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# Hesitant fuzzy soft sets over UP-algebras

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Abstract. This paper aims to extend the notion of hesitant fuzzy sets on UP-algebras to hesitant fuzzy soft sets over UP-algebras by merging the notions of hesitant fuzzy sets and soft sets. Further, we discuss the notions of hesitant fuzzy soft strongly UP-ideals, hesitant fuzzy soft UP-ideals, hesitant fuzzy soft UP-filters, and hesitant fuzzy soft UP-subalgebras of UP-algebras, and provide some properties.

2010 AMS Classification: 03G25

Keywords: UP-algebra, Hesitant fuzzy soft UP-subalgebra, Hesitant fuzzy soft UP-filter, Hesitant fuzzy soft UP-ideal, Hesitant fuzzy soft strongly UP-ideal.

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## 1. Introduction

The branch of the logical algebra, UP-algebras was introduced by Iampan [3] in 2017, and it is known that the class of KU-algebras [14] is a proper subclass of the class of UP-algebras. It have been examined by several researchers, for example, Somjanta et al. [19] introduced the notion of fuzzy sets in UP-algebras, the notion of intuitionistic fuzzy sets in UP-algebras was introduced by Kesorn et al. [7], the notion of $Q$-fuzzy sets in UP-algebras was introduced by Tanamoon et al. [20], Senapati et al. [17, 18] applied cubic set and interval-valued intuitionistic fuzzy structure in UP-algebras, etc.

A soft set over a universe set is a parametized family of subsets of the universe set. Molodtsov [8] introduced the notion of soft sets over a universe set in 1999.

A hesitant fuzzy set on a set is a function from a reference set to a power set of the unit interval. The notion of a hesitant fuzzy set on a set was first considered by Torra [21] in 2010. Recently hesitant fuzzy sets theory has been applied to the different algebraic structures (see [6, 11, 12, 13]). In UP-algebras, Mosrijai et al. [9] extended the notion of fuzzy sets in UP-algebras to hesitant fuzzy sets on UP-algebras, and Satirad et al. [16] considered level subsets of a hesitant fuzzy set on UP-algebras in 2017. The notion of partial constant hesitant fuzzy sets on UP-algebras was introduced by Mosrijai et al. [10] afterwards.

The notion of hesitant fuzzy soft sets that is a link between classical soft sets and hesitant fuzzy sets is introduced by Babitha and John [1] in 2013. There exists some researchers, such as Jun et al. [5], applied hesitant fuzzy soft set theory to some algebraic structures, which are BCK and BCI algebras.

In this paper, we extend the notion of hesitant fuzzy sets on UP-algebras to hesitant fuzzy soft sets over UP-algebras by merging the notion of hesitant fuzzy sets and soft sets. Further, we discuss the notion of hesitant fuzzy soft strongly UP-ideals, hesitant fuzzy soft UP-ideals, hesitant fuzzy soft UP-filters and hesitant fuzzy soft UP-subalgebras of UP-algebras, and provide some properties.

## 2. Basic results on UP-algebras

Before we begin our study, we will introduce the definition of a UP-algebra.

**Definition 2.1** ([3])**.** An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra, where $A$ is a nonempty set, $\cdot$ is a binary operation on $A$, and 0 is a fixed element of $A$ (i.e., a nullary operation), if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
(UP-2) $0 \cdot x = x$,
(UP-3) $x \cdot 0 = 0$,
(UP-4) $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

From [3], we know that the notion of UP-algebras is a generalization of KU-algebras.

**Example 2.2** ([15])**.** Let $X$ be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by : for all $A, B \in \mathcal{P}_\Omega(X)$,

$$A \cdot B = B \cap (A' \cup \Omega).$$

Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and we shall call it the generalized power UP-algebra of type 1 with respect to $\Omega$.

In particular, $(\mathcal{P}(X), \cdot, \varnothing)$ is the power UP-algebra of type 1.

**Example 2.3** ([15])**.** Let $X$ be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by : for all $A, B \in \mathcal{P}^\Omega(X)$,

$$A * B = B \cup (A' \cap \Omega).$$

Then $(\mathcal{P}^\Omega(X), *, \Omega)$ is a UP-algebra and we shall call it the generalized power UP-algebra of type 2 with respect to $\Omega$.

In particular, $(\mathcal{P}(X), *, X)$ is the power UP-algebra of type 2.

In a UP-algebra $A = (A, \cdot, 0)$, the following assertions are valid (see [3, 4]):

$$(\forall x \in A)(x \cdot x = 0), \tag{2.1}$$

$$(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0), \tag{2.2}$$

$$(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0), \tag{2.3}$$

$$(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0), \tag{2.4}$$

$$(\forall x, y \in A)(x \cdot (y \cdot x) = 0), \tag{2.5}$$

$$(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x), \tag{2.6}$$

$$(\forall x, y \in A)(x \cdot (y \cdot y) = 0), \tag{2.7}$$

$$(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0), \tag{2.8}$$

$$(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0), \tag{2.9}$$

$$(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0), \tag{2.10}$$

$$(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0), \tag{2.11}$$

$$(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0), \tag{2.12}$$

$$(\forall a, x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0). \tag{2.13}$$

In what follows, let $A$ denote UP-algebras unless otherwise specified.

