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Abstract. The concept of fuzzy multigroup is an algebraic structure of fuzzy multiset that generalizes both the theories of classical group and fuzzy group. Fuzzy multigroup constitutes an application of fuzzy multiset to the elementary theory of classical group. In this paper, we give a concise note on α-cuts of fuzzy multigroups and propose the notion of α-cuts homomorphism of fuzzy multigroups. Some properties of α-cuts homomorphism of fuzzy multigroups are explicated. The notions of quasi-surjective mapping, pre-surjective mapping, surjective α-cuts homomorphic mapping and α-cuts isomorphic mapping are established in α-cuts homomorphism of fuzzy multigroups with some number of results.

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Corresponding Author: P. A. Ejegwa (ocholohi@gmail.com)

1. Introduction

The theory of fuzzy sets proposed by [25] is a method for representing imprecision or uncertainty in a collection. For a classical set X, a fuzzy set over X, or a fuzzy subset of X, is characterized by a membership function μ which associates values from the closed unit interval $I = [0, 1]$ to members of X. Since the theory of groups was built from the notion of set theory, Rosenfeld [15] proposed the concept of fuzzy group as an algebraic structure of fuzzy set. In fact, fuzzy group is an application of fuzzy set to group theory. Several works have been done on the theory of fuzzy groups, for some details see [14, 16]. The idea of fuzzy groups has been extended to intuitionistic fuzzy soft groups induced by (t,s)-norm in [24].

Yager [23] applied the idea of multiset [19, 20, 22], which is an extension of set with repeated elements in a collection to propose fuzzy multiset. That is, fuzzy multiset allows repetition of membership degrees of elements in multiset framework.

In fact, fuzzy multiset generalizes fuzzy sets. With this, one can conveniently say that, every fuzzy set is a fuzzy multiset but the reverse is not necessarily true. Fuzzy multisets theory has been extensively studied and applied in real-life problems [2, 3, 8, 11, 12, 13, 18, 21].

The concept of fuzzy multigroups was proposed in [17] as an algebraic structure of fuzzy multisets that generalizes fuzzy groups. This algebraic structure is a multiset of $X \times [0, 1]$ satisfying some set of axioms, where X is a classical group. In fact, since fuzzy multiset is a generalization of fuzzy set, it then follows that fuzzy multigroup is an extension of fuzzy group. The concept of fuzzy multigroups constitutes an application of fuzzy multisets to the notion of group. Fuzzy multigroups and fuzzy groups are different generalizations of classical groups such that, every fuzzy group is a fuzzy multigroup but the converse is not always true. The notion of fuzzy submultigroups of fuzzy multigroups and some properties of fuzzy multigroups were explicated in [5]. The ideas of abelian fuzzy multigroups and order of fuzzy multigroups have been studied [1, 6], and the notion of normal fuzzy submultigroups of fuzzy multigroups was proposed with some number of results in [7]. The ideas of homomorphism and direct product in fuzzy multigroups context were extensively explored in [4, 9, 10].

This paper is motivated to establish a structure that bridges fuzzy multigroups and group theory. Hence the notion of α-cuts of fuzzy multigroups is germaine to connect fuzzy multigroups to groups. The concept of α-cuts homomorphism of fuzzy multigroups is informed because homomorphism in fuzzy multigroup context has been hitherto established. To say the least, α-cuts homomorphism of fuzzy multigroups links homomorphism of fuzzy multigroups to homomorphism in group theory. In this paper, we give a precise note on α-cuts of fuzzy multigroups to enhance the introduction of α-cuts homomorphism of fuzzy multigroups. Some properties of α-cuts homomorphism of fuzzy multigroups are explicated with some number of results.

The paper is organized as follows: In Section 2, some preliminary definitions and results on fuzzy multisets and fuzzy multigroups are reviewed. Section 3 introduces the concept of α-cuts of fuzzy multigroups to enhance the study of α-cuts homomorphism of fuzzy multigroups. In Section 4, the idea of α-cuts homomorphism of fuzzy multigroups is explored with some number of results. Meanwhile, Section 5 discusses upper α-cut homomorphic properties of fuzzy multigroups. Section 6 draws conclusion to the paper and suggests areas of future works.

2. Preliminaries

In this section, we review some existing definitions and results for the sake of completeness and reference.

Definition 2.1 ([23])**.** Assume X is a set of elements. Then, a fuzzy bag/multiset A drwan from X can be characterized by a count membership function CM_A such that

$$CM_A : X \to Q,$$

where Q is the set of all crisp bags or multisets from the unit interval $I = [0, 1]$.

From [21], a fuzzy multiset can also be characterized by a high-order function. In particular, a fuzzy multiset A can be characterized by a function

$$CM_A : X \to N^I \text{ or } CM_A : X \to [0,1] \to N,$$

where $I = [0,1]$ and $N = \mathbb{N} \cup \{0\}$.

