0.8,0.5,0.8\}},\frac{v_2}{\{0.9,0.8,0.8\}}\right\}\end{array}\right\},$$

$$\tilde{N}\left(a_1,\ b_2\right)=\left\{\begin{array}{c}\left\{\frac{h_1}{\{0.7,0.8,0.9\}},\frac{h_2}{\{0.4,0.7,0.2\}},\frac{h_3}{\{0.9,0.7,0.5\}},\frac{h_4}{\{0.3,0.2,0.3\}}\right\},\\ \left\{\frac{c_1}{\{0.7,0.7,0.4\}},\frac{c_2}{\{0.7,0.6,,0.8\}},\frac{c_3}{\{0.9,0.4,0.5\}}\right\},\left\{\frac{v_1}{\{0.8,0.9,0.7\}},\frac{v_2}{\{0.9,0.7,0.9\}}\right\}\end{array}\right\},$$

$$\tilde{N}\left(a_1,\ b_3\right)=\left\{\begin{array}{c}\left\{\frac{h_1}{\{0.5,0.8,0.8\}},\frac{h_2}{\{0.7,0.9,0.3\}},\frac{h_3}{\{0.9,0.7,0.5\}},\frac{h_4}{\{0.6,0.5,0.2\}}\right\},\\ \left\{\frac{c_1}{\{0.7,0.8,0.9\}},\frac{c_2}{\{0.7,0.6,0.5\}},\frac{c_3}{\{0.9,0.8,0.3\}}\right\},\left\{\frac{v_1}{\{0.9,0.8,0.8\}},\frac{v_2}{\{0.9,0.7,0.8\}}\right\}\end{array}\right\},$$

$$\tilde{N}\left(a_2,\ b_1\right)=\left\{\begin{array}{c}\left\{\frac{h_1}{\{0.7,0.9,0.9\}},\frac{h_2}{\{0.4,0.6,0.1\}},\frac{h_3}{\{0.9,0.8,0.5\}},\frac{h_4}{\{0.6,0.5,0.2\}}\right\},\\ \left\{\frac{c_1}{\{0.7,0.6,0.2\}},\frac{c_2}{\{0.7,0.6,0.8\}},\frac{c_3}{\{0.9,0.8,0.8\}}\right\},\left\{\frac{v_1}{\{0.8,0.9,0.8\}},\frac{v_2}{\{0.9,0.8,0.9\}}\right\}\end{array}\right\},$$

$$\tilde{N}\left(a_2,\ b_2\right)=\left\{\begin{array}{c}\left\{\frac{h_1}{\{0.7,0.8,0.9\}},\frac{h_2}{\{0.3,0.7,0.2\}},\frac{h_3}{\{0.8,0.7,0.3\}},\frac{h_4}{\{0.3,0.2,0.3\}}\right\},\\ \left\{\frac{c_1}{\{0.7,0.7,0.4\}},\frac{c_2}{\{0.2,0.4,0.8\}},\frac{c_3}{\{0.7,0.9,0.9\}}\right\},\left\{\frac{v_1}{\{0.8,0.9,0.7\}},\frac{v_2}{\{0.6,0.7,0.9\}}\right\}\end{array}\right\},$$

$$\tilde{N}\left(a_2,\ b_3\right)=\left\{\begin{array}{c}\left\{\frac{h_1}{\{0.7,0.7,0.9\}},\frac{h_2}{\{0.7,0.9,0.3\}},\frac{h_3}{\{0.8,0.6,0.4\}},\frac{h_4}{\{0.6,0.5,0.2\}}\right\},\\ \left\{\frac{c_1}{\{0.7,0.8,0.9\}},\frac{c_2}{\{0.2,0.5,0.8\}},\frac{c_3}{\{0.9,0.9,0.8\}}\right\},\left\{\frac{v_1}{\{0.9,0.9,0.6\}},\frac{v_2}{\{0.5,0.6,0.9\}}\right\}\end{array}\right\}.$$