On a UP-algebra $A = (A, \cdot, 0)$, we define a binary relation $\leq$ on $A$ [3] as follows: for any $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0.$$

**Definition 2.4** ([3]). A subset $S$ of $A$ is called a UP-subalgebra of $A$, if the constant 0 of $A$ is in $S$ and $(S, \cdot, 0)$ itself forms a UP-algebra.

Iampan [3] proved the useful criteria that a nonempty subset $S$ of $A$ is a UP-subalgebra of $A$ if and only if $S$ is closed under the $\cdot$ multiplication on $A$.

**Definition 2.5** ([2, 3, 19]). A subset $S$ of $A$ is called:

(1) a UP-filter of $A$, if
  (i) the constant 0 of $A$ is in $S$, and
  (ii) for any $x, y \in A, x \cdot y \in S$ and $x \in S$ imply $y \in S$,

(2) a UP-ideal of $A$, if
  (i) the constant 0 of $A$ is in $S$, and
  (ii) for any $x, y, z \in A, x \cdot (y \cdot z) \in S$ and $y \in S$ imply $x \cdot z \in S$,

(3) a strongly UP-ideal of $A$, if
  (i) the constant 0 of $A$ is in $S$, and
  (ii) for any $x, y, z \in A, (z \cdot y) \cdot (z \cdot x) \in S$ and $y \in S$ imply $x \in S$.

Guntasow et al. [2] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra $X$ is the only one strongly UP-ideal of itself.

## 3. Basic results on hesitant fuzzy sets

**Definition 3.1** ([21])**.** Let $X$ be a reference set. A hesitant fuzzy set on $X$ is defined in term of a function $\mathrm{h_H}$ that when applied to $X$ return a subset of $[0,1]$, that is, $\mathrm{h_H}\colon X \to \mathcal{P}([0,1])$. A hesitant fuzzy set $\mathrm{h_H}$ can also be viewed as the following mathematical representation:

$$\mathrm{H} := \{(x, \mathrm{h_H}(x)) \mid x \in X\}$$

where $\mathrm{h_H}(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the elements $x \in X$ to the set H.

**Definition 3.2** ([9])**.** A hesitant fuzzy set H on $A$ is called:

(1) a hesitant fuzzy UP-subalgebra of $A$, if it satisfies the following property: for any $x, y \in A$, $\mathrm{h_H}(x \cdot y) \supseteq \mathrm{h_H}(x) \cap \mathrm{h_H}(y)$,
(2) a hesitant fuzzy UP-filter of $A$, if it satisfies the following properties: for any $x, y \in A$,
  (i) $\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x)$, and
  (ii) $\mathrm{h_H}(y) \supseteq \mathrm{h_H}(x \cdot y) \cap \mathrm{h_H}(x)$,
(3) a hesitant fuzzy UP-ideal of $A$, if it satisfies the following properties: for any $x, y, z \in A$,
  (i) $\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x)$, and
  (ii) $\mathrm{h_H}(x \cdot z) \supseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cap \mathrm{h_H}(y)$,
(4) a hesitant fuzzy strongly UP-ideal of $A$, if it satisfies the following properties: for any $x, y, z \in A$,
  (i) $\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x)$, and
  (ii) $\mathrm{h_H}(x) \supseteq \mathrm{h_H}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h_H}(y)$.

Mosrijai et al. [9] proved that the notion of hesitant fuzzy UP-subalgebras of UP-algebras is a generalization of hesitant fuzzy UP-filters, the notion of hesitant fuzzy UP-filters of UP-algebras is a generalization of hesitant fuzzy UP-ideals, and the notion of hesitant fuzzy UP-ideals of UP-algebras is a generalization of hesitant fuzzy strongly UP-ideals.

**Theorem 3.3** ([9])**.** *A hesitant fuzzy set* H *on* $A$ *is a hesitant fuzzy strongly UP-ideal of* $A$ *if and only if it is a constant hesitant fuzzy set on* $A$*.*

**Proposition 3.4.** *Let* H *be a hesitant fuzzy UP-filter (and also hesitant fuzzy UP-ideal, hesitant fuzzy strongly UP-ideal) of* $A$*. Then for any* $x, y \in A$,

$$x \leq y \text{ implies } \mathrm{h_H}(x) \subseteq \mathrm{h_H}(y) \subseteq \mathrm{h_H}(x \cdot y).$$

