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Common fixed point theorems for compatible and weakly compatible maps in fuzzy cone metric spaces

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Abstract. In this paper, we present some common fixed theorems for compatible and weakly compatible four self-mappings in fuzzy cone metric spaces in which h is a continuous self-map. We extend and improve some results given in the literature.

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1. Introduction

In 1965, the theory of fuzzy set was introduced by Zadeh [34]. While in [12], the fuzzy metric space was defined by Kramosil and Michalek. Later on, the stronger form of the fuzzy metric space was given by George and Veeramani [5]. Some more related results are studied for the theory of fixed point in fuzzy metric spaces (e.g, [6, 7, 11, 21, 22, 24, 29, 30]).

Initially, Jungck and Rhoads [10] introduced weakly compatible maps and proved some results in the context of metric space. While, in 2007, Som [33] generalized the results of [19, 20, 32] and proved common fixed point theorems for continuous self-mappings in fuzzy metric spaces. Some more compatible mapping results are studied in (see [3, 4, 9, 13, 14, 20, 23, 25, 26] the references are therein).

Recently, Oner et. al. in [18], defined the new concept of fuzzy cone metric space and proved some basic properties and a Banach contraction theorem with the assumption of Cauchy sequences. In [27], Rehman and Li generalize the result of Oner et. al. [18] and proved some fixed point theorems for single-valued maps in fuzzy cone metric spaces without the assumption of Cauchy sequences. Some more results are in (e.g, see [2, 15, 16, 17, 28]).

The aim of this research work is to obtain some common fixed point results for compatible and weakly compatible self-mappings satisfying the more generalize form of the fuzzy cone Banach contraction theorem in fuzzy cone metric spaces. We prove the generalize results for four self-mappings with a continuous self-map h, as well as without continuity of h with the condition of M_f triangular. The illustrative examples are also given in the paper.

2. Preliminaries

Definition 2.1 ([31])**.** A binary operation $*: [0,1]\times[0,1]\to[0,1]$ is called a continuous t-norm, if it satisfying the following conditions:
(i) $*$ is commutative, associative and continuous,
(ii) $1*a=a$ for all $a\in[0,1]$,
(iii) $a*b\le c*d$, whenever $a\le c$ and $b\le d$, for each $a,b,c,d\in[0,1]$.

The basic t-norms; i.e, minimum, product and Lukasiewicz continuous t-norms are defined as (see [31]):

$$a*b=\min\{a,b\},\quad a*b=ab,\quad \text{and}\quad a*b=\max\{a+b-1,0\}.$$

Definition 2.2 ([8])**.** A subset P of a real Banach space $\mathbf{E}$ is called a cone, provided that
(i) P is closed, nonempty and $P\neq\{\theta\}$,
(ii) if $a,b\in[0,\infty)$ and $x,y\in P$, then $ax+by\in P$,
(iii) if both $x\in P$ and $-x\in P$, then $x=\theta$.

For a given cone $P\subset\mathbf{E}$, a partial ordering $\preceq$ on $\mathbf{E}$ via P is defined by $x\preceq y$ if only if $y-x\in P$. $x\prec y$ stands for $x\preceq y$ and $x\neq y$, while $x\ll y$ stands for $y-x\in int(P)$. All the cones have nonempty interior.

Definition 2.3 ([18])**.** A three-tuple $(X,M_f,*)$ is said to be a fuzzy cone metric space, if P is a cone of $\mathbf{E}$, X is an arbitrary set, $*$ is a continuous t-norm and M_f is a fuzzy set on $X^2\times int(P)$ satisfying the conditions: for all $x,y,z\in X$ and $s,t\in int(P)$,
(i) $M_f(x,y,s)>0$ and $M_f(x,y,s)=1\Leftrightarrow x=y$,
(ii) $M_f(x,y,s)=M_f(y,x,s)$,
(iii) $M_f(x,y,s)*M_f(y,z,s)\le M_f(x,z,s+t)$,
(iv) $M_f(x,y,.): int(P)\to[0,1]$ is continuous.

Note that if $\mathbf{E}=\mathbb{R}$, $P=[0,\infty)$ and $a*b=ab$, then every fuzzy metric space becomes a fuzzy cone metric space. Throughout paper, $\mathbf{N}$ denotes a set of natural numbers.

Definition 2.4 ([18])**.** Let $(X,M_f,*)$ be a fuzzy cone metric space, $x\in X$ and (x_i) be a sequence in X. Then
(i) (x_i) is said to converge to x, if for $s\gg\theta$ and $r\in(0,1)$, there exists $i_1\in\mathbf{N}$ such that $M_f(x_i,x,s)>1-r$, for all $i\ge i_1$. We denote this by $\lim_{i\to\infty}x_i=x$ or $x_i\to x$ as $i\to\infty$.
(ii) (x_i) is said to be a Cauchy sequence, if for $r\in(0,1)$ and $s\gg\theta$, there exists $i_1\in\mathbf{N}$ such that $M_f(x_i,x_j,s)>1-r$, for all $i,j\ge i_1$.

(iii) $(X, M_f, *)$ is said to be complete, if every Cauchy sequence is convergent in X.

(iv) (x_i) is said to be fuzzy cone contractive, if there exists $\mu \in (0, 1)$ such that

$$\frac{1}{M_f(x_i, x_{i+1}, s)} - 1 \leq \mu \left(\frac{1}{M_f(x_{i-1}, x_i, s)} - 1 \right),$$

for all $s \gg \theta$, $i \geq 1$.

Definition 2.5 ([27])**.** Let $(X, M_f, *)$ be a fuzzy cone metric space. The fuzzy cone metric M_f is triangular, if

$$\frac{1}{M_f(x, z, s)} - 1 \leq \left(\frac{1}{M_f(x, y, s)} - 1 \right) + \left(\frac{1}{M_f(y, z, s)} - 1 \right),$$

for all $x, y, z \in X$ and each $s \gg \theta$.

Lemma 2.6 ([18])**.** *Let $x \in X$ in a fuzzy cone metric space $(X, M_f, *)$ and (x_i) be a sequence in X. Then (x_i) converges to x if and only if $M_f(x_i, x, s) \to 1$ as $i \to \infty$, for each $s \gg \theta$.*

Definition 2.7 ([18])**.** Let $(X, M_f, *)$ be a fuzzy cone metric space and $B : X \to X$. Then B is said to be fuzzy cone contractive, if there exists $\mu \in (0, 1)$ such that

$$\frac{1}{M_f(Bx, By, s)} - 1 \leq \mu \left(\frac{1}{M_f(x, y, s)} - 1 \right), \tag{2.1}$$

for each $x, y \in X$ and $s \gg \theta$. μ is called the contraction constant of B.

For more detail, we shall refer the readers to study [18, 27].

Definition 2.8. A pair of self-mappings (B, h) of a fuzzy cone metric space $(X, M_f, *)$ is said to be compatible, if $\lim_{i \to \infty} M_f(hBx_i, Bhx_i, s) = 1$, for $s \gg \theta$, whenever (x_i) is a sequence in X such that $\lim_{i \to \infty} hx_i = \lim_{i \to \infty} Bx_i = u$, for some $u \in X$.

Definition 2.9 ([1])**.** Let B and h be self-maps on a set X (i.e., $B, h : X \to X$). If $u = Bv = hv$, for some $v \in X$, then v is called a coincidence point of B and h, and u is called a point of coincidence of B and h. The self mappings B and h are said to be weakly compatible, if they commutes at their coincidence point, i.e. $Bv = hv$ for some $v \in X$, then $Bhv = hBv$.