Theorem 3.35. *Let $\left(\widetilde{\Gamma},\ A\right)$, $(\tilde{G},\ B)$ and $(\tilde{H},\ C)$ be the hesitant fuzzy soft multisets over U. The associative law of hesitant fuzzy soft multisets holds as follows:*

(1) $\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\bigwedge}\left(\tilde{G},\ B\right)\right)\widetilde{\bigwedge}\left(\tilde{H},\ C\right)=\ \left(\widetilde{\Gamma},\ A\right)\widetilde{\bigwedge}\left((\tilde{G},\ B)\widetilde{\bigwedge}\left(\tilde{H},\ C\right)\right),$

(2) $\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\bigvee}\,(\tilde{G},\ B)\right)\widetilde{\bigvee}\left(\tilde{H},\ C\right)=\ \left(\widetilde{\Gamma},\ A\right)\widetilde{\bigvee}\left((\tilde{G},\ B)\widetilde{\bigvee}\left(\tilde{H},\ C\right)\right).$

Proof. For any $a\in A,\ b\in B$ and $c\in C$, since

$$\left(\widetilde{\Gamma}\,(a)\cap\tilde{G}\,(b)\right)\cap\tilde{H}\,(c)=\widetilde{\Gamma}\,(a)\cap\left(\tilde{G}\,(b)\cap\tilde{H}\,(c)\right),$$

we can conclude that

$$\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\bigwedge}\,(\tilde{G},\ B)\right)\widetilde{\bigwedge}\left(\tilde{H},\ C\right)=\ \left(\widetilde{\Gamma},\ A\right)\widetilde{\bigwedge}\left((\tilde{G},\ B)\widetilde{\bigwedge}\left(\tilde{H},\ C\right)\right).$$

In the same vein, we can conclude that

$$\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\bigvee}\,(\tilde{G},\ B)\right)\widetilde{\bigvee}\left(\tilde{H},\ C\right)=\ \left(\widetilde{\Gamma},\ A\right)\widetilde{\bigvee}\left((\tilde{G},\ B)\widetilde{\bigvee}\left(\tilde{H},\ C\right)\right).$$

□

The following "De Morgan-like" results hold.

Theorem 3.36. (1) $\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\bigwedge}\,(\tilde{G},\ B)\right)^{\mathbf{c}}=\ \left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}}\widetilde{\bigvee}\left(\tilde{G},\ B\right)^{\mathbf{c}}.$

(2) $\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\bigvee}\,(\tilde{G},\ B)\right)^{\mathbf{c}}=\ \left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}}\widetilde{\bigwedge}\left(\tilde{G},\ B\right)^{\mathbf{c}}.$

Proof. (1) Let $\left(\widetilde{\Gamma},\ A\right)\widetilde{\bigwedge}\left(\tilde{G},\ B\right)=(H,\ \ A\times B)$,
where for all $(\alpha_1,\ \alpha_2)\in A\times B$, we have

$$H\,(\alpha_1,\ \alpha_2)=\widetilde{\Gamma}\,(\alpha_1)\cap\tilde{G}(\alpha_2).$$

Now, $\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\bigwedge}\left(\tilde{G},\ B\right)\right)^{C}=\ (H,\ A\times B)^{C}=(H^{C},\ \neg(A\times B))$,
for all $(\neg\alpha_1,\ \neg\alpha_2)\in\neg(A\times B)$, we have

$$H^{C}\,(\neg\alpha_1,\ \neg\alpha_2)=\ (H\,(\alpha_1,\ \alpha_2))^{C}=\ (\widetilde{\Gamma}\,(\alpha_1)\cap\tilde{G}(\alpha_2))^{C}=\widetilde{\Gamma}^{C}\,(\neg\alpha_1)\cup\tilde{G}^{C}(\neg\alpha_2).$$

On the other hand, let $\left(\widetilde{\Gamma},\ A\right)^{C}\widetilde{\bigvee}\left(\tilde{G},\ B\right)^{C}=\left(\widetilde{\Gamma}^{C},\neg A\ \right)\widetilde{\bigvee}(\tilde{G}^{C},\neg B\ )=(J,\ \neg A\times\neg B)$.
For all $(\neg\alpha_1,\ \neg\alpha_2)\in(\neg A\times\neg B)$, we have

$$J\,(\neg\alpha_1,\ \neg\alpha_2)=\ \widetilde{\Gamma}^{C}\,(\neg\alpha_1)\cup\tilde{G}^{C}(\neg\alpha_2).$$

Since $H^{C}\,(\neg\alpha_1,\ \neg\alpha_2)=\ J\,(\neg\alpha_1,\ \neg\alpha_2)$, (1) has been established.