*Proof.* Let $x, y \in A$ be such that $x \leq y$. Then $x \cdot y = 0$. Thus

$$\mathrm{h_H}(y) \supseteq \mathrm{h_H}(x \cdot y) \cap \mathrm{h_H}(x) = \mathrm{h_H}(0) \cap \mathrm{h_H}(x) = \mathrm{h_H}(x).$$

By (2.5), we have $y \leq x \cdot y$. So $\mathrm{h_H}(y) \subseteq \mathrm{h_H}(x \cdot y)$. □

## 4. Hesitant fuzzy soft UP-subalgebras

**Definition 4.1** ([1])**.** Let $X$ be a reference set (or an initial universe set) and $P$ be a set of parameters. Let $\mathrm{HFS}(X)$ be the set of all hesitant fuzzy sets on $X$ and $Y$

be a nonempty subset of $P$. A pair $(\widetilde{\mathrm{H}}, Y)$ is called a hesitant fuzzy soft set over $X$, where $\widetilde{\mathrm{H}}$ is a mapping given by:

$$\widetilde{\mathrm{H}}\colon Y \to \mathrm{HFS}(X), p \mapsto \widetilde{\mathrm{H}}[p].$$

**Definition 4.2.** Let $Y$ be a nonempty subset of $P$. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $A$ is called a hesitant fuzzy soft UP-subalgebra based on $p \in Y$ (we shortly call a $p$-hesitant fuzzy soft UP-subalgebra) of $A$, if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on $A$ is a hesitant fuzzy UP-subalgebra of $A$. If $(\widetilde{\mathrm{H}}, Y)$ is a $p$-hesitant fuzzy soft UP-subalgebra of $A$ for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of $A$.

**Theorem 4.3.** *If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of $A$, then it satisfies the property: for any $p \in Y$ and $x \in A$,*

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x). \tag{4.1}$$

*Proof.* Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of $A$ and let $p \in Y$ and $x \in A$. Then $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-subalgebra of $A$. Thus,

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot x) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

□

**Example 4.4.** Let $(\mathcal{P}_{\varnothing}(\{a, b\}), \cdot, \varnothing)$ is the power UP-algebra of type 1 which a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | $\varnothing$ | $\{a\}$ | $\{b\}$ | $X$ |
|---|---|---|---|---|
| $\varnothing$ | $\varnothing$ | $\{a\}$ | $\{b\}$ | $X$ |
| $\{a\}$ | $\varnothing$ | $\varnothing$ | $\{b\}$ | $\{b\}$ |
| $\{b\}$ | $\varnothing$ | $\{a\}$ | $\varnothing$ | $\{a\}$ |
| $X$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\varnothing$ |

Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $\mathcal{P}_{\varnothing}(\{a, b\})$ by the following table:

| $\widetilde{\mathrm{H}}$ | $\varnothing$ | $\{a\}$ | $\{b\}$ | $X$ |
|---|---|---|---|---|
| $p_1$ | $\{0.3, 0.4\}$ | $\{0.3\}$ | $\{0.4\}$ | $\varnothing$ |
| $p_2$ | $[0.6, 0.9]$ | $\{0.9\}$ | $[0.6, 0.9]$ | $[0.6, 0.9]$ |
| $p_3$ | $(0.3, 0.8)$ | $[0.3, 0.5]$ | $\{0.4, 0.5\}$ | $\{0.4, 0.5\}$ |
| $p_4$ | $[0, 1)$ | $[0, 1)$ | $[0, 1)$ | $[0, 1)$ |

Then $(\widetilde{\mathrm{H}}, Y)$ satisfies the property (4.1), but not a hesitant fuzzy soft UP-subalgebra of $A$ based on parameter $p_2$. Indeed,

$$\begin{aligned}\mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(\{b\} \cdot X) = \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(\{a\}) = \{0.9\} &\not\supseteq [0.6, 0.9] \\ &= [0.6, 0.9] \cap [0.6, 0.9] \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(\{b\}) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(X).\end{aligned}$$

**Theorem 4.5.** *Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over $A$ which satisfies the condition: for any $p \in Y$ and $x, y, z \in A$,*

$$z \leq x \cdot y \text{ implies } \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(z) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x). \tag{4.2}$$

*Then $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of $A$.*

*Proof.* Let $p \in Y$ and $x, y \in A$. Then by (2.5) and (UP-3), we have $x \cdot (y \cdot (x \cdot y)) = x \cdot 0 = 0$. Thus $x \leq y \cdot (x \cdot y)$. It follows form (4.2) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

So, $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-subalgebra of $A$. Hence, $(\widetilde{\mathrm{H}}, Y)$ is a $p$-hesitant fuzzy soft UP-subalgebra of $A$. Since $p$ is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of $A$. □

**Corollary 4.6.** *If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over $A$ which satisfies the condition (4.2), then it satisfies the property (4.1).*

*Proof.* It is straightforward form Theorems 4.5 and 4.3. □

**Theorem 4.7.** *If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of $A$ and $N$ is a nonempty subset of $Y$, then $(\widetilde{\mathrm{H}}|_N, N)$ is a hesitant fuzzy soft UP-subalgebra of $A$.*

