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On triple sequence space of Bernstein-Stancu operator of rough $I_\lambda-$statistical convergence of weighted $g\,(A)$

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Abstract. We introduce and study some basic properties of rough $I_\lambda-$statistical convergent of weight $g\,(A)$, where $g:\mathbb{N}^3\to[0,\infty)$ is a function statisying $g\,(m,n,k)\to\infty$ and $g\,(m,n,k)\not\to 0$ as $m,n,k\to\infty$ and A represent the RH-regular matrix and also prove the Korovkin approximation theorem by using the notion of weighted A-statistical convergence of weight $g\,(A)$ limits of a triple sequence of Bernstein-Stancu polynomials.

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1. Introduction

The idea of rough convergence was first introduced by Phu [13, 14, 15] in finite dimensional normed spaces,showed that the set LIM_x^r is bounded, closed, convex and introduced the notion of rough Cauchy sequence also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r. Pal et al. [12] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

Aytar [1] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained to statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [2] studied that the $r-$limit set of the sequence is equal to intersection of these sets and that $r-$core of the sequence is equal to the union of these sets. Dündar and Cakan [9] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence. The notion of $I-$convergence of a triple sequence which is based on the structure of the ideal I of subsets of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence and also Zhan et al. [10, 11, 19, 20, 21] studied various rough sets.

Our primary interest in the present paper is to obtain a general Korovkin-type approximation theorem for triple sequences of positive linear operators of two variables from $H_w(K)$ to $C_w(K)$ via statistical A-summability.

Let A be any three dimensional matrix. For a given triple sequence $x = (x_{mnk})$, the $A-$transform of x, denoted by $Ax := \left((Ax)_{ij\ell}\right)$, given by

$$(Ax)_{i,j,\ell} = \sum_{(m,n,k)\in\mathbb{N}^3} a_{i,j,\ell,m,n,k} x_{mnk} \tag{1.1}$$

provided the triple series converges in Pringsheim's sense for every $(i, j, \ell) \in \mathbb{N}^3$.

A three dimensional matrix $A = (a_{i,j,\ell,m,n,k})$ is said to be RH-regular it maps every bounded $P-$convergent sequence into a $P-$convergent sequence with the same $P-$limit. Any three dimensional matrix $A = (a_{i,j,\ell,m,n,k})$ is RH-regular if and only if

(i) $P - \lim_{i,j} a_{i,j,\ell,m,n,k} = 0$ for each $(m, n, k) \in \mathbb{N}^3$,
(ii) $P - \lim_{i,j,\ell} \sum_{(m,n,k)\in\mathbb{N}^3} a_{i,j,\ell,m,n,k} = 1$,
(iii) $P - \lim_{i,j,\ell} \sum_{m\in\mathbb{N}} a_{i,j,\ell,m,n,k} = 0$ for each $n, k \in \mathbb{N}$,
(iv) $P - \lim_{i,j,\ell} \sum_{n\in\mathbb{N}} a_{i,j,\ell,m,n,k} = 0$ for each $m, k \in \mathbb{N}$,
(v) $P - \lim_{i,j,\ell} \sum_{k\in\mathbb{N}} a_{i,j,\ell,m,n,k} = 0$ for each $m, n \in \mathbb{N}$,
(vi) $\sum_{(m,n,k)\in\mathbb{N}^3} |a_{i,j,\ell,m,n,k}|$ is $P-$convergent for every $(i, j, \ell) \in \mathbb{N}^3$,
(vii) there exist finite positive integers A and B such that $\sum_{m,n,k>B} |a_{i,j,\ell,m,n,k}| < A$ holds for every $(i, j, \ell) \in \mathbb{N}^3$.

Now let $A = (a_{i,j,\ell,m,n,k})$ be a non-negative RH-regular matrix, and $K \subset \mathbb{N}^3$. Then the $A-$density of K is given by

$$\delta_2^A \{K\} := P - \lim_{i,j,\ell} \sum_{(m,n,k)\in K(\epsilon)} a_{i,j,\ell,m,n,k}$$

where

$$K(\epsilon) := \left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - L| \geq \epsilon\right\}$$

provided that the limit on the right-hand side exists in Pringsheim's sense. A real triple sequence $x = (x_{mnk})$ is said to be $A-$statistically convergent to a number L if, for every $\epsilon > 0$,

$$\delta_2^A \left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - L| \geq \epsilon\right\} = 0.$$

In this case, we write $st_2^A - \lim_{m,n,k} = L$.

Let K be a subset of the set of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and let us denote the set $K_{ik\ell} = \{(m, n, k) \in K : m \leq i, n \leq j, k \leq \ell\}$. Then the natural density of K is given by

$$\delta(K) = \lim_{i,j,\ell\to\infty} \frac{|K_{ij\ell}|}{ij\ell},$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

Throughout the paper, $\mathbb{R}^3$ denotes the real three dimensional space with metric (X,d). Consider a triple sequence space $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}^3, m,n,k \in \mathbb{N}$.

A triple sequence space $x = (x_{mnk})$ is said to be statistically convergent to $\bar{0} = (0,0,0) \in \mathbb{R}^3$, written as $st - \lim x = \bar{0}$, provided that the set

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk}| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, $\bar{0}$ is called the statistical limit of the triple sequence of x.

If a triple sequence is statistically convergent, then for every $\epsilon > 0$, infinitely many terms of the sequence may remain outside the $\epsilon-$ neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence $x = (x_{mnk})$ satisfies some property P for all m,n,k except a set of natural density zero, then we say that the triple sequence space x satisfies P for "almost all (m,n,k)" and we abbreviate this by "a.a. (m,n,k)".

Let $\left(x_{m_i n_j k_\ell}\right)$ be a subsequence of $x = (x_{mnk})$. If the natural density of the set $K = \left\{(m_i n_j k_\ell) \in \mathbb{N}^3 : (i,j,\ell) \in \mathbb{N}^3\right\}$ is different from zero, then $\left(x_{m_i n_j k_\ell}\right)$ is called a non-thin sub sequence of a triple sequence of x.

$c \in \mathbb{R}^3$ is called a statistical cluster point of a triple sequence space $x = (x_{mnk})$ provided that the natural density of the set

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - c| < \epsilon\right\}$$

is different from zero for every $\epsilon > 0$. We denote the set of all statistical cluster points of the subsequence x by Γ_x.

A triple sequence $x = (x_{mnk})$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk}|^{1/m+n+k} \geq M\right\}\right) = 0.$$

First applied the concept of $(p,q)-$calculus in approximation theory and introduced the $(p,q)-$analogue of Bernstein operators. Later, based on $(p,q)-$integers, some approximation results for Bernstein-Stancu, Bernstein- Kantorovich, $(p,q)-$Lorentz, Bleimann-Butzer, Hahn operators and Bernstein-Shurer operators etc.

Motivated by the above mentioned work on $(p,q)-$approximation and its application, in this paper we study statistical approximation properties of Bernstein-Stancu operators based on $(p,q)-$integers.

Now we recall some basic definitions about $(p,q)-$integers. For any $u,v,w \in \mathbb{N}^3$, the $(p,q)-$integer $[uvw]_{p,q}$ is defined by

$$[0]_{p,q} := 0 \text{ and } [uvw]_{p,q} = \frac{p^{uvw} - q^{uvw}}{p-q} \text{ if } u,v,w \geq 1,$$

where $0 < q < p \leq 1$. The $(p,q)-$factorial is defined by

$$[0]_{p,q}! := 1 \text{ and } [uvw]!_{p,q} = [1]_{p,q}[2]_{p,q} \cdots [uvw]_{p,q} \text{ if } u,v,w \geq 1 \text{ and } u,v,w,m,n,k \in \mathbb{N}.$$

Also the $(p,q)-$binomial coefficient is defined by

$$\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q}=\frac{[u]!_{p,q}}{[m]!_{p,q}\,[u-m]!_{p,q}}\ \frac{[v]!_{p,q}}{[n]!_{p,q}\,[v-n]!_{p,q}}\ \frac{[w]!_{p,q}}{[k]!_{p,q}\,[w-k]!_{p,q}}$$

for all $u,v,w,m,n,k\in\mathbb{N}$ and $u\geq m,\ v\geq n,\ w\geq k$.