(2) Let $\left(\widetilde{\Gamma},\ A\right)\widetilde{\bigvee}\left(\tilde{G},\ B\right)=(H,\ \ A\times B)$ and for all $(\alpha_1,\ \alpha_2)\in A\times B$. Then we have,

$$H\,(\alpha_1,\ \alpha_2)=\widetilde{\Gamma}\,(\alpha_1)\cup\tilde{G}(\alpha_2).$$

Now, $\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\bigvee}\left(\tilde{G},\ B\right)\right)^{C}=\ (H,\ A\times B)^{C}=(H^{C},\ \neg(A\times B))$, for all $(\neg\alpha_1,\ \neg\alpha_2)\in\neg(A\times B)$, we have

$$H^{C}\,(\neg\alpha_1,\ \neg\alpha_2)=\ (H\,(\alpha_1,\ \alpha_2))^{C}=\ (\widetilde{\Gamma}\,(\alpha_1)\cup\tilde{G}(\alpha_2))^{C}=\widetilde{\Gamma}^{C}\,(\neg\alpha_1)\cap\tilde{G}^{C}(\neg\alpha_2).$$

On the other hand, let $\left(\widetilde{\Gamma},\ A\right)^{C}\widetilde{\bigwedge}\left(\tilde{G},\ B\right)^{C}=\left(\widetilde{\Gamma}^{C},\neg A\ \right)\widetilde{\bigwedge}(\tilde{G}^{C},\neg B\ )=(K,\ \neg A\times\neg B)$.
For all $(\neg\alpha_1,\ \neg\alpha_2)\in(\neg A\times\neg B)$, we have $K\,(\neg\alpha_1,\ \neg\alpha_2)=\ \widetilde{\Gamma}^{C}\,(\neg\alpha_1)\cap\tilde{G}^{C}(\neg\alpha_2)$.
Since $H^{C}\,(\neg\alpha_1,\ \neg\alpha_2)=K\,(\neg\alpha_1,\ \neg\alpha_2)$, (2) has been established. □

Theorem 3.37. *Let* $\left(\widetilde{\Gamma},\ A\right)$ *and* $(\tilde{G},\ B)$ *be two hesitant fuzzy soft multisets over* U. *Then*

(1) $\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}(\tilde{G},\ B)\right)^{\mathbf{c}}\widetilde{\subseteq}\left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}}\widetilde{\cup}(G,\ B)^{\mathbf{c}}$,

(2) $\left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}}\Cap(G,\ B)^{\mathbf{c}}\ \widetilde{\subseteq}\left(\left(\widetilde{\Gamma},\ A\right)\Cap(\tilde{G},\ B)\right)^{\mathbf{c}}$.

Proof. (1) Let $\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}\left(\tilde{G},\ B\right)=(\tilde{M},\ C)$, where $C=A\cup B$ and $\forall\ e\in C$,

$$\tilde{M}\left(e\right)=\begin{cases}\widetilde{\Gamma}\left(e\right) & \text{if } e\in A/B\\ G\left(e\right) & \text{if } e\in B/A\\ \widetilde{\Gamma}\left(e\right)\cup G\left(e\right) & \text{if } e\in A\cap B.\end{cases}$$

Then, $\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}(\tilde{G},\ B)\right)^{\mathbf{c}}=(\tilde{M},\ C)^{\mathbf{c}}$=$(\tilde{M}^{\mathbf{c}},\ C)$ and

$$\tilde{M}^{c}(e)=\begin{cases}\widetilde{\Gamma}^{c}\left(e\right) & \text{if } e\in A/B\\ \tilde{G}^{c}\left(e\right) & \text{if } e\in B/A\\ \widetilde{\Gamma}^{c}\left(e\right)\cap\tilde{G}^{c}\left(e\right) & \text{if } e\in A\cap B.\end{cases}$$