By [12], it implies that $CM_A(x)$ for $x \in X$ is given as

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ...\},$$

where $\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ... \in [0,1]$ such that $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x) \geq ...$, whereas in a finite case, we write

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x)\},$$

for $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x)$.

A fuzzy multiset A can be represented in the form

$$A = \{\langle \frac{CM_A(x)}{x} \rangle \mid x \in X\} \text{ or } A = \{\langle x, CM_A(x) \rangle \mid x \in X\}.$$

In a simple term, a fuzzy multiset A of X is characterized by the count membership function $CM_A(x)$ for $x \in X$, that takes the value of a multiset of a unit interval $I = [0,1]$ [2, 13].

We denote the set of all fuzzy multisets by $FMS(X)$.

Example 2.2. Assume that $X = \{a, b, c\}$ is a set. Then for $CM_A(a) = \{1, 0.5, 0.5\}$, $CM_A(b) = \{0.9, 0.7, 0\}$, $CM_A(c) = \{0, 0, 0\}$, A is a fuzzy multiset of X written as

$$A = \{\langle \frac{1, 0.5, 0.5}{a} \rangle, \langle \frac{0.9, 0.7, 0}{b} \rangle, \langle \frac{0, 0, 0}{c} \rangle\}.$$

Definition 2.3 ([11]). Let $A, B \in FMS(X)$. Then, A is called a fuzzy submultiset of B written as $A \subseteq B$ if $CM_A(x) \leq CM_B(x)\ \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy submultiset of B and denoted as $A \subset B$.

Definition 2.4 ([21]). Let $\{A_i\}_{i \in I}$ be a family of fuzzy multisets over X. Then

(i) $CM_{\bigcap_{i\in I} A_i}(x) = \bigwedge_{i \in I} CM_{A_i}(x)\ \forall x \in X$,

(ii) $CM_{\bigcup_{i\in I} A_i}(x) = \bigvee_{i \in I} CM_{A_i}(x)\ \forall x \in X$,

where $\bigwedge$ and $\bigvee$ denote minimum and maximum operations.

Definition 2.5 ([11]). Let $A, B \in FMS(X)$. Then, we say A and B are comparable to each other if and only if $A \subseteq B$ and $B \subseteq A$, and $A = B \Leftrightarrow CM_A(x) = CM_B(x)\ \forall x \in X$. Clearly, the comparability of two fuzzy multisets of X implies equality.

Example 2.6. Supposing $A = \{\langle \frac{1, 0.5, 0.5}{a} \rangle, \langle \frac{0.9, 0.7, 0}{b} \rangle, \langle \frac{0, 0, 0}{c} \rangle\}$ and $B = \{\langle \frac{1, 0.5, 0.5}{a} \rangle, \langle \frac{0.9, 0.7, 0}{b} \rangle\}$ are fuzzy multisets of $X = \{a, b, c\}$. Then A and B are comparable to each other or equal.

Definition 2.7 ([17]). Let X be a group. A fuzzy multiset A over X is called a fuzzy multigroup of X if the count membership function of A, that is,

$$CM_A : X \to [0,1]$$

satisfies the following conditions:

(i) $CM_A(xy) \geq CM_A(x) \wedge CM_A(y)\ \forall x, y \in X$,
(ii) $CM_A(x^{-1}) = CM_A(x)\ \forall x \in X$.

By implication, a fuzzy multiset A over X is called a fuzzy multigroup of a group X, if

$$CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y), \forall x, y \in X.$$

It follows immediately from the definition that,

$$CM_A(e) \geq CM_A(x)\ \forall x \in X,$$

where e is the identity element of X. We denote the set of all fuzzy multigroups of X by $FMG(X)$.

Example 2.8. Let $X = \{1, -1, i, -i\}$ be group. Then, the fuzzy multiset A of X, that is,

$$A = \{\langle \frac{1, 0.8}{1} \rangle, \langle \frac{0.7, 0.6}{-1} \rangle, \langle \frac{0.6, 0.5}{i} \rangle, \langle \frac{0.6, 0.5}{-i} \rangle\}$$

is a fuzzy multigroup of X satisfying the conditions of Definition 2.7.

Definition 2.9 ([5]). Let $\{A_i\}_{i\in I}, I = 1, ..., n$ be an arbitrary family of fuzzy multigroups of X. Then, $\{A_i\}_{i\in I}$ is said to have inf/sup assuming chain if either $A_1 \subseteq A_2 \subseteq ... \subseteq A_n$ or $A_1 \supseteq A_2 \supseteq ... \supseteq A_n$, respectively.