The formula for $(p,q)-$binomial expansion is as follows:

$$\begin{aligned}&(ax+by)_{p,q}^{uvw}\\&=\sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}p^{\frac{(u-m)(u-m-1)+(v-n)(v-n-1)+(w-k)(w-k-1)}{2}}q^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}\\&\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q}a^{(u-m)+(v-n)+(w-k)}b^{m+n+k}x^{(u-m)+(v-n)+(w-k)}y^{m+n+k},\end{aligned}$$

$$(x+y)_{p,q}^{uvw}=(x+y)\,(px+qy)\,(p^2x+q^2y)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x+q^{(u-1)+(v-1)+(w-1)}y\right),$$

$$(1-x)_{p,q}^{uvw}=(1-x)\,(p-qx)\,(p^2-q^2x)\cdots\left(p^{(u-1)+(v-1)+(w-1)}-q^{(u-1)+(v-1)+(w-1)}x\right),$$

and

$$(x)_{p,q}^{mnk}=x\,(px)\,(p^2x)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x\right)=p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}.$$

The Bernstein operator of order (r,s,t) is given by

$$B_{rst}\,(f,x)=\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}\,(1-x)^{(m-r)+(n-s)+(k-t)}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

The $(p,q)-$Bernstein operators are defined as follows:

$$\begin{aligned}&B_{rst,p,q}\,(f,x)\\&=\frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{2}}}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k}p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}x^{m+n+k}\\&\prod_{u_1=0}^{(r-m-1)}(p^{u_1}-q^{u_1}x)\prod_{u_2=0}^{(s-n-1)}(p^{u_2}-q^{u_2}x)\prod_{u_3=0}^{(t-k-1)}(p^{u_3}-q^{u_3}x)\\&f\left(\frac{[m]_{p,q}\,[n]_{p,q}\,[k]_{p,q}}{p^{(m-r)+(n-s)+(k-t)}\,[r]_{p,q}\,[s]_{p,q}\,[t]_{p,q}+\mu}\right),\ x\in[0,1]\end{aligned}$$

Also, we have

$$(1-x)_{p,q}^{rst}=\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}(-1)^{m+n+k}\,p^{\frac{(r-m)(r-m-1)+(s-n)(s-n-1)+(t-k)(t-k-1)}{6}}.$$

$$q^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}}\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}\qquad(*)$$

$(p,q)-$Bernstein Stancu operators are defined as follows:

$$S_{rst,p,q}(f,x)$$
$$= \frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{6}}} \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} \binom{r}{m}\binom{s}{n}\binom{t}{k} p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}} x^{m+n+k}$$

(**)

$$\prod_{u_1=0}^{(r-m-1)}\left(p^{u_1}-q^{u_1}x\right)\prod_{u_2=0}^{(s-n-1)}\left(p^{u_2}-q^{u_2}x\right)\prod_{u_3=0}^{(t-k-1)}\left(p^{u_3}-q^{u_3}x\right)$$
$$f\left(\frac{p^{(r-m)+(s-n)+(t-k)}[m]_{p,q}[n]_{p,q}[k]_{p,q}+\eta}{[r]_{p,q}[s]_{p,q}[t]_{p,q}+\mu}\right),\ x\in[0,1]$$

Note that for $m=n=0$, $(p,q)-$Bernstein-Stancu operators given by (*) reduce into $(p,q)-$Bernstein-Stancu operators. Also for $p=1$, $(p,q)-$Bernstein-Stancu operators given by (**) turn out to be $q-$Bernstein-Stancu operators.

Throughout the paper, $\mathbb{R}$ denotes the real numbers with metric (X,d). Consider a triple sequence of Bernstein-Stancu polynomials $(B_{mnk}(f,x))$ such that $(B_{mnk}(f,x))\in\mathbb{R}$, $m,n,k\in\mathbb{N}$.

Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is said to be statistically convergent to $0\in\mathbb{R}$, written as $st-\lim\ x=0$, provided that the set

$$K_\epsilon := \left\{(m,n,k)\in\mathbb{N}^3 : |S_{rst,p,q}(f,x)-(f,x)|\geq\epsilon\right\}$$

has natural density zero for any $\epsilon>0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein-Stancu polynomials. i.e., $\delta(K_\epsilon)=0$. That is,

$$\lim_{rst\to\infty}\frac{1}{pqj}\left|\{m\leq p, n\leq q, k\leq j : |S_{rst,p,q}(f,x)-(f,x)|\geq\epsilon\}\right|=0.$$

In this case, we write $\delta-\lim S_{rst,p,q}(f,x)=(f,x)$ or $S_{rst,p,q}(f,x)\to^{S_S}(f,x)$.

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

In this paper, we introduce the notion of Bernstein stancu polynomials of rough $\lambda-$statistically $\rho-$Cauchy convergence. Defining the set of Bernstein-Stancu polynomials of rough $\lambda-$statistical limit points of a sequence, we obtain $\lambda-$ statistical convergence criteria associated with this set of rough $\lambda-$statistically $\rho-$Cauchy sequence.

A triple sequence (real or complex) can be defined as a function $x:\mathbb{N}^3\to\mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated initially by *Sahiner et al.* [16, 17], *Esi et al.* [3, 4, 5, 6], *Dutta et al.* [7], *Subramanian et al.* [18], *Debnath et al.* [8] and many others. Throughout the paper let r be a nonnegative real number.

2. Definitions and preliminaries

Throughout the paper $\mathbb{R}^3$ denotes the real three dimensional case with the metric. Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}^3$; $m, n, k \in \mathbb{N}$. The following definition are obtained:

Definition 2.1. Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f, x))$ is said to be statistically convergent to $f(x)$ denoted by $(S_{rst,p,q}(f, x)) \rightarrow^{st-\lim x} f(x)$, if for any $\epsilon > 0$ we have $d(A(\epsilon)) = 0$, where

$$A(\epsilon) = \left\{(m, n, k) \in \mathbb{N}^3 : |(S_{rst,p,q}(f, x)) - f(x)| \geq \epsilon\right\}.$$

Definition 2.2. Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f, x))$ is said to be statistically convergent to $f(x)$ denoted by $(S_{rst,p,q}(f, x)) \rightarrow^{st-\lim x} f(x)$, provided that the set

$$\left\{(m, n, k) \in \mathbb{N}^3 : |(S_{rst,p,q}(f, x)) - f(x)| \geq \epsilon\right\},$$

has natural density zero for every $\epsilon > 0$.

In this case, $f(x)$ is called the statistical limit of the sequence of Berstein-Stancu polynomials.

Definition 2.3. Let f be a continuous function defined on the closed interval $[0, 1]$ and r be a non-negative real number. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f, x))$ in a metric space $(X, |.,.|)$ is said to be $r-$convergent to $f(x)$, denoted by $(S_{rst,p,q}(f, x)) \rightarrow^{r} f(x)$, if for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for all $m, n, k \geq N_\epsilon$ we have

$$|(S_{rst,p,q}(f, x)) - f(x)| < r + \epsilon$$

In this case, $(S_{rst,p,q}(f, x))$ is called an $r-$limit of $f(x)$.

Remark 2.4. We consider $r-$limit set $(S_{rst,p,q}(f, x))$ which is denoted by $LIM^r_{(S_{rst,p,q}(f,x))}$ and is defined by

$$LIM^r_{(S_{rst,p,q}(f,x))} = \left\{(S_{rst,p,q}(f, x)) \in X : (S_{rst,p,q}(f, x)) \rightarrow^{r} f(x)\right\}.$$

Definition 2.5. Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f, x))$ is said to be $r-$convergent, if $LIM^r_{(S_{rst,p,q}(f,x))} \neq \phi$ and r is called a rough convergence degree of $(S_{rst,p,q}(f, x))$. If $r = 0$, then it is ordinary convergence of triple sequence of Bernstein-Stancu polynomials.

Definition 2.6. Let f be a continuous function defined on the closed interval $[0, 1]$ and r be a non-negative real number. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f, x))$ in a metric space $(X, |.,.|)$ is said to be $r-$statistically convergent to $f(x)$, denoted by $(S_{rst,p,q}(f, x)) \rightarrow^{r-st_3} f(x)$, if for any $\epsilon > 0$, we have $d(A(\epsilon)) = 0$, where

$$A(\epsilon) = \left\{(m, n, k) \in \mathbb{N}^3 : |S_{rst,p,q}(f, x) - f(x)| \geq r + \epsilon\right\}.$$

In this case, $f(x)$ is called $r-$ statistical limit of $B_{mnk}f(x)$. If $r = 0$, then it is ordinary statistical convergent of triple sequence of Bernstein polynomials.

Definition 2.7. A class I of subsets of a nonempty set X is said to be an ideal in X, provided that

(i) $\phi \in I$,
(ii) $A, B \in I$ implies $A \bigcup B \in I$,
(iii) $A \in I$, $B \subset A$ implies $B \in I$,
I is called a nontrivial ideal if $X \notin I$.

Definition 2.8. A nonempty class F of subsets of a nonempty set X is said to be a filter in X, provided that

(i) $\phi \in F$,
(ii) $A, B \in F$ implies $A \bigcap B \in F$,
(iii) $A \in F, A \subset B$ implies $B \in F$.

Definition 2.9. I is a non trivial ideal in $X \neq \phi$, then the class

$$F(I) = \{M \subset X : M = X - A \text{ for some } A \in I\}$$

is a filter on X and it is called the filter associated with I.

Definition 2.10. A non trivial ideal I in X is called admissible, if $\{x\} \in I$ for each $x \in X$.

Note 2.11. *If I is an admissible ideal, then usual convergence in X implies I convergence in X.*

Remark 2.12. If I is an admissible ideal, then usual rough convergence implies rough $I-$ convergence.

Definition 2.13. Let f be a continuous function defined on the closed interval $[0, 1]$ and r be a non-negative real number. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f, x))$ in a metric space $(X, |.,.|)$ is said to be rough ideal convergent of weight g or $rI_\lambda-$convergent to $f(x)$, denoted by $S_{rst,p,q} \to^{rI_\lambda^g} f(x)$, if for any $\epsilon > 0$ we have

$$\left\{(p, q, j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |S_{rst,p,q}(f, x) - f(x)| \geq r + \epsilon\right\} \in I.$$

In this case $(S_{rst,p,q}(f, x))$ is called $rI_\lambda-$limit of (f, x) and a triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f, x))$ is called rough $I_\lambda-$convergent weight g to $f(x)$ with r as roughness of degree. If $r = 0$ then it is ordinary $I_\lambda-$convergent of weight g.

Note 2.14. *Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f, x))$ is not $I_\lambda-$convergent of weight g in usual sense and $|S_{rst,p,q}(f, x) - S_{rst,p,q}(f, y)| \leq r$ for all $(m, n, k) \in \mathbb{N}^3$ or*

$$\left\{(p, q, j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |S_{rst,p,q}(f, x) - S_{rst,p,q}(f, y)| \geq r\right\} \in I,$$

for some $r > 0$. Then the triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f, x))$ is rI_λ^g-convergent of weight g.