Similarly, $\left(\widetilde{\Gamma},\ A\right)^{c}\widetilde{\cup}(G,\ B)^{c}=\left(\widetilde{\Gamma}^{c},\ A\right)\widetilde{\cup}\left(\tilde{G}^{c},\ B\right)=(\tilde{N},\ C)$,
where $C=A\cup B$ and $\forall\ e\in C$,

$$\tilde{N}\left(\mathbf{e}\right)=\begin{cases}\widetilde{\Gamma}^{c}\left(e\right) & \text{if } e\in A/B\\ \tilde{G}^{c}\left(e\right) & \text{if } e\in B/A\\ \widetilde{\Gamma}^{c}\left(e\right)\cup\tilde{G}^{c}\left(e\right) & \text{if } e\in\ A\cap B.\end{cases}$$

Thus, $\tilde{M}^{\mathbf{c}}(e)\subseteq\tilde{N}\left(e\right)$, $\forall\ e\in C$. So, $\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}(\tilde{G},\ B)\right)^{\mathbf{c}}\widetilde{\subseteq}\left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}}\widetilde{\cup}\ (G,\ B)^{\mathbf{c}}$.

(2) Consider that $\left(\widetilde{\Gamma},\ A\right)\Cap\left(\tilde{G},\ B\right)=(\tilde{P},\ C)$,
where $C=A\cap B$ and $\forall\ e\in C$,

$$\tilde{P}\left(e\right)=\ \widetilde{\Gamma}(e)\cap\tilde{G}(e).$$

Then, $\left(\left(\widetilde{\Gamma},\ A\right)\Cap(\tilde{G},\ B)\right)^{\mathbf{c}}=(\tilde{P},\ C)^{\mathbf{c}}$=$(\tilde{P}^{\mathbf{c}},\ C)$ and $\tilde{P}^{\mathbf{c}}\left(e\right)=\widetilde{\Gamma}^{\mathbf{c}}\left(e\right)\ \cup\tilde{G}^{\mathbf{c}}\left(e\right)$.
Similarly, let $\left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}}\Cap(G,\ B)^{\mathbf{c}}=\left(\widetilde{\Gamma}^{\mathbf{c}},A\right)\Cap\left(\tilde{G}^{\mathbf{c}},B\right)$=$(\tilde{Q},C)$, where $C=A\cap B$ and $\forall\ e\in C$,

$$\tilde{Q}\left(e\right)=\widetilde{\Gamma}^{\mathbf{c}}\left(e\right)\cap\tilde{G}^{\mathbf{c}}\left(e\right).$$

Obviously, $\forall\ e\in C$, $\tilde{Q}\left(e\right)\ \subseteq\tilde{P}^{\mathbf{c}}\left(e\right)$. So,

$$\left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}}\Cap\ (G,\ B)^{\mathbf{c}}\ \widetilde{\subseteq}\left(\left(\widetilde{\Gamma},\ A\right)\Cap(\tilde{G},\ B)\right)^{\mathbf{c}}.$$

□

Theorem 3.38. *Let* $\left(\widetilde{\Gamma},\ A\right)$ *and* $(\tilde{G},\ B)$ *be two hesitant fuzzy soft multisets over* U. *Then*

(1) $\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}(\tilde{G},\ A)\right)^{\mathbf{c}}=\ \left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}}\Cap(G,\ A)^{\mathbf{c}}$,

(2) $\left(\left(\widetilde{\Gamma},\ A\right)\Cap(\tilde{G},\ A)\right)^{\mathbf{c}}=\left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}}\widetilde{\cup}(G,\ A)^{\mathbf{c}}$.