Proposition 2.10 ([1])**.** *Let B and h be weakly compatible self-maps of a set X. If B and h have a unique point of coincidence $u = Bv = hv$, then u is the unique common fixed point of B and h.*

"A self-mapping B in a complete fuzzy cone metric space in which the contractive sequence are Cauchy and hold (2.1), then B has a unique fixed point in X" is known as a fuzzy cone Banach contraction theorem, which is obtained in ([18]).

3. Major section

In this section, we present some single-valued common fixed point theorems for compatible and weakly compatible mappings in fuzzy cone metric space $(X, M_f, *)$. Now we state and prove our first main result.

Theorem 3.1. *Suppose that $A, B, h, g : X \to X$ be four self-mappings and M is triangular in a complete fuzzy cone metric space $(X, M_f, *)$ satisfies,*

$$\begin{aligned}\frac{1}{M_f(Ax,By,s)}-1 \leq\ & a_1\left(\frac{1}{M_f(hx,gy,s)}-1\right)+a_2\left(\frac{1}{M_f(hx,Ax,s)}-1\right)\\ &+a_3\left(\frac{1}{M_f(gy,By,s)}-1\right)+a_4\left(\frac{1}{M_f(hx,By,s)}-1\right)\\ &+a_5\left(\frac{1}{M_f(gy,Ax,s)}-1\right)\end{aligned} \tag{3.1}$$

for all $x, y \in X$, $s \gg \theta$, where $a_1, a_2, a_3, a_4, a_5 \in [0, \infty)$ with $a_1+a_2+a_3+a_4+a_5 < 1$ and $a_2 = a_3$ or $a_4 = a_5$. If $A(X) \subseteq g(X)$, $B(X) \subseteq h(X)$ and h is continuous, (h, A) is compatible and (g, B) is weakly compatible. Then A, B, h, and g have a unique common fixed point in X.

Proof. Fix $x_0 \in X$ and by using the condition $A(X) \subseteq g(X)$, $B(X) \subseteq h(X)$, choose a sequence (x_i) in X such that

$$y_{2i+1} = gx_{2i+1} = Ax_{2i} \quad \text{and} \quad y_{2i+2} = hx_{2i+2} = Bx_{2i+1}, \quad \text{for all } i \geq 0.$$

Now, by (3.1), for $s \gg \theta$,

$$\begin{aligned}&\frac{1}{M_f(gx_{2i+1},hx_{2i+2},s)}-1=\frac{1}{M_f(Ax_{2i},Bx_{2i+1},s)}-1\\
&\leq a_1\left(\frac{1}{M_f(hx_{2i},gx_{2i+1},s)}-1\right)+a_2\left(\frac{1}{M_f(hx_{2i},Ax_{2i},s)}-1\right)\\
&\quad+a_3\left(\frac{1}{M_f(gx_{2i+1},Bx_{2i+1},s)}-1\right)+a_4\left(\frac{1}{M_f(hx_{2i},Bx_{2i+1},s)}-1\right)\\
&\quad+a_5\left(\frac{1}{M_f(gx_{2i+1},Ax_{2i},s)}-1\right)\\
&\leq a_1\left(\frac{1}{M_f(hx_{2i},gx_{2i+1},s)}-1\right)+a_2\left(\frac{1}{M_f(hx_{2i},gx_{2i+1},s)}-1\right)\\
&\quad+a_3\left(\frac{1}{M_f(gx_{2i+1},hx_{2i+2},s)}-1\right)+a_4\left(\frac{1}{M_f(hx_{2i},hx_{2i+2},s)}-1\right)\\
&\quad+a_5\left(\frac{1}{M_f(gx_{2i+1},gx_{2i+1},s)}-1\right)\\
&\leq a_1\left(\frac{1}{M_f(hx_{2i},gx_{2i+1},s)}-1\right)+a_2\left(\frac{1}{M_f(hx_{2i},gx_{2i+1},s)}-1\right)\\
&\quad+a_3\left(\frac{1}{M_f(gx_{2i+1},hx_{2i+2},s)}-1\right)\\
&\quad+a_4\left(\frac{1}{M_f(hx_{2i},gx_{2i+1},s)}-1+\frac{1}{M_f(gx_{2i+1},hx_{2i+2},s)}-1\right).\end{aligned}$$

After simplification, for $s \gg \theta$, we can get

(3.2)
$$\frac{1}{M_f(gx_{2i+1}, hx_{2i+2}, s)} - 1 \leq \lambda \left(\frac{1}{M_f(hx_{2i}, gx_{2i+1}, s)} - 1 \right), \quad \text{where } \lambda = \frac{a_1 + a_2 + a_4}{1 - (a_3 + a_4)}.$$

Similarly, again by (3.1), for $s \gg \theta$,

$$\begin{aligned}
\frac{1}{M_f(hx_{2i+2}, gx_{2i+3}, s)} - 1 &= \frac{1}{M_f(Ax_{2i+2}, Bx_{2i+1}, s)} - 1 \\
&\leq a_1 \left(\frac{1}{M_f(hx_{2i+1}, gx_{2i+2}, s)} - 1 \right) + a_2 \left(\frac{1}{M_f(hx_{2i+2}, Ax_{2i+2}, s)} - 1 \right) \\
&\quad + a_3 \left(\frac{1}{M_f(gx_{2i+1}, Bx_{2i+1}, s)} - 1 \right) + a_4 \left(\frac{1}{M_f(hx_{2i+2}, Bx_{2i+1}, s)} - 1 \right) \\
&\quad + a_5 \left(\frac{1}{M_f(gx_{2i+1}, Ax_{2i+2}, s)} - 1 \right) \\
&\leq a_1 \left(\frac{1}{M_f(hx_{2i+1}, gx_{2i+2}, s)} - 1 \right) + a_2 \left(\frac{1}{M_f(hx_{2i+2}, gx_{2i+3}, s)} - 1 \right) \\
&\quad + a_3 \left(\frac{1}{M_f(gx_{2i+1}, hx_{2i+2}, s)} - 1 \right) + a_4 \left(\frac{1}{M_f(hx_{2i+2}, hx_{2i+2}, s)} - 1 \right) \\
&\quad + a_5 \left(\frac{1}{M_f(gx_{2i+1}, gx_{2i+3}, s)} - 1 \right) \\
&\leq a_1 \left(\frac{1}{M_f(hx_{2i+1}, gx_{2i+2}, s)} - 1 \right) + a_2 \left(\frac{1}{M_f(hx_{2i+2}, gx_{2i+3}, s)} - 1 \right) \\
&\quad + a_3 \left(\frac{1}{M_f(gx_{2i+1}, hx_{2i+2}, s)} - 1 \right) \\
&\quad + a_5 \left(\frac{1}{M_f(gx_{2i+1}, hx_{2i+2}, s)} - 1 + \frac{1}{M_f(hx_{2i+2}, gx_{2i+3}, s)} - 1 \right).
\end{aligned}$$

After simplification, for $s \gg \theta$, we can get

(3.3)
$$\frac{1}{M_f(hx_{2i+2}, gx_{2i+3}, s)} - 1 \leq \mu \left(\frac{1}{M_f(gx_{2i+1}, hx_{2i+2}, s)} - 1 \right), \quad \text{where } \mu = \frac{a_1 + a_3 + a_5}{1 - (a_2 + a_5)}.$$