Note 2.15. *It is clear that rI_λ^g-limit of $f(x)$ is not necessarily unique.*

Definition 2.16. Consider rI_λ^g- limit set of $f(x)$, which is denoted by

$$I_\lambda^g - LIM^r_{S_{rst,p,q}(f,x)} = \left\{ f(x) \in X : S_{rst,p,q}(f,x) \to^{rI_\lambda^g} f(x) \right\}.$$

Then the triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is said to be $rI_\lambda-$convergent of weight g, if $I_\lambda^g - LIM^r_{S_{rst,p,q}(f,x)} \neq \phi$ and r is called a rough $I_\lambda-$convergence of weight g degree of $S_{rst,p,q}(f,x)$.

Definition 2.17. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ in a metric space (X,d). A point $f(x) \in X$ is said to be an I_λ^g-accumulation point ,if for each $\epsilon > 0$, the set

$$\left\{ (p,q,j) \in \mathbb{N}^3 : d\left(\left(S_{rst,p,q}(f,x) \right), f(x) \right) = \frac{1}{g(\lambda_{pqj})} \left| S_{rst,p,q}(f,x) - f(x) \right| < \epsilon \right\} \notin I.$$

A point $f(x) \in X$ is said to be I_λ^g- analytic, if there exists a positive real number M such that

$$\left\{ (p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} \left| S_{rst,p,q}(f,x) \right|^{1/m+n+k} \geq M \right\} \in I.$$

Definition 2.18. A point $f(x) \in X$ is said to be an I_λ^g-accumulation point and Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ in a metric space (X,d) if and only if for each $\epsilon > 0$ the set

$$\left\{ (p,q,j) \in \mathbb{N}^3 : d\left(\left(S_{rst,p,q}(f,x) \right), f(x) \right) = \frac{1}{g(\lambda_{pqj})} \left| S_{rst,p,q}(f,x) - f(x) \right| < \epsilon \right\} \notin I.$$

We denote the set of all I_λ^g-accumulation points of $(S_{rst,p,q}(f,x))$ by $I_\lambda^g\left(\Gamma_{S_{rst,p,q}(f,x)}\right)$.

Definition 2.19. Let f be a continuous function defined on the closed interval $[0,1]$. For a triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ of real numbers, the notions of ideal limit superior and ideal limit inferior are defined as follows:

$$I_\lambda^g - \limsup S_{rst,p,q}(f,x) = \begin{cases} \sup\ B_x, & \text{if } B_{S_{rst,p,q}(f,x)} \neq \phi, \\ -\infty, & \text{if } B_{S_{rst,p,q}(f,x)} = \phi \end{cases},$$

and

$$I_\lambda^g - \liminf \left(S_{rst,p,q}(f,x) \right) = \begin{cases} \inf A_{S_{rst,p,q}(f,x)}, & \text{if } A_{S_{rst,p,q}(f,x)} \neq \phi, \\ +\infty, & \text{if } A_{S_{rst,p,q}(f,x)} = \phi \end{cases},$$

where

$$A_{S_{rst,p,q}(f,x)} = \left\{ a \in \mathbb{R} : \left\{ (m,n,k) \in \mathbb{N}^3 : S_{rst,p,q}(f,x) < a \right\} \notin I \right\}$$

and

$$B_{S_{rst,p,q}(f,x)} = \left\{ b \in \mathbb{R} : \left\{ (m,n,k) \in \mathbb{N}^3 : S_{rst,p,q}(f,x) > b \right\} \notin I \right\}.$$

Definition 2.20. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is said to be rough $I_\lambda-$convergent of weight g, if $I_\lambda^g - LIM^r S_{rst,p,q}(f,x) \neq \phi$. It is clear that if $I_\lambda^g - LIM^r S_{rst,p,q}(f,x) \neq \phi$ for a triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ of real numbers, then we have

$$I_\lambda^g - LIM^r S_{rst,p,q}(f,x) = [I_\lambda^g - \text{limsup}\, S_{rst,p,q}(f,x) - r, I_\lambda^g - \text{liminf}\, S_{rst,p,q}(f,x) + r]\,.$$

Remark 2.21. Let $\lambda = (\lambda_{pqj})_{(p,q,j)\in\mathbb{N}^3}$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{(pqj)+1} \leq \lambda_{pqj} + 1, \lambda_{111} = 1.$$

The collection of such sequences λ will be denoted by η. We define the generalized de la Valée-Pousin mean of weight g by

$$t_{pqj}(x) = \frac{1}{g(\lambda_{pqj})} \sum_{(m,n,k)\in I_{pqj}} x_{mnk}.$$

where $I_{rst} = [(pqj) - \lambda_{pqj+1}, pqj]$.

Let $r = (r_{mnk})$ be a triple sequence of nonnegative numbers such that $r_{000} > 0$ and

$$u_{pqj} = \sum_{m,n,k}^{p,q,j} u_{mnk} \to \infty. \tag{2.1}$$

$$t_{pqj}(x) = \frac{1}{g(\lambda_{pqj})} \frac{1}{u_{pqj}} \sum_{(m,n,k)\in I_{pqj}} u_{pqj} x_{mnk},\ p,q,j = 0,1,2,\ldots$$

where $I_{rst} = [(pqj) - \lambda_{pqj+1}, pqj]$.

Definition 2.22. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is said to be $[V,\lambda](I)^g -$ summable to $f(x)$, if

$$I - \lim_{pqj} t_{pqj}(S_{rst,p,q}(f,x)) \to f(x)\,.$$

i.e., for any $\delta > 0$,

$$\left\{(p,q,j) \in \mathbb{N}^3 : |t_{pqj}(S_{rst,p,q}(f,x)) - f(x)| \geq \delta\right\} \in I$$

and it is denoted by $[V,\lambda](I)^g$.

Definition 2.23. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is said to be $I_\lambda-$statistically convergent of weight g, if for every $\epsilon > 0$ and $\delta > 0$,

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} |\{(m,n,k) \in I_{pqj} : |(S_{rst,p,q}(f,x)) - f(x)| \geq r + \epsilon\}| \geq \delta\right\} \in I.$$

In this case, we write $(I_\lambda)^g - \lim S_{rst,p,q}(f,x) = f(x)$. Or $S_{rst,p,q}(f,x) \to f(x)(I_\lambda)^g$.

Definition 2.24. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is said to be $I_\lambda-$statistically convergent of weight g, if for every $\epsilon > 0$,

$$\lim_{pqj} \frac{1}{g(\lambda_{pqj})} \frac{1}{u_{pqj}} |\{(m,n,k) \in I_{pqj} : |(S_{rst,p,q}(f,x)) - f(x)| \geq r + \epsilon\}| \in I.$$

In this case, we write $(I_\lambda)^{g_{\bar{N}}} - \lim S_{rst,p,q}(f,x) = f(x)$ or $S_{rst,p,q}(f,x) \to f(x)(I_\lambda)^{g_{\bar{N}}}$.

Remark 2.25. If $u_{pqj} = 1$ for all m, n, k, then $\left(\bar{N}, u_{pqj}\right)-$summable is reduced to $[C, 1, 1, 1]-$summable or Cesáro summable and statistically convergent of weight g is reduced to statistical convergence of Bernstein-Stancu polynomials of triple sequence.

3. A Korovkin-type approximation theorem by $g(A)-$statistical converence

Let $C_B(K)$ the space of all continuous and bounded real valued functions on $K = [0,\infty) \times [0,\infty) \times [0,\infty)$. This space is equipped with the supremum norm

$$\|f\| = \sup_{(x,y,z)\in K} S_{rst,p,q}\left|f, (x,y,z)\right|, \ \left(f \in C_B(K)\right).$$

Consider the triple space of $H_w(K)$ of all real valued functions of Bernstein-Stancu polynomials of f on K satisfying

$$\begin{aligned}&\left|S_{rst,p,q}\left(f, (u,v,w)\right) - S_{rst,p,q}\left(f, (x,y,z)\right)\right| \\ &\leq w\left(\left|\frac{u}{1+u} - \frac{x}{1+x}\right|, \left|\frac{v}{1+v} - \frac{y}{1+y}\right|, \left|\frac{w}{1+w} - \frac{z}{1+z}\right|\right)\end{aligned}$$

where w be a function of the type of the modulus of continuity given by, for $\delta, \delta_1, \delta_2, \delta_3 > 0$,

(1) w is non-negative increasing function on K with respect to $\delta_1, \delta_2, \delta_3$,
(2) $w(\delta, \delta_1 + \delta_2 + \delta_3) \leq w(\delta, \delta_1) + w(\delta, \delta_2) + w(\delta, \delta_3)$,
(3) $w(\delta_1 + \delta_2 + \delta_3, \delta) \leq w(\delta_1, \delta) + w(\delta_2, \delta) + w(\delta_3, \delta)$,
(4) $\lim_{\delta_1,\delta_2,\delta_3\to 0} w(\delta_1, \delta_2, \delta_3) = 0$.

The Bernstein-Stancu polynomials of $S_{rst,p,q}(f) \in H_w(K)$ satisfies the inequality

$$S_{rst,p,q}\left|\left(f, (x,y,z)\right)\right| \leq S_{rst,p,q}\left(f, (0,0,0)\right) + w(1,1,1), x, y, z \geq 0$$

and hence it is bounded on K.Therefore $H_w(K) \subset C_B(K)$.