*Proof.* Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of $A$ and $\varnothing \neq N \subseteq Y$. Then $(\widetilde{\mathrm{H}}, Y)$ is a $p$-hesitant fuzzy soft UP-subalgebra of $A$, for all $p \in Y$. Since $N \subseteq Y$, we have $(\widetilde{\mathrm{H}}|_N, N)$ is a $p$-hesitant fuzzy soft UP-subalgebra of $A$, for all $p \in N$. Then, $(\widetilde{\mathrm{H}}|_N, N)$ is a hesitant fuzzy soft UP-subalgebra of $A$. □

The following example shows that there exists a nonempty subset $N$ of $Y$ such that $(\widetilde{\mathrm{H}}|_N, N)$ is a hesitant fuzzy soft UP-subalgebra of $A$, but $(\widetilde{\mathrm{H}}, Y)$ is not a hesitant fuzzy soft UP-subalgebra of $A$.

**Example 4.8.** Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 0 | 0 | 2 | 3 | 4 |
| 2 | 0 | 0 | 0 | 3 | 4 |
| 3 | 0 | 0 | 0 | 0 | 4 |
| 4 | 0 | 0 | 0 | 0 | 0 |

Then $(A, \cdot, 0)$ is a UP-algebra. Let $Y = \{p_1, p_2, p_3, p_4, p_5\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $A$ as the following table:

| $\widetilde{\mathrm{H}}$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p_1$ | $\{0.5, 0.6, 0.7\}$ | $\{0.5\}$ | $\{0.6\}$ | $\{0.6\}$ | $\varnothing$ |
| $p_2$ | $[0.4, 0.6]$ | $(0.4, 0.6)$ | $(0.5, 0.6)$ | $(0.5, 0.55)$ | $\{0.5\}$ |
| $p_3$ | $\{0.1, 0.2, 0.3\}$ | $\{0.1, 0.2\}$ | $\{0.1, 0.2\}$ | $\{0.2\}$ | $\{0.2\}$ |
| $p_4$ | $[0.7, 1)$ | $[0.7, 1]$ | $\{0.7\}$ | $\{0.5, 0.7\}$ | $[0.5, 0.7]$ |
| $p_5$ | $\{0.9\}$ | $\{0.9\}$ | $\{0.9\}$ | $\{0.9\}$ | $\{0.9\}$ |

Then $\widetilde{\mathrm{H}}[p_4]$ is not a hesitant fuzzy UP-subalgebra of $A$. Indeed,

$$\begin{aligned}\mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(1\cdot 1)=\mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(0)=[0.7,1)&\not\supseteq[0.7,1]\\&=[0.7,1]\cap[0.7,1]\\&=\mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(1)\cap\mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(1).\end{aligned}$$

Thus, $(\widetilde{\mathrm{H}},Y)$ is not a hesitant fuzzy soft UP-subalgebra of $A$. But if we choose $N=\{p_1,p_2,p_3,p_5\}$, then $(\widetilde{\mathrm{H}}|_N,N)$ is a hesitant fuzzy soft UP-subalgebra of $A$.

## 5. Hesitant fuzzy soft UP-filters

**Definition 5.1.** Let $Y$ be a nonempty subset of $P$. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}},Y)$ over $A$ is called a hesitant fuzzy soft UP-filter based on $p\in Y$ (we shortly call a $p$-hesitant fuzzy soft UP-filter) of $A$, if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p]:=\{(a,\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a))\mid a\in A\}$$

on $A$ is a hesitant fuzzy UP-filter of $A$. If $(\widetilde{\mathrm{H}},Y)$ is a $p$-hesitant fuzzy soft UP-filter of $A$ for all $p\in Y$, we state that $(\widetilde{\mathrm{H}},Y)$ is a hesitant fuzzy soft UP-filter of $A$.

From [9], we known that every hesitant fuzzy UP-filter of $A$ is a hesitant fuzzy UP-subalgebra. Then we have the following theorem:

**Theorem 5.2.** *Every $p$-hesitant fuzzy soft UP-filter of $A$ is a $p$-hesitant fuzzy soft UP-subalgebra.*

*Proof.* Assume that $(\widetilde{\mathrm{H}},Y)$ is a $p$-hesitant fuzzy soft UP-filter of $A$. Then $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-filter of $A$. Thus $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-subalgebra of $A$. So, $(\widetilde{\mathrm{H}},Y)$ is a $p$-hesitant fuzzy soft UP-subalgebra of $A$. □

The following example shows that the converse of Theorem 5.2 is not true, in general.