Definition 2.10 ([4]). Let X and Y be groups and let $f : X \to Y$ be a homomorphism. Suppose A and B are fuzzy multigroups of X and Y, respectively. Then, f induces a homomorphism from A to B which satisfies

(i) $CM_A(f^{-1}(y_1y_2)) \geq CM_A(f^{-1}(y_1)) \wedge CM_A(f^{-1}(y_2))\ \forall y_1, y_2 \in Y$,
(ii) $CM_B(f(x_1x_2)) \geq CM_B(f(x_1)) \wedge CM_B(f(x_2))\ \forall x_1, x_2 \in X$,

where

(i) the image of A under f, denoted by $f(A)$, is a fuzzy multiset over Y defined by

$$CM_{f(A)}(y) = \begin{cases} \bigvee_{x\in f^{-1}(y)} CM_A(x), & f^{-1}(y) \neq \varnothing \\ 0, & \text{otherwise}, \end{cases}$$

for each $y \in Y$.

(ii) the inverse image of B under f, denoted by $f^{-1}(B)$, is a fuzzy multiset over X defined by

$$CM_{f^{-1}(B)}(x) = CM_B(f(x))\ \forall x \in X.$$

Theorem 2.11 ([4]). *Let f be a homomorphic mapping from a group X onto a group Y.*
(1) *For $A, B \in FMG(X)$, if $A \subseteq B$, then $f(A) \subseteq f(B)$.*
(2) *For $A, B \in FMG(Y)$, if $A \subseteq B$, then $f^{-1}(A) \subseteq f^{-1}(B)$.*

Proposition 2.12 ([4]). *Let X, Y be two groups and $f : X \to Y$ be a homomorphism. If $A \in FMG(X)$ and $B \in FMG(Y)$, respectively, then $f(A) \in FMG(Y)$ and $f^{-1}(B) \in FMG(X)$.*

3. Some basic notions of alpha-cuts of fuzzy multigroups

In this section, we propose the notion of α-cuts of fuzzy multigroups and explore some of its basic properties. With these, α-cuts homomorphism of fuzzy multigroups are established.

Definition 3.1. Let $A \in FMG(X)$. Then for $\alpha \in [0,1]$, the sets $A_{[\alpha]}$ and $A_{(\alpha)}$ defined by

$$A_{[\alpha]} = \{x \in X \mid CM_A(x) \geq \alpha\}$$

and

$$A_{(\alpha)} = \{x \in X \mid CM_A(x) > \alpha\}$$

are called strong and weak upper α-cuts of A.

Whenever the count membership values of x is greater than or equal to α, that is,

$$CM_A(x) = \{\mu^1, \mu^2, ..., \mu^n\} \geq \alpha,$$

the strong upper α-cut of A exist for such $x \in X$. Likewise the weak upper α-cut of A can be listed.

For example, let $X = \{1, -1, i, -i\}$ be group. Then

$$A = \{\langle \frac{1, 0.8}{1} \rangle, \langle \frac{0.7, 0.6}{-1} \rangle, \langle \frac{0.6, 0.5}{i} \rangle, \langle \frac{0.6, 0.5}{-i} \rangle\}$$

is a fuzzy multigroup of X. Let $\alpha = 0.4, 0.6$. Then

$$A_{[0.4]} = \{1, -1, i, -i\}$$
$$A_{[0.6]} = \{1, -1\}$$

and

$$A_{(0.4)} = \{1, -1, i, -i\}$$
$$A_{(0.6)} = \{1\}.$$

Definition 3.2. Let $A \in FMG(X)$. Then for $\alpha \in [0,1]$, the sets $A^{[\alpha]}$ and $A^{(\alpha)}$ defined by

$$A^{[\alpha]} = \{x \in X \mid CM_A(x) \leq \alpha\}$$

and

$$A^{(\alpha)} = \{x \in X \mid CM_A(x) < \alpha\}$$

are called strong and weak lower α-cuts of A.

The strong and weak lower α-cuts of A can be constructed similarly as in the case of strong and weak upper α-cuts of A.

Remark 3.3. Let $A \in FMG(X)$ and take any $\alpha \in [0,1]$ such that $A_{[\alpha]}$ and $A^{[\alpha]}$ exist. Then, it follows that
(1) $A_{(\alpha)} \subseteq A_{[\alpha]}$ and $A^{(\alpha)} \subseteq A^{[\alpha]}$,
(2) $A_{[\alpha]} = B_{[\alpha]}, A_{(\alpha)} = B_{(\alpha)}, A^{[\alpha]} = B^{[\alpha]}$ and $A^{(\alpha)} = B^{(\alpha)}$ iff $A = B$.

For the purpose of this work, we are interested in the strong alpha–cuts of fuzzy multigroups since $A_{(\alpha)} \subseteq A_{[\alpha]}$ and $A^{(\alpha)} \subseteq A^{[\alpha]}$. Again from henceforth, we are concerned with the situation where $A_{[\alpha]}$ and $A^{[\alpha]}$ exist.

Proposition 3.4. *Let $A, B \in FMG(X)$ and $\alpha, \alpha_1, \alpha_2 \in [0,1]$. Then we have*
(1) $A_{[\alpha_1]} \subseteq A_{[\alpha_2]}$ *iff* $\alpha_1 \geq \alpha_2$,
(2) $A \subseteq B$ *iff* $A_{[\alpha]} \subseteq B_{[\alpha]}$ $\forall \alpha \in [0,1]$.

