Annals of Fuzzy Mathematics and Informatics
Volume 19, No. 1, (February 2020) pp. 21–43
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2020.19.1.21

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 19, No. 1, February 2020

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Received 31 March 2019; Revised 27 April 2019; Accepted 4 June 2019

Abstract. Various relations of fuzzy types has been applied to decision makings, graph theories, congruence problems in algebras and quotient spaces in topological spaces, etc. Then in this paper, we deal with cubic relations as one of various fuzzy type's relations. First, we define a cubic point and obtained some of its properties. Second, we introduce the notions of cubic reflexive [resp. symmetric and transitive] relations, and we define the composition of two cubic relations and the inverse of a cubic relation. And investigate some of each properties and give some examples. Third, we define the concepts of cubic equivalence relations, cubic equivalence classes and cubic partitions, and obtain some of each properties, respectively. Finally, we define a $<\widetilde{a},\alpha>$-level relation of a cubic relation and study some of its properties.

2010 AMS Classification: 03E72

Keywords: Cubic set, Cubic point, Cubic relation, Cubic equivalence relation, Cubic partition, Cubic quotient set, $<\widetilde{a},\alpha>$-level relation.

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1. Introduction

In 1971, Zadeh [29] studied a fuzzy relation by using the concept of fuzzy sets introduced by himself [28] as the generalization of an ordinary set. After that time, many researchers [3, 4, 5, 6, 8, 10, 20] investigated variously fuzzy relations.

In 2012, Jun et al. [14] introduced the concept of a cubic set. After then, Kang and Kim [19] defined a mapping of cubic set and studied some of its properties. Jun et al. [13] studied cubic subgroups. Zhan et al. [21] investigated H_v-LA-semigroups by using cubic sets. Jun and Khan [12] studied cubic ideals in semigroups. Senapati et al. [26] investigated cubic subalgebras and ideals of B-algebras (Refer to [1, 2]). Jun and Lee [15] studied cubic ideals in BCK/BCI-algebras (See additionally [16, 18]). Yaqoob et al. [27] investigated cubic KU-ideals of KU-algebras. Chinnadurai et al. [7] dealt with some characterizations of cubic sets. Zeb et al. [31] defined a cubic topology and investigated some of its properties. Rashid et al. [22] dealt with decision-making problems by using cubic sets. Smarandache et al. [18] extended the concept of cubic sets to neutrosophic sets and studied some of its properties.

Various relations of fuzzy types has been applied to decision makings, graph theories, congruence problems in algebras and quotient spaces in topological spaces, etc. Then in this paper, we deal with cubic relations as one of various fuzzy type's relations. First, we define a cubic point and obtained some of its properties. Second, we introduce the concepts of cubic reflexive [resp. symmetric and transitive] relations, and we define the composition of two cubic relations and the inverse of a cubic relation. And we study some of each properties and give some examples. Third, we define the notions of cubic equivalence relations, cubic equivalence classes and cubic partitions, and obtain some of each properties, respectively (In particular, see Corollary 5.13 and Proposition 5.14). Finally, we define a $<\widetilde{a},\alpha>$-level relation of a cubic relation and investigate some of its properties.

2. Preliminaries

In this section, we list some basic definitions needed in the next sections (See [9, 11, 14, 23, 24, 25, 28, 29, 30]). Throughout this paper, I denotes the closed unit interval $[0,1]$.

Definition 2.1 ([28])**.** Let X be a nonempty set. Then a a mapping $\lambda : X \to I$ is called a fuzzy set in X. The collection of all fuzzy sets in X is denoted by I^X. In particular, 0 and 1 denote the fuzzy empty set and the fuzzy whole set in X, respectively.

Definition 2.2 ([28])**.** Let $\lambda,\ \mu \in I^X$ and let $(\lambda_j)_{j\in J}$ be any family of fuzzy sets in X. Then the inclusion of λ and μ, denoted by $\lambda \leq \mu$, the union and the intersection of λ and μ, denoted by $\lambda \wedge \mu$ and $\lambda \vee \mu$, the union and the intersection of $(\lambda_j)_{j\in J}$, denoted by $\bigwedge_{j\in J}\lambda_j$ and $\bigvee_{j\in J}\lambda_j$, and the complement of λ, denoted by λ^c are defined as follows, respectively: for each $x \in X$,

(i) $\lambda \leq \mu \Longleftrightarrow \lambda(x) \leq \mu(x)$,
(ii) $(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x)$, $(\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x)$,
(iii) $(\bigwedge_{j\in J}\lambda_j)(x) = \bigwedge_{j\in J}\lambda_j(x)$, $(\bigvee_{j\in J}\lambda_j)(x) = \bigvee_{j\in J}\lambda_j(x)$,
(iv) $\lambda^c(x) = 1 - \lambda(x)$.

Definition 2.3 ([23])**.** $\lambda \in I^X$ is called a fuzzy point with the support $x \in X$ and the value $\alpha \in I$ with $\alpha > 0$, denoted by $\lambda = x_\alpha$, if for each $y \in X$,

$$x_\alpha = \begin{cases} \alpha & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$$

The set of all fuzzy points in X is denoted by $F_P(X)$.

For each $x_\alpha \in F_P(X)$ and $\lambda \in I^X$, x_α is said to belong to λ, denoted by $x_\alpha \in \lambda$, if $\alpha \leq \lambda(x)$.

It is clear that $\lambda = \bigcup_{x_\alpha\in\lambda} x_\alpha$, for each $\lambda \in I^X$.

Definition 2.4 ([29])**.** Let X, Y be any sets. Then $\lambda \in I^{X\times Y}$ is called a fuzzy relation from X to Y. In particular, $\lambda \in I^{X\times X}$ is called a fuzzy relation on X.

Definition 2.5 ([29]). Let $\lambda,\ \mu \in I^{X\times X}$. Then the composition of λ and μ, denoted by $\mu\circ\lambda$ and the inverse f λ, denoted by λ^{-1}, are defined as follows, respectively: for each $(x,y)\in X\times X$,

(i) $(\mu\circ\lambda)(x,y)=\bigvee_{z\in X}[\lambda(x,z)\wedge\mu(z,y)]$,

(ii) $\lambda^{-1}(x,y)=\lambda(y,x)$.

Definition 2.6 ([29]). $\lambda\in I^{X\times X}$ is called a fuzzy equivalence relation on X, if

(i) it is reflexive, i.e., $\lambda(x,x)=1$ for each $x\in X$,

(ii) it is symmetric, i.e., $\lambda^{-1}=\lambda$,

(iii) it is transitive, i.e., $\lambda\circ\lambda\leq\lambda$.

We denote the set of all fuzzy equivalence relations on X as $FR_E(X)$.

For any $\lambda\in FR_E(X)$ and for any $a\in X$, the fuzzy equivalence class of a by λ, denoted by λ_a, is a fuzzy set in X defined as follows: for each $x\in X$,

$$\lambda_a(x)=\lambda(a,x).$$

The set of all closed subintervals of I is denoted by $[I]$ (See [14]), and members of $[I]$ are called interval numbers and are denoted by $\widetilde{a},\ \widetilde{b},\ \widetilde{c}$, etc., where $\widetilde{a}=[a^-,a^+]$ and $0\leq a^-\leq a^+\leq 1$. In particular, if $a^-=a^+$, then we write as $\widetilde{a}=\mathbf{a}$.

We define relations $\succeq$, $\preceq$ and $=$ on $[I]$ as follows:

$$(\forall\ \widetilde{a},\ \widetilde{b}\in[I])(\widetilde{a}\succeq\widetilde{b}\Longleftrightarrow a^-\geq b^- \text{ and } a^+\geq b^+),$$
$$(\forall\ \widetilde{a},\ \widetilde{b}\in[I])(\widetilde{a}\preceq\widetilde{b}\Longleftrightarrow a^-\leq b^- \text{ and } a^+\leq b^+),$$
$$(\forall\ \widetilde{a},\ \widetilde{b}\in[I])(\widetilde{a}=\widetilde{b}\Longleftrightarrow \widetilde{a}\succeq\widetilde{b} \text{ and } \widetilde{a}\preceq\widetilde{b}), i.e.,$$
$$(\forall\ \widetilde{a},\ \widetilde{b}\in[I])(\widetilde{a}=\widetilde{b}\Longleftrightarrow a^-=b^- \text{ and } a^+=b^+).$$

To say $\widetilde{a}\succ\widetilde{b}$ (resp. $\widetilde{a}\prec\widetilde{b}$), we mean $\widetilde{a}\succeq\widetilde{b}$ and $\widetilde{a}\neq\widetilde{b}$ (resp. $\widetilde{a}\preceq\widetilde{b}$ and $\widetilde{a}\neq\widetilde{b}$).

For any $\widetilde{a},\ \widetilde{b}\in[I]$, their minimum and maximum, denoted by $\widetilde{a}\breve{\wedge}\widetilde{b}$ and $\widetilde{a}\breve{\vee}\widetilde{b}$ are defined as follows:

$$\widetilde{a}\breve{\wedge}\widetilde{b}=[a^-\wedge b^-,a^+\wedge b^+],$$
$$\widetilde{a}\breve{\vee}\widetilde{b}=[a^-\vee b^-,a^+\vee b^+].$$

Let $(\widetilde{a_j})_{j\in J}\subset[I]$. Then its inf and sup, denoted by $\breve{\bigwedge}_{j\in J}\widetilde{a_j}$ and $\breve{\bigvee}_{j\in J}\widetilde{a_j}$, are defined as follows:

$$\breve{\bigwedge}_{j\in J}\widetilde{a_j}=[\bigwedge_{j\in J}a_j^-,\bigwedge_{j\in J}a_j^+],$$
$$\breve{\bigvee}_{j\in J}\widetilde{a_j}=[\bigvee_{j\in J}a_j^-,\bigvee_{j\in J}a_j^+].$$

For each $\widetilde{a}\in[I]$, its complement, denoted by $\widetilde{a}^c$, is defined as follows:

$$\widetilde{a}^c=[1-a^+,1-a^-].$$

Definition 2.7 ([9, 24, 30]). Let X be a nonempty set. Then a mapping $A:X\to[I]$ is called an interval-valued fuzzy set (briefly, an IVF set) in X. Let $[I]^X$ denote the set of all IVF sets in X. For each $A\in[I]^X$ and $x\in X$, $A(x)=[A^-(x),A^+(x)]$ is called the degree of membership of an element x to A, where $A^-,A^+\in I^X$ are called a lower fuzzy set and an upper fuzzy set in X, respectively. For each $A\in[I]^X$, we write $A=[A^-,A^+]$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy whole set in X, respectively. We define relations $\subset$ and $=$ on $[I]^X$ as follows:

$$(\forall\ A,\ B \in [I]^X)(A \subset B \Longleftrightarrow (x \in X)(A(x) \lesssim B(x)),$$

$$(\forall\ A,\ B \in [I]^X)(A = B \Longleftrightarrow (x \in X)(A(x) = B(x)).$$

Definition 2.8 ([9, 24, 30])**.** Let X be a nonempty set, let $A \in [I]^X$ and let $(A_j)_{j\in J}$ be any subfamily of $[I]^X$. Then the complement of A, denoted by A^c, and the intersection and the union of $(A_j)_{j\in J}$, denoted by $\breve{\bigcap}_{j\in J}A_j$ and $\breve{\bigcup}_{j\in J}A_j$ are defined as follows, respectively: for each $x \in X$,

$$A^c(x) = [1 - A^+(x), 1 - A^-(x)],$$

$$(\breve{\bigcap}_{j\in J}A_j)(x) = \breve{\bigwedge}_{j\in J}A_j(x),$$

$$(\breve{\bigcup}_{j\in J}A_j)(x) = \breve{\bigvee}_{j\in J}A_j(x).$$

Definition 2.9 ([24])**.** $A \in [I]^X$ is called an interval-valued fuzzy point (briefly, an IVF point) with the support $x \in X$ and the value $\widetilde{a} \in [I]$ with $a^+ > 0$, denoted by $A = x_{\widetilde{a}}$, if for each $y \in X$,

$$x_{\widetilde{a}} = \begin{cases} \widetilde{a} & \text{if } y = x \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

The set of all IVF points in X is denoted by $IVF_P(X)$.

For each $x_{\widetilde{a}} \in IVF_P(X)$ and $A \in [I]^X$, $x_{\widetilde{a}}$ is said to belong to A, denoted by $x_{\widetilde{a}} \in A$, if $\widetilde{a} \lesssim A(x)$.

It is clear that $A = \bigcup_{x_{\widetilde{a}}\in A} x_{\widetilde{a}}$, for each $A \in [I]^X$.

Definition 2.10 ([25])**.** For two sets X, Y, $R \in [I]^{X\times Y}$ is called an interval-valued fuzzy relation (briefly, IVF relation) from X to Y. In particular, $R \in [I]^{X\times X}$ is called an IVF relation on X.

Definition 2.11 ([25])**.** For each $R \in [I]^{X\times X}$, the inverse of R, denoted by R^{-1}, defined as follows: $(x, y) \in X \times X$,

$$R^{-1}(x, y) = R(y, x).$$

Definition 2.12 ([11])**.** For any $R,\ S \in [I]^{X\times X}$, the composition of R and S, denoted by $S \circ R$, is defined as follows: for each $(x, y) \in X \times X$,

$$(S \circ R)(x, y) = \breve{\bigvee}_{z\in X}[R(x, z)\breve{\wedge}S(z, y)].$$

Definition 2.13 ([11])**.** $R \in [I]^{X\times X}$ is called an interval-valued fuzzy equivalence relation on X, if

(i) it is reflexive, i.e., $R(x, x) = \mathbf{1}$ for each $x \in X$,
(ii) it is symmetric, i.e., $R^{-1} = R$,
(iii) it is transitive, i.e., $R \circ R \subset R$.