**Example 5.3.** Let $A=\{0,1,2,3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 0 | 2 | 3 |
| 2 | 0 | 0 | 0 | 3 |
| 3 | 0 | 0 | 0 | 0 |

Then $(A,\cdot,0)$ is a UP-algebra. Let $Y=\{p_1,p_2,p_3,p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}},Y)$ over $A$ by the following table:

| $\widetilde{\mathrm{H}}$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p_1$ | $\{0.5,0.6\}$ | $\{0.5\}$ | $\{0.6\}$ | $\varnothing$ |
| $p_2$ | $[0.4,0.6]$ | $(0.4,0.6)$ | $[0.4,0.6]$ | $[0.4,0.6]$ |
| $p_3$ | $(0.2,0.7)$ | $[0.3,0.5]$ | $\{0.4,0.5\}$ | $\{0.5\}$ |
| $p_4$ | $\{0.8\}$ | $\{0.8\}$ | $\{0.8\}$ | $\{0.8\}$ |

Then $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-subalgebra of $A$, but not a hesitant fuzzy soft UP-filter of $A$ based on parameter $p_1$. Indeed,

$$\begin{aligned}\mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(1) = \{0.5\} &\not\supseteq \{0.6\} \\ &= \{0.5, 0.6\} \cap \{0.6\} \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(0) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(2) \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(2 \cdot 1) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(2).\end{aligned}$$

**Theorem 5.4.** *A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $A$ is a hesitant fuzzy soft UP-filter of $A$ if and only if it satisfies the condition (4.2).*

*Proof.* Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-filter of $A$. Let $p \in Y$ and let $x, y, z \in A$ be such that $z \leq x \cdot y$. Then $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-filter of $A$. By Proposition 3.4, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y)$. Thus,

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(z) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

Conversely, assume that $(\widetilde{\mathrm{H}}, Y)$ satisfies the condition (4.2). Let $p \in Y$ and let $x \in A$. By Corollary 4.6, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)$. Let $x, y \in A$. Then by (2.1), we have $(x \cdot y) \cdot (x \cdot y) = 0$. Thus $x \cdot y \leq x \cdot y$. It follows from (4.2) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

So, $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-filter of $A$. Hence, $(\widetilde{\mathrm{H}}, Y)$ is a $p$-hesitant fuzzy soft UP-filter of $A$. Since $p$ is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-filter of $A$. □

**Theorem 5.5.** *Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over $A$ which satisfies the condition: for any $p \in Y$ and $w, x, y, z \in A$,*

$$x \leq w \cdot (y \cdot z) \text{ implies } \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y). \tag{5.1}$$

*Then it is a hesitant fuzzy soft UP-filter of $A$.*

*Proof.* Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over $A$ which satisfies the condition (5.1). Let $p \in Y$ and let $x, y \in A$. Then by (2.1), we have $0 \cdot ((x \cdot y) \cdot (x \cdot y)) = 0 \cdot 0 = 0$. Thus $0 \leq (x \cdot y) \cdot (x \cdot y)$. It follows form (5.1) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0 \cdot y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

So, $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-filter of $A$. □

**Corollary 5.6.** *If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over $A$ which satisfies the condition (5.1), then it satisfies the condition (4.2).*

*Proof.* It is straightforward form Theorems 5.5 and 5.4. □

**Theorem 5.7.** *Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over $A$ which satisfies the condition: for any $p \in Y$ and $w, x, y, z \in A$,*

$$w \leq x \cdot (y \cdot z) \text{ implies } \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y). \tag{5.2}$$

*Then it is a hesitant fuzzy soft UP-filter of $A$.*

*Proof.* Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over $A$ which satisfies the condition (5.2). Let $p \in Y$ and let $x, y \in A$. Then by (2.1) and (UP-2), we have $(x \cdot y) \cdot (0 \cdot (x \cdot y)) = (x \cdot y) \cdot (x \cdot y) = 0$. Thus $x \cdot y \leq 0 \cdot (x \cdot y)$. It follows form (5.2) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0 \cdot y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

So, $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-filter of $A$. □

**Corollary 5.8.** *If* $(\widetilde{\mathrm{H}}, Y)$ *is a hesitant fuzzy soft set over* $A$ *which satisfies the condition* (5.2)*, then it satisfies the condition* (4.2).

*Proof.* It is straightforward form Theorems 5.7 and 5.4. □

## 6. Hesitant fuzzy soft UP-ideals

**Definition 6.1.** Let $Y$ be a nonempty subset of $P$. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $A$ is called a hesitant fuzzy soft UP-ideal based on $p \in Y$ (we shortly call a $p$-hesitant fuzzy soft UP-ideal) of $A$ if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on $A$ is a hesitant fuzzy UP-ideal of $A$. If $(\widetilde{\mathrm{H}}, Y)$ is a $p$-hesitant fuzzy soft UP-ideal of $A$ for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-ideal of $A$.