Proof. Straightforward □

Remark 3.5. Let $A, B \in FMG(X)$ and $\alpha, \alpha_1, \alpha_2 \in [0,1]$. Then the following hold:
(1) $A^{[\alpha_1]} \subseteq A^{[\alpha_2]}$ iff $\alpha_1 \geq \alpha_2$,
(2) $A \subseteq B$ iff $A^{[\alpha]} \subseteq B^{[\alpha]}$ $\forall \alpha \in [0,1]$.

Proposition 3.6. *Let $A \in FMG(X)$. For any $\alpha_1, \alpha_2 \in [0,1]$ such that $\alpha_1 \leq \alpha_2$. Then we have $A_{(\alpha_2)} \subseteq A_{[\alpha_2]} \subseteq A_{(\alpha_1)}$ and $A^{(\alpha_1)} \subseteq A^{(\alpha_2)} \subseteq A^{[\alpha_2]}$.*

Proof. Straightforward □

Theorem 3.7. *Let $A \in FMG(X)$. Then $A_{[\alpha]}, \alpha \in [0,1]$ is a subgroup of X for all $\alpha \leq CM_A(e)$, where e is the identity element of X.*

Proof. Let $x, y \in A_{[\alpha]}$. Then $CM_A(x) \geq \alpha$ and $CM_A(y) \geq \alpha$. Since $A \in FMG(X)$, we get

$$\begin{aligned} CM_A(xy^{-1}) &\geq (CM_A(x) \wedge CM_A(y)) \geq \alpha \\ &= CM_A(x) \geq \alpha \wedge CM_A(y) \geq \alpha. \end{aligned}$$

Thus $xy^{-1} \in A_{[\alpha]}$. So $A_{[\alpha]}, \alpha \in [0,1]$ is a subgroup of X for all $\alpha \leq CM_A(e)$. Also, $A_{(\alpha)}, \alpha \in [0,1]$ is a subgroup of X, if $\alpha < CM_A(e)$. □

Remark 3.8. Let $A \in FMG(X)$. Then $A^{[\alpha]}$, for $\alpha \in [0,1]$ is a subgroup of X for all $\alpha \geq CM_A(e)$, where e is the identity element of X.

4. Alpha-cuts homomorphism of fuzzy muligroups

Having established α-cuts of fuzzy muligroups, we proceed to propose and explore α-cuts homomorphism of fuzzy muligroups in this section.

Definition 4.1. Let X, Y be groups, $A \in FMG(X)$, $B \in FMG(Y)$ and $f : X \to Y$ be a homomorphic mapping. If f is a homomorphic mapping from $A_{[\alpha]}$ to $B_{[\alpha]}$ for any $\alpha \in [0,1]$, then f is called an upper α-cut homomorphic mapping from A to B.

Definition 4.2. Let X, Y be groups, $A \in FMG(X)$, $B \in FMG(Y)$ and $f : X \to Y$ be a homomorphic mapping. If f is a homomorphic mapping from $A^{[\alpha]}$ to $B^{[\alpha]}$ for at least one $\alpha \in [0,1]$, then f is called a lower α-cut homomorphic mapping from A to B.

Proposition 4.3. *Let $f : X \to Y$ be a homomorphism, $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. For any $\alpha \in [0,1]$, we have*
(1) $f(A_{[\alpha]}) \subseteq (f(A))_{[\alpha]}$,
(2) $f^{-1}(B_{[\alpha]}) = (f^{-1}(B))_{[\alpha]}$,
(3) $f(A_{(\alpha)}) \subseteq f(A_{[\alpha]}) \subseteq (f(A))_{[\alpha]}$,
(4) $f^{-1}(B_{(\alpha)}) \subseteq f^{-1}(B_{[\alpha]}) = (f^{-1}(B))_{[\alpha]}$.

Proof. Suppose $A \in FMG(X)$ and $B \in FMG(Y)$, where $f : X \to Y$ is a homomorphism.

(1) Let $y \in f(A_{[\alpha]})$. Then $\exists\, x \in A_{[\alpha]}$ such that $f(x) = y$ and

$$CM_A(x) \geq \alpha, \alpha \in [0, 1].$$

Thus we get

$$CM_A(f^{-1}(y)) \geq \alpha, \alpha \in [0, 1] \text{ implies } CM_{f(A)}(y) \geq \alpha, \alpha \in [0, 1]$$

. So $y \in (f(A))_{[\alpha]}$. Hence $f(A_{[\alpha]}) \subseteq (f(A))_{[\alpha]}$.