We denote the set of all fuzzy equivalence relations on X as $IVR_E(X)$.

For any $R \in IVR_E(X)$ and for any $a \in X$, the fuzzy equivalence class of a by R, denoted by R_a, is an interval-valued fuzzy set in X defined as follows: for each $x \in X$,

$$R_a(x) = R(a, x).$$

Definition 2.14 ([14])**.** Let X be a nonempty set. Then a complex mapping $\mathcal{A} =< A, \lambda >: X \to [I] \times I$ is called a cubic set in X.

A cubic set $\mathcal{A} =< A, \lambda >$ in which $A(x) = \mathbf{0}$ and $\lambda(x) = 1$ (resp. $A(x) = \mathbf{1}$ and $\lambda(x) = 0$) for each $x \in X$ is denoted by $\ddot{0}$ (resp. $\ddot{1}$).

A cubic set $\mathcal{B} =< B, \mu >$ in which $B(x) = \mathbf{0}$ and $\mu(x) = 0$ (resp. $B(x) = \mathbf{1}$ and $\mu(x) = 1$) for each $x \in X$ is denoted by $\hat{0}$ (resp. $\hat{1}$). In this case, $\hat{0}$ (resp. $\hat{1}$) will be called a cubic empty (resp. whole) set in X.

We denote the set of all cubic sets in X as $([I] \times I)^X$.

Definition 2.15 ([14])**.** Let X be a nonempty set and let $\mathcal{A} =< A, \lambda >\in ([I] \times I)^X$. Then $\mathcal{A}$ is called :

(i) an internal cubic set (briefly, ICS) in X, if $A^-(x) \leq \lambda(x) \leq A^+(x)$ for each $x \in X$,

(ii) an external cubic set (briefly, ECS) in X, if $\lambda(x) \notin (A^-(x), A^+(x))$ for each $x \in X$.

3. Cubic points

In this section, we obtain further properties of operations on cubic sets. Next, we define a cubic point and study some of its properties.

Definition 3.1 ([14])**.** Let $\mathcal{A} =< A, \lambda >$, $\mathcal{B} =< B, \mu >\in ([I] \times I)^X$. Then we define the following relations:

(i) (Equality) $\mathcal{A} = \mathcal{B} \Leftrightarrow A = B$ and $\lambda = \mu$,

(ii) (P-order) $\mathcal{A} \sqsubset \mathcal{B} \Leftrightarrow A \subset B$ and $\lambda \leq \mu$,

(iii) (R-order) $\mathcal{A} \Subset \mathcal{B} \Leftrightarrow A \subset B$ and $\lambda \geq \mu$

Definition 3.2 ([14])**.** Let $\mathcal{A} =< A, \lambda >$, $\mathcal{B} =< B, \mu >\in ([I] \times I)^X$ and let $(\mathcal{A}_j)_{j \in J} = (< A_j, \lambda_j >)_{j \in J} \subset ([I] \times I)^X$. Then the complement $\mathcal{A}^c$ of $\mathcal{A}$, P-union $\sqcup$, P-intersection $\sqcap$, R-union $\Cup$ and R-intersection $\Cap$ are defined as follows, respectively: for each $x \in X$,

(i) (Complement) $\mathcal{A}^c(x) =< A^c(x), \lambda^c(x) >$,

(ii) (P-union) $(\mathcal{A} \sqcup \mathcal{B})(x) =< (A \breve{\cup} B)(x), (\lambda \vee \mu)(x) >$,

$$(\sqcup_{j \in J} \mathcal{A}_j)(x) =< (\breve{\bigcup}_{j \in J} A_j)(x), (\bigvee_{j \in J} \lambda_j)(x) >,$$

(iii) (P-intersection) $(\mathcal{A} \sqcap \mathcal{B})(x) =< (A \breve{\cap} B)(x), (\lambda \wedge \mu)(x) >$,

$$(\sqcap_{j \in J} \mathcal{A}_j)(x) =< (\breve{\bigcap}_{j \in J} A_j)(x), (\bigwedge_{j \in J} \lambda_j)(x) >,$$

(iv) (R-union) $(\mathcal{A} \Cup \mathcal{B})(x) =< (A \breve{\cup} B)(x), (\lambda \wedge \mu)(x) >$,

$$(\Cup_{j \in J} \mathcal{A}_j)(x) =< (\breve{\bigcup}_{j \in J} A_j)(x), (\bigwedge_{j \in J} \lambda_j)(x) >,$$

(v) (R-intersection) $(\mathcal{A} \Cap \mathcal{B})(x) =< (A \breve{\cap} B)(x), (\lambda \vee \mu)(x) >$,

$$(\Cap_{j \in J} \mathcal{A}_j)(x) =< (\breve{\bigcap}_{j \in J} A_j)(x), (\bigvee_{j \in J} \lambda_j)(x) >.$$

It is well-known [14] that the followings hold:

(1) $\check{0}^c = \check{1}$, $\check{1}^c = \check{0}$, $\ddot{0}^c = \ddot{1}$ and $\ddot{1}^c = \ddot{0}$,

(2) for each $\mathcal{A} \in ([I] \times I)^X$, $(\mathcal{A}^c)^c = \mathcal{A}$,
(3) for any $(\mathcal{A}_j)_{j\in J} \subset ([I] \times I)^X$,

$$(\sqcup_{j\in J}\mathcal{A}_j)^c = \sqcap_{j\in J}\mathcal{A}_j^c,\ (\sqcap_{j\in J}\mathcal{A}_j)^c = \sqcup_{j\in J}\mathcal{A}_j^c,$$
$$(\Cup_{j\in J}\mathcal{A}_j)^c = \Cap_{j\in J}\mathcal{A}_j^c,\ (\Cap_{j\in J}\mathcal{A}_j)^c = \Cup_{j\in J}\mathcal{A}_j^c.$$

Remark 3.3. For any $\mathcal{A} =< A, \lambda >\in ([I] \times I)^X$, $\mathcal{A} \sqcup \mathcal{A}^c \neq \check{1}$ and $\mathcal{A} \sqcap \mathcal{A}^c \neq \check{0}$, in general. Let $\mathcal{A} =< \mathbf{0.5}, 0.5 >$. Then clearly, $\mathcal{A} \sqcup \mathcal{A}^c =< \mathbf{0.5}, 0.5 >\neq \check{1}$ and $\mathcal{A} \sqcap \mathcal{A}^c =< \mathbf{0.5}, 0.5 >\neq \check{0}$.

The followings are the immediate results of Definition 3.2.

Proposition 3.4. *Let* $\mathcal{A} =< A, \lambda >,\ \mathcal{B} =< B, \mu >,\ \mathcal{C} =< C, \nu >\in ([I] \times I)^X$, *let* $(\mathcal{A}_j)_{j\in J} = (< A_j, \lambda_j >)_{j\in J} \subset ([I] \times I)^X$.
(1) $\mathcal{A} \sqcup \mathcal{A} = \mathcal{A}$, $\mathcal{A} \sqcap \mathcal{A} = \mathcal{A}$, $\mathcal{A} \Cup \mathcal{A} = \mathcal{A}$, $\mathcal{A} \Cap \mathcal{A} = \mathcal{A}$.
(2) $\mathcal{A} \sqcup \mathcal{B} = \mathcal{B} \sqcup \mathcal{A}$, $\mathcal{A} \sqcap \mathcal{B} = \mathcal{B} \sqcap \mathcal{A}$, $\mathcal{A} \Cup \mathcal{B} = \mathcal{B} \Cup \mathcal{A}$, $\mathcal{A} \Cap \mathcal{B} = \mathcal{B} \Cap \mathcal{A}$.
(3) $\mathcal{A} \sqcup (\mathcal{B} \sqcup \mathcal{C}) = (\mathcal{A} \sqcup \mathcal{B}) \sqcup \mathcal{C}$, $\mathcal{A} \sqcap (\mathcal{B} \sqcap \mathcal{C}) = (\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{C}$,
$\mathcal{A} \Cup (\mathcal{B} \Cup \mathcal{C}) = (\mathcal{A} \Cup \mathcal{B}) \Cup \mathcal{C}$, $\mathcal{A} \Cap (\mathcal{B} \Cap \mathcal{C}) = (\mathcal{A} \Cap \mathcal{B}) \Cap \mathcal{C}$.
(4) $\mathcal{A} \sqcup (\mathcal{B} \sqcap \mathcal{C}) = (\mathcal{A} \sqcup \mathcal{B}) \sqcap (\mathcal{A} \sqcup \mathcal{C})$, $\mathcal{A} \sqcap (\mathcal{B} \sqcup \mathcal{C}) = (\mathcal{A} \sqcap \mathcal{B}) \sqcup (\mathcal{A} \sqcap \mathcal{C})$,
$\mathcal{A} \Cup (\mathcal{B} \Cap \mathcal{C}) = (\mathcal{A} \Cup \mathcal{B}) \Cap (\mathcal{A} \Cup \mathcal{C})$, $\mathcal{A} \Cap (\mathcal{B} \Cup \mathcal{C}) = (\mathcal{A} \Cap \mathcal{B}) \Cup (\mathcal{A} \Cap \mathcal{C})$.
$(4)'$ $\mathcal{A} \sqcup (\sqcap_{j\in J}\mathcal{A}_j) = \sqcap_{j\in J}(\mathcal{A} \sqcup \mathcal{A}_j)$, $\mathcal{A} \sqcap (\sqcup_{j\in J}\mathcal{A}_j) = \sqcup_{j\in J}(\mathcal{A} \sqcap \mathcal{A}_j)$,
$\mathcal{A} \Cup (\Cap_{j\in J}\mathcal{A}_j) = \Cap_{j\in J}(\mathcal{A} \Cup \mathcal{A}_j)$, $\mathcal{A} \Cap (\Cup_{j\in J}\mathcal{A}_j) = \Cup_{j\in J}(\mathcal{A} \Cap \mathcal{A}_j)$.

From the above Proposition 3.4, we can see that $(([I] \times I)^X, \sqcup, \sqcap, \check{0}, \check{1})$ forms a Boolean algebra except the property of Remark 3.3.

Definition 3.5. Let $\mathcal{A} =< A, \lambda >\in ([I] \times I)^X$, let $\widetilde{a} \in [I]$ with $a^+ > 0$ and let $\alpha \in I$ with $\alpha > 0$. Then $\mathcal{A} =< A, \lambda >$ is called a cubic point in X with the support $x \in X$ and the value $< \widetilde{a}, \alpha >$, denoted by $x_{<\widetilde{a},\alpha>}$, if for each $y \in X$,

$$x_{<\widetilde{a},\alpha>} = \begin{cases} < \widetilde{a}, \alpha > & \text{if } y = x \\ < \mathbf{0}, 0 > & \text{otherwise.} \end{cases}$$

The set of all cubic points in X is denoted by $C_P(X)$.

Definition 3.6. Let $x_{<\widetilde{a},\alpha>} \in C_P(X)$ and let $\mathcal{A} =< A, \lambda >\in ([I] \times I)^X$.
(i) $x_{<\widetilde{a},\alpha>}$ is said to belong to $\mathcal{A}$ by P-order type, denoted by $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A}$, if
$\widetilde{a} \preceq A(x)$ and $\alpha \leq \lambda(x)$, i.e., $x_{\widetilde{a}} \in A$ and $x_\alpha \in \lambda$.
(ii) $x_{<\widetilde{a},\alpha>}$ is said to belong to $\mathcal{A}$ by R-order type, denoted by $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A}$, if
$\widetilde{a} \preceq A(x)$ and $\alpha \geq \lambda(x)$, i.e., $x_{\widetilde{a}} \in A$ and $x_{1-\alpha} \in \lambda^c$.

Theorem 3.7. *Let* $x_{<\widetilde{a},\alpha>} \in C_P(X)$ *and let* $\mathcal{A} =< A, \lambda >,\ \mathcal{B} =< B, \mu >\in ([I] \times I)^X$.
(1) $\mathcal{A} \sqsubset \mathcal{B}$ *if and only if* $x_{<\widetilde{a},\alpha>} \in_P \mathcal{B}$, *for each* $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A}$.
(2) $\mathcal{A} \Subset \mathcal{B}$ *if and only if* $x_{<\widetilde{a},\alpha>} \in_R \mathcal{B}$, *for each* $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A}$.

Proof. (1) Suppose $\mathcal{A} \sqsubset \mathcal{B}$ and let $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A}$. Then $\widetilde{a} \preceq A(x)$ and $\alpha \leq \lambda(x)$. Since $\mathcal{A} \sqsubset \mathcal{B}$, $A \subset B$ and $\lambda \leq \mu$. Thus $A(x) \precsim B(x)$ and $\lambda(x) \leq \mu(x)$. So $\widetilde{a} \preceq B(x)$ and $\alpha \leq \mu(x)$. Hence $x_{<\widetilde{a},\alpha>} \in_P \mathcal{B}$.
The converse is straightforward.
(2) The proof is similar to (1). ∎

Proposition 3.8. *Let* $\mathcal{A} =< A, \lambda >,\ \mathcal{B} =< B, \mu >\in ([I] \times I)^X$*, let* $(\mathcal{A}_j)_{j\in J} = (< A_j, \lambda_j >)_{j\in J} \subset ([I] \times I)^X$ *and let* $x_{<\widetilde{a},\alpha>} \in C_P(X)$*.*

(1) *If* $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A}$ *or* $x_{<\widetilde{a},\alpha>} \in_P \mathcal{B}$*, then* $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A} \sqcup \mathcal{B}$*.*

$(1)'$ *If there is* $j \in J$ *such that* $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A}_j$*, then* $x_{<\widetilde{a},\alpha>} \in_P \sqcup_{j\in J}\mathcal{A}_j$*.*

(2) *If* $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A}$ *or* $x_{<\widetilde{a},\alpha>} \in_R \mathcal{B}$*, then* $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A} \Cup \mathcal{B}$*.*

$(2)'$ *If there is* $j \in J$ *such that* $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A}_j$*, then* $x_{<\widetilde{a},\alpha>} \in_R \Cup_{j\in J}\mathcal{A}_j$*.*

Proof. (1) Suppose $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A}$ or $x_{<\widetilde{a},\alpha>} \in_P \mathcal{B}$. Then $x_{\widetilde{a}} \in A$ and $x_\alpha \in \lambda$ or $x_{\widetilde{a}} \in B$ and $x_\alpha \in \mu$. Thus $\widetilde{a} \preceq (A\breve{\cup}B)(x)$ and $\alpha \leq (\lambda \vee \mu)(x)$. So $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A} \sqcup \mathcal{B}$.