Now, by induction, from (3.2) and (3.3), we obtain that

$$\begin{aligned}
\frac{1}{M_f(gx_{2i+1}, hx_{2i+2}, s)} - 1 &\leq \lambda \left(\frac{1}{M_f(hx_{2i}, gx_{2i+1}, s)} - 1 \right) \\
&\leq \lambda\mu \left(\frac{1}{M_f(gx_{2i-1}, hx_{2i}, s)} - 1 \right) \\
&\leq \lambda\mu\lambda \left(\frac{1}{M_f(hx_{2i-2}, gx_{2i-1}, s)} - 1 \right) \\
&\leq \cdots \leq \lambda(\mu\lambda)^i \left(\frac{1}{M_f(hx_0, gx_1, s)} - 1 \right), \quad \text{for } s \gg \theta. \qquad (3.4)
\end{aligned}$$

And,

$$\begin{aligned}\frac{1}{M_f(hx_{2i+2}, gx_{2i+3}, s)} - 1 &\leq \mu\left(\frac{1}{M_f(gx_{2i+1}, hx_{2i+2}, s)} - 1\right)\\ &\leq \mu\lambda\left(\frac{1}{M_f(hx_{2i}, gx_{2i+1}, s)} - 1\right)\\ &\leq \cdots \leq (\mu\lambda)^{i+1}\left(\frac{1}{M_f(hx_0, gx_1, s)} - 1\right), \quad \text{for } s \gg \theta. \qquad (3.5)\end{aligned}$$

(Case i): If $a_2 = a_3$, then

$$\lambda\mu = \frac{a_1 + a_2 + a_4}{1 - (a_3 + a_4)} \cdot \frac{a_1 + a_3 + a_5}{1 - (a_2 + a_5)} = \frac{a_1 + a_2 + a_4}{1 - (a_2 + a_5)} \cdot \frac{a_1 + a_3 + a_5}{1 - (a_3 + a_4)} < 1 \cdot 1 = 1. \qquad (3.6)$$

(Case ii): If $a_4 = a_5$, then

$$\lambda\mu = \frac{a_1 + a_2 + a_4}{1 - (a_3 + a_4)} \cdot \frac{a_1 + a_3 + a_5}{1 - (a_2 + a_5)} < 1 \cdot 1 = 1. \qquad (3.7)$$

Since M_f is triangular, for $j > i \geq i_0$, we have

$$\begin{aligned}\frac{1}{M_f(y_{2i+1}, y_{2j+1}, s)} - 1 &\leq \left(\frac{1}{M_f(y_{2i+1}, y_{2i+2}, s)} - 1\right) + \cdots + \left(\frac{1}{M_f(y_{2m}, y_{2m+1}, s)} - 1\right)\\ &\leq \left(\lambda\sum_{k=i}^{j-1}(\lambda\mu)^k + \sum_{k=i+1}^{j}(\lambda\mu)^k\right)\left(\frac{1}{M_f(y_0, y_1, s)} - 1\right)\\ &\leq \left(\frac{\lambda(\lambda\mu)^i}{1-\lambda\mu} + \frac{(\lambda\mu)^{i+1}}{1-\lambda\mu}\right)\left(\frac{1}{M_f(y_0, y_1, s)} - 1\right)\\ &= (1+\mu)\frac{\lambda(\lambda\mu)^i}{1-\lambda\mu}\left(\frac{1}{M_f(y_0, y_1, s)} - 1\right), \quad \text{for } s \gg \theta.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{1}{M_f(y_{2i}, y_{2j+1}, s)} - 1 &\leq (1+\lambda)\frac{(\lambda\mu)^i}{1-\lambda\mu}\left(\frac{1}{M_f(y_0, y_1, s)} - 1\right), \quad \text{for } s \gg \theta,\\ \frac{1}{M_f(y_{2i}, y_{2j}, s)} - 1 &\leq (1+\lambda)\frac{(\lambda\mu)^i}{1-\lambda\mu}\left(\frac{1}{M_f(y_0, y_1, s)} - 1\right), \quad \text{for } s \gg \theta,\\ \frac{1}{M_f(y_{2i+1}, y_{2j}, s)} - 1 &\leq (1+\mu)\frac{\lambda(\lambda\mu)^i}{1-\lambda\mu}\left(\frac{1}{M_f(y_0, y_1, s)} - 1\right), \quad \text{for } s \gg \theta.\end{aligned}$$

Then for $j > i$,

$$\begin{aligned}\frac{1}{M_f(y_{2i+1}, y_{2j+1}, s)} - 1 &\leq \max\left\{(1+\lambda)\frac{(\lambda\mu)^i}{1-\lambda\mu}, (1+\mu)\frac{\lambda(\lambda\mu)^i}{1-\lambda\mu}\right\}\left(\frac{1}{M_f(y_0, y_1, s)} - 1\right)\\ &\to 0, \quad \text{as } i \to \infty.\end{aligned}$$

Which shows that a sequence $(y_i)_{i\geq 0}$ is a Cauchy sequence. Since, by the completeness of X, $\exists\ v \in X$ such that $y_i \to v$ as $i \to \infty$, for its the subsequence we obtain,

$$gx_{2i+1} \to v,\ hx_{2i+2} \to v,\ Ax_{2i} \to v \text{ and } Bx_{2i+1} \to v. \qquad (3.8)$$

Since h is a continuous self-mapping on X, such that

$$h(gx_{2i+1}) \to hv,\ h(hx_{2i+2}) \to hv,\ h(Ax_{2i}) \to hv \text{ and } h(Bx_{2i+1}) \to hv.$$

$h(Ax_{2i}) \to h(v)$ and (A, h) is compatible. Then, we have that

$$\lim_{i\to\infty} M_f(A(hx_{2i}), h(Ax_{2i}), s) = \lim_{i\to\infty} M_f(A(hx_{2i}), hv, s) = 1,\ \lim_{i\to\infty} M_f(h(Ax_{2i}), hv, s) = 1, \tag{3.9}$$

for $s \gg \theta$.

Now, we have to show that $hv = v$, by Definition 2.3 (iii),

$$M_f(hv, v, 2s) \geq M_f(hv, A(hx_{2i}), s) * M_f(A(hx_{2i}), v, s), \quad \text{for } s \gg \theta. \tag{3.10}$$

As a pair (A, h) is compatible, by definition of $*$, and from (3.8), (3.9) and (3.10), we have

$$M_f(hv, v, 2s) \geq \lim_{i\to\infty} (M_f(hv, A(hx_{2i}), s) * M_f(A(hx_{2i}), v, s)) = 1 * 1 = 1,$$

for $s \gg \theta$. Hence, $M_f(hv, v, 2s) = 1$, for $s \gg \theta$, and $hv = v$. Next, we shall show that $Av = v$, again by Definition 2.3 (iii),

$$M_f(Av, v, 2s) \geq M_f(Av, h(Ax_{2i}), s) * M_f(h(Ax_{2i}), v, s), \quad \text{for } s \gg \theta. \tag{3.11}$$

Now again by definition of $*$, and from (3.8), (3.9) and (3.11), we have

$$M_f(Av, v, 2s) \geq \lim_{i\to\infty} (M_f(Av, h(Ax_{2i}), s) * M_f(h(Ax_{2i}), v, s)) = 1 * 1 = 1,$$

for $s \gg \theta$. Then $M_f(Av, v, 2s) = 1$, for $s \gg \theta$ and $Av = v$. Thus $v = hv = Av$.