We also use the following Bernstein-Stancu polynomials of test functions

$$\begin{aligned}&S_{rst,p,q}\left(f_{000}, (u,v,w)\right) = 1, S_{rst,p,q}\left(f_{111}, (u,v,w)\right) = \frac{u}{1+u}, \\ &S_{rst,p,q}\left(f_{222}, (u,v,w)\right) = \frac{v}{1+v}, S_{rst,p,q}\left(f_{333}, (u,v,w)\right) = \frac{w}{1+w}\end{aligned}$$

and

$$S_{rst,p,q}\left(f_{444}, (u,v,w)\right) = \left(\frac{u}{1+u}\right)^2 + \left(\frac{v}{1+v}\right)^2 + \left(\frac{w}{1+w}\right)^2.$$

Definition 3.1. Let f be a continuous function defined on the closed interval $[0,1]$ and $A = \left(a_{i,j,\ell,m,n,k}\right)$ be a nonnegative RH-regular matrix. A triple sequence of Bernstein-Stancu polynomials $\left(S_{rst,p,q}(f,x)\right)$ is said to be $I_\lambda-$ statistically convergent of weight g, if for every $\epsilon > 0$,

$$\lim_{pqj} \sum_{(m,n,k)\in E(u,\epsilon)} a_{i,j,\ell,m,n,k} = 0, \tag{3.1}$$

$$E(u,\epsilon)=\left\{(p,q,j)\in\mathbb{N}^3:\frac{1}{g(\lambda_{pqj})}\frac{1}{u_{pqj}}\left|\{(m,n,k)\in I_{pqj}:\right.\right.$$
$$\left.\left.|S_{rst,p,q}(f,x)-f(x)|\geq r+\epsilon\}\right|\right\}\in I.$$

In this case, we write $(I_\lambda)_A^{g_{\bar{N}}}-\lim S_{rst,p,q}(f,x)=f(x)$ or $S_{rst,p,q}(f,x)\to f(x)\,(I_\lambda)_A^{g_{\bar{N}}}$.

4. Main Results

Theorem 4.1. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials of real numbers of $(S_{rst,p,q}(f,x)):[0,1]\to[0,1]$*

$$(I_\lambda)^{g_{\bar{N}}}-\lim|S_{rst,p,q}(f,x)-f(x)|=0 \tag{4.1}$$

if and only if

$$(I_\lambda)^{g_{\bar{N}}}-\lim\left|S_{rst,p,q}\left(v^i,x\right)-f\left(x^i\right)\right|=0,\ \text{for } 0\leq i\leq 8. \tag{4.2}$$

Proof. Following the proof of Theorem, we obtain

$$E\subset E_i\subset E_{i+1},\ \text{for } 1\leq\ i\leq 8 \tag{4.3}$$

and then

$$\delta(I_\lambda)^{g_{\bar{N}}}(E)\subset\sum_{i=1}^{8}\delta(I_\lambda)^{g_{\bar{N}}}(E_i).$$

Equations (4.2) gives that $(I_\lambda)^{g_{\bar{N}}}-\lim|S_{rst,p,q}(f,x)-f(x)|=0$. □

Theorem 4.2. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials of real numbers of $(S_{rst,p,q}(f,x)):[0,1]\to[0,1]$ which satisfies (4.1) to (4.3) of Theorem 4.1 and the following condition holds:*

$$(I_\lambda)^{g_{\bar{N}}}-\lim|S_{rst,p,q}(1,x)-1|=0 \tag{4.4}$$

Then,

$$\lim_{pqj}\frac{1}{g(\lambda_{pqj})}\frac{1}{u_{pqj}}\sum_{m,n,k=0}^{p,q,j}u_{mnk}\{|S_{rst,p,q}(f,x)-f(x)|\geq r+\epsilon\}\in I. \tag{4.5}$$

Proof. It follows (4.4) that $|S_{rst,p,q}(1,x)|\leq C'$, for some constant $C'>0$ and for all $m,n,k\in\mathbb{N}$. Then we obtain

$$\begin{aligned}u_{mnk}\{|S_{rst,p,q}(f,x)-f(x)|\}&\leq u_{mnk}(|f(x)|\,|S_{rst,p,q}(1,x)|+|f(x)|)\\&\leq u_{mnk}C\left(C'+1\right)\end{aligned} \tag{4.6}$$

In right hand side of (4.6) is constant. Thus we get

$$(I_\lambda)^{g_{\bar{N}}}-\lim|S_{rst,p,q}(f,x)-f(x)|=0.$$

□

5. Statistical $A-$summability

In this section we define statistical A-summability of a triple sequence of Bernstein-Stancu polynomials of RH-regular summability matrix and prove that it is stronger than $A-$Bernstein-Stancu polynomials of rough statistical convergence for analytic triple sequences.

Definition 5.1. Let f be a continuous function defined on the closed interval $[0,1]$ and $A = (a_{i,j,\ell,m,n,k})$ be a nonnegative regular summability matrix of a triple sequence of Bernstein-Stancu polynomials $S_{rst,p,q}(f,x)$ is said to be rough statistically A summable convergent to $f(x)$, denoted by $S_{rst,p,q}(f,x) \rightarrow^{st_A - \lim x} f(x)$, if for every $\epsilon > 0$, provided that the set

$$\delta_2\left(\left\{(m,n,k) \in \mathbb{N}^3 : \left|S_{rst,p,q}(f,x) - f(x)\right| \geq r + \epsilon\right\}\right) = 0.$$

Then, if x is Bernstein-Stancu polynomials of rough statistically $A-$summable to $f(x)$, then for every $\epsilon > 0$,

$$P - \lim_{pqj} \frac{1}{pqj}\left(\left\{(m,n,k) \in \mathbb{N}^3 : \left|S_{rst,p,q}(f,x) - f(x)\right| \geq r + \epsilon\right\}\right) = 0.$$

Thus, the triple sequence of Bernstein-Stancu polynomials of $S_{rst,p,q}(f,x)$ is rough statistically $A-$summable to $f(x)$ if and only if $S_{rst,p,q}(f,Ax)$ is rough statistically convergent to $f(x)$.

Theorem 5.2. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ of real numbers is analytic and $A-$rough statistically convergent to $f(x)$ then it is rough statiscally $A-$summable convergent to $f(x)$ but not conversely.*

Proof. Let $S_{rst,p,q}(f,x)$ be analytic and $A-$rough statistically convergent to $f(x)$, and

$$K(\epsilon) = \left\{(m,n,k) \in \mathbb{N}^3 : \left|S_{rst,p,q}(f,x) - f(x)\right| \geq r + \epsilon\right\}.$$

Then

$$\begin{aligned}
\left|S_{rst,p,q}(f,x) - f(x)\right|^{1/m+n+k} &= \left|\sum_{(m,n,k)=1}^{\infty} a_{i,j,\ell,m,n,k}\left(S_{rst,p,q}(f,x)\right)^{1/m+n+k} - f(x)\right| \\
&= \left|\sum_{(m,n,k)=1}^{\infty} a_{i,j,\ell,m,n,k}\left(S_{rst,p,q}(f,x) - f(x)\right)^{1/m+n+k}\right. \\
&\quad \left. + f(x)\left(\sum_{(m,n,k)=1}^{\infty} a_{i,j,\ell,m,n,k} - 1\right)\right|
\end{aligned}$$

$$\leq \left| \sum_{(m,n,k)=1}^{\infty} a_{i,j,\ell,m,n,k} \left(S_{rst,p,q}\left(f,x\right) - f\left(x\right) \right)^{1/m+n+k} \right|$$
$$+ \left| f\left(x\right) \right| \left| \sum_{(m,n,k)=1}^{\infty} a_{i,j,\ell,m,n,k} - 1 \right|$$
$$\leq \left| \sum_{(m,n,k \in K(\epsilon))} a_{i,j,\ell,m,n,k} \left(S_{rst,p,q}\left(f,x\right) - f\left(x\right) \right)^{1/m+n+k} \right|$$
$$+ \left| \sum_{(m,n,k \notin K(\epsilon))} a_{i,j,\ell,m,n,k} \left(S_{rst,p,q}\left(f,x\right) - f\left(x\right) \right)^{1/m+n+k} \right|$$
$$+ \left| f\left(x\right) \right| \left| \sum_{(m,n,k)=1}^{\infty} a_{i,j,\ell,m,n,k} - 1 \right|.$$

Using the definition of triple sequence of Bernstein-Stancu polynomials of A-rough statistical convergence and the conditions of RH-regularity of A, we get $P - \lim_{pqj} \left| S_{rst,p,q}\left(f,x\right) - f\left(x\right) \right| = 0$ from the arbitraryness of $\epsilon > 0$. Thus

$$S_{rst,p,q}\left(f,x\right) \rightarrow^{st_A - limx} f\left(x\right).$$

To show that the converse is not true in general, we give the following examples:
(i) $A = \left(a_{i,j,\ell,m,n,k}\right)$ be $C\left[1,1,1\right]$, the six dimensional Cesáro matrix

$$a_{i,j,\ell,m,n,k} = \begin{cases} \left(1/mnk\right), & \text{if } i \leq m, j \leq n, \ell \leq k \\ 0, & \text{otherwise} \end{cases}$$

and let $x = \left(x_{mnk}\right)$ be defined

$$\left(S_{rst,p,q}\left(f,x\right)\right)^{1/m+n+k} = \left(-1\right)^{mnk} \text{ for all } m,n,k.$$

Then triple sequence of Bernstein-Stancu polynomials of $S_{rst,p,q}\left(f,x\right)$ is $C\left[1,1,1\right]$ summable (and hence statistical $C\left[1,1,1\right]-$summable) to zero but not $C\left[1,1,1\right]$ statistically convergent.