(2) For every x, $x \in f^{-1}(B_{[\alpha]}) \Leftrightarrow f(x) \in B_{[\alpha]} \Leftrightarrow CM_B(f(x)) \geq \alpha, \alpha \in [0, 1]$. Then by Definitions 2.10 and 4.1, we see that

$$CM_{f^{-1}(B)}(x) = CM_B(f(x)) \geq \alpha, \alpha \in [0, 1],$$

that is, $x \in (f^{-1}(B))_{[\alpha]}$. Thus $f^{-1}(B_{[\alpha]}) = (f^{-1}(B))_{[\alpha]}$.

(3) Since $A_{(\alpha)} \subseteq A_{[\alpha]}$, $f(A_{(\alpha)}) \subseteq f(A_{[\alpha]})$ by Theorem 2.11. Then the result follows from (1).

(4) Also, $B_{(\alpha)} \subseteq B_{[\alpha]}$. Then $f^{-1}(A_{(\alpha)}) \subseteq f^{-1}(A_{[\alpha]})$ by the same reasons as in (3). The proof is completed by (2). □

Corollary 4.4. *Let $f : X \to Y$ be a homomorphism. Suppose $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. Then for at least one $\alpha \in [0, 1]$,*

(1) $f(A^{[\alpha]}) \subseteq (f(A))^{[\alpha]}$,
(2) $f^{-1}(B^{[\alpha]}) = (f^{-1}(B))^{[\alpha]}$,
(3) $f(A^{(\alpha)}) \subseteq f(A^{[\alpha]}) \subseteq (f(A))^{[\alpha]}$,
(4) $f^{-1}(B^{(\alpha)}) \subseteq f^{-1}(B^{[\alpha]}) = (f^{-1}(B))^{[\alpha]}$.

Proof. Similar to Proposition 4.3. □

Theorem 4.5. *Let $f : X \to Y$ be a homomorphism, $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. Then $(f(A))_{[\alpha]} = f(A_{[\alpha]})$ for any $\alpha \in [0, 1]$ if and only if for each $y \in Y$ there exists $x_0 \in f^{-1}(y)$ such that $CM_{f(A)}(y) = CM_A(x_0)$.*

Proof. Suppose $(f(A))_{[\alpha]} = f(A_{[\alpha]})$. For arbitrary $y \in Y$, let $CM_{f(A)}(y) = \alpha$. Then $y \in (f(A))_{[\alpha]} = f(A_{[\alpha]})$. It follows that there exist $x_0 \in A_{[\alpha]}$ such that $y = f(x_0)$. Thus we have $x_0 \in f^{-1}(y)$ which satisfies $CM_A(x_0) \geq \alpha$. So we have

$$CM_A(x_0) \geq CM_{f(A)}(y) = \bigvee_{x \in f^{-1}(y)} CM_A(x) \geq CM_A(x_0).$$

Hence $CM_{f(A)}(y) = CM_A(x_0)$.

Conversely, suppose for each $y \in Y$, there exists $x_0 \in f^{-1}(y)$ such that $CM_{f(A)}(y) = CM_A(x_0)$. For $\alpha \in [0, 1]$, let $y \in (f(A))_{[\alpha]}$. We show that $y \in f(A_{[\alpha]})$. Since

$$CM_A(x_0) = CM_{f(A)}(y) = \bigvee_{x \in f^{-1}(y)} CM_A(x) \geq \alpha,$$

we have $f(x_0) = y$. Then $x_0 \in A_{[\alpha]}$ implies $y \in f(A_{[\alpha]})$. Thus $(f(A))_{[\alpha]} = f(A_{[\alpha]})$. □

Corollary 4.6. *Let $f : X \to Y$ be a homomorphic mapping, $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. Then $(f(A))^{[\alpha]} = f(A^{[\alpha]})$ for at least one $\alpha \in [0,1]$ if and only if for each $y \in Y$ there exists $x_0 \in f^{-1}(y)$ such that $CM_{f(A)}(y) = CM_A(x_0)$.*

Proof. Similar to Theorem 4.5. □

Definition 4.7. Let X, Y be groups, $f : X \to Y$ and $A \in FMG(X)$. Then for every $y \in Y$, if there exists $x_0 \in f^{-1}(y)$ such that $CM_{f(A)}(y) = CM_A(x_0)$, then f is said to be quasi-surjective.

Lemma 4.8. *Let X, Y be groups, $f : X \to Y$ and $A \in FMG(X)$. Then for at least one $\alpha \in [0,1]$, we have $(f(A))_{[\alpha]} = f(A_{[\alpha]})$ or $(f(A))^{[\alpha]} = f(A^{[\alpha]})$ if and only if f is quasi-surjective.*

Proof. Combining Theorem 4.5, Corollary 4.6 and Definition 4.7, the result follows. □

Theorem 4.9. *Let X, Y be groups, $A \in FMG(X), B \in FMG(Y)$ and $f : X \to Y$ be quasi-surjective. Then f is an upper α-cut homomorphic mapping from A to B if and only if f is a homomorphic mapping from X to Y, and $(f(A))_{[\alpha]} \subseteq B_{[\alpha]}$ for any $\alpha \in [0,1]$.*