Now we shall show that $Bv = gv$. As $A(X) \subseteq g(X)$ and $\exists\ u \in X$ such that $v = Av = gu$, by view of (3.1), for $s \gg \theta$,

$$\begin{aligned}
\frac{1}{M_f(Bu, gu, s)} - 1 &= \frac{1}{M_f(Av, Bu, s)} - 1 \\
&\leq a_1\left(\frac{1}{M_f(hv, gu, s)} - 1\right) + a_2\left(\frac{1}{M_f(hv, Av, s)} - 1\right) \\
&\quad + a_3\left(\frac{1}{M_f(gu, Bu, s)} - 1\right) + a_4\left(\frac{1}{M_f(hv, Bu, s)} - 1\right) \\
&\quad + a_5\left(\frac{1}{M_f(gu, Av, s)} - 1\right) \\
&= a_1\left(\frac{1}{M_f(hv, v, s)} - 1\right) + a_2\left(\frac{1}{M_f(v, hv, s)} - 1\right) \\
&\quad + a_3\left(\frac{1}{M_f(gu, Bu, s)} - 1\right) + a_4\left(\frac{1}{M_f(gu, Bu, s)} - 1\right) \\
&\quad + a_5\left(\frac{1}{M_f(gu, gu, s)} - 1\right) \\
&= (a_3 + a_4)\left(\frac{1}{M_f(gu, Bu, s)} - 1\right).
\end{aligned}$$

Notice that $(a_3 + a_4) < 1$ since $(a_1 + a_2 + a_3 + a_4 + a_5) < 1$, then $M_f(gu, Bu, s) = 1$, that is, $Bu = gu = v$ and by the weak compatibility of B and g, implies that

$$gv = g(Bu) = B(gu) = Bv.$$

Now we shall show that $Bv = v$, then by view of (3.1), for $s \gg \theta$,

$$\begin{aligned}
\frac{1}{M_f(Bv,v,s)} - 1 &= \frac{1}{M_f(Bv,Av,s)} - 1 \\
&\leq a_1\left(\frac{1}{M_f(hv,gv,s)} - 1\right) + a_2\left(\frac{1}{M_f(hv,Av,s)} - 1\right) \\
&\quad + a_3\left(\frac{1}{M_f(gv,Bv,s)} - 1\right) + a_4\left(\frac{1}{M_f(hv,Bv,s)} - 1\right) \\
&\quad + a_5\left(\frac{1}{M_f(gv,Av,s)} - 1\right) \\
&= a_1\left(\frac{1}{M_f(v,Bv,s)} - 1\right) + a_2\left(\frac{1}{M_f(hv,hv,s)} - 1\right) \\
&\quad + a_3\left(\frac{1}{M_f(gv,gv,s)} - 1\right) + a_4\left(\frac{1}{M_f(v,Bv,s)} - 1\right) \\
&\quad + a_5\left(\frac{1}{M_f(Bv,v,s)} - 1\right) \\
&= (a_1 + a_4 + a_5)\left(\frac{1}{M_f(v,Bv,s)} - 1\right).
\end{aligned}$$

Notice that $(a_1 + a_4 + a_5) < 1$ since $(a_1 + a_2 + a_3 + a_4 + a_5) < 1$, $M_f(v, Bv, s) = 1$, that is, $v = Bv$, which further implies that $gv = v$. Hence, $hv = gv = Av = Bv = v$, proved that v is the common fixed point of the four self-mappings A, B, g and h in X.

Uniqueness: Let there is another point $z \in X$ such that $hz = gz = Az = Bz = z$. Then by (3.1), for $s \gg \theta$,

$$\begin{aligned}
\frac{1}{M_f(z,v,s)} - 1 &= \frac{1}{M_f(Az,Bv,s)} - 1 \\
&\leq a_1\left(\frac{1}{M_f(hz,gv,s)} - 1\right) + a_2\left(\frac{1}{M_f(hz,Az,s)} - 1\right) \\
&\quad + a_3\left(\frac{1}{M_f(gv,Bv,s)} - 1\right) + a_4\left(\frac{1}{M_f(hz,Bv,s)} - 1\right) \\
&\quad + a_5\left(\frac{1}{M_f(gv,Az,s)} - 1\right) \\
&= (a_1 + a_4 + a_5)\left(\frac{1}{M_f(z,v,s)} - 1\right), \quad \text{for } s \gg \theta.
\end{aligned}$$

Since $(a_1 + a_4 + a_5) < 1$, it follows that $M_f(z, v, s) = 1$, that is, $z = v$. Thus we proved that the common fixed point of A, B, h, and g is unique. □

Example 3.2. Let $X = [0, 1]$, $*$ is a continuous t-norm and $M_f : X^2 \times (0, \infty) \to [0, 1]$ be defined as

$$M_f(x, y, s) = s/(s + |x - y|),$$

$\forall\, x, y \in X$ and $s > 0$. Then easily one can verify that M_f is triangular and $(X, M_f, *)$ is a complete fuzzy cone metric space. The mappings $A, B, h, g : X \to X$ can be defined as: for each $x \in X$,

$$Ax = Bx = \begin{cases} \frac{1}{2}\left(\frac{2x}{3} + \frac{1}{4}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

And

$$gx = hx = \begin{cases} \frac{2x}{3} + \frac{1}{4}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Since $A(X) = B(X)$ and $g(X) = h(X)$, we have $A(X) \subseteq g(X)$ or $B(X) \subseteq h(X)$. Then from (3.1), we have that

$$\begin{aligned} \frac{1}{M_f(Ax, By, s)} - 1 = \frac{|Ax - By|}{s} &= \frac{|x - y|}{3s} \\ &\leq a_1 \cdot \left(\frac{1}{M_f(hx, gy, s)} - 1\right) + a_2 \cdot \left(\frac{1}{M_f(hx, Ax, s)} - 1\right) \\ &\quad + a_3 \cdot \left(\frac{1}{M_f(gy, By, s)} - 1\right) + a_4 \cdot \left(\frac{1}{M_f(hx, By, s)} - 1\right) \\ &\quad + a_5 \cdot \left(\frac{1}{M_f(gy, Ax, s)} - 1\right), \end{aligned}$$

for all $x, y \in X$ and $s \gg \theta$. Thus all the condition of Theorem 3.1 is satisfied with $a_1 = \frac{1}{2}$, $a_2 = a_3 = \frac{1}{6}$ and $a_4 = a_5 = 0$, and the unique common fixed point of the mappings A, B, h and g in X is 0.

Corollary 3.3. *Suppose that $A, B, h, g : X \to X$ be four self-mappings and M_f is triangular in a complete fuzzy cone metric space $(X, M_f, *)$ satisfies,*

$$\begin{aligned} \frac{1}{M_f(Ax, By, s)} - 1 \leq a_1 &\left(\frac{1}{M_f(hx, gy, s)} - 1\right) + a_2 \left(\frac{1}{M_f(hx, Ax, s)} - 1\right) \\ &+ a_3 \left(\frac{1}{M_f(gy, By, s)} - 1\right), \end{aligned} \tag{3.12}$$

for all $x, y \in X$, $s \gg \theta$, where $a_1, a_2, a_3 \in [0, \infty)$ with $a_1 + a_2 + a_3 < 1$. If $A(X) \subseteq g(X)$, $B(X) \subseteq h(X)$ and h is continuous, (h, A) is compatible and (g, B) is weakly compatible. Then A, B, h, and g have a unique common fixed point in X.