(ii) Define $A = \left(a_{i,j,\ell,m,n,k}\right)$ by

$$a_{i,j,\ell,m,n,k} = \begin{cases} 1/m^3n^3k^3 \\ \text{if } m=n=k, i \leq m, j \leq n, \ell \leq k \text{ and } m,n,k \text{ are even} \\ 1/\left(m^3-m\right)\left(n^3-n\right)\left(k^3-k\right) \\ \text{if } m=n=k, i \neq j \neq k, i \leq m, j \leq n, \ell \leq k \text{ and } m,n,k \text{ are odd} \\ 0 \\ \text{otherwise} \end{cases}$$

and define the triple analytic sequence of Bernstein-Stancu polynomials of $S_{rst,p,q}(f,x)$

$$[S_{rst,p,q}(f,x)]^{1/m+n+k} = \begin{cases} 1, \\ \text{if } m,n,k \text{ are odd and for all } m,n,k \\ 0, \\ \text{otherwise} \end{cases}$$

We can easily verify that A is RH-regular, that is, conditions RH(i)–RH (vi) hold. Moreover, for the sequence defined above,

$$\sum_{(m,n,k)=1}^{\infty} a_{i,j,\ell,m,n,k}[[S_{rst,p,q}(f,x)]^{1/m+n+k} = \begin{cases} 1/3 \\ \text{if and } m,n,k \text{ are even} \\ (m+1)(n+1)(k+1)/27mnk \\ \text{if } m,n,k \text{ are odd} \\ 0 \\ \text{ otherwise} \end{cases}$$

Thus it is clear that the triple sequence of Bernstein-Stancu polynomials of $S_{rst,p,q}(f,x)$ is not $A-$summable and hence is not $A-$rough statistically convergent but$S_{rst,p,q}(f,x) \rightarrow^{st_A - \lim x} f(x) = 0$. So triple sequence of Bernstein-Stancu polynomials of $S_{rst,p,q}(f,x)$ is rough statistically $A-$summable to zero. □

6. Korovkin-type approximation theorem

Theorem 6.1. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein stancu polynomials $(S_{rst,p,q}(f,x))$ of real numbers of $C_B(K)$ into itself. Then for all $f \in C_B(K)$,*

$$P - \lim_{pqj} \|S_{rst,p,q}(f,x) - f(x)\|_{C_B(K)} = 0$$

if and only if

$$P - \lim_{pqj} \|S_{rst,p,q}(f_{uvw},x) - f_{uvw}(x)\|_{C_B(K)} = 0, \ (u,v,w = 0,1,2,3,\ldots)$$

where

$$S_{rst,p,q}(f_{000},(u,v,w)) = 1, S_{rst,p,q}(f_{111},(u,v,w)) = \frac{u}{1+u},$$
$$S_{rst,p,q}(f_{222},(u,v,w)) = \frac{v}{1+v}, S_{rst,p,q}(f_{333},(u,v,w)) = \frac{w}{1+w}$$

and

$$S_{rst,p,q}(f_{444},(u,v,w)) = \left(\frac{u}{1+u}\right)^2 + \left(\frac{v}{1+v}\right)^2 + \left(\frac{w}{1+w}\right)^2.$$

Proof. It is routine verification. Therefore the proof is omitted. □

Theorem 6.2. *Let f be a continuous function defined on the closed interval $[0,1]$. Let $A = (a_{i,j,\ell,m,n,k})$ be a nonnegative RH-regular summability matrix and a triple sequence of Bernstein stancu polynomials $(S_{rst,p,q}(f,x))$ of real numbers of $C_B(K)$ into itself. Then for all $f \in C_B(K)$,*

$$st_{3A} - \lim_{pqj} \|S_{rst,p,q}(f,x) - f(x)\|_{C_B(K)} = 0$$

if and only if

$$st_{3A} - \lim_{pqj} \left\| S_{rst,p,q}\left(f_{uvw}, x\right) - f_{uvw}\left(x\right) \right\|_{C_B(K)} = 0, \ (u, v, w = 0, 1, 2, 3, \ldots)$$

where

$$S_{rst,p,q}\left(f_{000}, (u, v, w)\right) = 1, S_{rst,p,q}\left(f_{111}, (u, v, w)\right) = \frac{u}{1+u},$$
$$S_{rst,p,q}\left(f_{222}, (u, v, w)\right) = \frac{v}{1+v}, S_{rst,p,q}\left(f_{333}, (u, v, w)\right) = \frac{w}{1+w}$$

and

$$S_{rst,p,q}\left(f_{444}, (u, v, w)\right) = \left(\frac{u}{1+u}\right)^2 + \left(\frac{v}{1+v}\right)^2 + \left(\frac{w}{1+w}\right)^2.$$

Proof. It is routine verification. Therefore the proof is omitted. □

Theorem 6.3. *Let f be a continuous function defined on the closed interval $[0, 1]$. Let $A = (a_{i,j,\ell,m,n,k})$ be a nonnegative RH-regular summability matrix and a triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f, x))$ of real numbers of $C_B(K)$ into itself. Then for all $f \in C_B(K)$,*

$$st_3 - \lim_{pqj} \left\| \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}\left(f, x\right) - f\left(x\right) \right\|_{C_B(K)} = 0. \tag{6.1}$$

if and only if

$$st_3 - \lim_{pqj} \left\| \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}(f_{uvw}, x) - f_{uvw}\left(x\right) \right\|_{C_B(K)} = 0, \ (u, v, w = 0, 1, 2, 3, \ldots) \tag{6.2}$$

where

$$S_{rst,p,q}\left(f_{000}, (u, v, w)\right) = 1, S_{rst,p,q}\left(f_{111}, (u, v, w)\right) = \frac{u}{1+u},$$
$$S_{rst,p,q}\left(f_{222}, (u, v, w)\right) = \frac{v}{1+v}, S_{rst,p,q}\left(f_{333}, (u, v, w)\right) = \frac{w}{1+w}$$

and

$$S_{rst,p,q}\left(f_{444}, (u, v, w)\right) = \left(\frac{u}{1+u}\right)^2 + \left(\frac{v}{1+v}\right)^2 + \left(\frac{w}{1+w}\right)^2.$$

Proof. Condition (6.2) follows immediately from condition (6.1), since each $f_{uvw} \in C_B(K)\,(u, v, w = 0, 1, 2, 3, \ldots)$. Let us prove the converse. By the continuity of f on compact set K, we can write $|f(x, y, z)| \leq M$, where $M = \|f\|_{C_B(K)}$. Also since $f \in C_B(K)$, for every $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(u, v, w) - f(x, y, z)| < r + \epsilon$ for all $(u, v, w) \in K$ satisfying $\left|\frac{u}{1+u} - \frac{x}{1+x}\right| < \delta$; $\left|\frac{v}{1+v} - \frac{y}{1+y}\right| < \delta$ and

$\left|\frac{w}{1+w}-\frac{z}{1+z}\right|<\delta$. Then we get

$$\begin{aligned} &|f(u,v,w)-f(x,y,z)| \\ &< r+\epsilon+\frac{3M}{\delta^2}\left\{\left(\frac{u}{1+u}-\frac{x}{1+x}\right)^2+\left(\frac{v}{1+v}-\frac{y}{1+y}\right)^2+\left(\frac{w}{1+w}-\frac{z}{1+z}\right)^2\right\}. \end{aligned} \tag{6.3}$$

Thus from (6.3), we obtain for any $m,n,k\in\mathbb{N}$ so that

$$\begin{aligned} &\left\|\sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}(f,x)-f(x)\right\| \\ &\leq \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}\left(|f(u,v,w)-f(x,y,z)|,(x,y,z)\right) \\ &\quad+|f(x,y,z)|\left|\sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}\left(f_{000},(x,y,z)\right)-f_{000}(x,yz)\right| \\ &\leq \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}\left(r+\epsilon+\frac{3M}{\delta^2}\left\{\left(\frac{u}{1+u}-\frac{x}{1+x}\right)^2\right.\right. \\ &\quad\left.\left.+\left(\frac{v}{1+v}-\frac{y}{1+y}\right)^2+\left(\frac{w}{1+w}-\frac{z}{1+z}\right)^2\right\},(x,y,z)\right)+ \\ &\quad|f(x,y,z)|\left|\sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}\left(f_{000},(x,y,z)\right)-f_{000}(x,yz)\right| \\ &\leq \epsilon+(\epsilon+M)\left|\sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}\left(f_{000},(x,y,z)\right)-f_{000}\right|+ \\ &\quad\frac{3M}{\delta^2}\left|\sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}\left(f_{333},(x,y,z)\right)-f_{333}(x,yz)\right|+ \\ &\quad 3|x|\left|\sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}\left(f_{111},(x,y,z)\right)-f_{111}(x,yz)\right|+ \\ &\quad 3|y|\left|\sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}\left(f_{222},(x,y,z)\right)-f_{222}(x,y,z)\right|+ \\ &\quad 3|z|\left|\sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}\left(f_{333},(x,y,z)\right)-f_{333}(x,y,z)\right|+ \end{aligned}$$

$$\left(\left(\frac{x}{1+x}\right)^2+\left(\frac{y}{1+y}\right)^2+\left(\frac{z}{1+z}\right)^2\right)$$
$$\left|\sum_{(m,n,k)=1}^{\infty}a_{ij\ell}^{mnk}S_{rst,p,q}\left(f_{000},(x,y,z)\right)-f_{000}\left(x,y,z\right)\right|$$