Proof. Suppose f is an upper α-cut homomorphic mapping from A to B. Then for every $\alpha \in [0,1]$, we can infer that f is a homomorphic mapping from $A_{[\alpha]}$ to $B_{[\alpha]}$. Actually, $X = A_{[0]}, Y = B_{[0]}$. Thus f is an homomorphic mapping from X to Y. As f is quasi-surjective, in light of Lemma 4.8, we get

$$(f(A))_{[\alpha]} = f(A_{[\alpha]}) \subseteq B_{[\alpha]}.$$

Conversely, suppose f is a homomorphic mapping from X to Y and $(f(A))_{[\alpha]} \subseteq B_{[\alpha]}$ for $\alpha \in [0,1]$. Then for all $\alpha \in [0,1]$, because f is quasi-surjective, for any $x \in A_{[\alpha]} \subseteq X$, we have

$$f(x) \in f(A_{[\alpha]}) = (f(A))_{[\alpha]} \subseteq B_{[\alpha]}.$$

Thus f is a homomorphism from $A_{[\alpha]}$ to $B_{[\alpha]}$. Since f is a homomorphic mapping from X to Y,

$$f(xy) = f(x)f(y) \text{ holds for arbitrary } x, y \in A_{[\alpha]} \subseteq X,$$

where $f(x), f(y) \in B_{[\alpha]}$. Thus by Theorem 3.7, this indicates that $B_{[\alpha]}$ is a subgroup of Y. So $f(x)f(y) \in B_{[\alpha]}$, that is to say, f preserves the operation.
Synthesizing this discussion, f is a homomorphic mapping from $A_{[\alpha]}$ to $B_{[\alpha]}$. Hence, by Definition 4.1, we obtain that f is an upper α-cut homomorphic mapping from A to B. □

Corollary 4.10. *Let X, Y be groups such that $f : X \to Y$ is quasi-surjective, $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. Then f is a lower α-cut homomorphic mapping from A to B if and only if f is a homomorphic mapping from X to Y, and $(f(A))^{[\alpha]} \subseteq B^{[\alpha]}$ for at least one $\alpha \in [0,1]$.*

Proof. Similar to Theorem 4.9. □

Definition 4.11. Let X and Y be groups, $f : X \to Y$, $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. If f is a surjective homomorphic mapping from $A_{[\alpha]}$ to $B_{[\alpha]}$ for any $\alpha \in [0, 1]$, then f is called a surjective upper α-cut homomorphic mapping from A to B.

Theorem 4.12. *Let X and Y be groups, A and B be fuzzy multigroups of X and Y, respectively, and $f : X \to Y$ with f quasi-surjective. Then f is a surjective upper α-cut homomorphic mapping from A to B if and only if f is a surjective homomophic mapping from X to Y with $f(A_{[\alpha]}) = B_{[\alpha]}$ for any $\alpha \in [0, 1]$.*

Proof. Suppose f is a surjective upper α-cut homomorphic mapping from A to B. Then by Definition 4.11, for any $\alpha \in [0, 1]$, it follows that f is a surjective homomorphic mapping from $A_{[\alpha]}$ to $B_{[\alpha]}$. Observe that, $X = A_{[0]}$ and $Y = B_{[0]}$, evidently, f is a surjective homomorphic mapping from X to Y. Clearly, we have $f(A_{[\alpha]}) \subseteq B_{[\alpha]}$. Similarly, $B_{[\alpha]} \subseteq f(A_{[\alpha]})$ is obvious. Thus $f(A_{[\alpha]}) = B_{[\alpha]}$.

Conversely, suppose for any $\alpha \in [0, 1]$ and $y \in B_{[\alpha]}$, $f(A_{[\alpha]}) = B_{[\alpha]}$ implies that $\exists\, x \in A_{[\alpha]}$ such that $f(x) = y$, that is, f is a surjection from $A_{[\alpha]}$ to $B_{[\alpha]}$. To prove that f preserves the operation follows from the converse proof of Theorem 4.5, so we omit it. Hence, for any $\alpha \in [0, 1]$, it follows that f is a surjective upper α-cut homomorphic mapping from A to B. □

Definition 4.13. Let X and Y be groups, $f : X \to Y$, $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. If f is a surjective homomorphic mapping from $A^{[\alpha]}$ to $B^{[\alpha]}$ for at least one $\alpha \in [0, 1]$, then f is called a surjective lower α-cut homomorphic mapping from A to B.

Corollary 4.14. *Let X and Y be groups, A and B be fuzzy multigroups of X and Y, respectively, and $f : X \to Y$ with f quasi-surjective. Then f is a surjective lower α-cut homomorphic mapping from A to B if and only if f is a surjective homomophic mapping from X to Y with $f(A^{[\alpha]}) = B^{[\alpha]}$ for at least one $\alpha \in [0, 1]$.*

Proof. Similar to Theorem 4.12. □

Definition 4.15. Let X and Y be groups, $A \in FMG(X)$, $B \in FMG(Y)$ and $f : X \to Y$. If f is an isomorphic mapping from $A_{[\alpha]}$ to $B_{[\alpha]}$ for any $\alpha \in [0, 1]$, then f is called an upper α-cut isomorphic mapping from A to B.