From [9], we known that every hesitant fuzzy UP-ideal of $A$ is a hesitant fuzzy UP-filter. Then we have the following theorem:

**Theorem 6.2.** *Every* $p$*-hesitant fuzzy soft UP-ideal of* $A$ *is a* $p$*-hesitant fuzzy soft UP-filter.*

*Proof.* Assume that $(\widetilde{\mathrm{H}}, Y)$ is a $p$-hesitant fuzzy soft UP-ideal of $A$. Then $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-ideal of $A$. Thus $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-filter of $A$. So, $(\widetilde{\mathrm{H}}, Y)$ is a $p$-hesitant fuzzy soft UP-filter of $A$. □

The following example shows that the converse of Theorem 6.2 is not true in general.

**Example 6.3.** Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 0 | 3 | 3 |
| 2 | 0 | 1 | 0 | 0 |
| 3 | 0 | 1 | 2 | 0 |

Then $(A, \cdot, 0)$ is a UP-algebra. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $A$ by the following table:

| $\widetilde{\mathrm{H}}$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p_1$ | $[0.2, 0.5]$ | $\{0.5\}$ | $\varnothing$ | $\{0.3, 0.4\}$ |
| $p_2$ | $[0.4, 0.6]$ | $(0.4, 0.6)$ | $\{0.5\}$ | $\{0.5\}$ |
| $p_3$ | $[0.5, 0.7]$ | $\{0.5, 0.6\}$ | $\{0.5, 0.6\}$ | $[0.5, 0.6)$ |
| $p_4$ | $\{0.1, 0.6\}$ | $\{0.1, 0.6\}$ | $\{0.1, 0.6\}$ | $\{0.1, 0.6\}$ |

Then $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-filter of $A$, but not a hesitant fuzzy soft UP-ideal of $A$ based on parameter $p_1$. Indeed,

$$\begin{aligned}\mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(3\cdot 2) = \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(2) = \varnothing &\not\supseteq \{0.5\}\\ &= [0.2, 0.5] \cap \{0.5\}\\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(0) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(1)\\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(3\cdot(1\cdot 2)) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(1).\end{aligned}$$

**Theorem 6.4.** *If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-ideal of $A$, then it satisfies the condition (5.1).*

*Proof.* Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-ideal of $A$. Let $p \in Y$ and let $w, x, y, z \in A$ be such that $x \leq w \cdot (y \cdot z)$. Then $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-ideal of $A$ and $x \cdot (w \cdot (y \cdot z)) = 0$. Thus

$$\begin{aligned}\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot(y\cdot z)) &\supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot(w\cdot(y\cdot z))) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w)\\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w).\end{aligned}$$

So, $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot(y\cdot z)) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)$. □

**Example 6.5.** Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the same Cayley table in Example 6.3. Then $(A, \cdot, 0)$ is a UP-algebra. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $A$ by the following table:

| $\widetilde{\mathrm{H}}$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p_1$ | $[0.2, 0.5]$ | $\{0.5\}$ | $\varnothing$ | $\{0.3, 0.4\}$ |
| $p_2$ | $[0.4, 0.6]$ | $(0.4, 0.6)$ | $\{0.5\}$ | $\{0.5\}$ |
| $p_3$ | $[0.5, 0.7]$ | $[0.5, 0.7)$ | $\varnothing$ | $\{0.7\}$ |
| $p_4$ | $\{0.7, 0.8, 0.9\}$ | $\{0.7, 0.8\}$ | $\{0.8\}$ | $\{0.8\}$ |

Then $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-filter of $A$, but does not satisfy the condition (5.1).

**Theorem 6.6.** *A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $A$ is a hesitant fuzzy soft UP-ideal of $A$ if and only if it satisfies the condition (5.2).*

*Proof.* Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-ideal of $A$. Let $p \in Y$ and $w, x, y, z \in A$ be such that $w \leq x \cdot (y \cdot z)$. Then $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-ideal of $A$. Thus by Proposition 3.4, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot(y\cdot z))$. So,

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot(y\cdot z)) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Conversely, assume that $(\widetilde{\mathrm{H}}, Y)$ satisfies the condition (5.2). Let $p \in Y$ and let $x \in A$. By Corollary 5.8 and 4.6, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)$. Let $x, y, z \in A$. Then by (2.1), we have $(x\cdot(y\cdot z))\cdot(x\cdot(y\cdot z)) = 0$. Thus $x\cdot(y\cdot z) \leq x\cdot(y\cdot z)$. It follows form (5.2) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x\cdot(y\cdot z)) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

So, $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-ideal of $A$. Hence, $(\widetilde{\mathrm{H}}, Y)$ is a $p$-hesitant fuzzy soft UP-ideal of $A$. Since $p$ is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-ideal of $A$. □

**Theorem 6.7.** *Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over $A$ which satisfies the condition: for any $p \in Y$ and $w, x, y, z \in A$,*

$$w \leq (z \cdot y) \cdot (z \cdot x) \text{ implies } \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y). \tag{6.1}$$