$$\leq r+\epsilon+\left(r+\epsilon+M+\frac{3M}{\delta^2}\left(C^2+D^2+E^2\right)\right)$$
$$\left|\sum_{(m,n,k)=1}^{\infty}a_{ij\ell}^{mnk}S_{rst,p,q}\left(f_{000},(x,y,z)\right)-f_{000}\left(x,y,z\right)\right|$$
$$+\frac{3M}{\delta^2}\left|\sum_{(m,n,k)=1}^{\infty}a_{ij\ell}^{mnk}S_{rst,p,q}\left(f_{333},(x,y,z)\right)-f_{333}\left(x,y,z\right)\right|+$$
$$\frac{3MC}{\delta^2}\left|\sum_{(m,n,k)=1}^{\infty}a_{ij\ell}^{mnk}S_{rst,p,q}\left(f_{111},(x,y,z)\right)-f_{111}\left(x,y,z\right)\right|+$$
$$\frac{3MD}{\delta^2}\left|\sum_{(m,n,k)=1}^{\infty}a_{ij\ell}^{mnk}S_{rst,p,q}\left(f_{222},(x,y,z)\right)-f_{222}\left(x,y,z\right)\right|+$$
$$\frac{3ME}{\delta^2}\left|\sum_{(m,n,k)=1}^{\infty}a_{ij\ell}^{mnk}S_{rst,p,q}\left(f_{333},(x,y,z)\right)-f_{333}\left(x,y,z\right)\right|,$$

where $C:=\max|x|\,,D:=\max|y|\,,E:=\max|z|$. So taking supremum over $(x,y,z)\in K$, we get

$$\left\|\sum_{(m,n,k)=1}^{\infty}a_{ij\ell}^{mnk}S_{rst,p,q}\left(f\right)-f\right\|\leq r+\epsilon+$$
$$B\sum_{(u,v,w)=0}^{3}\left\|\sum_{(m,n,k)=1}^{\infty}a_{ij\ell}^{mnk}S_{rst,p,q}\left(f_{uvw},(x,y,z)\right)-f_{uvw}\left(x,y,z\right)\right\|,$$

where

$$B:=\max\left\{r+\epsilon+M+\frac{3M}{\delta^2}\left(C^2+D^2+E^2\right),\frac{3M}{\delta^2},\frac{6MC}{\delta^2},\frac{6MD}{\delta^2},\frac{6ME}{\delta^2}\right\}.$$

Now for given $\rho>0$, choose $\epsilon>0$ such that $\epsilon<\rho$ and define

$$E:=\left\{(m,n,k)\in\mathbb{N}^3:\left\|\sum_{(m,n,k)=1}^{\infty}a_{ij\ell}^{mnk}S_{rst,p,q}\left(f,(x,y,z)\right)-f\left(x,y,z\right)\right\|\geq\rho\right\}$$

$$E_{uvw} := \left\{ (m,n,k) \in \mathbb{N}^3 : \left\| \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}\left(f_{uvw},(x,y,z)\right) - f_{uvw}\left(x,y,z\right) \right\| \geq \frac{\rho-\epsilon}{6B} \right\},\ u,v,w=0,1,2,3.$$

Then $E \subset \bigcup_{u,v,w=0}^{3} E_{uvw}$ and thus $\delta_2(E) \leq \sum_{(u,v,w)=0}^{3} \delta_2(E_{uvw})$. So by considering this inequality and using (6.2), we obtain (6.1). □

Example 6.4. Now, we will show that Theorem 6.3 is stronger than its classical and statistical forms. Let A be $C[1,1,1]$ and defined $x=(x_{mnk})$ by $x_{mnk}=(-1)^{mnk}$ for all m,n,k. Then triple sequence of Bernstein-Stancu polynomials is neither $P-$convergent nor $A-$rough statistically convergent but $st_3-\lim Ax=0$.

The Bernstein-Stancu operator of order (u,v,w) is given by

$$S_{uvw,p,q}\left(f,(x,y,z)\right) = \sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w} f\left(\frac{mnk}{uvw}\right)\binom{u}{m}\binom{v}{n}\binom{w}{k} x^{m+n+k}\left(1-x\right)^{(m-u)+(n-v)+(k-w)}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$, where $(x,y,z)\in K=[0,1]\times[0,1]\times[0,1]$; $f\in C_B(K)$. By using these operators, define the following positive linear operators on $C_B(K)$:

(6.4)

$$S_{rst,p,q}\left(f,(x,y,z)\right) = \left(1+x_{mnk}\right) S_{rst,p,q}\left(f,(x,y,z)\right), (x,y,z)\in K, f\in C(K).$$

Then observe that

$$\begin{aligned}
&S_{rst,p,q}\left(f_{000},(x,y,z)\right) = \left(1+x_{mnk}\right) f_{000}\left(x,y,z\right),\\
&S_{rst,p,q}\left(f_{111},(x,y,z)\right) = \left(1+x_{mnk}\right) f_{111}\left(x,y,z\right),\\
&S_{rst,p,q}\left(f_{222},(x,y,z)\right) = \left(1+x_{mnk}\right) f_{222}\left(x,y,z\right),\\
&S_{rst,p,q}\left(f_{333},(x,y,z)\right) = \left(1+x_{mnk}\right) f_{333}\left(x,y,z\right),\\
&S_{rst,p,q}\left(f_{444},(x,y,z)\right) = \left(1+x_{mnk}\right)\cdot\\
&\left(f_{444}\left(x,y,z\right) + \frac{\frac{x}{1+x}-\left(\frac{x}{1+x}\right)^2}{m} + \frac{\frac{y}{1+y}-\left(\frac{y}{1+y}\right)^2}{n} + \frac{\frac{z}{1+z}-\left(\frac{z}{1+z}\right)^2}{k}\right),
\end{aligned}$$

where

$$S_{rst,p,q}\left(f_{000},(x,y,z)\right)=1, B_{mnk}\left(f_{111},(x,y,z)\right)=\frac{x}{1+x},$$
$$S_{rst,p,q}\left(f_{222},(x,y,z)\right)=\frac{y}{1+y}, S_{rst,p,q}\left(f_{333},(x,y,z)\right)=\frac{z}{1+z}$$

and

$$S_{rst,p,q}\left(f_{444},(x,y,z)\right)=\left(\frac{x}{1+x}\right)^2+\left(\frac{y}{1+y}\right)^2+\left(\frac{z}{1+z}\right)^2.$$

Since $st_3 - \lim Ax = 0$, we obtain

$$st_3 - \lim_{pqj} \left\| \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}(f_{uvw}) - f_{uvw} \right\|_{C_B(K)} = 0$$

$$= st_3 - \lim_{pqj} \frac{1}{pqj} \left\| \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}(f_{uvw}) - f_{uvw} \right\|_{C_B(K)} = 0$$

for $u, v, w = 0, 1, 2, 3$. Hence by Theorem 6.3 we conclude that

$$st_3 - \lim_{pqj} \left\| \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}(f) - f \right\|_{C_B(K)} = 0$$

for any $f \in C_B(K)$.

However, since $P-$limit and the statistical limit of the triple sequence of Bernstein-Stancu polynomials of $(S_{rst,p,q}(f, x))$ is not zero, then for $u, v, w = 0, 1, 2, 3$, $\|S_{rst,p,q}(f_{uvw}) - f_{uvw}\|_{C_B(K)}$ is neither $P-$convergent nor statistically convergent to zero. So, Theorem 6.1 and Theorem 6.2 do not work for our operators defined by (6.4).

Theorem 6.5. *Let f be a continuous function defined on the closed interval $[0, 1]$. Let $A = (a_{i,j,\ell,m,n,k})$ be a nonnegative RH-regular summability matrix and a triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f, x))$ of real numbers from $[0, 1]$ into itself,*

$$(I_\lambda)_A^{g_{\bar{N}}} - \lim_{pqj} \frac{1}{g(\lambda_{pqj})} \frac{1}{u_{pqj}} \left\| \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}(f, x) - f(x) \right\| = 0 \tag{6.5}$$

if and only if

$$(I_\lambda)_A^{g_{\bar{N}}} - \lim_{pqj} \frac{1}{g(\lambda_{pqj})} \frac{1}{u_{pqj}} \left\| \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}(1, x) - 1 \right\| = 0 \tag{6.6}$$

$$(I_\lambda)_A^{g_{\bar{N}}} - \lim_{pqj} \frac{1}{g(\lambda_{pqj})} \frac{1}{u_{pqj}} \left\| \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}(v, x) - \frac{x}{1+x} \right\| = 0 \tag{6.7}$$

$$(I_\lambda)_A^{g_{\bar{N}}} - \lim_{pqj} \frac{1}{g(\lambda_{pqj})} \frac{1}{u_{pqj}} \left\| \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}(v^2, x) - \left(\frac{x}{1+x}\right)^2 \right\| = 0. \tag{6.8}$$

Proof. We obtain

$$E \subset E_1 \bigcup E_2 \bigcup E_3 \tag{6.9}$$

and so

$$\delta_{(I_\lambda)_A^{g_{\bar{N}}}}(E) \subset \delta_{(I_\lambda)_A^{g_{\bar{N}}}}(E_1) + \delta_{(I_\lambda)_A^{g_{\bar{N}}}}(E_2) + \delta_{(I_\lambda)_A^{g_{\bar{N}}}}(E_3). \tag{6.10}$$

Equations (6.6)-(6.8) give that

$$(I_\lambda)_A^{g_{\bar{N}}} - \lim_{pqj} \frac{1}{g(\lambda_{pqj})} \frac{1}{u_{pqj}} \left\| \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}(f,x) - f(x) \right\| = 0.$$

□

Definition 6.6. Let f be a continuous function defined on the closed interval $[0,1]$ and $A = (a_{i,j,\ell,m,n,k})$ be a nonnegative RH-regular matrix. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is said to be $I_\lambda-$weighted statistically convergent of weight g, if for every $\epsilon > 0$,

$$\lim_{pqj} \sum_{m,n,k \in E(u,\epsilon)} a_{i,j,\ell,m,n,k} = 0, \tag{6.11}$$

$$E(u,\epsilon) = \left\{ \lim_{pqj} \sum_{(m,n,k)=0}^{p,q,j} \frac{1}{g(\lambda_{pqj})} \frac{1}{u_{pqj}} \left| \{(m,n,k) \in I_{pqj} : u_{mnk} \left| S_{rst,p,q}(f,x) - f(x) \right| \geq r + \epsilon \} \right| = 0 \right\} \in I.$$

In this case, we write $(I_\lambda)_A^{g_{\bar{N}},u_{pqj}} - \lim S_{rst,p,q}(f,x) = f(x)$, or $S_{rst,p,q}(f,x) \to f(x) (I_\lambda)_A^{g_{\bar{N}},u_{pqj}}$.