$(1)'$ Suppose $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A}_j$, for some $j \in J$. Then $x_{\widetilde{a}} \in A_j$ and $x_\alpha \in \lambda_j$, for some $j \in J$. Thus $\widetilde{a} \preceq A_j(x)$ and $\alpha \leq \lambda_j(x)$. So $\widetilde{a} \preceq \breve{\bigvee}_{j\in J}A_j(x)$ and $\alpha \leq \bigvee_{j\in j}\lambda_j(x)$. Hence $x_{\widetilde{a}} \in (\breve{\bigcup}_{j\in J}A_j)(x)$ and $x_\alpha \in (\bigvee_{j\in j}\lambda_j)(x)$. Therefore $x_{<\widetilde{a},\alpha>} \in_P \sqcup_{j\in J}\mathcal{A}_j$.

(2) Suppose $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A}$ or $x_{<\widetilde{a},\alpha>} \in_R \mathcal{B}$. Then $x_{\widetilde{a}} \in A$ and $x_{1-\alpha} \in \lambda^c$ or $x_{\widetilde{a}} \in B$ and $x_{1-\alpha} \in \mu^c$. Thus $\widetilde{a} \in (A\breve{\cup}B)(x)$ and $1-\alpha \leq (\lambda \wedge \mu)^c(x)$. So $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A} \Cup \mathcal{B}$.

$(2)'$ Suppose $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A}_j$, for some $j \in J$. Then $x_{\widetilde{a}} \in A_j$ and $x_{1-\alpha} \in \lambda_j^c$, for some $j \in J$. Thus $\widetilde{a} \preceq A_j(x)$ and $1-\alpha \leq \lambda_j^c(x)$. So $\widetilde{a} \preceq \breve{\bigvee}_{j\in J}A_j(x)$ and $1-\alpha \leq (\bigwedge_{j\in j}\lambda_j)^c(x)$. Hence $x_{\widetilde{a}} \in (\breve{\bigcup}_{j\in J}A_j)(x)$ and $x_{1-\alpha} \in (\bigwedge_{j\in j}\lambda_j)^c(x)$. Therefore $x_{<\widetilde{a},\alpha>} \in_R \Cup_{j\in J}\mathcal{A}_j$. □

The converse of Proposition 3.8 need not to be true in general as shown in the following example.

Example 3.9. Let $X = \{a, b, c\}$ and let $A,\ B$ be two IVF sets in X defined by:

$$A(a) = [0.3, 0.9],\ A(b) = [0.2, 0.6],\ A(c) = [0.4, 0.8]$$

and

$$B(a) = [0.5, 0.7],\ B(b) = [0.4, 0.8],\ B(c) = [0.3, 0.6].$$

Let $\lambda,\ \mu$ be two fuzzy sets in X given by:

$$\lambda(a) = 0.4,\ \lambda(b) = 0.5,\ \lambda(c) = 0.7;\ \mu(a) = 0.6,\ \mu(b) = 0.8,\ \mu(c) = 0.6.$$

Then we can easily check that $\mathcal{A} =< A, \lambda >,\ \mathcal{B} =< B, \mu >\in ([I]\times I)^X$. Consider the cubic point $a_{<\widetilde{a},\alpha>}$, where $\widetilde{a} = [0.4, 0.8]$ and $\alpha = 0.5$. Then clearly, $a_{<\widetilde{a},\alpha>} \in_P \mathcal{A}\sqcup\mathcal{B}$ but $a_{<\widetilde{a},\alpha>} \notin_P \mathcal{A}$ and $a_{<\widetilde{a},\alpha>} \notin_P \mathcal{B}$. Also we can easily check that $a_{<\widetilde{a},\alpha>} \in_R \mathcal{A}\sqcup\mathcal{B}$ but $a_{<\widetilde{a},\alpha>} \notin_R \mathcal{A}$ and $a_{<\widetilde{a},\alpha>} \notin_R \mathcal{B}$.

Theorem 3.10. *Let* $\mathcal{A} =< A, \lambda >,\ \mathcal{B} =< B, \mu >\in ([I] \times I)^X$*, let* $(\mathcal{A}_j)_{j\in J} = (< A_j, \lambda_j >)_{j\in J} \subset ([I] \times I)^X$ *and let* $x_{<\widetilde{a},\alpha>} \in C_P(X)$*.*

(1) $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A} \sqcap \mathcal{B}$ *if and only if* $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A}$ *and* $x_{<\widetilde{a},\alpha>} \in_P \mathcal{B}$*.*

$(1)'$ $x_{<\widetilde{a},\alpha>} \in_P \sqcap_{j\in J}\mathcal{A}_j$ *if and only if* $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A}_j$*, for each* $j \in J$*.*

(2) $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A} \Cap \mathcal{B}$ *if and only if* $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A}$ *and* $x_{<\widetilde{a},\alpha>} \in_R \mathcal{B}$*.*

$(2)'$ $x_{<\widetilde{a},\alpha>} \in_R \Cap_{j\in J}\mathcal{A}_j$ *if and only if* $x_{<\widetilde{a},\alpha>} \in_R \mathcal{A}_j$*, for each* $j \in J$*.*

Proof. We will prove only (1). Suppose $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A} \sqcap \mathcal{B}$. Then $x_{\widetilde{a}} \in_P A\breve{\cap}B$ and $x_\alpha \in \lambda \wedge \mu$. Thus $\widetilde{a} \preceq (A\breve{\cap}B)(x) = A(x)\breve{\wedge}B(x)$ and $\alpha \leq \lambda(x) \wedge \mu(x)$. So $\widetilde{a} \preceq A(x),\ \alpha \leq \lambda(x)$ and $\widetilde{a} \preceq B(x),\ \alpha \leq \mu(x)$. Hence $x_{<\widetilde{a},\alpha>} \in_P \mathcal{A}$ and $x_{<\widetilde{a},\alpha>} \in_P \mathcal{B}$. The proof of the converse is easy. □

4. Cubic relations

In this section, we define a cubic relation and introduce P-order, R-order, P-intersection, R-intersection, P-union and R-union of any two cubic relations. Also we define the inverse of a cubic relation and the composition of two cubic relations, and we obtain some of each properties and give some examples.

Definition 4.1. Let $X,\ Y$ be two sets. Then $\mathcal{R} =< R, \lambda >$ is called a cubic relation from X to Y, if $R \in [I]^{X\times Y}$, i.e., R is an interval-valued fuzzy relation (briefly, IVF relation) from X to Y and $\lambda \in I^{X\times Y}$, i.e., λ is a fuzzy relation from X to Y.

A cubic relation $\mathcal{R} =< R, \lambda >$ in which $R(x, y) = \mathbf{0}$ and $\lambda(x, y) = 1$ (resp. $R(x, y) = \mathbf{1}$ and $\lambda(x, y) = 0$) for each $(x, y) \in X \times Y$ is denoted by $\ddot{0}$ (resp. $\ddot{1}$).

A cubic relation $\mathcal{S} =< S, \mu >$ in which $S(x, y) = \mathbf{0}$ and $\mu(x, y) = 0$ (resp. $S(x, y) = \mathbf{1}$ and $\mu(x, y) = 1$) for each $(x, y) \in X \times Y$ is denoted by $\hat{0}$ (resp. $\hat{1}$). In this case, $\hat{0}$ (resp. $\hat{1}$) will be called a cubic empty (resp. whole) relation from X to Y.

We will denote the set of all cubic relation from X to Y as $([I] \times I)^{X\times Y}$. If $\mathcal{R} =< R, \lambda >\in ([I] \times I)^{X\times X}$, $\mathcal{R}$ is called a cubic relation in (or on) X.

Example 4.2. Let $X = \{a, b, c\}$ be a set, let R be the IVF relation and λ be the fuzzy relation on X given, respectively by the following tables:

R	a	b	c
a	$[0.3, 0.7]$	$[0.4, 0.8]$	$[0.1, 0.6]$
b	$[0.1, 0.6]$	$[0, 1]$	$[0.2, 0.5]$
c	$[0.4, 0.9]$	$[0.3, 0.8]$	$[0, 1]$

Table 4.1

λ	a	b	c
a	0.6	0.4	0.7
b	0.8	0.5	0.9
c	0.4	0.7	0.6

Table 4.2

Then clearly, $\mathcal{R} =< R, \lambda >$ is a cubic relation on X.

Since cubic relations are cubic sets, we have the following definitions.

Definition 4.3. Let $X,\ Y$ be two sets and let $\mathcal{R} =< R, \lambda >,\ \mathcal{S} =< S, \mu >\in ([I] \times I)^{X\times Y}$. Then we define the following relations:
(i) (Equality) $\mathcal{R} = \mathcal{S} \Leftrightarrow R = S$ and $\lambda = \mu$,
(ii) (P-order) $\mathcal{R} \sqsubset \mathcal{S} \Leftrightarrow R \subset S$ and $\lambda \leq \mu$,
(iii) (R-order) $\mathcal{R} \Subset \mathcal{S} \Leftrightarrow R \subset S$ and $\lambda \geq \mu$

Definition 4.4. Let $X,\ Y$ be two sets and let $\mathcal{R} =< R, \lambda >,\ \mathcal{S} =< S, \mu >\in ([I] \times I)^{X\times Y}$ and let $(\mathcal{R}_j)_{j\in J} = (< R_j, \lambda_j >)_{j\in J} \subset ([I] \times I)^{X\times Y}$. Then the complement $\mathcal{R}^c$ of $\mathcal{R}$, P-union $\sqcup$, P-intersection $\sqcap$, R-union $\Cup$ and R-intersection $\Cap$ are defined as follows, respectively: for each $(x, y) \in X \times Y$,

(i) (Complement) $\mathcal{R}^c(x,y) =< R^c(x,y), \lambda^c(x,y) >$,
(ii) (P-union) $(\mathcal{R} \sqcup \mathcal{S})(x,y) =< (R\breve{\cup}S)(x,y), (\lambda \vee \mu)(x,y) >$,

$$(\sqcup_{j\in J}\mathcal{R}_j)(x,y) =< (\breve{\bigcup}_{j\in J}R_j)(x,y), (\bigvee_{j\in J}\lambda_j)(x,y) >,$$

(iii) (P-intersection) $(\mathcal{R} \sqcap \mathcal{S})(x,y) =< (R\breve{\cap}S)(x,y), (\lambda \wedge \mu)(x,y) >$,

$$(\sqcap_{j\in J}\mathcal{R}_j)(x,y) =< (\breve{\bigcap}_{j\in J}R_j)(x,y), (\bigwedge_{j\in J}\lambda_j)(x,y) >,$$

(iv) (R-union) $(\mathcal{R} \Cup \mathcal{S})(x,y) =< (R\breve{\cup}S)(x,y), (\lambda \wedge \mu)(x,y) >$,

$$(\Cup_{j\in J}\mathcal{R}_j)(x,y) =< (\breve{\bigcup}_{j\in J}R_j)(x,y), (\bigwedge_{j\in J}\lambda_j)(x,y) >,$$

(v) (R-intersection) $(\mathcal{R} \Cap \mathcal{S})(x) =< (R\breve{\cap}S)(x,y), (\lambda \vee \mu)(x,y) >$,

$$(\Cap_{j\in J}\mathcal{R}_j)(x,y) =< (\breve{\bigcap}_{j\in J}R_j)(x,y), (\bigvee_{j\in J}\lambda_j)(x,y) >.$$

We can easily see that the followings hold:
(1) $\breve{0}^c = \breve{1},\ \breve{1}^c = \breve{0},\ \ddot{0}^c = \ddot{1}$ and $\ddot{1}^c = \ddot{0}$,
(2) for each $\mathcal{R} \in ([I] \times I)^{X\times Y}$, $(\mathcal{R}^c)^c = \mathcal{R}$,
(3) for any $(\mathcal{R}_j)_{j\in J} \subset ([I] \times I)^{X\times Y}$,

$$(\sqcup_{j\in J}\mathcal{R}_j)^c = \sqcap_{j\in J}\mathcal{R}_j^c,\ (\sqcap_{j\in J}\mathcal{R}_j)^c = \sqcup_{j\in J}\mathcal{R}_j^c,$$
$$(\Cup_{j\in J}\mathcal{R}_j)^c = \Cap_{j\in J}\mathcal{R}_j^c,\ (\Cap_{j\in J}\mathcal{R}_j)^c = \Cup_{j\in J}\mathcal{R}_j^c,$$

(4) we have the similar properties to Proposition 3.4 in $([I] \times I)^{X\times Y}$.

Definition 4.5. Let $\mathcal{R} =< R, \lambda >\in ([I]\times I)^{X\times Y}$. Then the inverse of $\mathcal{R} =< R, \lambda >$, denoted by $\mathcal{R}^{-1} =< R^{-1}, \lambda^{-1} >$, is a cubic relation from Y to X defined as follows: for each $(x,y) \in X \times Y$,

$$\mathcal{R}^{-1}(x,y) = \mathcal{R}(y,x),\ \text{i.e.,}\ R^{-1}(x,y) = R(y,x) \text{ and } \lambda^{-1}(x,y) = \lambda(y,x).$$

The followings are the immediate results of Definitions 4.4 and 4.5.