*Then it is a hesitant fuzzy soft UP-ideal of $A$.*

*Proof.* Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over $A$ which satisfies the condition (6.1). Let $p \in Y$ and let $x, y, z \in A$. Then by (2.1) and (UP-3), we have $(x\cdot(y\cdot z))\cdot(((x\cdot z)\cdot y)\cdot((x\cdot z)\cdot(x\cdot z))) = (x\cdot(y\cdot z))\cdot(((x\cdot z)\cdot y)\cdot 0) = (x\cdot(y\cdot z))\cdot 0 = 0$. Thus $x \cdot (y \cdot z) \leq ((x \cdot z) \cdot y) \cdot ((x \cdot z) \cdot (x \cdot z))$. It follows form (6.1) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (y \cdot z)) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

So, $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-ideal of $A$. □

**Corollary 6.8.** *If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over $A$ which satisfies the condition (6.1), then it satisfies the conditions (5.1) and (5.2).*

*Proof.* It is straightforward form Theorems 6.7, 6.4, and 6.6. □

## 7. Hesitant fuzzy soft strongly UP-ideals

**Definition 7.1.** Let $Y$ be a nonempty subset of $P$. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $A$ is called a hesitant fuzzy soft strongly UP-ideal based on $p \in Y$ (we shortly call a $p$-hesitant fuzzy soft strongly UP-ideal) of $A$, if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on $A$ is a hesitant fuzzy strongly UP-ideal of $A$. If $(\widetilde{\mathrm{H}}, Y)$ is a $p$-hesitant fuzzy soft strongly UP-ideal of $A$ for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft strongly UP-ideal of $A$.

From [9], we known that every hesitant fuzzy strongly UP-ideal of $A$ is a hesitant fuzzy UP-ideal. Then we have the following theorem:

**Theorem 7.2.** *Every $p$-hesitant fuzzy soft strongly UP-ideal of $A$ is a $p$-hesitant fuzzy soft UP-ideal.*

*Proof.* Assume that $(\widetilde{\mathrm{H}}, Y)$ is a $p$-hesitant fuzzy soft strongly UP-ideal of $A$. Then $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy strongly UP-ideal of $A$. Thus $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy UP-ideal of $A$. So, $(\widetilde{\mathrm{H}}, Y)$ is a $p$-hesitant fuzzy soft UP-ideal of $A$. □

The following example shows that the converse of Theorem 7.2 is not true in general.

**Example 7.3.** Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 0 | 0 |
| 3 | 0 | 1 | 2 | 0 |

Then $(A, \cdot, 0)$ is a UP-algebra. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $A$ by the following table:

| $\widetilde{\mathrm{H}}$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p_1$ | $[0.2, 0.5]$ | $\{0.4\}$ | $(0.2, 0.4)$ | $(0.2, 0.4)$ |
| $p_2$ | $[0.9, 1]$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |
| $p_3$ | $[0, 0.2]$ | $(0, 0.2]$ | $[0, 0.2]$ | $[0, 0.2]$ |
| $p_4$ | $(0, 1)$ | $(0, 1)$ | $(0, 1)$ | $(0, 1)$ |

Then $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft UP-ideal of $A$, but not a hesitant fuzzy soft strongly UP-ideal of $A$ based on parameter $p_2$. Indeed,

$$\begin{aligned} \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(3) = \{1\} &\not\supseteq [0.9, 1] \\ &= [0.9, 1] \cap [0.9, 1] \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(0) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(0) \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}((1 \cdot 0) \cdot (1 \cdot 3)) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(0). \end{aligned}$$

By Theorems 5.2, 6.2, and 7.2 and Examples 5.3, 6.3, and 7.3, we have that the notion of $p$-hesitant fuzzy soft UP-subalgebras is a generalization of $p$-hesitant fuzzy soft UP-filters, the notion of $p$-hesitant fuzzy soft UP-filters is a generalization of $p$-hesitant fuzzy soft UP-ideals, and the notion of $p$-hesitant fuzzy soft UP-ideals is a generalization of $p$-hesitant fuzzy soft strongly UP-ideals.

**Theorem 7.4.** *A hesitant fuzzy soft set* $(\widetilde{\mathrm{H}}, Y)$ *over* $A$ *is a hesitant fuzzy soft strongly UP-ideal of* $A$ *if and only if it satisfies the condition* (6.1).