Theorem 6.7. *Let f be a continuous function defined on the closed interval $[0,1]$. Let $A = (a_{i,j,\ell,m,n,k})$ be a nonnegative RH-regular summability matrix and a triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ of real numbers from $[0,1]$ which satisfies (6.7)-(6.8) of Theorem 6.5 and the following condition holds:*

$$\lim_{pqj} \frac{1}{g(\lambda_{pqj})} \frac{1}{u_{pqj}} \left\| \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}(1,x) - 1 \right\| = 0. \tag{6.12}$$

Then,

$$\lim_{pqj} \sum_{(m,n,k)=0}^{p,q,j} \frac{1}{g(\lambda_{pqj})} \frac{1}{u_{pqj}} u_{mnk} \left\| S_{rst,p,q}(f,x) - f(x) \right\| = 0. \tag{6.13}$$

Proof. It follows from (6.12) that $\left\| \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}(1,x) - 1 \right\| \leq C^{'}$, for some constant $C^{'} > 0$ and for all $m,n,k \in \mathbb{N}$. Then we obtain

$$\begin{aligned} u_{mnk} \left\| S_{rst,p,q}(f,x) - f(x) \right\| &\leq u_{mnk} \left(\|f\| \left\| S_{rst,p,q}(f,x) - f(x) \right\| + \|f\| \right) \\ &\leq u_{mnk} C \left(C^{'} + 1 \right). \end{aligned} \tag{6.14}$$

Since right hand side of (6.14) is constant, $u_{mnk} \|S_{rst,p,q}(f,x) - f(x)\|$ is bounded. Since (6.12) implies (6.6), by Theorem 6.5, we get that

$$(6.15) \qquad (I_\lambda)_A^{g_{\bar{N}}} - \lim_{pqj} \frac{1}{g(\lambda_{pqj})} \frac{1}{u_{pqj}} \left\| \sum_{(m,n,k)=1}^{\infty} a_{ij\ell}^{mnk} S_{rst,p,q}(f,x) - f(x) \right\| = 0.$$

□

Definition 6.8. Let f be a continuous function defined on the closed interval $[0,1]$ and $A = (a_{i,j,\ell,m,n,k})$ be a nonnegative RH-regular matrix and let (u_{pqj}) be a positive nonincreasing sequence. A triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$ is said to be $I_\lambda-$weighted statistically convergent of weight g, if for every $\epsilon > 0$,

$$(6.16) \qquad \lim_{pqj} \frac{1}{u_{pqj}} \sum_{m,n,k \in E(u,\epsilon)} a_{i,j,\ell,m,n,k} = 0,$$

where

$$E(u,\epsilon) = \left\{ \frac{1}{g(\lambda_{pqj})} \{(m,n,k) \in I_{pqj} : u_{mnk} |S_{rst,p,q}(f,x) - f(x)| \geq r + \epsilon\} \right\} \in I.$$

In this case, we write $(I_\lambda)_A^{g_{\bar{N}}} - o(a_{mnk}) - \lim S_{rst,p,q}(f,x) = f(x)$. Or $S_{rst,p,q}(f,x) \to f(x)\,(I_\lambda)_A^{g_{\bar{N}}} - o(a_{m,n,k})$ as $m,n,k \to \infty$.

Theorem 6.9. *Let f be a continuous function defined on the closed interval $[0,1]$. Let $A = (a_{i,j,\ell,m,n,k})$ be a nonnegative RH-regular summability matrix. Suppose that (a_{mnk}), (b_{mnk}) and (c_{mnk}) are three positive nonincreasing sequences. Let a triple sequence of Bernstein-Stancu polynomials $(S_{rst,p,q}(f,x))$, $(S_{rst,p,q}(f,y))$ and $(S_{rst,p,q}(f,z))$ of real numbers from $[0,1]$ such that*

$$S_{rst,p,q}(f,x) - f(x) = (I_\lambda)_A^{g_{\bar{N}}} - o(a_{m,n,k}),$$
$$S_{rst,p,q}(f,y) - f(y) = (I_\lambda)_A^{g_{\bar{N}}} - o(b_{m,n,k})$$

and

$$S_{rst,p,q}(f,z) - f(z) = (I_\lambda)_A^{g_{\bar{N}}} - o(c_{m,n,k}).$$

Then (1) $(S_{rst,p,q}(f,x) - f_1(x)) \pm (S_{rst,p,q}(f,y) - f_2(y)) \pm (S_{rst,p,q}(f,z) - f_3(y)) = (I_\lambda)_A^{g_{\bar{N}}} - o(d_{m,n,k})$,

(2) $(S_{rst,p,q}(f,x) - f_1(x))(S_{rst,p,q}(f,y) - f_2(y))(S_{rst,p,q}(f,z) - f_3(z)) = (I_\lambda)_A^{g_{\bar{N}}} - o(d_{m,n,k})$,

(3) $\alpha(S_{rst,p,q}(f,x) - f_1(x)) = (I_\lambda)_A^{g_{\bar{N}}} - o(a_{m,n,k})$, *for any scalar* α,

where $d_{mnk} = \max[a_{mnk}, b_{mnk}, c_{mnk}]$.

Proof. (1) Suppose that

$$(***) \qquad \begin{aligned} S_{rst,p,q}(f,x) - f_1(x) &= (I_\lambda)_A^{g_{\bar{N}}} - o(a_{m,n,k}), \\ S_{rst,p,q}(f,y) - f_2(y) &= (I_\lambda)_A^{g_{\bar{N}}} - o(b_{m,n,k}), \\ S_{rst,p,q}(f,z) - f_3(z) &= (I_\lambda)_A^{g_{\bar{N}}} - o(c_{m,n,k}). \end{aligned}$$

Given $\epsilon > 0$, define

$$E^{'} = \left\{ \frac{1}{g\left(\lambda_{pqj}\right)} \left\{ u_{mnk} \left| \left(S_{rst,p,q}\left(f,x\right) - f_1\left(x\right)\right) \pm \left(S_{rst,p,q}\left(f,y\right) - f_2\left(y\right)\right) \pm \left(S_{rst,p,q}\left(f,z\right) - f_3\left(z\right)\right) \right| \geq r + \epsilon \right\} \right\},$$

$$E^{''} = \left\{ \frac{1}{g\left(\lambda_{pqj}\right)} \left\{ (m,n,k) \in I_{pqj} : u_{mnk} \left| S_{rst,p,q}\left(f,x\right) - f_1\left(x\right) \right| \geq r + \frac{\epsilon}{2} \right\} \right\},$$
$$E^{'''} = \left\{ \frac{1}{g\left(\lambda_{pqj}\right)} \left\{ (m,n,k) \in I_{pqj} : u_{mnk} \left| S_{rst,p,q}\left(f,y\right) - f_2\left(y\right) \right| \geq r + \frac{\epsilon}{2} \right\} \right\},$$
$$E^{''''} = \left\{ \frac{1}{g\left(\lambda_{pqj}\right)} \left\{ (m,n,k) \in I_{pqj} : u_{mnk} \left| S_{rst,p,q}\left(f,z\right) - f_3\left(z\right) \right| \geq r + \frac{\epsilon}{2} \right\} \right\}.$$

It is easy to see that

$$E^{'} \subset E^{''} \bigcup E^{'''} \bigcup E^{''''}. \tag{6.17}$$

This yields that

$$\frac{1}{d_{pqj}} \sum_{m,n,k \in E'} a_{i,j,\ell,m,n,k} \leq \frac{1}{d_{pqj}} \sum_{m,n,k \in E''} a_{i,j,\ell,m,n,k} + \frac{1}{d_{pqj}} \sum_{m,n,k \in E'''} a_{i,j,\ell,m,n,k} + \frac{1}{d_{pqj}} \sum_{m,n,k \in E'''';} a_{i,j,\ell,m,n,k} \tag{6.18}$$

holds, for all $(p,q,j) \in \mathbb{N}$. Since $d_{mnk} = \max\left[a_{mnk}, b_{mnk}, c_{mnk}\right]$, (6.19) gives that

$$\frac{1}{d_{pqj}} \sum_{m,n,k \in E'} a_{i,j,\ell,m,n,k} \leq \frac{1}{a_{pqj}} \sum_{m,n,k \in E''} a_{i,j,\ell,m,n,k} + \frac{1}{b_{pqj}} \sum_{m,n,k \in E'''} a_{i,j,\ell,m,n,k} + \frac{1}{c_{pqj}} \sum_{m,n,k \in E''''} a_{i,j,\ell,m,n,k} \tag{6.19}$$