Corollary 3.4. *Suppose that $A, B, h, g : X \to X$ be four self-mappings and M is triangular in a complete fuzzy cone metric space $(X, M_f, *)$ satisfies,*

$$\begin{aligned} \frac{1}{M_f(Ax, By, s)} - 1 \leq a_1 &\left(\frac{1}{M_f(hx, gy, s)} - 1\right) + a_2 \left(\frac{1}{M_f(hx, By, s)} - 1\right) \\ &+ a_3 \left(\frac{1}{M_f(gy, Ax, s)} - 1\right), \end{aligned} \tag{3.13}$$

for all $x, y \in X$, $s \gg \theta$, where $a_1, a_2, a_3 \in [0, \infty)$ with $a_1 + 2(a_2 + a_3) < 1$. If $A(X) \subseteq g(X)$, $B(X) \subseteq h(X)$ and h is continuous, (h, A) is compatible and (g, B) is weakly compatible. Then A, B, h, and g have a unique common fixed point in X.

Corollary 3.5. *Suppose that $A, B, h, g : X \to X$ be four self-mappings and M is triangular in a complete fuzzy cone metric space $(X, M_f, *)$ satisfies,*

$$\frac{1}{M_f(Ax, By, s)} - 1 \leq a\left(\frac{1}{M_f(hx, gy, s)} - 1\right), \tag{3.14}$$

for all $x, y \in X$, $s \gg \theta$ and $a \in [0, 1)$. If $A(X) \subseteq g(X)$, $B(X) \subseteq h(X)$ and h is continuous, (h, A) is compatible and (g, B) is weakly compatible. Then A, B, h, and g have a unique common fixed point in X.

Example 3.6. As from Example 3.2, the mappings $A, B, h, g : X \to X$ can be defined as

$$Ax = x/(x+6),\ By = y/(y+10),\ hx = x/3 \quad \text{and} \quad gy = y/5,$$

for every $x, y \in X$. Then from (3.14), we have that

$$\begin{aligned}\frac{1}{M_f(Ax, By, s)} - 1 &= \left|\frac{Ax - By}{s}\right| = \frac{1}{s}\left|\frac{x}{x+6} - \frac{y}{y+10}\right| \\ &= \frac{1}{s}\left|\frac{10x - 6y}{(x+6)(y+10)}\right| \\ &\leq \frac{1}{s}\left|\frac{10x - 6y}{60}\right| \\ &= \frac{1}{2s}\left|\frac{x}{3} - \frac{y}{5}\right| = \frac{1}{2}\left(\frac{1}{M_f(hx, gy, s)} - 1\right).\end{aligned}$$

Thus all the condition of Corollary 3.5 is satisfied with $a = 1/2$ and A, B, h, g have a unique common fixed point 0 in X.

If we choose $A = B$ and $h = g$, then directly we can get the Corollary 3.7.

Corollary 3.7. *Suppose that $B, h : X \to X$ be two self-maps and M_f is triangular in a complete fuzzy cone metric space $(X, M_f, *)$ satisfies,*

$$\begin{aligned}\frac{1}{M_f(Bx, By, s)} - 1 \leq\ & a_1\left(\frac{1}{M_f(hx, hy, s)} - 1\right) + a_2\left(\frac{1}{M_f(hx, Bx, s)} - 1\right) \\ & + a_3\left(\frac{1}{M_f(hy, By, s)} - 1\right) + a_4\left(\frac{1}{M_f(hx, By, s)} - 1\right) \\ & + a_5\left(\frac{1}{M_f(hy, Bx, s)} - 1\right),\end{aligned} \tag{3.15}$$

for all $x, y \in X$, $s \gg \theta$, where $a_1, a_2, a_3, a_4, a_5 \in [0, \infty)$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, and $a_2 = a_3$ or $a_4 = a_5$. If $B(X) \subseteq h(X)$, h is continuous and (B, h) is weakly compatible. Then B and h have a unique common fixed point in X.

In the following Theorem 3.8, we need not the continuity of h whereas the completeness of X is replaced with the completeness of $B(X)$ or $h(X)$.

Theorem 3.8. *Suppose that $B, h : X \to X$ be two self-maps and M_f is triangular in a fuzzy cone metric space $(X, M_f, *)$ satisfies,*

$$\begin{aligned} \frac{1}{M_f(Bx,By,s)} - 1 \leq\ & a_1\left(\frac{1}{M_f(hx,hy,s)} - 1\right) + a_2\left(\frac{1}{M_f(hx,Bx,s)} - 1\right) \\ & + a_3\left(\frac{1}{M_f(hy,By,s)} - 1\right) + a_4\left(\frac{1}{M_f(hx,By,s)} - 1\right) \\ & + a_5\left(\frac{1}{M_f(hy,Bx,s)} - 1\right), \end{aligned} \tag{3.16}$$

for all $x, y \in X$, $s \gg \theta$, where $a_1, a_2, a_3, a_4, a_5 \in [0, \infty)$ with $a_1+a_2+a_3+a_4+a_5 < 1$, and $a_2 = a_3$ or $a_4 = a_5$. If $B(X) \subseteq h(X)$, $B(X)$ or $h(X)$ is complete and (B, h) is weakly compatible. Then B and h have a unique common fixed point in X.

Proof. As same as, in the proof of Theorem 3.1, we construct a Cauchy sequence (y_i) in $h(X)$ such that

$$y_{2i+1} = hx_{2i+1} = Bx_{2i} \quad \text{and} \quad y_{2i+2} = hx_{2i+2} = Bx_{2i+1}, \quad \text{for } i \geq 0.$$

Since $h(X)$ is complete, and $\exists\, u, v \in X$ such that $y_{2i+1} \to u = hv$ as $i \to \infty$,

$$\lim_{i\to\infty} M_f(y_{2i+1}, u, s) = \lim_{i\to\infty} M_f(hx_{2i+1}, u, s) = 1, \quad \text{for } s \gg \theta. \tag{3.17}$$

Since M_f is triangular,

$$\frac{1}{M_f(hv,Bv,s)} - 1 \leq \left(\frac{1}{M_f(hv,y_{2i+2},s)} - 1\right) + \left(\frac{1}{M_f(y_{2i+2},Bv,s)} - 1\right), \quad \text{for } s \gg \theta. \tag{3.18}$$

Then by view of (3.16) and (3.17), for $s \gg \theta$,

$$\begin{aligned} \frac{1}{M_f(y_{2i+2},Bv,s)} - 1 &= \frac{1}{M_f(Bx_{2i+1},Bv,s)} - 1 \\ &\leq a_1\left(\frac{1}{M_f(hx_{2i+1},hv,s)} - 1\right) + a_2\left(\frac{1}{M_f(hx_{2i+1},Bx_{2i+1},s)} - 1\right) \\ &\quad + a_3\left(\frac{1}{M_f(hv,Bv,s)} - 1\right) + a_4\left(\frac{1}{M_f(hx_{2i+1},Bv,s)} - 1\right) \\ &\quad + a_5\left(\frac{1}{M_f(hv,Bx_{2i+1},s)} - 1\right) \\ &= a_1\left(\frac{1}{M_f(hx_{2i+1},hv,s)} - 1\right) + a_2\left(\frac{1}{M_f(hx_{2i+1},hx_{2i+2},s)} - 1\right) \\ &\quad + a_3\left(\frac{1}{M_f(hv,Bv,s)} - 1\right) + a_4\left(\frac{1}{M_f(hx_{2i+1},Bv,s)} - 1\right) \\ &\quad + a_5\left(\frac{1}{M_f(hv,hx_{2i+2},s)} - 1\right) \\ &\to (a_3+a_4)\left(\frac{1}{M_f(hv,Bv,s)} - 1\right), \quad \text{as } i \to \infty. \end{aligned}$$

Thus

$$\limsup_{i\to\infty}\left(\frac{1}{M_f(y_{2i+2},Bv,s)}-1\right)\leq(a_3+a_4)\left(\frac{1}{M_f(hv,Bv,s)}-1\right),\quad \text{as } i\to\infty.$$