*Proof.* Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft strongly UP-ideal of $A$. Let $p \in Y$ and let $w, x, y, z \in A$ be such that $w \leq (z \cdot y) \cdot (z \cdot x)$. Then $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy strongly UP-ideal of $A$. Thus by Proposition 3.4, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}((z \cdot y) \cdot (z \cdot x))$. So,

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Conversely, assume that $(\widetilde{\mathrm{H}}, Y)$ satisfies the condition (6.1). Let $p \in Y$ and let $x \in A$. By Corollary 6.8, 5.8 and 4.6, respectively, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)$. Let $x, y, z \in A$. Since $((z \cdot y) \cdot (z \cdot x)) \cdot ((z \cdot y) \cdot (z \cdot x)) = 0$, we have $(z \cdot y) \cdot (z \cdot x) \leq (z \cdot y) \cdot (z \cdot x)$. Then it follows from (6.1) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Thus, $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy strongly UP-ideal of $A$. So, $(\widetilde{\mathrm{H}}, Y)$ is a $p$-hesitant fuzzy soft strongly UP-ideal of $A$. Since $p$ is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft strongly UP-ideal of $A$. □

**Theorem 7.5.** *Let $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over $A$ such that $\varnothing \neq N \subseteq Y$. Then the following statements are hold:*

1. *if $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of $A$, then $(\widetilde{\mathrm{H}}|_N, N)$ is a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of $A$,*
2. *There exists $(\widetilde{\mathrm{H}}|_N, N)$ is a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of $A$, but $(\widetilde{\mathrm{H}}, Y)$ is not a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of $A$.*

*Proof.* (1) Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of $A$. In the same way as Theorem 4.7, we can show that $(\widetilde{\mathrm{H}}|_N, N)$ is a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of $A$.

(2) By Example 7.3 (resp., Example 6.3, Example 5.3), if we choose $N = \{p_4\}$ (resp., $\{p_3, p_4\}$, $\{p_3, p_4\}$) , then $(\widetilde{\mathrm{H}}|_N, N)$ is a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of $A$, but $(\widetilde{\mathrm{H}}, Y)$ is not a hesitant fuzzy soft strongly UP-ideal (resp., hesitant fuzzy soft UP-ideal, hesitant fuzzy soft UP-filter) of $A$. □

**Definition 7.6.** Let $Y$ be a nonempty subset of $P$. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $A$ is called a constant hesitant fuzzy soft set based on $p \in Y$ (we shortly call a $p$-constant hesitant fuzzy soft set) over $A$, if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on $A$ is a constant hesitant fuzzy set on $A$. If $(\widetilde{\mathrm{H}}, Y)$ is a $p$-constant hesitant fuzzy soft set over $A$ for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is a constant hesitant fuzzy soft set over $A$.

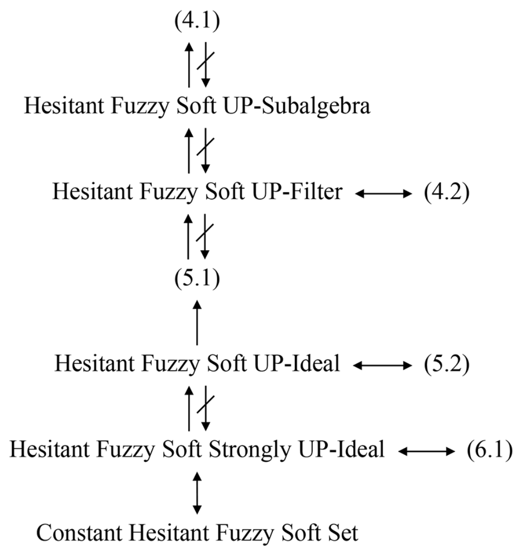
**Theorem 7.7.** *A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $A$ is a hesitant fuzzy soft strongly UP-ideal of $A$ if and only if is a constant hesitant fuzzy soft set over $A$.*

*Proof.* Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft strongly UP-ideal of $A$ and let $p \in Y$. Then $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy strongly UP-ideal of $A$. By Theorem 3.3, we obtain $\widetilde{\mathrm{H}}[p]$ is a constant hesitant fuzzy set on $A$. Thus $(\widetilde{\mathrm{H}}, Y)$ is a $p$-constant hesitant fuzzy soft set over $A$. Since $p$ is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is a constant hesitant fuzzy soft set over $A$.

Conversely, let $p \in Y$. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a constant hesitant fuzzy soft set over $A$. Then $\widetilde{\mathrm{H}}[p]$ is a constant hesitant fuzzy set on $A$. By Theorem 3.3, we have $\widetilde{\mathrm{H}}[p]$ is a hesitant fuzzy strongly UP-ideal of $A$. Since $p$ is arbitrary, we state that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft strongly UP-ideal of $A$. □

## 8. Conclusions and Future Work

In this paper, we have introduced the notion of hesitant fuzzy soft sets which is a new extension of hesitant fuzzy sets over UP-algebras and the notions of hesitant fuzzy soft UP-subalgebras, hesitant fuzzy soft UP-filters, hesitant fuzzy soft UP-ideals and hesitant fuzzy soft strongly UP-ideals of UP-algebras and investigated some of its important properties. Then we have the diagram of hesitant fuzzy soft sets over UP-algebras below.




In our future study of UP-algebras, may be the following topics should be considered:

- To get more results in hesitant fuzzy soft sets over UP-algebras.
- To define hesitant anti-fuzzy soft sets over UP-algebras.
- To define operations of hesitant (anti-)fuzzy soft sets over UP-algebras.

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