Proposition 4.6. *Let $\mathcal{R} =< R, \lambda >,\ \mathcal{S} =< S, \mu >\in ([I] \times I)^{X\times Y}$.*
(1) *If $\mathcal{R} \sqsubset \mathcal{S}$, then $\mathcal{R}^{-1}\sqsubset\mathcal{S}^{-1}$.*
(2) *If $\mathcal{R} \Subset \mathcal{S}$, then $\mathcal{R}^{-1} \Subset \mathcal{S}^{-1}$.*
(3) *$(\mathcal{R}^{-1})^{-1} = \mathcal{R}$.*
(4) *$(\mathcal{R} \sqcup \mathcal{S})^{-1} = \mathcal{R}^{-1} \sqcap \mathcal{S}^{-1}$, $(\mathcal{R} \sqcap \mathcal{S})^{-1} = \mathcal{R}^{-1} \sqcup \mathcal{S}^{-1}$.*
(5) *$(\mathcal{R} \Cup \mathcal{S})^{-1} = \mathcal{R}^{-1} \Cap \mathcal{S}^{-1}$, $(\mathcal{R} \Cap \mathcal{S})^{-1} = \mathcal{R}^{-1} \Cup \mathcal{S}^{-1}$.*

Definition 4.7. Let $\mathcal{R} =< R, \lambda >\in ([I]\times I)^{X\times Y}$ and let $\mathcal{S} =< S, \mu >\in ([I]\times I)^{Y\times Z}$. Then the P-composition of $\mathcal{R}$ and S, denoted by $\mathcal{S} \circ_P \mathcal{R}$, is a cubic relation from X to Z defined by: for each $(x,z) \in X \times Z$,

$$(\mathcal{S} \circ_P \mathcal{R})(x,z) = \sqcup_{y\in Y}[\mathcal{R}(x,y) \sqcap \mathcal{S}(y,z)]$$
$$=< \breve{\bigvee}_{y\in Y}[R(x,y)\breve{\wedge}S(y,z)], \breve{\bigvee}_{y\in Y}[\lambda(x,y)\breve{\wedge}\mu(y,z)] > .$$

Example 4.8. Let $X = \{a, b, c\}$ be a set, let $\mathcal{R} =< R, \lambda >$ and $\mathcal{S} =< S, \mu >$ be the cubic relations on X given, respectively by the following tables:

$\mathcal{R}$	a	b	c
a	$< [0.3, 0.7], 0.6 >$	$< [0.4, 0.8], 0.4 >$	$< [0.1, 0.6], 0.7 >$
b	$< [0.1, 0.6], 0.8 >$	$< [0, 1], 0.5 >$	$< [0.2, 0.5], 0.9 >$
c	$< [0.4, 0.9], 0.4 >$	$< [0.3, 0.8], 0.7 >$	$< [0, 1], 0.6 >$

Table 4.3

Then we can easily check that $\mathcal{S} \circ_P \mathcal{R}$ has the following table:

$\mathcal{S}$	a	b	c
a	$< [0.4, 0.8], 0.9 >$	$< [0.2, 0.6], 0.7 >$	$< [0.2, 0.6], 0.8 >$
b	$< [0.3, 0.6], 0.7 >$	$< [0.2, 0.7], 0.5 >$	$< [0.4, 0.8], 0.9 >$
c	$< [0.3, 0.8], 0.4 >$	$< [0.4, 0.7], 0.6 >$	$< [0.1, 0.6], 0.6 >$

Table 4.4

$\mathcal{S} \circ_P \mathcal{R}$	a	b	c
a	$< [0.3, 0.7], 0.6 >$	$< [0.2, 0.7], 0.6 >$	$< [0.4, 0.8], 0.6 >$
b	$< [0.2, 0.6], 0.8 >$	$< [0.2, 0.7], 0.7 >$	$< [0.1, 0.8], 0.8 >$
c	$< [0.4, 0.8], 0.7 >$	$< [0.2, 0.7], 0.6 >$	$< [0.3, 0.8], 0.7 >$

Table 4.5

The followings are the immediate results of Definitions 4.3, 4.4, 4.5, 4.5 and 4.7.

Proposition 4.9. *Let $\mathcal{R} =< R, \lambda >$, $\mathcal{S} =< S, \mu >$ and $\mathcal{T} =< T, \nu >$ be cubic relations and suppose each composition holds. Then*

(1) $\mathcal{T} \circ_P (\mathcal{S} \circ_P \mathcal{R}) = (\mathcal{T} \circ_P \mathcal{S}) \circ_P \mathcal{R}$,
(2) $\mathcal{T} \circ_P (\mathcal{R} \sqcup \mathcal{S}) = (\mathcal{T} \circ_P \mathcal{R}) \sqcup (\mathcal{T} \circ_P \mathcal{S})$, $\mathcal{T} \circ_P (\mathcal{R} \sqcap \mathcal{S}) = (\mathcal{T} \circ_P \mathcal{R}) \sqcap (\mathcal{T} \circ_P \mathcal{S})$,
(3) *If* $\mathcal{R} \sqsubset \mathcal{S}$, *then* $\mathcal{T} \circ_P \mathcal{R} \subset \mathcal{T} \circ_P \mathcal{S}$,
(4) $(\mathcal{S} \circ_P \mathcal{R})^{-1} = \mathcal{R}^{-1} \circ_P \mathcal{S}^{-1}$.

Definition 4.10. Let $\mathcal{R} =< R, \lambda > \in ([I] \times I)^{X \times X}$. Then R is said to be:

(i) reflexive, if $\mathcal{R}(x, x) =< \mathbf{1}, 1 >$, for each $x \in X$,
(ii) irreflexive, if $\mathcal{R}(x, x) =< \mathbf{0}, 0 >$, for each $x \in X$,
(iii) symmetric, if $\mathcal{R}^{-1}(x, y) = \mathcal{R}(x, y)$, for each $(x, y) \in X \times X$,
(iv) antisymmetric, if for each $(x, y) \in X \times X$, $\mathcal{R}(x, y) \neq< \mathbf{0}, 0 >$ and $\mathcal{R}(y, x) \neq< \mathbf{0}, 0 >$, then $x = y$,
(v) transitive, if $\mathcal{R} \circ_P \mathcal{R} \sqsubset \mathcal{R}$.

Example 4.11. Let $X = \{a, b, c\}$.

(1) Let $\mathcal{R}_1 =< R_1, \lambda_1 >$ be a cubic relation on X given by the following table:

$\mathcal{R}_1$	a	b	c
a	$< \mathbf{1}, 1 >$	$< [0.4, 0.8], 0.4 >$	$< [0.1, 0.6], 0.7 >$
b	$< [0.1, 0.6], 0.8 >$	$< \mathbf{1}, 1 >$	$< [0.2, 0.5], 0.9 >$
c	$< [0.4, 0.9], 0.4 >$	$< [0.3, 0.8], 0.7 >$	$< \mathbf{1}, 1 >$

Table 4.6

Then clearly, $\mathcal{R}_1$ is reflexive.

(2) Let $\mathcal{R}_2 =< R_2, \lambda_2 >$ be a cubic relation on X given by the following table:

$\mathcal{R}_2$	a	b	c
a	$< \mathbf{0}, 0 >$	$< [0.4, 0.8], 0.4 >$	$< [0.1, 0.6], 0.7 >$
b	$< [0.1, 0.6], 0.8 >$	$< \mathbf{0}, 0 >$	$< [0.2, 0.5], 0.9 >$
c	$< [0.4, 0.9], 0.4 >$	$< [0.3, 0.8], 0.7 >$	$< \mathbf{0}, 0 >$

Table 4.7

Then clearly, $\mathcal{R}_2$ is irreflexive.

(3) Let $\mathcal{R}_3 =< R_3, \lambda_3 >$ be a cubic relation on X given by the following table:

$\mathcal{R}_3$	a	b	c
a	$< \mathbf{1}, 1 >$	$< [0.1, 0.6], 0.8 >$	$< [0.1, 0.6], 0.7 >$
b	$< [0.1, 0.6], 0.8 >$	$< \mathbf{1}, 1 >$	$< [0.3, 0.8], 0.7 >$
c	$< [0.1, 0.6], 0.7 >$	$< [0.3, 0.8], 0.7 >$	$< \mathbf{1}, 1 >$

Table 4.8

Then clearly, $\mathcal{R}_3$ is reflexive and symmetric.

(4) Let $\mathcal{R}_4 =< R_4, \lambda_4 >$ be a cubic relation on X given by the following table:

$\mathcal{R}_4$	a	b	c
a	$< \mathbf{1}, 1 >$	$< [0.3, 0.8], 0.7 >$	$< [0.1, 0.7], 0.7 >$
b	$< [0.1, 0.6], 0.8 >$	$< \mathbf{1}, 1 >$	$< [0.1, 0.7], 0.9 >$
c	$< [0.4, 0.9], 1 >$	$< [0.3, 0.8], 0.7 >$	$< \mathbf{1}, 1 >$

Table 4.9

Then we can easily check that $\mathcal{R}_4$ is reflexive and transitive.

Definition 4.12. The cubic identity relation on a set X, denoted by $\mathcal{I}_X$ (in short, $\mathcal{I}$), is a cubic relation on X defined by: for each $(x, y) \in X \times X$,

$$\mathcal{I}(x, y) = \begin{cases} < \mathbf{1}, 1 > & \text{if } x = y \\ < \mathbf{0}, 0 > & \text{if } x \neq y. \end{cases}$$

It is clear that $\mathcal{I} = \mathcal{I}^{-1}$ and $\mathcal{R}$ is symmetric if and only if $\mathcal{R} = \mathcal{R}^{-1}$, for each $\mathcal{R} =< R, \lambda >\in [I]^{X \times X}$.

From the above Definition and Definition 4.10 (i), it is obvious that if $\mathcal{R}$ is reflexive, then $\mathcal{I} \sqsubset \mathcal{R}$.

The following is the immediate result of Definitions 4.4 and 4.10.

Proposition 4.13. *Let $\mathcal{R} =< R, \lambda >,\ \mathcal{S} =< S, \mu >\in ([I] \times I)^{X \times X}$.*

(1) *$\mathcal{R}$ is reflexive [resp. irreflexive] if and only if $\mathcal{R}^{-1}$ is reflexive [resp. irreflexive].*

(2) *If $\mathcal{R}$ is reflexive, then $\mathcal{R} \sqcup \mathcal{S}$ is reflexive.*

(3) *If $\mathcal{R}$ is irreflexive, then $\mathcal{R} \sqcup \mathcal{S}$ is irreflexive if and only if $\mathcal{S}$ is irreflexive.*

(4) *If $\mathcal{R}$ is reflexive, then $\mathcal{R} \sqcap \mathcal{S}$ is reflexive if and only if $\mathcal{S}$ is reflexive.*

(5) *If $\mathcal{R}$ is irreflexive, then $\mathcal{R} \sqcap \mathcal{S}$ is irreflexive.*

Proposition 4.14. *Let $\mathcal{R} =< R, \lambda >,\ \mathcal{S} =< S, \mu >\in ([I] \times I)^{X \times X}$. If $\mathcal{R}$ and $\mathcal{S}$ are reflexive, then so $\mathcal{S} \circ_P \mathcal{R}$.*

Proof. Let $x \in X$. Then by the hypothesis,

$$(\mathcal{S} \circ_P \mathcal{R})(x, x) =< \breve{\bigvee}_{y \in X} [R(x, y) \breve{\wedge} S(y, x)], \bigvee_{y \in X} [\lambda(x, y) \wedge \mu(y, z)] > .$$

On the other hand,

$$\breve{\bigvee}_{y \in X} [R(x, y) \breve{\wedge} S(y, x) \succeq R(x, x) \breve{\wedge} S(x, x) \\ = \mathbf{1}$$

and

$\bigvee_{y\in X}[\lambda(x,y)\wedge\mu(y,z)\geq\lambda(x,x)\wedge\mu(x,x)$
$= 1.$

Thus $(\mathcal{S}\circ_P\mathcal{R})(x,x) =< \mathbf{1}, 1 >$. So $\mathcal{S}\circ_P\mathcal{R}$ is reflexive. □

The following is the immediate result of Definition 4.10 and Proposition 4.6 (4) and (5).

Proposition 4.15. *Let $\mathcal{R} =< R,\lambda >$, $\mathcal{S} =< S,\mu >\in([I]\times I)^{X\times X}$. If $\mathcal{R}$ and $\mathcal{S}$ are symmetric, then $\mathcal{R}\sqcup\mathcal{R}$, $\mathcal{R}\sqcap\mathcal{R}$, $\mathcal{R}\Cup\mathcal{R}$ and $\mathcal{R}\Cap\mathcal{R}$ are symmetric.*

Remark 4.16. $\mathcal{R}$ and $\mathcal{S}$ are symmetric but $\mathcal{S}\circ_P\mathcal{R}$ is not symmetric, in general

Example 4.17. Let $X = \{a, b, c\}$. Consider two cubic relations $\mathcal{R} =< R,\lambda >$ and $\mathcal{S} =< S,\mu >$ on X given by:

$\mathcal{R}$	a	b	c
a	$< [0.3, 0.6], 0.6 >$	$< [0.2, 0.7], 0.8 >$	$< [0.1, 0.5], 0.7 >$
b	$< [0.2, 0.7], 0.8 >$	$< [0.5, 1], 0.4 >$	$< [0.3, 0.7], 0.8 >$
c	$< [0.1, 0.5], 0.7 >$	$< [0.3, 0.7], 0.8 >$	$< [0.2, 0.6], 0.6 >$

Table 4.10

$\mathcal{S}$	a	b	c
a	$< [0.2, 0.5], 0.6 >$	$< [0.3, 0.7], 0.6 >$	$< [0.4, 0.8], 0.9 >$
b	$< [0.3, 0.7], 0.6 >$	$< \mathbf{1}, 1 >$	$< [0.2, 0.8], 0.7 >$
c	$< [0.4, 0.8], 0.9 >$	$< [0.2, 0.8], 0.7 >$	$< [0.3, 0.5], 0.6 >$

Table 4.11

Then clearly, $\mathcal{R}$ and $\mathcal{S}$ are symmetric. But

$$(\mathcal{S}\circ_P\mathcal{R})(b,c) =< [0.3, 0.8], 0.8 >\neq< [0.3, 0.7], 0.8 >= (\mathcal{S}\circ_P\mathcal{R})(c,b).$$

Thus $\mathcal{S}\circ_P\mathcal{R}$ is not symmetric.