Now, from (3.17) and (3.18), we can get

$$\frac{1}{M_f(hv,Bv,s)}-1\leq(a_3+a_4)\left(\frac{1}{M_f(hv,Bv,s)}-1\right),\quad \text{for } s\gg\theta. \tag{3.19}$$

Since $a_3+a_4<1$, $M_f(hv,Bv,s)=1$. So $u=hv=Bv$. By the weakly compatibility of (B,h), we have

$$Bu=B(hv)=h(Bv)=hu.$$

Hence by view of (3.16), for $s\gg\theta$,

$$\begin{aligned}
&\frac{1}{M_f(Bu,u,s)}-1=\frac{1}{M_f(Bu,Bv,s)}-1\\
&\leq a_1\left(\frac{1}{M_f(hu,hv,s)}-1\right)+a_2\left(\frac{1}{M_f(hu,Bu,s)}-1\right)\\
&\quad+a_3\left(\frac{1}{M_f(hv,Bv,s)}-1\right)+a_4\left(\frac{1}{M_f(hu,Bv,s)}-1\right)\\
&\quad+a_5\left(\frac{1}{M_f(hv,Bu,s)}-1\right)\\
&=a_1\left(\frac{1}{M_f(Bu,u,s)}-1\right)+a_2\left(\frac{1}{M_f(hu,hu,s)}-1\right)\\
&\quad+a_3\left(\frac{1}{M_f(hv,hv,s)}-1\right)+a_4\left(\frac{1}{M_f(Bu,u,s)}-1\right)\\
&\quad+a_5\left(\frac{1}{M_f(u,Bu,s)}-1\right)\\
&=(a_1+a_4+a_5)\left(\frac{1}{M_f(Bu,u,s)}\right).
\end{aligned}$$

Since $a_1+a_4+a_5<1$, $M_f(Bu,u,s)=1$, for $s\gg\theta$. Thus $u=Bu=hu$. So u is the common fixed point of B and h.

Uniqueness: Let there is another point $z \in X$ such that $z = Bz = hz$. Then by (3.16), for $s \gg \theta$,

$$\begin{aligned}
\frac{1}{M_f(z,v,s)} - 1 &= \frac{1}{M_f(Bz,Bv,s)} - 1 \\
&\leq a_1\left(\frac{1}{M_f(hz,hv,s)} - 1\right) + a_2\left(\frac{1}{M_f(hz,Bz,s)} - 1\right) \\
&\quad + a_3\left(\frac{1}{M_f(hv,Bv,s)} - 1\right) + a_4\left(\frac{1}{M_f(hz,Bv,s)} - 1\right) \\
&\quad + a_5\left(\frac{1}{M_f(hv,Bz,s)} - 1\right) \\
&= (a_1 + a_4 + a_5)\left(\frac{1}{M_f(z,v,s)} - 1\right), \quad \text{for } s \gg \theta.
\end{aligned}$$

Since $(a_1 + a_4 + a_5) < 1$, it follows that $M_f(z,v,s) = 1$, that is, $z = v$. Thus we proved that the common fixed point of B and h is unique. □

Corollary 3.9. *Suppose that $B, h : X \to X$ be two self-mappings and M_f is triangular in a fuzzy cone metric space $(X, M_f, *)$, $B(X) \subseteq h(X)$ and satisfies that*

$$\begin{aligned}
\frac{1}{M_f(B^ix,B^iy,s)} - 1 \leq{}& a_1\left(\frac{1}{M_f(hx,hy,s)} - 1\right) + a_2\left(\frac{1}{M_f(hx,B^ix,s)} - 1\right) \\
&+ a_3\left(\frac{1}{M_f(hy,B^iy,s)} - 1\right) + a_4\left(\frac{1}{M_f(hx,B^iy,s)} - 1\right) \\
&+ a_5\left(\frac{1}{M_f(hy,B^ix,s)} - 1\right),
\end{aligned} \tag{3.20}$$

for all $x, y \in X$, $s \gg \theta$, where $a_1, a_2, a_3, a_4, a_5 \in [0, \infty)$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, and $a_2 = a_3$ or $a_4 = a_5$. Then B and h have a unique common fixed point in X, if $B(h) = h(B)$ and holds one of the following:

(C:1) *X is complete and h is continuous,*
(C:2) *$h(X)$ is complete,*
(C:3) *$B(X)$ is complete.*

Proof. By Corollary 3.7 and Theorem 3.8, we obtain $u \in X$ such that

$$hu = B^iu = u. \tag{3.21}$$

Then, from (3.20), for $s \gg \theta$,

$$\begin{aligned}
&\frac{1}{M_f(Bu,u,s)} - 1 = \frac{1}{M_f(B(B^i u), B^i u, s)} - 1 = \frac{1}{M_f(B^i(Bu), B^i u, s)} - 1 \\
&\leq a_1\left(\frac{1}{M_f(h(Bu), hu, s)} - 1\right) + a_2\left(\frac{1}{M_f(h(Bu), B^i(Bu), s)} - 1\right) \\
&\quad + a_3\left(\frac{1}{M_f(hu, B^i u, s)} - 1\right) + a_4\left(\frac{1}{M_f(h(Bu), B^i u, s)} - 1\right) \\
&\quad + a_5\left(\frac{1}{M_f(hu, B^i(Bu), s)} - 1\right) \\
&= a_1\left(\frac{1}{M_f(B(hu), hu, s)} - 1\right) + a_2\left(\frac{1}{M_f(B(hu), B(B^i u), s)} - 1\right) \\
&\quad + a_3\left(\frac{1}{M_f(hu, u, s)} - 1\right) + a_4\left(\frac{1}{M_f(B(hu), u, s)} - 1\right) \\
&\quad + a_5\left(\frac{1}{M_f(hu, B(B^i u), s)} - 1\right) \\
&= a_1\left(\frac{1}{M_f(Bu, u, s)} - 1\right) + a_2\left(\frac{1}{M_f(Bu, Bu, s)} - 1\right) \\
&\quad + a_3\left(\frac{1}{M_f(hu, u, s)} - 1\right) + a_4\left(\frac{1}{M_f(Bu, u, s)} - 1\right) \\
&\quad + a_5\left(\frac{1}{M_f(u, Bu, s)} - 1\right) \\
&= (a_1 + a_4 + a_5)\left(\frac{1}{M_f(Bu, u, s)} - 1\right).
\end{aligned}$$

Since $a_1 + a_4 + a_5 < 1$, $M_f(Bu, u, s) = 1$, for $s \gg \theta$. Thus $u = Bu = hu$. So u is the common fixed point of B and h.

Uniqueness: Let there is another point $z \in X$ such that $z = Bz = hz$ and $z = B^i z = hz$ as in (3.21). Then by (3.20), for $s \gg \theta$,

$$\begin{aligned}
\frac{1}{M_f(z,v,s)} - 1 &= \frac{1}{M_f(B^i z, B^i v, s)} - 1 \\
&\leq a_1\left(\frac{1}{M_f(hz, hv, s)} - 1\right) + a_2\left(\frac{1}{M_f(hz, B^i z, s)} - 1\right) \\
&\quad + a_3\left(\frac{1}{M_f(hv, B^i v, s)} - 1\right) + a_4\left(\frac{1}{M_f(hz, B^i v, s)} - 1\right) \\
&\quad + a_5\left(\frac{1}{M_f(hv, B^i z, s)} - 1\right) \\
&= (a_1 + a_4 + a_5)\left(\frac{1}{M_f(z, v, s)} - 1\right), \quad \text{for } s \gg \theta.
\end{aligned}$$

Since $(a_1 + a_4 + a_5) < 1$, it follows that $M_f(z, v, s) = 1$, that is, $z = v$. Thus we proved that the common fixed point of B and h is unique. □

Next, we prove a new-type of fuzzy cone contraction theorem in fuzzy cone metric spaces.