Theorem 4.16. *Let X and Y be groups, $A \in FMG(X), B \in FMG(Y)$ and $f : X \to Y$ with f quasi-surjective. Then f is an upper α-cut isomorphic mapping from A to B if and only if f is an isomorphic mapping from X to Y with $f(A_{[\alpha]}) = B_{[\alpha]}$ for any $\alpha \in [0, 1]$.*

Proof. The Proof follows by combining Theorems 4.9 and 4.12. □

Definition 4.17. Let $f : X \to Y$ be homomorphism, $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. If f is an isomorphic mapping from $A^{[\alpha]}$ to $B^{[\alpha]}$ for at least one $\alpha \in [0, 1]$, then f is called a lower α-cut isomorphic mapping from A to B.

Corollary 4.18. *Let $f : X \to Y$ be homomorphism with f quasi-surjective, $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. Then f is a lower α-cut isomorphic mapping from A to B if and only if f is an isomorphic mapping from X to Y with $f(A^{[\alpha]}) = B^{[\alpha]}$ for at least one $\alpha \in [0, 1]$.*

Proof. The Proof follows by combining Corollaries 4.10 and 4.14. □

Theorem 4.19. *Let $f : X \to Y$ be an isomorphism, $A \in FMG(X)$ and $B \in FMG(Y)$. Then $f(A_{[\alpha]})$ and $f^{-1}(B_{[\alpha]})$ are subgroups of Y and X, respectively, for all $\alpha \leq (CM_A(e), CM_B(e'))$, where $\alpha \in [0,1]$ and e, e' are the identities of X and Y, respectively.*

Proof. By Theorem 3.7, it is clear that $A_{[\alpha]}$ is a subgroup of X. We show that $f(A_{[\alpha]})$ is a subgroup of Y. Let $y_1, y_2 \in f(A_{[\alpha]})$ be any two elements. Then $CM_{f(A)}(y_1) \geq \alpha$ and $CM_{f(A)}(y_2) \geq \alpha$. By Proposition 4.3, $f(A_{[\alpha]}) \subseteq (f(A))_{[\alpha]}$, $\alpha \in [0,1]$. Thus $\exists\, x_1, x_2 \in X$ such that

$$CM_A(x_1) = CM_{f(A)}(y_1) \geq \alpha \text{ and } CM_A(x_2) = CM_{f(A)}(y_2) \geq \alpha$$

imply

$$CM_A(x_1) \geq \alpha \text{ and } CM_A(x_2) \geq \alpha.$$

So

$$CM_A(x_1) \wedge CM_A(x_2) \geq \alpha.$$

Again, $CM_A(x_1x_2^{-1}) \geq CM_A(x_1) \wedge CM_A(x_2) \geq \alpha \Rightarrow CM_A(x_1x_2) \geq \alpha$. Hence

$$\begin{aligned} & x_1x_2^{-1} \in A_{[\alpha]} \\ \Leftrightarrow\ & f(x_1x_2^{-1}) \in f(A_{[\alpha]}) \subseteq (f(A))_{[\alpha]} \\ \Leftrightarrow\ & f(x_1)f(x_2^{-1}) \in (f(A))_{[\alpha]} = f(x_1)(f(x_2))^{-1} \in (f(A))_{[\alpha]} \\ \Leftrightarrow\ & y_1y_2^{-1} \in (f(A))_{[\alpha]}. \end{aligned}$$

Therefore $f(A_{[\alpha]})$ is a subgroup of Y.

The proof of the second part is similar. □

Corollary 4.20. *Let $f : X \to Y$ be an isomorphism, $A \in FMG(X)$ and $B \in FMG(Y)$. Then $f(A^{[\alpha]})$ is a subgroup of Y and $f^{-1}(B^{[\alpha]})$ is a subgroup of X for all $\alpha \geq (CM_A(e), CM_B(e'))$, where $\alpha \in [0,1]$ and e, e' are the identities of X and Y, respectively.*

Proof. Similar to Theorem 4.19. □

Corollary 4.21. *If $f : X \to Y$ be homomorphism of group X onto group Y and $\{A_i\}_{i\in I}$ be family of fuzzy multigroups of X. Then for all $\alpha \leq (CM_{A_i}(e), CM_{B_i}(e'))$, where $\alpha \in [0,1]$ and e, e' are the identities of X and Y, respectively,*