Then taking limit $p,q,j \to \infty$ in (6.19) together with (***), we obtain

$$\lim_{pqj} \frac{1}{d_{pqj}} \sum_{m,n,k \in E'} a_{i,j,\ell,m,n,k} = 0. \tag{6.20}$$

Thus,

$$\left(S_{rst,p,q}\left(f,x\right) - f_1\left(x\right)\right) \pm \left(S_{rst,p,q}\left(f,y\right) - f_2\left(y\right)\right) \pm \left(S_{rst,p,q}\left(f,z\right) - f_3\left(z\right)\right) = \left(I_\lambda\right)_A^{g_{\bar{N}}} - o\left(c_{m,n,k}\right).$$

Similarly, we can prove (2) and (3). Since continuity of f on the interval $[0,1]$, we write

$$w\left(f,\delta\right) = \sup\left\{ \left|f\left(x\right) - f\left(y\right)\right|, \left|f\left(y\right) - f\left(z\right)\right|, \left|f\left(z\right) - f\left(x\right)\right| : x,y,z \in [0,1], \left|x-y\right|, \left|y-z\right|, \left|z-x\right| < \delta \right\}.$$

It is well known that

$$\begin{aligned}&\left|f(x)-f(y)\right|,\left|f(y)-f(z)\right|,\left|f(z)-f(x)\right|\\&\leq w(f,\delta)\left(\frac{\left|\frac{x}{1+x}-\frac{y}{1+y}\right|}{\delta}+1,\frac{\left|\frac{y}{1+y}-\frac{z}{1+z}\right|}{\delta}+1,\frac{\left|\frac{z}{1+z}-\frac{x}{1+x}\right|}{\delta}+1\right).\end{aligned}$$

□

Theorem 6.10. *Let f be a continuous function defined on the closed interval $[0,1]$. Let $A=(a_{i,j,\ell,m,n,k})$ be a nonnegative RH-regular matrix. Let a triple sequence of Bernstein-Stancu polynomials $S_{rst,p,q}:[0,1]\to[0,1]$ satisifies the conditions*

(i) $(I_\lambda)_A^{g_{\bar{N}}}-\lim_{pqj}\frac{1}{g(\lambda_{pqj})}\frac{1}{u_{pqj}}\left\|\sum_{(m,n,k)=1}^{\infty}a_{ij\ell}^{mnk}S_{rst,p,q}(1,x)-1\right\|=(I_\lambda)_A^{g_{\bar{N}}}-o(a_{m,n,k})$,

(ii) $w(f,\lambda_{mnk})=(I_\lambda)_A^{g_{\bar{N}}}-o(b_{m,n,k})$,

(iii) $w(f,\lambda_{mnk})=(I_\lambda)_A^{g_{\bar{N}}}-o(c_{m,n,k})$ *with* $\lambda_{mnk}=\sqrt{S_{rst,p,q}(\varphi_x:x)}$ *and* $\varphi_x(y)=\left(\frac{y}{1+y}-\frac{x}{1+x}\right)^2$, $\varphi_x(z)=\left(\frac{z}{1+z}-\frac{y}{1+y}-\frac{x}{1+x}\right)^3$, *where (a_{mnk}), (b_{mnk}) and (c_{mnk}) are three positive nonincreasing sequences, then*

$$\|S_{rst,p,q}(f,x)-f(x)\|=(I_\lambda)_A^{g_{\bar{N}}}-o(d_{m,n,k}),\tag{6.21}$$

where $d_{mnk}=\max\{a_{mnk},b_{mnk},c_{mnk}\}$.

Proof. Consider

$$\begin{aligned}&\left|S_{rst,p,q}(f,x)-f(x)\right|\\
\leq&S_{rst,p,q}\left(\left|f(x)-f(y)\right|,\left|f(y)-f(z)\right|,\left|f(z)-f(x)\right|,x\right)+\left|f(x)\right|\left|S_{rst,p,q}(1,x)-1\right|\\
\leq&S_{rst,p,q}\left(1+\frac{\left|\frac{x}{1+x}\right|}{\delta},1+\frac{\left|\frac{y}{1+y}\right|}{\delta},1+\frac{\left|\frac{z}{1+z}\right|}{\delta}\right)w(f,\delta)+\left|f(x)\right|\left|S_{rst,p,q}(1,x)-1\right|\\
\leq&S_{rst,p,q}\left(1+\frac{\left|\left(\frac{x}{1+x}-\frac{y}{1+y}\right)^2\right|}{\delta^2},1+\frac{\left|\left(\frac{y}{1+y}-\frac{z}{1+z}\right)\right|}{\delta^2},1+\frac{\left|\left(\frac{z}{1+z}-\frac{x}{1+x}\right)^2\right|}{\delta^2}\right)w(f,\delta)\\
&+\left|f(x)\right|\left|S_{rst,p,q}(1,x)-1\right|\\
\leq&S_{rst,p,q}\left(1+\frac{\left|\left(\frac{x}{1+x}-\frac{y}{1+y}-\frac{z}{1+z}\right)^3\right|}{\delta^3},1+\frac{\left|\left(\frac{x}{1+x}-\frac{y}{1+y}-\frac{z}{1+z}\right)\right|}{\delta^3},\right.\\
&\left.1+\frac{\left|\left(\frac{x}{1+x}-\frac{y}{1+y}-\frac{z}{1+z}\right)^2\right|}{\delta^3}\right)w(f,\delta)+\left|f(x)\right|\left|S_{rst,p,q}(1,x)-1\right|\\
\leq&\left|S_{rst,p,q}(1,x)-1\right|w(f,\delta)+\left|f(x)\right|\left|S_{rst,p,q}(1,x)-1\right|+w(f,\delta)\\
&+\frac{1}{\delta^2}S_{rst,p,q}(\varphi_x:x)w(f,\delta)+w(f,\delta)+\frac{1}{\delta^3}S_{rst,p,q}(\varphi_x:x)w(f,\delta).\end{aligned}$$

Choose $\delta = \lambda_{mnk} = \sqrt{S_{rst,p,q}(\varphi_x : x)}$, then we obtain

$$\begin{aligned} \|S_{rst,p,q}(f,x) - f(x)\| \le T\, \|S_{rst,p,q}(1,x) - 1\| + 3w(f,\lambda_{mnk}) + \\ \|S_{rst,p,q}(1,x) - 1\| + w(f,\lambda_{mnk}), \end{aligned} \tag{6.22}$$

where $T = \|f\|$. For a given $\epsilon > 0$, we define the following sets:

$$\begin{aligned} E^{'} &= \left\{(m,n,k) \in \mathbb{N}^3 : u_{mnk}\, \|S_{rst,p,q}(f,x) - f(x)\| \ge r + \epsilon\right\}, \\ E^{''} &= \left\{(m,n,k) \in \mathbb{N}^3 : u_{mnk}\, \|S_{rst,p,q}(1,x) - 1\| \ge r + \frac{\epsilon}{4T}\right\}, \\ E^{'''} &= \left\{(m,n,k) \in \mathbb{N}^3 : u_{mnk} w(f,\lambda_{mnk}) \ge r + \frac{\epsilon}{9}\right\}, \\ E^{''''} &= \left\{(m,n,k) \in \mathbb{N}^3 : u_{mnk} w(f,\lambda_{mnk})\, \|S_{rst,p,q}(1,x) - 1\| \ge r + \frac{\epsilon}{4}\right\}. \end{aligned}$$

It follows from (6.22) that

$$\begin{aligned} \frac{1}{d_{pqj}} \sum_{m,n,k \in E'} a_{i,j,\ell,m,n,k} \le \frac{1}{d_{pqj}} \sum_{m,n,k \in E''} a_{i,j,\ell,m,n,k} + \\ \frac{1}{d_{pqj}} \sum_{m,n,k \in E'''} a_{i,j,\ell,m,n,k} + \frac{1}{d_{pqj}} \sum_{m,n,k \in E'''';} a_{i,j,\ell,m,n,k} \end{aligned} \tag{6.23}$$

holds for all $(p,q,j) \in \mathbb{N}$. Since $d_{mnk} = \max[a_{mnk}, b_{mnk}, c_{mnk}]$, (6.23) gives that

$$\begin{aligned} \frac{1}{d_{pqj}} \sum_{m,n,k \in E'} a_{i,j,\ell,m,n,k} \le \frac{1}{a_{pqj}} \sum_{m,n,k \in E''} a_{i,j,\ell,m,n,k} + \\ \frac{1}{b_{pqj}} \sum_{m,n,k \in E'''} a_{i,j,\ell,m,n,k} + \frac{1}{c_{pqj}} \sum_{m,n,k \in E''''} a_{i,j,\ell,m,n,k} \end{aligned} \tag{6.24}$$

Taking limit $p,q,j \to \infty$ in (6.24), we obtain

$$\lim_{pqj} \frac{1}{d_{pqj}} \sum_{m,n,k \in E'} a_{i,j,\ell,m,n,k} = 0. \tag{6.25}$$

Hence $\|B_{mnk}(f,x) - f(x)\| = (I_\lambda)_A^{g_{\bar{N}}} - o(d_{m,n,k})$. □

7. Conclusion

In this paper, we have discussed various general topological and algebraic properties of statistical convergent of weight $g(A)$ and also have proved Korovkin approximation theorem by using the notion of weight $g(A)$ limits of a triple sequence space of Bernstein-Stancu polynomials. All the results will certainly motivate young researchers.

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