Theorem 3.10. *Suppose that* $B, h : X \to X$ *be two self-maps and* M_f *is triangular in a complete fuzzy cone metric space* $(X, M_f, *)$ *satisfies,*

$$\frac{1}{M_f(Bx,By,s)} - 1 \leq a_1\left(\frac{1}{M_f(hx,hy,s)} - 1\right) + a_2\left(\frac{1}{M_f(hx,Bx,s)} - 1\right) + a_3\left(\frac{1}{M_f(hy,By,s) * M_f(hy,Bx,s)} - 1\right), \tag{3.22}$$

for all $x, y \in X$, $s \gg \theta$, *and* $a_1, a_2, a_3 \in [0, \infty)$ *with* $a_1 + a_2 + a_3 < 1$*. If* $B(X) \subseteq h(X)$*, and a pair* (B, h) *is weakly compatible. Then* B *and* h *have a unique common fixed point in* X*.*

Proof. Fix $x_0 \in X$ and by condition $B(X) \subseteq h(X)$, choose a sequence (x_i) in X such that

$$y_{i+1} = hx_{i+1} = Bx_i, \quad \text{for all } i \geq 0.$$

Now, by (3.22), for $s \gg \theta$,

$$\begin{aligned}\frac{1}{M_f(hx_i,hx_{i+1},s)} - 1 &= \frac{1}{M_f(Bx_{i-1},Bx_i,s)} - 1\\ &\leq a_1\left(\frac{1}{M_f(hx_{i-1},hx_i,s)} - 1\right) + a_2\left(\frac{1}{M_f(hx_{i-1},Bx_{i-1},s)} - 1\right)\\ &+ a_3\left(\frac{1}{M_f(hx_i,Bx_i,s) * M_f(hx_i,Bx_{i-1},s)} - 1\right)\\ &\leq a_1\left(\frac{1}{M_f(hx_{i-1},hx_i,s)} - 1\right) + a_2\left(\frac{1}{M_f(hx_{i-1},hx_i,s)} - 1\right)\\ &+ a_3\left(\frac{1}{M_f(hx_i,hx_{i+1},s) * 1} - 1\right),\end{aligned}$$

After simplification, we can get

$$\frac{1}{M_f(hx_i,hx_{i+1},s)} - 1 \leq \mu\left(\frac{1}{M_f(hx_{i-1},hx_i,s)} - 1\right), \quad \text{where } \mu = \frac{a_1 + a_2}{1 - a_3} < 1.$$

Continuing the same process, for $s \gg \theta$, we can get the following

$$\frac{1}{M_f(hx_i,hx_{i+1},s)} - 1 \leq \mu\left(\frac{1}{M_f(hx_{i-1},hx_i,s)} - 1\right) \leq \cdots \leq \mu^i\left(\frac{1}{M_f(hx_0,hx_1,s)} - 1\right),$$

which shows that (hx_i) is a fuzzy cone contractive sequence. Thus

$$\lim_{i\to\infty} M_f(hx_i,hx_{i+1},s) = 1, \quad \text{for } s \gg \theta. \tag{3.23}$$

Since M_f is triangular, for $j > i \geq i_0$,

$$\begin{aligned}\frac{1}{M(hx_i, hx_j, s)} - 1 &\leq \left(\frac{1}{M(hx_i, hx_{i-1}, s)} - 1\right) + \cdots + \left(\frac{1}{M(hx_{j-1}, hx_j, s)} - 1\right)\\ &\leq \left(\mu^i + \mu^{i+1} + \cdots + \mu^{j-1}\right)\left(\frac{1}{M(hx_0, hx_1, s)} - 1\right)\\ &\leq \frac{\mu^i}{1-\mu}\left(\frac{1}{M(hx_0, hx_1, s)} - 1\right)\\ &\to 0, \quad \text{as } i \to \infty,\end{aligned}$$

which shows that (hx_i) is a Cauchy sequence. Since by the completeness of X, $\exists\, u, v \in X$ such that $y_i = hx_i \to u = hv$ as $i \to \infty$,

$$\lim_{i\to\infty} M_f(hx_i, u, s) = 1, \quad \text{for } s \gg \theta. \tag{3.24}$$

Since M_f is triangular,

$$\frac{1}{M_f(hv, Bv, s)} - 1 \leq \left(\frac{1}{M_f(hv, hx_{i+1}, s)} - 1\right) + \left(\frac{1}{M_f(hx_{i+1}, Bv, s)} - 1\right), \quad \text{for } s \gg \theta. \tag{3.25}$$

Then, by view of (3.22), (3.23) and (3.24),

$$\begin{aligned}&\frac{1}{M_f(hx_{i+1}, Bv, s)} - 1 = \frac{1}{M_f(Bx_i, Bv, s)} - 1\\ &\leq a_1\left(\frac{1}{M_f(hx_i, hv, s)} - 1\right) + a_2\left(\frac{1}{M_f(hx_i, Bx_i, s)} - 1\right)\\ &\quad + a_3\left(\frac{1}{M_f(hv, hx_i, s) * M(hv, Bv, s)} - 1\right)\\ &= a_1\left(\frac{1}{M_f(hx_i, hv, s)} - 1\right) + a_2\left(\frac{1}{M_f(hx_i, hx_{i+1}, s)} - 1\right)\\ &\quad + a_3\left(\frac{1}{M_f(hv, hx_i, s) * M(hv, Bv, s)} - 1\right)\\ &\to a_3\left(\frac{1}{M(hv, Bv, s)} - 1\right), \quad \text{as } i \to \infty.\end{aligned}$$

Thus

$$\limsup_{i\to\infty}\left(\frac{1}{M_f(hx_{i+1}, Bv, s)} - 1\right) \leq a_3\left(\frac{1}{M(hv, Bv, s)} - 1\right), \quad \text{for } s \gg \theta.$$

Now, this together with (3.24) and (3.25),

$$\frac{1}{M(hv, Bv, s)} - 1 \leq a_3\left(\frac{1}{M(hv, Bv, s)} - 1\right), \quad \text{for } s \gg \theta.$$

Noticing that $a_3 < 1$, since $a_1 + a_2 + a_3 < 1$, $M(hv, Bv, s) = 1$, for $s \gg \theta$. So $u = hv = Bv$. Now, by the weak compatibility of (B, h), we have

$$Bu = B(hv) = h(Bv) = hu.$$

Then, by view of (3.22), for $s \gg \theta$,

$$\begin{aligned}
&\frac{1}{M_f(Bu,u,s)} - 1 = \frac{1}{M_f(Bu,Bv,s)} - 1 \\
&\leq a_1\left(\frac{1}{M_f(hu,hv,s)} - 1\right) + a_2\left(\frac{1}{M_f(hu,Bu,s)} - 1\right) \\
&\quad + a_3\left(\frac{1}{M_f(hv,Bv,s) * M_f(hv,Bu,s)} - 1\right) \\
&= (a_1 + a_3)\left(\frac{1}{M_f(Bu,u,s)} - 1\right).
\end{aligned}$$

Since $a_1 + a_3 < 1$, $M_f(Bu,u,s) = 1$, for $s \gg \theta$. Thus $u = Bu = hu$. So u is the common fixed point of B and h.