The following gives the condition for its being symmetric.

Theorem 4.18. *Let $\mathcal{R} =< R,\lambda >$, $\mathcal{S} =< S,\mu >\in([I]\times I)^{X\times X}$ and let $\mathcal{R}$ and $\mathcal{S}$ be symmetric. Then $\mathcal{S}\circ_P\mathcal{R}$ is symmetric if and only if $\mathcal{S}\circ_P\mathcal{R} = \mathcal{R}\circ_P\mathcal{S}$.*

Proof. Suppose $\mathcal{S}\circ_P\mathcal{R}$ is symmetric. Then

$\mathcal{S}\circ_P\mathcal{R} = \mathcal{S}^{-1}\circ\mathcal{R}^{-1}$ [Since $\mathcal{R}$ and $\mathcal{S}$ are symmetric]
$= (\mathcal{R}\circ_P\mathcal{S})^{-1}$ [By Proposition 4.9 (4)]
$= \mathcal{R}\circ_P\mathcal{S}$. [By the hypothesis]

Conversely, suppose $\mathcal{S}\circ_P\mathcal{R} = \mathcal{R}\circ_P\mathcal{S}$. Then

$(\mathcal{S}\circ_P\mathcal{R})^{-1} = \mathcal{R}^{-1}\circ_P\mathcal{S}^{-1}$ [By Proposition 4.9 (4)]
$= \mathcal{R}\circ_P\mathcal{S}$ [Since $\mathcal{R}$ and $\mathcal{S}$ are symmetric]
$= \mathcal{S}\circ_P\mathcal{R}$. [By the hypothesis]

Thus $\mathcal{S}\circ_P\mathcal{R}$ is symmetric. □

The following is the immediate result of the above Theorem.

Corollary 4.19. *If $\mathcal{R} =< R, \lambda >\in ([I] \times I)^{X\times X}$ is symmetric, then $\mathcal{R}^n$ is symmetric, for all positive integer n, where $\mathcal{R}^n = \mathcal{R} \circ_P \mathcal{R} \circ_P \cdots$ n times.*

Proposition 4.20. *If $\mathcal{R} =< R, \lambda >\in ([I] \times I)^{X\times X}$ is transitive, then so are $\mathcal{R}^{-1}$ and $\mathcal{R}^2$.*

Proof. Let $(x, y) \in X \times X$. Then

$$\begin{aligned}(\mathcal{R}^{-1} \circ_P \mathcal{R}^{-1})(x, y) &= (\mathcal{R} \circ_P \mathcal{R})^{-1}(x, y)\\ &= (\mathcal{R} \circ_P \mathcal{R})(y, x)\\ &\sqsubset \mathcal{R}(y, x) \text{ [Since } \mathcal{R} \text{ is transitive]}\\ &= \mathcal{R}^{-1}(x, y).\end{aligned}$$

Thus $\mathcal{R}^{-1} \circ_P \mathcal{R}^{-1} \sqsubset \mathcal{R}^{-1}$. So $\mathcal{R}^{-1}$ is transitive.

The proof of the second part is similar. □

Proposition 4.21. *Let $\mathcal{R} =< R, \lambda >$, $\mathcal{S} =< S, \mu >\in ([I] \times I)^{X\times X}$. If $\mathcal{R}$ and $\mathcal{S}$ are transitive, then so is $\mathcal{R} \sqcap \mathcal{S}$.*

Proof. Let $(x, y) \in X \times X$. Then

$$\begin{aligned}&[(\mathcal{R} \sqcap \mathcal{S}) \circ_P (\mathcal{R} \sqcap \mathcal{S})](x, y)\\ &= \sqcup_{z\in X}[(\mathcal{R} \sqcap \mathcal{S})(x, z) \sqcap (\mathcal{R} \sqcap \mathcal{S})(z, y)]\\ &= \sqcup_{z\in X}[(\mathcal{R}(x, z) \sqcap \mathcal{S}(x, z)) \sqcap (\mathcal{R}(z, y) \sqcap \mathcal{S}(z, y))]\\ &= \sqcup_{z\in X}([\mathcal{R}(x, z) \sqcap \mathcal{R}(z, y)] \sqcap [\mathcal{S}(x, z) \sqcap \mathcal{S}(z, y)])\\ &= (\sqcup_{z\in X}[\mathcal{R}(x, z) \sqcap \mathcal{R}(z, y)]) \sqcap (\sqcup_{z\in X}[\mathcal{S}(x, z) \sqcap \mathcal{S}(z, y)])\\ &= (\mathcal{R} \circ_P \mathcal{R})(x, y) \sqcap (\mathcal{S} \circ_P \mathcal{S})(x, y)\\ &\sqsubset \mathcal{R}(x, y) \sqcap \mathcal{S}(x, y) \text{ [Since } \mathcal{R} \text{ and } \mathcal{S} \text{ are transitive]}\\ &= (\mathcal{R} \sqcap \mathcal{S})(x, y).\end{aligned}$$

Thus $(\mathcal{R} \sqcap \mathcal{S}) \circ_P (\mathcal{R} \sqcap \mathcal{S}) \sqsubset \mathcal{R} \sqcap \mathcal{S}$. So $\mathcal{R} \sqcap \mathcal{S}$ is transitive. □

Remark 4.22. For two cubic transitive relations $\mathcal{R} =< R, \lambda >$ and $\mathcal{S} =< S, \mu >$ on X, $\mathcal{R} \sqcup \mathcal{S}$ is not transitive, in general.

Example 4.23. Let $X = \{a, b, c\}$, let $\mathcal{R}$ and $\mathcal{S}$ be two hesitant fuzzy transitive relations on X given in Table 4.12 and Table 4.13, respectively:

$\mathcal{R}$	a	b	c
a	$< [0.2, 0.6], 0.7 >$	$< [0.1, 0.5], 0.5 >$	$< [0.1, 0.6], 0.5 >$
b	$< [0.3, 0.6], 0.8 >$	$< [0.2, 0.7], 0.6 >$	$< [0.3, 0.6], 0.7 >$
c	$< [0.4, 0.9], 0.6 >$	$< [0.2, 0.5], 0.5 >$	$< \mathbf{1}, 1 >$

Table 4.12

$\mathcal{S}$	a	b	c
a	$< \mathbf{1}, 1 >$	$< [0.1, 0.5], 0.6 >$	$< [0.1, 0.5], 0.7 >$
b	$< [0.1, 0.5], 0.6 >$	$< [0.4, 0.9], 0.8 >$	$< [0.4, 0.9], 0.6 >$
c	$< 0.1, 0.5], 0.7 >$	$< [0.4, 0.9], 0.6 >$	$< [0.4, 0.9], 0.7 >$

Table 4.13

Then $\mathcal{R} \sqcup \mathcal{S}$ is the cubic relation on X given Table 4.14:
Thus $[(\mathcal{R} \sqcup \mathcal{S}) \circ_P (\mathcal{R} \sqcup \mathcal{S})](b, a) =< [0.4, 0.9], 0.8 >$. So $[0.4, 0.9] \not\preceq< [0.3, 0.6]$. Hence $(\mathcal{R} \sqcup \mathcal{S}) \circ_P (\mathcal{R} \sqcup \mathcal{S}) \not\sqsubset \mathcal{R} \sqcup \mathcal{S}$. Therefore $\mathcal{R} \sqcup \mathcal{S}$ is not transitive.

$\mathcal{R} \sqcup \mathcal{S}$	a	b	c
a	$< \mathbf{1}, 1 >$	$< [0.1, 0.5], 0.6 >$	$< [0.1, 0.6], 0.7 >$
b	$< [0.3, 0.6], 0.8 >$	$< [0.4, 0.9], 0.8 >$	$< [0.4, 0.9], 0.7 >$
c	$< [0.4, 0.9], 0.7 >$	$< [0.4, 0.9], 0.6 >$	$< \mathbf{1}, 1 >$

Table 4.14

5. Cubic equivalence relations

In this section, we define a cubic equivalence relation, a cubic quotient set and a cubic partition, and prove that a cubic quotient set forms a cubic partition and a cubic partition forms a cubic cubic equivalence relation. Also we define a level relation of a cubic relation and obtain some of its properties.

Definition 5.1. $\mathcal{R} =< R, \lambda >\in ([I] \times I)^{X \times X}$ is called a cubic:

(i) tolerance relation on X, if it is reflexive and symmetric,

(ii) similarity (or equivalence) relation on X, if it is reflexive, symmetric and transitive,

(iii) order relation on X, if it is reflexive, antisymmetric and transitive.

We will denote the set of all cubic tolerance [resp. equivalence and order] relations on X as $CR_T(X)$ [resp. $CR_E(X)$ and $CR_O(X)$].

Example 5.2. (1) Let $X = \{a, b, c\}$. Consider the cubic relation $\mathcal{R} =< R, \lambda >$ on X given by:

$\mathcal{R} =< R, \lambda >$	a	b	c
a	$< \mathbf{1}, 1 >$	$< [0.3, 0.8], 0.6 >$	$< [0.2, 0.7], 0.9 >$
b	$< [0.3, 0.8], 0.6 >$	$< \mathbf{1}, 1 >$	$< [0.4, 0.9], 1 >$
c	$< [0.2, 0.7], 0.9 >$	$< [0.4, 0.9], 1 >$	$< \mathbf{1}, 1 >$

Table 5.1

Then we can easily check that $\mathcal{R}$ is a cubic tolerance relation on X.

(2) Let $X = \{a, b, c\}$, let $\mathcal{R}$ be the cubic relation on X given in Table 5.2 :

$\mathcal{R}$	a	b	c
a	$< \mathbf{1}, 1 >$	$< [0.2, 0.7], 0.8 >$	$< [0.2, 0.7], 0.7 >$
b	$< [0.2, 0.7], 0.8 >$	$< \mathbf{1}, 1 >$	$< [0.4, 0.9], 0.7 >$
c	$< [0.2, 0.7], 0.7 >$	$< [0.4, 0.9], 0.7 >$	$< \mathbf{1}, 1 >$

Table 5.2

Then we can easily check that $\mathcal{R}$ is a cubic equivalence relation on X.

The following is the immediate result of Definition 5.1.

Remark 5.3. From Definition 5.1, we can easily see that $\mathcal{R} =< R, \lambda >\in CR_E(X)$ if and only if $R \in IVR_E(X)$ and $\lambda \in FR_E(X)$.

The following is the immediate result of Propositions 4.13, 4.15 and 4.21.

Proposition 5.4. *Let* $(\mathcal{R}_j)_{j \in J}$ *be any subfamily of* $CR_T(X)$ [resp. $CR_E(X)$ and $CR_O(X)$]. *Then* $\sqcap_{j \in J} R_j \in CR_T(X)$ [*resp.* $CR_E(X)$ *and* $CR_O(X)$].

Proposition 5.5. *Let* $\mathcal{R} =< R, \lambda >\in CR_E(X)$. *Then* $\mathcal{R} = \mathcal{R} \circ_P \mathcal{R}$.

Proof. Since $\mathcal{R}$ is transitive, $\mathcal{R} \circ_P \mathcal{R} \sqsubset \mathcal{R}$. Then it is sufficient to show that $\mathcal{R} \sqsubset \mathcal{R} \circ_P \mathcal{R}$. Let $(x, y) \in X \times X$. Then

$$(\mathcal{R} \circ_P \mathcal{R})(x, y) =< \breve{\bigvee}_{z\in Y}[R(x, y)\breve{\wedge}R(y, z)], \bigvee_{z\in Y}[\lambda(x, z) \wedge \lambda(z, y)] > .$$

On the other hand,

$$\begin{aligned}\breve{\bigvee}_{z\in Y}[R(x, z)\breve{\wedge}R(z, y)] &\succeq R(x, y)\breve{\wedge}R(y, y)\\ &= R(x, y)\breve{\wedge}\mathbf{1} \text{ [Since } \mathcal{R} \text{ is transitive]}\\ &= R(x, y)\end{aligned}$$

and

$$\begin{aligned}\bigvee_{z\in Y}[\lambda(x, z) \wedge \lambda(z, y)] &\geq \lambda(x, y) \wedge \lambda(y, y)\\ &= \lambda(x, y) \wedge 1 \text{ [Since } \mathcal{R} \text{ is transitive]}\\ &= \lambda(x, y).\end{aligned}$$

Thus $\mathcal{R} \sqsubset \mathcal{R} \circ_P \mathcal{R}$. So $\mathcal{R} = \mathcal{R} \circ_P \mathcal{R}$. □

Theorem 5.6. *Let* $\mathcal{R} =< R, \lambda >$, $\mathcal{S} =< S, \mu >\in CR_E(X)$. *Then* $\mathcal{S} \circ_P \mathcal{R} \in CR_E(X)$ *if and only if* $\mathcal{S} \circ_P \mathcal{R} = \mathcal{R} \circ_P \mathcal{S}$.