Proof. (1) Let $\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}\left(\tilde{G},\ A\right)=(\tilde{R},\ A)$ and $\forall e\in A,\tilde{R}\,(e)=\ \widetilde{\Gamma}(e)\cup\tilde{G}(e)$. Then $\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}(\tilde{G},\ A)\right)^{\mathbf{c}}=(\tilde{R},\ A)^{\mathbf{c}}=\left(\tilde{R}^{\mathbf{c}},\ A\right)$ and $\forall e\in A$,

$$\tilde{R}^{\mathbf{c}}\,(e)=\ \widetilde{\Gamma}^{c}(e)\cap\tilde{G}^{c}(e).$$

Now let $\left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}}\Cap(G,\ A)^{\mathbf{c}}=\left(\widetilde{\Gamma}^{c},\ A\right)\Cap\left(\tilde{G}^{c},\mathrm{A}\right)=(\widetilde{\mathrm{S}},\ A)$ and $\forall e\in A$,

$$\widetilde{\mathrm{S}}\,(e)=\widetilde{\Gamma}^{c}(e)\cap\tilde{G}^{c}(e)\ .$$

Then, $\widetilde{\mathrm{S}}\,(e)=\tilde{R}^{\mathbf{c}}\,(e)$. Thus, $\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}(\tilde{G},\ A)\right)^{\mathbf{c}}=\left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}}\Cap(G,\ A)^{\mathbf{c}}$.

(2) Let $\left(\widetilde{\Gamma},\ A\right)\Cap\left(\tilde{G},\ A\right)=\ \left(\tilde{F},\ A\right)$ and $\forall e\in A$,

$$\tilde{F}\,(e)=\ \widetilde{\Gamma}(e)\cap\tilde{G}(e)\ .$$

Then, $\left(\left(\widetilde{\Gamma},\ A\right)\Cap(\tilde{G},\ A)\right)^{\mathbf{c}}=(\tilde{F},\ A)^{\mathbf{c}}=\left(\tilde{F}^{\mathbf{c}},\ A\right)$ and $\forall e\in A$,

$$\tilde{F}^{\mathbf{c}}\,(e)=\widetilde{\Gamma}^{c}(e)\cup\tilde{G}^{c}(e).$$

Similarly, let $\left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}}\widetilde{\cup}(G,\ A)^{\mathbf{c}}=\ \left(\widetilde{\Gamma}^{c},\ A\right)\widetilde{\cup}\left(\tilde{G}^{c},\mathrm{A}\right)=(\widetilde{\mathrm{T}},\ A)$ and $\forall e\in A$,

$$\widetilde{\mathrm{T}}\,(e)=\widetilde{\Gamma}^{c}(e)\cup\tilde{G}^{c}(e).$$

Since $\widetilde{\mathrm{T}}\,(e)=\tilde{F}^{\mathbf{c}}\,(e)$, $\left(\left(\widetilde{\Gamma},\ A\right)\Cap(\tilde{G},\ A)\right)^{\mathbf{c}}=\left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}}\widetilde{\cup}(G,\ A)^{\mathbf{c}}$. □

Theorem 3.39. *Let* $\left(\widetilde{\Gamma},\ A\right)$ *and* $(\tilde{G},\ B)$ *be two hesitant fuzzy soft multisets over* U*. Then*

(1) $\left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}}\Cap(G,\ B)^{\mathbf{c}}\widetilde{\subseteq}\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}(\tilde{G},\ B)\right)^{\mathbf{c}}$,

(2) $\left(\left(\widetilde{\Gamma},\ A\right)\Cap(\tilde{G},\ B)\right)^{\mathbf{c}}\widetilde{\subseteq}\left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}}\widetilde{\cup}(G,\ B)^{\mathbf{c}}$.