Uniqueness: Let there is another point $z \in X$ such that $z = Bz = hz$. Then by (3.22) for $s \gg \theta$,

$$\begin{aligned}
&\frac{1}{M_f(z,v,s)} - 1 = \frac{1}{M_f(Bz,Bv,s)} - 1 \\
&\leq a_1\left(\frac{1}{M_f(hz,hv,s)} - 1\right) + a_2\left(\frac{1}{M_f(hz,Bz,s)} - 1\right) \\
&\quad + a_3\left(\frac{1}{M(hv,Bv,s) * M(hv,Bz,s)}\right) \\
&= (a_1 + a_3)\left(\frac{1}{M_f(z,z,s)} - 1\right), \quad \text{for } s \gg \theta,
\end{aligned}$$

Since $(a_1 + a_3) < 1$, it follows that $M_f(z,v,s) = 1$, that is, $z = v$. Thus we proved that the common fixed point of B and h is unique. □

Corollary 3.11. *Suppose that $B, h : X \to X$ be two self-maps and M_f is triangular in a complete fuzzy cone metric space $(X, M_f, *)$ satisfies,*

(3.26)

$$\frac{1}{M_f(Bx,By,s)} - 1 \leq a_1\left(\frac{1}{M_f(hx,hy,s)} - 1\right) + a_2\left(\frac{1}{M_f(hy,By,s) * M_f(hy,Bx,s)} - 1\right),$$

for all $x, y \in X$, $s \gg \theta$, and $a_1, a_2 \in [0, \infty)$ with $a_1 + a_2 < 1$. If $B(X) \subseteq h(X)$, and a pair (B, h) is weakly compatible. Then B and h have a unique common fixed point in X.

4. Conclusions

We used the concept of compatible and weakly compatible self-mappings in fuzzy cone metric spaces and proved some generalized common fixed point theorems for four self-mappings in fuzzy cone metric spaces. We proved different contractive type results for self-mappings with the continuity of a self-map h, that is, Theorem 3.1 and without continuity for a pair of weakly compatible self-mappings are Theorem 3.8 and Theorem 3.10 which generalized and extended the results of [18, 27, 28]. So, one can study this concept for a family of mappings in fuzzy cone metric spaces for different contractive type mappings to improve and extend many results.

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References

[1] M. Abbas and G. Jungck, Common fixed point results for non-commuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl. 341 (2008) 416–420.
[2] A. M. Ali and G. R. Kanna, Intuitionistic fuzzy cone metric spaces and fixed point theorems, Int. J. Math. Appl. 5 (2017) 25–36.
[3] N. Cakic, Z. Kadelburg, S. Radenovic and A. Razani, Common fixed point results in cone metric spaces for a family of weakly compatible maps, Advances and Applications in Mathematical Sciences 1 (2009) 183–207.
[4] L. Ciric, A. Razani, S. Radenovic and J. S. Ume, Common fixed point theorems for families of weakly compatible maps, Computers and Mathematics with Applications, 55 (2008) 2533–2543.
[5] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets Systems 64 (1994) 395–399.
[6] M. Grabiec, Fixed point in fuzzy metric spaces, Fuzzy Sets Systems 27 (1988) 385–389.
[7] O. Hadzic and E. Pap, Fixed point theorem for multivalued mappings in probabilistic metric spaces and an applications in fuzzy metric spaces, Fuzzy Sets Systems 127 (2002) 333–344.
[8] L. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007) 1468–1476.
[9] M. Imdad and J. Ali, Some Common Fixed Point Theorems in Fuzzy Metric Spaces, Mathematical Communications 11 (2006) 153–163.
[10] G. Jungck and B. E. Rhoades, Fixed points for setvalued functions without continuity, Indian J. Pure Appl. Math. 29 (1998) 227–238.
[11] F. Kiany and A. Amini-Harandi, Fixed point and endpoint theorems for set-valued fuzzy contraction maps in fuzzy metric spaces, Fixed Point Theo. Appl. 94 (2011) 1–9.
[12] O. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetika 11 (1975) 336–344.
[13] S. Kutukcu, D. Turkoglu and C. Yildiz, Common fixed points of compatible maps of type (β) on fuzzy metric spaces, Communications of the Korean Society 21 (2006) 89–100.
[14] P. P. Murthy, S. Kumar and K. Tas, Common fixed points of self maps satisfying an integral type contractive condition in fuzzy metric spaces, Math. Comm. 15 (2010) 521–537.
[15] T. Oner, Some topological properties of fuzzy cone metric spaces, J. Nonlinear Sci. Appl. 9 (2016) 799–805.
[16] T. Oner, On some results in fuzzy cone metric spaces, Int. J. Adv. Comput. Eng. Network 4 (2016) 37–39.
[17] T. Oner, On the metrizability of fuzzy cone metric spaces, Int. J. of Manag. Appl. Sci. 2 (2016) 133–135.
[18] T. Oner, M. B. Kandemire and B. Tanay, Fuzzy cone metric spaces, J. Nonlinear Sci. Appl. 8 (2015) 610–616.
[19] R. P. Pant, Common fixed points of noncommuting mappings, J. Math. Anal. Appl. 188 (1994) 436–440.
[20] B. D. Pant and S. Chauhan, Common fixed point theorems for two pairs of weakly compatible mappings in menger spaces and fuzzy metric spaces, Scientific Studies and Research. Series Mathematics and Informatics 21 (2011) 81–96.
[21] A. Razani, A contraction theorem in fuzzy metric spaces, Fixed Point Theory and Applications 3 (2005) 257–265.
[22] A. Razani, Existence of fixed point for the nonexpansive mapping of intuitionistic fuzzy metric spaces, Chaos, Solitons and Fractals 30 (2006) 367–373.

[23] A. Razani and M. Samanipour, Common fixed point theorems for families of weakly compatible maps in a 2-metric space, Applied Mathematics and Information Sciences 2 (2008) 275–89.
[24] A. Razani and M. Shirdaryazdi, Some results on fixed points in the fuzzy metric space, J. Appl. Math. Comput. 20 (2006) 401–408.
[25] A. Razani and M. Shirdaryazdi, A common fixed point theorem of compatible maps in Menger space, Chaos, Solitons and Fractals 37 (2007) 26–34.
[26] A. Razani and M. Yazdi, Two common fixed point theorems for compatible mappings, Int. J. Nonlinear Anal. Appl. 2 (2011) 7–18.
[27] S. U. Rehman and H. X. Li, Fixed point theorems in fuzzy cone metric spaces, J. Nonlinear Sci. Appl. 10 (2017) 5763–5769.
[28] S. U. Rehman, Y. Li, S. Jabeen and T. Mahmood, Common fixed point theorems for a pair of self-mappings in fuzzy cone metric spaces, Abstract and Applied analysis 2019 10 pages.
[29] R. Saadati, A. Razani and H. Adibi, A common fixed point theorem in L-fuzzy metric spaces, Chaos, Solitons and Fractals 33 (2007) 358–363.
[30] Z. Sadeghi, S. M. Vaezpour, C. Park, R. Saadati and C. Vetro, Set-valued mappings in partially ordered fuzzy metric spaces, J. Inequal. Appl. 157 (2014) 1–17.
[31] B. Schweizer and A. Sklar, Statical metric spaces, Pac. J. Math. 10 (1960) 314–334.
[32] T. Som, Some fixed point theorems on metric and Banach spaces, J. Pure Appl. Math. 16 (1985) 575–585.
[33] T. Som, Some results on common fixed point in fuzzy metric spaces, Sooc. J. Math. 33 (2007) 553–561.
[34] L. A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.

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