Proof. Suppose $\mathcal{S} \circ_P \mathcal{R} \in CR_E(X)$. Then by Theorem 4.18, $\mathcal{S} \circ_P \mathcal{R} = \mathcal{R} \circ_P \mathcal{S}$.

Conversely, suppose $\mathcal{S}\circ_P\mathcal{R} = \mathcal{R}\circ_P\mathcal{S}$. Then by Theorem 4.18, $\mathcal{S}\circ_P\mathcal{R}$ is symmetric. Also by Proposition 4.14, $\mathcal{S}\circ_P\mathcal{R}$ is reflexive. Thus it is sufficient to prove that $\mathcal{S}\circ_P\mathcal{R}$ is transitive.

$$\begin{aligned}(\mathcal{S} \circ_P \mathcal{R}) \circ_P (\mathcal{S} \circ_P \mathcal{R}) &= \mathcal{S} \circ_P (\mathcal{R} \circ_P \mathcal{S}) \circ_P \mathcal{R} \text{ [By Proposition 4.9 (1)]}\\ &= \mathcal{S} \circ_P (\mathcal{S} \circ_P \mathcal{R}) \circ_P \mathcal{R} \text{ [By the hypothesis]}\\ &= (\mathcal{S} \circ_P \mathcal{S}) \circ_P (\mathcal{R} \circ_P \mathcal{R})\\ &\sqsubset \mathcal{S} \circ_P \mathcal{R}. \text{ [Since } \mathcal{R} \text{ and } \mathcal{R} \text{ are transitive]}\end{aligned}$$

So $\mathcal{S} \circ_P \mathcal{R} \in CR_E(X)$. □

Proposition 5.7. *Let* $\mathcal{R} =< R, \lambda >$, $\mathcal{S} =< S, \mu >\in CR_E(X)$. *If* $\mathcal{R} \sqcup \mathcal{S} = \mathcal{S} \circ_P \mathcal{R}$, *then* $\mathcal{R} \sqcup \mathcal{S} \in CR_E(X)$.

Proof. Suppose $\mathcal{R} \sqcup \mathcal{S} = \mathcal{S} \circ_P \mathcal{R}$. Since $\mathcal{R}$ and $\mathcal{S}$ are reflexive, by Proposition 4.13 (2), $\mathcal{R} \sqcup \mathcal{S}$ is reflexive. Since $\mathcal{R}$ and $\mathcal{S}$ are symmetric, by Proposition 4.15, $\mathcal{R} \sqcup \mathcal{S}$ is symmetric. Thus by Theorem 4.18, $\mathcal{S} \circ_P \mathcal{R} = \mathcal{R} \circ_P \mathcal{S}$. So by Theorem 5.6, $\mathcal{S} \circ_P \mathcal{R} \in CR_E(X)$. Hence $\mathcal{R} \sqcup \mathcal{S} \in CR_E(X)$. □

Proposition 5.8. *Let* $f : X \to Y$ *be a mapping and let* $\mathcal{R} =< R, \lambda >\in CR_E(Y)$. *We define the mapping* $f^{-1}(\mathcal{R}) : X\times X \to [I]\times I$ *as follows: for each* $(x, y) \in X\times X$,

$$f^{-1}(\mathcal{R})(x, y) = \mathcal{R}(f(x), f(y)) =< R(f(x), f(y)), \lambda(f(x), f(y)) > .$$

Then $f^{-1}(\mathcal{R}) \in CR_E(X)$.

In this case, $f^{-1}(\mathcal{R})$ is called the preimage of $\mathcal{R}$ under f.

Proof. From the definition of $f^{-1}(\mathcal{R})$, it is obvious that $f^{-1}(\mathcal{R})$ is a cubic relation on X. Since $\mathcal{R}$ is reflexive and symmetric, by the definition of $f^{-}(\mathcal{R})$, $f^{-}(\mathcal{R})$ is reflexive and symmetric. Let $(x, z) \in X \times X$. Then

$$\begin{aligned}&[f^{-1}(\mathcal{R}) \circ_P f^{-1}(\mathcal{R})](x, z)\\ &=< \breve{\bigvee}_{y\in X}[f^{-1}(\mathcal{R})(x, y)\breve{\wedge}f^{-1}(\mathcal{R})(y, z)], \bigvee_{y\in Y}[f^{-1}(\lambda)(x, y) \wedge f^{-1}(\lambda)(y, z)] >\end{aligned}$$

$=< \check{\bigvee}_{y\in X}[R(f(x),f(y))\check{\wedge}R(f(y),f(z))], \bigvee_{y\in X}[\lambda(f(x),f(y))\wedge\lambda(f(y),f(z))] >$
[By the definition of $f^{-1}(R)$]
$= [\mathcal{R}\circ_P\mathcal{R}](f(x),f(z))$
$\sqsubset \mathcal{R}(f(x),f(z))$ [Since R is transitive]
$= f^{-1}(\mathcal{R})(x,z)$.
Thus $f^{-1}(\mathcal{R})$ is transitive. So $f^{-1}(\mathcal{R})\in CR_E(X)$. □

Definition 5.9. Let $\mathcal{R} =< R,\lambda >\in CR_E(X)$ and let $a\in X$. Then the cubic equivalence class of a by $\mathcal{R}$, denoted by $\mathcal{R}_a$, is a mapping $\mathcal{R}_a : X\to[I]\times I$ defined as follows: for each $x\in X$,

$$\mathcal{R}_a(x) = \mathcal{R}(a,x) =< R_a(x),\lambda_a(x) >,$$

where $R_a : X\to[I]$ and $\lambda_a : X\to I$ are the mappings defined as follows, respectively: for each $x\in X$,

$$R_a(x) = R(a,x) \text{ and } \lambda_a(x) = \lambda(a,x).$$

In fact, for $\mathcal{R}_a\in([I]\times I)^X$, R_a [resp. λ_a] is the interval-valued fuzzy equivalence class of a by R [resp. is the fuzzy equivalence class of a by λ].

We will denote the set of all cubic equivalence classes by $\mathcal{R}$ as $X/\mathcal{R}$ and it will be called the cubic quotient set of X by $\mathcal{R}$.

The following is the immediate result of Definitions 3.1 and 4.3.

Theorem 5.10. *Let $\mathcal{R} =< R,\lambda >$, $\mathcal{S} =< S,\mu >\in CR_E(X)$. Then $\mathcal{R}\sqsubset\mathcal{S}$ if and only if $\mathcal{R}_x\sqsubset\mathcal{S}_x$, for each $x\in X$.*

Theorem 5.11. *Let $\mathcal{R} =< R,\lambda >\in CR_E(X)$ and let $x,\ y\in X$. Then*
(1) $\mathcal{R}_x\neq\hat{0}$,
(2) $\mathcal{R}_x\sqcap\mathcal{R}_y=\hat{0}$ *if and only if* $R(x,y) =< \mathbf{0},0 >$,
(3) $\mathcal{R}_x=\mathcal{R}_y$ *if and only if* $\mathcal{R}(x,y) =< \mathbf{1},1 >$,
(4) $\sqcup_{x\in X}\mathcal{R}_x=\hat{1}$.

Proof. (1) Since $\mathcal{R}$ is reflexive, $\mathcal{R}_x(x) = R(x,x) =< \mathbf{1},1 >\neq< \mathbf{0},0 >$. Then $\mathcal{R}_x\neq\hat{0}$.
(2) Suppose $\mathcal{R}_x\sqcap\mathcal{R}_y=\hat{0}$ and let $z\in X$. Then

$$(\mathcal{R}_x\sqcap\mathcal{R}_y)(z) = \mathcal{R}_x(z)\sqcap\mathcal{R}_y(z) =< R(x,z)\check{\wedge}R(y,z),\lambda(x,z)\wedge\lambda(y,z) >=< \mathbf{0},0 > .$$

Thus
$\mathcal{R}(x,y) = (\mathcal{R}\circ_P\mathcal{R})(x,y)$ [By Theorem 5.5]
$=< \check{\bigvee}_{z\in X}[R(x,z)\check{\wedge}R(z,y)], \bigvee_{z\in X}[\lambda(x,z)\wedge\lambda(z,y)]$
$=< \check{\bigvee}_{z\in X}[R(x,z)\check{\wedge}R(y,z)], \bigvee_{z\in X}[\lambda(x,z)\wedge\lambda(y,z)]$
[Since R is symmetric]
$= \sqcup_{z\in X}[R(x,z)\sqcap R(y,z)]$
$= \sqcup_{z\in X}[(\mathcal{R}_x\sqcap\mathcal{R}_y)(z)]$
$=< \mathbf{0},0 >$.
The proof of the converse is clear.
(3) The proof is similar to (2).
(4) Let $y\in X$. Then

$$(\sqcup_{x\in X}R_x)(y) = \sqcup_{x\in X}R_x(y) =< \check{\bigvee}_{x\in X}R(x,y), \bigvee_{x\in X}\lambda(x,y) > .$$

On the other hand, $\breve{\bigvee}_{x\in X}R(x,y) \succeq R(y,y) = \mathbf{1}$ and $\bigvee_{x\in X}\lambda(x,y) \geq \lambda(y,y) = 1$. Thus $(\sqcup_{x\in X}\mathcal{R}_x)(y) =< \mathbf{1}, 1 >$. So $\sqcup_{x\in X}\mathcal{R}_x = \hat{1}$. □

Definition 5.12. Let $\Sigma = (\mathcal{A}_j)_{j\in J} =< (A_j)_{j\in J}, (\lambda_j)_{j\in J} >$ be any subfamily of $([I]\times I)^X$. Then Σ is called a cubic partition of X, if it satisfies the following conditions:

(i) $\mathcal{A}_j \neq \hat{0}$, for each $j \in J$,
(ii) either $\mathcal{A}_j = \mathcal{A}_k$ or $\mathcal{A}_j \sqcap \mathcal{A}_k = \hat{0}$, for any $j,\ k \in J$,
(iii) $\sqcup_{j\in J}\mathcal{A}_j = \hat{1}$.

The following is the immediate result of Theorem 5.11 and Definition 5.12.

Corollary 5.13. *Let $\mathcal{R} \in CR_E(X)$. Then $X/\mathcal{R}$ is a cubic partition of X.*

Proposition 5.14. *Let Σ be a cubic partition of X and let $\mathcal{R}(\Sigma) : X\times X \to [I]\times I$ be the mapping defined by: for each $(x,y) \in X\times X$,*

$$\mathcal{R}(\Sigma)(x,y) = \sqcup_{\mathcal{A}\in\Sigma}[\mathcal{A}(x) \sqcap \mathcal{A}(y)].$$

Then $\mathcal{R}(\Sigma) \in CR_E(X)$.

Proof. Let $x \in X$. Then by Definition 5.12 (iii),

$$\mathcal{R}(\Sigma)(x,x) = \sqcup_{\mathcal{A}\in\Sigma}[\mathcal{A}(x)\sqcap\mathcal{A}(x)] = \sqcup_{\mathcal{A}\in\Sigma}\mathcal{A}(x) = [\sqcup_{\mathcal{A}\in\Sigma}\mathcal{A}](x) =< \mathbf{1}, 1 > .$$

Thus $\mathcal{R}(\Sigma)$ is reflexive.

From the definition of $\mathcal{R}(\Sigma)$, it is obvious that $\mathcal{R}(\Sigma)$ is symmetric. Let $(x,y) \in X\times X$. Then

$$\begin{aligned}
&[\mathcal{R}(\Sigma)\circ_P \mathcal{R}(\Sigma)](x,y)\\
&= \sqcup_{z\in X}[\mathcal{R}(\Sigma)(x,z)\sqcap\mathcal{R}(\Sigma)(z,y)]\\
&= \sqcup_{z\in X}(\sqcup_{\mathcal{A}\in\Sigma}[\mathcal{A}(x)\sqcap\mathcal{A}(z)]\sqcap\sqcup_{\mathcal{B}\in\Sigma}[\mathcal{B}(z)\sqcap\mathcal{B}(y)])\\
&= \sqcup_{z\in X}(\sqcup_{\mathcal{A},\ \mathcal{B}\in\Sigma}[\mathcal{A}(x)\sqcap(\mathcal{A}(z)\sqcap\mathcal{B}(z))\sqcap\mathcal{B}(y)])
\end{aligned}$$

sqsubset $\sqcup_{\mathcal{A}\in\Sigma}\ (\sqcup_{z\in X}\mathcal{A}(z))\ \sqcup_{\mathcal{A}\in\Sigma}\ [\mathcal{A}(x)\sqcap\mathcal{A}(y)]$

$$\begin{aligned}
&[\text{By Definition 5.12 (ii)}, \mathcal{A}(y) = \mathcal{B}(y)]\\
&= \hat{1}\ \sqcup_{\mathcal{A}\in\Sigma}\ [\mathcal{A}(x)\sqcap\mathcal{B}(y)]\ [\text{By Definition 5.12 (iii)}]\\
&= \sqcup_{\mathcal{A}\in\Sigma}[\mathcal{A}(x)\sqcap\mathcal{B}(y)]\\
&= \sqcup_{\mathcal{A}\in\Sigma}[\mathcal{A}(x)\sqcap\mathcal{A}(y)]\\
&= \mathcal{R}(\Sigma)(x,y).
\end{aligned}$$

Thus $\mathcal{R}(\Sigma)$ is transitive. So $\mathcal{R}(\Sigma) \in CR_E(X)$. □

Proposition 5.15. *Let $\mathcal{R} =< R, \lambda >,\ \mathcal{S} =< S, \mu > \in CR_E(X)$ such that $\mathcal{R} \sqsubset \mathcal{S}$. We define the mapping $\mathcal{S}/\mathcal{R} : X/\mathcal{R} \to [I]\times I$ as follows: for each $(x,y) \in X\times X$,*

$$[\mathcal{S}/\mathcal{R}](\mathcal{R}_x, \mathcal{R}_y) = \mathcal{S}(x,y).$$

Then $\mathcal{S}/\mathcal{R} \in CR_E(X/\mathcal{R})$.