Proof. (1) Let $\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}\left(\tilde{G},\ B\right)=(\tilde{K},\ C)$, where $C=A\cup B$, and $\forall\ e\in C$,

$$\tilde{K}\,(e)=\begin{cases}\widetilde{\Gamma}\,(e) & \text{if } e\in A/B\\ G\,(e) & \text{if } e\in B/A\\ \widetilde{\Gamma}\,(e)\cup G\,(e) & \text{if } e\in A\cap B.\end{cases}$$

Then, $\left(\left(\widetilde{\Gamma},\ A\right)\widetilde{\cup}(\tilde{G},\ B)\right)^{c}=(\tilde{K},\ C)^{c}=(\tilde{K}^{c},C)$ and

$$\tilde{K}^{c}(e)=\begin{cases}\widetilde{\Gamma}^{c}\,(e) & \text{if } e\in A/B\\ \tilde{G}^{c}\,(e) & \text{if } e\in B/A\\ \widetilde{\Gamma}^{c}\,(e)\cap\tilde{G}^{c}\,(e) & \text{if } e\in A\cap B.\end{cases}$$

Similarly, let $\left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}} \Cap (G,\ B)^{\mathbf{c}}=\left(\widetilde{\Gamma}^{\mathbf{c}},\ A\right) \Cap \left(\tilde{G}^{\mathbf{c}},\ B\right)=(\tilde{V},\ C)$,
where $C = A \cap B$, and $\forall\ e \in C$,

$$\tilde{V}\,(e) = \widetilde{\Gamma}^{\mathbf{c}}\,(e) \cap \tilde{G}^{\mathbf{c}}\,(e)\,.$$

It is obvious that $\tilde{V}\,(e) \subseteq \tilde{K}^{\mathbf{c}}(e)$. Thus, $\left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}} \Cap (G,\ B)^{\mathbf{c}} \,\widetilde{\subseteq}\, \left(\left(\widetilde{\Gamma},\ A\right) \widetilde{\cup} (\tilde{G},\ B)\right)^{\mathbf{c}}$.

Let $\left(\widetilde{\Gamma},\ A\right) \Cap \left(\tilde{G},\ B\right) = (\tilde{T},\ C)$, where $C = A \cap B$ and $\forall\ e \in C$,

$$\tilde{T}\,(e) = \ \widetilde{\Gamma}(e) \cap \tilde{G}(e).$$

Then $\left(\left(\widetilde{\Gamma},\ A\right) \Cap (\tilde{G},\ B)\right)^{\mathbf{c}} = (\tilde{T},\ C)^{\mathbf{c}} = (\tilde{T}^{\mathbf{c}}, C)$ and $\forall\ e \in C$,

$$\tilde{T}^{\mathbf{c}}\,(e) = \widetilde{\Gamma}^{\mathbf{c}}\,(e) \cup \tilde{G}^{\mathbf{c}}\,(e)\,.$$

Similarly, let $\left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}} \widetilde{\cup} (G,\ B)^{\mathbf{c}} = \left(\widetilde{\Gamma}^{\mathbf{c}}, A\right) \widetilde{\cup}\ \ (G^{\mathbf{c}}, B) = (\tilde{W},\ C)$, where $C = A \cup B$. Then $\forall\ e \in C$,

$$\tilde{W}\,(e) = \begin{cases} \widetilde{\Gamma}^{\mathbf{c}}\,(e) & \text{if } e \in A/B \\ \tilde{G}^{\mathbf{c}}\,(e) & \text{if } e \in B/A \\ \widetilde{\Gamma}^{\mathbf{c}}\,(e) \cup \tilde{G}^{\mathbf{c}}\,(e) & \text{if } e \in A \cap B. \end{cases}$$

Thus, $\tilde{T}^{\mathbf{c}}\,(e) \subseteq \tilde{W}\,(e)$. So $\left(\left(\widetilde{\Gamma},\ A\right) \Cap (\tilde{G},\ B)\right)^{\mathbf{c}} \widetilde{\subseteq} \left(\widetilde{\Gamma},\ A\right)^{\mathbf{c}} \widetilde{\cup} \left(\tilde{G},\ B\right)^{\mathbf{c}}$. □

4. Conclusion

In this research paper, we defined the notion of hesitant fuzzy soft multiset which combines the hesitant fuzzy soft set and soft multiset. We defined the basic operations such as complement, AND product, OR product, union, restricted union, intersection, restricted intersection on hesitant fuzzy soft multisets. The basic properties of the various operations were investigated. Some important theorems based on the operations were proved in details with relevant examples. De Morgan's inclusions and laws were stated and proved in the background of hesitant fuzzy soft multiset.

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