In this case, $\mathcal{S}/\mathcal{R}$ will be called the quotient of $\mathcal{S}$ by $\mathcal{R}$.

Proof. From the definition of $\mathcal{S}/\mathcal{R}$, it is clear that $\mathcal{S}/\mathcal{R}$ is a cubic relation on $X/\mathcal{R}$. Since $\mathcal{S}$ is reflexive and symmetric, by the definition of $\mathcal{S}/\mathcal{R}$, $\mathcal{S}/\mathcal{R}$ is reflexive and symmetric. Let $(x,z) \in X\times X$. Then

$$[\mathcal{S}/\mathcal{R}\circ_P\mathcal{S}/\mathcal{R}](R_x, R_z) = \sqcup_{y\in X}[\mathcal{S}/\mathcal{R}(\mathcal{R}_x,\mathcal{R}_y)\sqcap\mathcal{S}/\mathcal{R}(\mathcal{R}_y,\mathcal{R}_z)]$$

$= \sqcup_{y \in X}[\mathcal{S}(x,y) \sqcap \mathcal{S}(y,z)]$ [By the definition of $\mathcal{S}/\mathcal{R}$]
$\sqsubset \mathcal{S}(x,z)$ [Since $\mathcal{S}$ is transitive]
$= [\mathcal{S}/\mathcal{R}](\mathcal{R}_x, \mathcal{R}_z)$.
Thus $\mathcal{S}/\mathcal{R}$ is transitive. So $\mathcal{S}/\mathcal{R} \in CR_E(X/\mathcal{R})$. □

The following is the immediate result of Proposition 4.14.

Corollary 5.16. *Let $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T} \in CR_E(X)$. If $\mathcal{R} \sqsubset \mathcal{S} \sqsubset \mathcal{T}$, then $\mathcal{S}/\mathcal{R} \sqsubset \mathcal{T}/\mathcal{R}$.*

Proposition 5.17. *Let $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T} \in CR_E(X)$. Suppose $\mathcal{R} \sqsubset \mathcal{S} \sqsubset \mathcal{T}$. Then*
(1) *$\mathcal{R} \sqsubset \mathcal{S} \circ_P \mathcal{T}$,*
(2) *if $\mathcal{S} \circ_P \mathcal{T} \in CR_E(X)$, then $(\mathcal{S}/\mathcal{R}) \circ_P (\mathcal{T}/\mathcal{R}) = (\mathcal{S} \circ_P \mathcal{T})/\mathcal{R}$,*
(3) *$(\mathcal{S}/\mathcal{R}) \circ_P (\mathcal{T}/\mathcal{R}) \in CR_E(X/\mathcal{R})$.*

Proof. Let $(x,y) \in X \times X$. Then
$(\mathcal{S} \circ_P \mathcal{T})(x,y) = \sqcup_{z \in X}[\mathcal{T}(x,z) \sqcap \mathcal{S}(z,y)]$
$\sqsupset \sqcup_{z \in X}[\mathcal{R}(x,z) \sqcap \mathcal{R}(z,y)]$ [Since $\mathcal{R} \sqsubset \mathcal{S} \sqsubset \mathcal{T}$]
$= (\mathcal{R} \circ_P \mathcal{R})(x,y)$
$= \mathcal{R}(x,y)$. [By Proposition 5.5]
Thus $\mathcal{R} \sqsubset \mathcal{S} \circ_P \mathcal{T}$.
(2) Suppose $\mathcal{S} \circ_P \mathcal{T} \in CR_E(X)$ and let $(x,y) \in X \times X$. Then
$[(\mathcal{S}/\mathcal{R}) \circ_P (\mathcal{T}/\mathcal{R})](\mathcal{R}_x, \mathcal{R}_y) = \sqcup_{z \in X}[\mathcal{T}/\mathcal{R}(\mathcal{R}_x, \mathcal{R}_z) \sqcap \mathcal{S}/\mathcal{R}(\mathcal{R}_z, \mathcal{R}_y)]$
$= \sqcup_{z \in X}[\mathcal{T}(x,z) \sqcap \mathcal{S}(z,y)]$
$= (\mathcal{S} \circ_P \mathcal{T})(x,y)$
$= [(\mathcal{S} \circ_P \mathcal{T})/\mathcal{R}](\mathcal{R}_x, \mathcal{R}_y)$.
Thus $(\mathcal{S}/\mathcal{R}) \circ_P (\mathcal{T}/\mathcal{R}) = (\mathcal{S} \circ_P \mathcal{T})/\mathcal{R}$.
(3) The proof is clear. □

Proposition 5.18. *Let $f : X \to Y$ be a mapping. We define the mapping $\mathcal{R}$: $X \times X \to [I] \times I$ as follows: for each $(x,y) \in X \times X$,*

$$\mathcal{R}(x,y) = \begin{cases} < \mathbf{1}, 1 > & \text{if } f(x) = f(y) \\ < \mathbf{0}, 0 > & \text{otherwise.} \end{cases}$$

Then $\mathcal{R} \in CR_E(X)$.

In this case, $\mathcal{R}$ is called the cubic equivalence relation determined by f and we will denote $\mathcal{R}$ by $\mathcal{R}_f$.

Proof. From the definition of $\mathcal{R}$, it is clear that R is a cubic relation on X. For each $x \in X$, it is clear that $f(x) = f(x)$. Then by the definition of $\mathcal{R}$, $\mathcal{R}(x,x) =< \mathbf{1}, 1 >$. Thus $\mathcal{R}$ is reflexive. It is obvious that $\mathcal{R}$ is symmetric. Let $(x,z) \in X \times X$. Then
$(\mathcal{R} \circ_P \mathcal{R})(x,z) = \sqcup_{y \in X}[\mathcal{R}(x,y) \sqcap \mathcal{R}(y,z)]$
$\sqsupset \mathcal{R}(x,x) \sqcap \mathcal{R}(x,z)$
$=< \mathbf{1}, 1 > \sqcap \mathcal{R}(x,z) = \mathcal{R}(x,z)$.
Suppose $f(x) = f(z)$, then $\mathcal{R}(x,z) =< \mathbf{1}, 1 >$. Thus $(\mathcal{R} \circ_P \mathcal{R})(x,z) =< \mathbf{1}, 1 >$.
Suppose $f(x) \neq f(z)$, then $(\mathcal{R} \circ_P \mathcal{R})(x,z) = \mathcal{R}(x,z) =< \mathbf{0}, 0 >$. Thus in either cases, $\mathcal{R} \circ_P \mathcal{R} = \mathcal{R}$. So $\mathcal{R} \in CR_E(X)$. □

Definition 5.19. Let $\mathcal{R} =< R, \lambda >\in ([I] \times I)^{X \times X}$ and let $< \widetilde{a}, \alpha >\in [I] \times I$ such that $\widetilde{a} \neq \mathbf{0}$ and $\alpha \neq 0$. Then the $< \widetilde{a}, \alpha >$-level relation of $\mathcal{R}$, denoted by $[\mathcal{R}]_{<\widetilde{a}, \alpha>}$, is a subset of $X \times X$ defined as follows:

$$[\mathcal{R}]_{<\widetilde{a},\alpha>} = \{(x,y) \in X \times X : R(x,y) \succeq \widetilde{a},\ \lambda(x,y) \geq \alpha\}.$$

It is obvious that if $\widetilde{a} \preceq \widetilde{b}$ and $\alpha \leq \beta$, then $[\mathcal{R}]_{<\widetilde{a},\alpha>} \sqsupset [\mathcal{R}]_{<\widetilde{b},\beta>}$.

Proposition 5.20. *Let* $\mathcal{R} =< R, \lambda >\in ([I] \times I)^{X\times X}$ *and let* $< \widetilde{a}, \alpha >\in [I] \times I$ *such that* $\widetilde{a} \neq \mathbf{0}$ *and* $\alpha \neq 0$*. Then*
(1) $([\mathcal{R}]_{<\widetilde{a},\alpha>})^{-1} = [\mathcal{R}^{-1}]_{<\widetilde{a},\alpha>}$,
(2) $[\mathcal{R} \circ_P \mathcal{R}]_{<\widetilde{a},\alpha>} = [\mathcal{R}]_{<\widetilde{a},\alpha>} \circ [\mathcal{R}]_{<\widetilde{a},\alpha>}$.

Proof. (1) Let $x,\ y \in X$. Then

$$\begin{aligned}(x,y) \in ([\mathcal{R}]_{<\widetilde{a},\alpha>})^{-1} &\Leftrightarrow (y,x) \in [\mathcal{R}]_{<\widetilde{a},\alpha>}\\ &\Leftrightarrow R^{-1}(x,y) = R(y,x) \succeq \widetilde{a} \text{ and } \lambda^{-1}(x,y) = \lambda(y,x) \geq \alpha\\ &\Leftrightarrow (x,y) \in [\mathcal{R}^{-1}]_{<\widetilde{a},\alpha>}.\end{aligned}$$

(2) Let $x,\ y \in X$. Then

$$\begin{aligned}&(x,y) \in [\mathcal{R}]_{<\widetilde{a},\alpha>} \circ [\mathcal{R}]_{<\widetilde{a},\alpha>}\\ \Leftrightarrow\ &\text{there is } z \in X \text{ such that } (x,z) \in [\mathcal{R}]_{<\widetilde{a},\alpha>},\ (z,y) \in [\mathcal{R}]_{<\widetilde{a},\alpha>}\\ \Leftrightarrow\ &\text{there is } z \in X \text{ such that } R(x,z) \succeq \widetilde{a},\ \lambda(x,z) \geq \alpha;\ R(z,y) \succeq \widetilde{a},\ \lambda(z,y) \geq \alpha\\ \Leftrightarrow\ &(\mathcal{R} \circ_P \mathcal{R})(x,y) = \sqcup_{z\in X}[\mathcal{R}(x,z) \sqcap \mathcal{R}(z,y)] \sqsupset< \widetilde{a}, \alpha >\\ \Leftrightarrow\ &(x,y) \in [\mathcal{R} \circ_P \mathcal{R}]_{<\widetilde{a},\alpha>}.\end{aligned}$$

□

Proposition 5.21. *Let* $\mathcal{R} =< R, \lambda >\in ([I] \times I)^{X\times X}$ *and let* $([\mathcal{R}]_{<\widetilde{a},\alpha>})_{<\widetilde{a},\alpha>\in[I]\times I}$ *be the family of all* $< \widetilde{a}, \alpha >$*-level relations of* $\mathcal{R}$*. Then* $([\mathcal{R}]_{<\widetilde{a},\alpha>})_{<\widetilde{a},\alpha>\in[I]\times I}$ *is descending and for each* $< \widetilde{a}, \alpha >\in [I] \times I$ *with* $\widetilde{a} \neq \mathbf{0}$ *and* $\alpha \neq 0$*,*

$$[\mathcal{R}]_{<\widetilde{a},\alpha>} = \sqcap_{\widetilde{b}\prec\widetilde{a},\ \beta<\alpha}[\mathcal{R}]_{<\widetilde{b},\beta>}.$$

Proof. For any $< \widetilde{a}, \alpha >,\ < \widetilde{b}, \beta >\in [I] \times I$, suppose $< \widetilde{b}, \beta >\sqsubset< \widetilde{a}, \alpha >$. Then clearly, $[\mathcal{R}]_{<\widetilde{a},\alpha>} \sqsubset [\mathcal{R}]_{<\widetilde{b},\beta>}$. Thus $([\mathcal{R}]_{<\widetilde{a},\alpha>})_{<\widetilde{a},\alpha>\in[I]\times I}$ is a descending family of ordinary relations on X. So

$$[\mathcal{R}]_{<\widetilde{a},\alpha>} \sqsubset \sqcap_{\widetilde{b}\prec\widetilde{a},\ \beta<\alpha}[\mathcal{R}]_{<\widetilde{b},\beta>},$$

for each $< \widetilde{a}, \alpha >\in [I] \times I$ such that $\widetilde{a} \neq \mathbf{0}$ and $\alpha \neq 0$.

Assume that $(x,y) \notin [\mathcal{R}]_{<\widetilde{a},\alpha>}$. Then by the definition of $[\mathcal{R}]_{<\widetilde{a},\alpha>}$, either $R(x,y) \prec \widetilde{a}$ or $\lambda(x,y) < \alpha$. Thus there is $< \mathbf{0}, 0 >\neq< \widetilde{b}, \beta >\in [I] \times I$ such that $R(x,y) \prec \widetilde{b} \prec \widetilde{a}$ or $\lambda(x,y) < \beta < \alpha$. So $(x,y) \notin \sqcup_{\widetilde{b}\prec\widetilde{a},\ \beta<\alpha}[\mathcal{R}]_{<\widetilde{b},\beta>}$. Hence $\sqcup_{\widetilde{b}\prec\widetilde{a},\ \beta<\alpha}[\mathcal{R}]_{<\widetilde{b},\beta>} \sqsubset [\mathcal{R}]_{<\widetilde{a},\alpha>}$. Therefore $[\mathcal{R}]_{<\widetilde{a},\alpha>} = \sqcap_{\widetilde{b}\prec\widetilde{a},\ \beta<\alpha}[\mathcal{R}]_{<\widetilde{b},\beta>}$. □

Proposition 5.22. *Let* $\mathcal{R} =< R, \lambda >\in ([I] \times I)^{X\times X}$ *and let* $[\mathcal{R}]_{<\widetilde{a},\alpha>} \neq \phi$*, for each* $< \widetilde{a}, \alpha >\in [I] \times I$*. If* $\mathcal{R} \in CR_E(X)$*, then* $[\mathcal{R}]_{<\widetilde{a},\alpha>}$ *is an equivalence relation on* X. In this case, $[\mathcal{R}]_{<\widetilde{a},\alpha>}$ will be called the $< \widetilde{a}, \alpha >$-level equivalence relation on X by $\mathcal{R}$.

Proof. Suppose $\mathcal{R} \in CR_E(X)$. Then clearly, $\mathcal{R}(x,x) =< \mathbf{1}, 1 >$, for each $x \in X$. Thus $R(x,x) \succeq \widetilde{a}$ and $\lambda(x,x) \geq \alpha$. So $(x,x) \in [\mathcal{R}]_{<\widetilde{a},\alpha>}$. Hence $[\mathcal{R}]_{<\widetilde{a},\alpha>}$ is reflexive.

Now let $(x,y),\ (y,z) \in [\mathcal{R}]_{<\widetilde{a},\alpha>}$. Then clearly, $R(x,y) \succeq \widetilde{a},\ \lambda(x,y) \geq \alpha$ and $R(y,z) \succeq \widetilde{a},\ \lambda(y,z) \geq \alpha$. Thus

$$\begin{aligned}R(x,z) &= (R \circ R)(x,z) \text{ [Since } R \text{ is interval-valued fuzzy transitive]}\\ &= \bigvee_{a\in X}[R(x,a) \curlywedge R(a,z)]\end{aligned}$$

$\succeq R(x,y)\check{\wedge}R(y,z)$
$\succeq \widetilde{a}.$

Similarly, we have $\lambda(x,z) \geq \alpha$. So $(x,z) \in [\mathcal{R}]_{<\widetilde{a},\alpha>}$. Hence $[\mathcal{R}]_{<\widetilde{a},\alpha>}$ is transitive.

Finally let $(x,y) \in [\mathcal{R}]_{<\widetilde{a},\alpha>}$. Then clearly, $R(x,y) \succeq \widetilde{a}$ and $\lambda(x,y) \geq \alpha$. Since $\mathcal{R}$ is cubic symmetric, $R(y,x) = R^{-1}(x,y) = R(x,y) \succeq \widetilde{a}$ and $\lambda(y,x) = \lambda^{-1}(x,y) = \lambda(x,y) \geq \alpha$. Thus $(y,x) \in [\mathcal{R}]_{<\widetilde{a},\alpha>}$. So $[\mathcal{R}]_{<\widetilde{a},\alpha>}$ is symmetric. Hence $[\mathcal{R}]_{<\widetilde{a},\alpha>}$ is an equivalence relation on X.

□

Definition 5.23. Let $R,\ S \subset X \times X$ and let $<\widetilde{a},\alpha> \in [I] \times I$ such that $\widetilde{a} \neq \mathbf{0}$ and $\alpha \neq 0$. We define two mappings $\widetilde{R} : X \times X \to [I]$ and $\lambda_S : X \times X \to I$, respectively as follows: for each $(x,y) \in X \times X$,

$$\widetilde{R}(x,y) = \begin{cases} \mathbf{1} & \text{if } (x,y) \in R \text{ and } x = y \\ \widetilde{a} & \text{if } (x,y) \in R \text{ and } x \neq y \\ \mathbf{0} & \text{otherwise} \end{cases}$$

and

$$\lambda_S(x,y) = \begin{cases} 1 & \text{if } (x,y) \in S \text{ and } x = y \\ \alpha & \text{if } (x,y) \in S \text{ and } x \neq y \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.24. *Let $<\widetilde{a},\alpha> \in [I] \times I$ such that $\widetilde{a} \neq \mathbf{0}$ and $\alpha \neq 0$. If R and S are equivalence relations on a set X, then $\mathcal{R} = <\widetilde{R}, \lambda_S> \in CR_E$.*

Proof. Since R and S are reflexive, $(x,x) \in R$ and $(x,x) \in S$. Then $\widetilde{R}(x,x) = \mathbf{1}$ and $\lambda_S(x,x) = 1$. Thus $\mathcal{R}$ is cubic reflexive.

Now let $(x,y) \in X \times X$ such that $x \neq y$.

Case 1: If $(x,y) \in R$ and $(x,y) \in S$, then $\widetilde{R}(x,y) = \widetilde{a}$ and $\lambda_S(x,y) = \alpha$. Since R and S are symmetric, $(y,x) \in R$ and $(y,x) \in S$. Thus $R(y,x) = \widetilde{a}$ and $\lambda_S(y,x) = \alpha$. So $\mathcal{R}(x,y) = \mathcal{R}(y,x) = <\widetilde{a},\alpha>$.

Case 2: If $(x,y) \notin R$ and $(x,y) \in S$, then $\widetilde{R}(x,y) = \mathbf{0}$ and $\lambda_S(x,y) = \alpha$. Since R and S are symmetric, $(y,x) \notin R$ and $(y,x) \in S$. Thus $\widetilde{R}(y,x) = \mathbf{0}$ and $\lambda_S(y,x) = \alpha$. So $\mathcal{R}(x,y) = \mathcal{R}(y,x) = <\mathbf{0},\alpha>$.

Case 3: If $(x,y) \in R$ and $(x,y) \notin S$, then the proof is similar to Case 2.

Case 4: If $(x,y) \notin R$ and $(x,y) \notin S$, then $\widetilde{R}(x,y) = \mathbf{0}$ and $\lambda_S(x,y) = 0$. Since R and S are symmetric, $(y,x) \notin R$ and $(y,x) \notin S$. Thus $\widetilde{R}(y,x) = \mathbf{0}$ and $\lambda_S(y,x) = 0$. So $\mathcal{R}(x,y) = \mathcal{R}(y,x) = <\mathbf{0},0>$. Hence in either cases, $\mathcal{R}$ is cubic symmetric.

Finally let $x,\ z \in X$ and suppose $x = z$. Then clearly,

$$\mathcal{R}(x,z) = <\mathbf{1},1> \sqsupset (\mathcal{R} \circ_P \mathcal{R})(x,z).$$

Thus

Case 1: If $(x,z) \in R$ and $(x,z) \in S$, then clearly, $\widetilde{R}(x,z) = \widetilde{a}$ and $\lambda_S(x,z) = \alpha$. Since R and S are transitive, $(x,z) \in R \circ R$ and $(x,z) \in S \circ S$. Thus there is $y \in X$ such that $(x,y) \in R$, $(y,z) \in R$ and $(x,y) \in S$, $(y,z) \in S$. So

$$(\widetilde{R} \circ \widetilde{R})(x,z) = \check{\bigvee}_{y \in X} [\widetilde{R}(x,y) \check{\wedge} \widetilde{R}(y,z) = \widetilde{a} = \widetilde{R}(x,z)$$

and

$$(\lambda_S \circ \lambda_S)(x,z) = \bigvee_{y \in X} [\lambda_S(x,y) \wedge (y,z) = \alpha = \lambda_S(x,z).$$

Case 2: If $(x,z) \notin R$ and $(x,z) \in S$, then clearly, $\widetilde{R}(x,z) = \mathbf{0}$ and $\lambda_S(x,z) = \alpha$. Since R is transitive, either $(x,y) \notin R$ or $(y,z) \notin R$, for each $y \in X$. Thus either $\widetilde{R}(x,y) = \mathbf{0}$ or $\widetilde{R}(y,z) = \mathbf{0}$, for each $y \in X$. So $\widetilde{R}(x,z) = \mathbf{0} = (\widetilde{R} \circ \widetilde{R})(x,z)$. By the proof of the above Case 1, $(\lambda_S \circ \lambda_S)(x,z) = \alpha = \lambda_S(x,z)$.

Case 3: If $(x,z) \in R$ and $(x,z) \notin S$, then the proof is similar to the above Case 2.

Case 4: If $(x,z) \notin R$ and $(x,z) \notin S$, then we can show that the followings hold:

$$(\widetilde{R} \circ \widetilde{R})(x,z) = \mathbf{0} = \widetilde{R}(x,z) \text{ and } (\lambda_S \circ \lambda_S)(x,z) = 0 = \lambda_S(x,z).$$

Hence in either cases, $\mathcal{R} \in CR_E(X)$. This completes the proof. □

6. Conclusions

We introduced the concepts of cubic reflexive [resp. symmetric and transitive] relation, and we define the composition of two cubic relations and the inverse of a cubic relation. And we investigated some of each properties and gave some examples. Also we defined a cubic equivalence relation, a cubic equivalence class and a cubic partition, and obtained some of its properties, respectively (In particular, see Corollary 5.13 and Proposition 4.14). Furthermore, we defined a $< \widetilde{a}, \alpha >$-level relation of a cubic relation and studied some of its properties. In the future, we expect that one can apply the concept of cubic sets to decision makings, graph theories, congruences in algebras, quotient spaces in topological spaces, etc.

Acknowledgements. The authors would like to thank the anonymous reviewers for careful and kind advices and valuable suggestions.

Funding: This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2018R1D1A1B07049321)

References

[1] S. S. Ahn and J. M. Ko, Cubic subalgebras and filters of CI-algebras, Honam Math. J. 36 (1) (2014) 43–54.
[2] M. Akram, N. Yaqoob and M. Gulistan, Cubic KU-subalgebras, Int. J. Pure Appl. Math. 89 (5) (2013) 659–665.
[3] M. K. Chakraborty and Mili Das, Studies in fuzzy relations over fuzzy subsets, Fuzzy sets and Systems 9 (1983) 79–89.
[4] M. K. Chakraborty and Mili Das, On fuzzy equivalence I, Fuzzy sets and Systems 11 (1983) 185–193.
[5] M. K. Chakraborty and Mili Das, On fuzzy equivalence II, Fuzzy sets and Systems 11 (1983) 299–307.
[6] M. K. Chakraborty and Subhra Sarkar, Fuzzy antisymmetry and order, Fuzzy sets and Systems 21 (1987) 169–182.

[7] V. Chinnadurai, K. Suganya and A. Swaminathan, Some characterizations on cubic sets, International Journal of Multidisciplinary Research and Modern Education (IJMRME) II (II) (2016) 258–264.
[8] K. A. Dib and Nabil L. Youssef, Fuzzy Cartesian product, fuzzy relations and fuzzy functions, Fuzzy sets and Systems 41 (1991) 299–315.
[9] M. B. Gorzalczany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, Fuzzy sets and Systems 21 (1987) 1–17.
[10] K. C. Gupta and R. K. Gupta, Fuzzy equivalence relation redefined, Fuzzy sets and Systems 79 (1996) 227–233.
[11] K. Hur, J. G. Lee and J. Y. Choi, Interval-valued fuzzy relations, JKIIS 19 (3) (2009) 425–432.
[12] Y. B. Jun and A. Khan, Cubic ideals in semigroups, Honam Math. J. 35 (4) (2013) 607–623.
[13] Y. B. Jun, S. T. Jun and M. S. Kim, Cubic subgroups, Ann. Fuzzy Math. Inform. 2 (1) (2011) 9–15.
[14] Y. B. Jun, C. S. Kim and K. O. Yang, Cubic sets, Ann. Fuzzy Math. Inform. 4 (1) (2012) 83–98.
[15] Y. B. Jun and K. J. Lee, Closed cubic ideals and cubic 0-subalgebras in BCK/BCI-algebras, Appl. Math. Sci. 4 (65–68) (2009) 3395–3402.
[16] Y. B. Jun, K. J. Lee and M. S. Kang, Cubic structures applied to ideals of BCI-algebras, Comput. Math. Appl. 62 (9) (2011) 3334–3342.
[17] Y. B. Jun, S. Smarandache and C. S. Kim, Neutrosophic cubic sets, New Mathematics and Natural Computation (2015) 1–15.
[18] Y. B. Jun, S. Z. Song and S. J. Kim Cubic interval-valued intuitionistic fuzzy sets and their application in BCK/BCI-algebras, Axioms 2018, 7, 7; doi:10.3390/axioms7010007.
[19] J. G. Kang and C. S. Kim, Mapping of cubic sets, Commun. Korean Math. Soc. 31 (3) (2016) 423–431.
[20] K. C. Lee, Fuzzy equivalence relations and fuzzy functions, International Journal of Fuzzy Logic and Intelligent Systems 9 (1) (2009) 20–29.
[21] Xue-Ling Ma, Jianming Zhan, Madad Khan, Muhammad Gulistan and Naveed Yaqoob, Generalized cubic relations in H_v-LA-semigroups, Journal of Discrete Mathematical Sciences and Cryptography 21 (3) (2018) 607–630.
[22] T. Mahmood, S. Abdullah, S. R. Rashid and M. Bila, Multicriteria decision making based on cubic set, Journal of New Theory 16 (2017) 01–09.
[23] Pu Pao-Ming and Liu Ying-Ming, Fuzzy topology I. Neighborhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. 76 (1980) 571–599.
[24] T. K. Mondal and S. K. Samanta, Topology of interval-valued fuzzy sets, Indian J. Pure Appl. Math. 30 (1) (1999) 23–38.
[25] M. K. Roy and R. Biswas, I-v fuuzy relations and Sanchz's approach for medical diagnois, Fuzzy sets and Systems 47 (1992) 35–38.
[26] T. Senapati, C. S. Kim, M. Bhowmik and M. Pal, Cubic subalgebras and cubic closed ideals of B-algebras, Fuzzy Inf. Eng. 7 (2) (2015) 129–149.
[27] N. Yaqoob, S. M. Mostafa and M. A. Ansari, On cubic KU-ideals of KU-algebras, ISRN Algebra (2013), Art. ID 935905, 10 pages.
[28] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[29] L. A. Zadeh, Similarity relations and fuzzy ordering, Inform. Sci. 3 (1971) 177–200.
[30] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, Inform. Sci. 8 (1975) 199–249.
[31] A. Zeb, S. Abdullah, M. Khan and A. Majid, Cubic topoloiy, International Journal of Computer and Information Security (IJCSIS) 14 (8) (2016) 659–669.

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