(1) *$f(\bigcap_{i\in I} A_{i[\alpha]})$ is a subgroup of Y,*
(2) *$f^{-1}(\bigcap_{i\in I} B_{i[\alpha]})$ is a subgroup of X,*
(3) *$f(\bigcup_{i\in I} A_{i[\alpha]})$ is a subgroup of Y provided $\{A_i\}_{i\in I}$ have sup/inf assuming chain,*
(4) *$f^{-1}(\bigcup_{i\in I} B_{i[\alpha]})$ is a subgroup of X provided $\{B_i\}_{i\in I}$ have sup/inf assuming chain.*

Proof. Similar to Theorem 4.19. □

Corollary 4.22. *If $f : X \to Y$ be homomorphism of group X onto group Y and $\{A_i\}_{i\in I}$ be family of fuzzy multigroups of X. Then for all $\alpha \geq (CM_{A_i}(e), CM_{B_i}(e'))$, where $\alpha \in [0,1]$ and e, e' are the identities of X and Y, respectively,*

(1) *$f(\bigcap_{i\in I} A_i^{[\alpha]})$ is a subgroup of Y,*
(2) *$f^{-1}(\bigcap_{i\in I} B_i^{[\alpha]})$ is a subgroup of X,*

(3) $f(\bigcup_{i\in I} A_i^{[\alpha]})$ *is a subgroup of* Y *provided* $\{A_i\}_{i\in I}$ *have sup/inf assuming chain,*

(4) $f^{-1}(\bigcup_{i\in I} B_i^{[\alpha]})$ *is a subgroup of* X *provided* $\{B_i\}_{i\in I}$ *have sup/inf assuming chain.*

Proof. Similar to Theorem 4.19. □

5. Upper alpha-cut homomorphic properties of fuzzy multigroups

This section focuses on upper α-cut homomorphic properties of fuzzy multigroups. We define a pre-surjective mapping f, introduce analogous concept of nested set and obtain some results.

Definition 5.1. Let X, Y be sets, $f : X \to Y$ be a mapping and $A \in FMS(X)$, respectively. If for every $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 < \alpha_2$, we have $(f(A))_{[\alpha_2]} \subseteq f(A_{[\alpha_1]})$, then f is called pre-surjective or it is said that f posses the pre-surjective property.

Theorem 5.2. *Let* $f : X \to Y$ *be a mapping,* $A \in FMS(X)$ *and* $\alpha_1, \alpha_2 \in [0,1]$ *satisfies* $\alpha_1 < \alpha_2$*. Then* f *is pre-surjective if and only if for every* $y \in (f(A))_{[\alpha_2]}$*, there exists* $x \in A_{[\alpha_1]}$ *such that* $f(x) = y$*.*

Proof. By hypothesis and Definition 5.1, it follows that f is pre-surjective $\Leftrightarrow$

$$(f(A))_{[\alpha_2]} \subseteq f(A_{[\alpha_1]}) \Leftrightarrow y \in (f(A))_{[\alpha_2]} \Rightarrow y \in f(A_{[\alpha_1]}) \Leftrightarrow \exists\, x \in A_{[\alpha_1]}$$

such that $f(x) = y$. □

Definition 5.3. Let $h : N \to P(X), \alpha \mapsto h(\alpha) \in P(X)$ be a mapping, T be an index set. Then h is called a nested set on X, if the following conditions are satisfied:

(i) $\alpha_1 < \alpha_2 \Rightarrow h(\alpha_2) \subseteq h(\alpha_1)$,

(ii) $\bigcap_{t\in T} h(\alpha_t) \subseteq \bigcap\{h(\alpha) \mid \alpha < \bigvee_{t\in T} \alpha_t\}$.

We depicts the sets that posses such conditions on X by $N(X)$.

Theorem 5.4. *Let* $f : X \to Y$ *be a mapping,* $A \in FMS(X)$ *and for all* $\alpha \in [0,1]$*, let* $h(\alpha) = f(A_{[\alpha]})$*. Then* $h \in N(Y)$ *if and only if* f *is pre-surjective.*

Proof. Suppose $h \in N(Y)$. In order to prove that f is pre-surjective, we only need to show that $(f(A))_{[\alpha_2]} \subseteq f(A_{[\alpha_1]})$, where $\alpha_1, \alpha_2 \in [0,1]$ and $\alpha_1 < \alpha_2$. In fact, for any $y \in (f(A))_{[\alpha_2]}$, we get

$$CM_{f(A)}(y) = \bigvee_{x\in f^{-1}(y)} CM_A(x) \geq \alpha_2.$$

Putting $T = \{t \in T \mid f(t) = y\}$ and $CM_A(t) = \alpha_t$, we have

$$\bigvee_{t\in T} \alpha_t = CM_{f(A)}(y) \geq \alpha_2.$$

For $t \in T$, we have $t \in A_{[\alpha]}$ with $y = f(t)$. Then $y \in f(A_{[\alpha_t]})$.
Since the mapping h is an analogous of nested set on Y, by Definition 5.3, it is straightforward to get

$$y \in \bigcap_{t\in T} f(A_{[\alpha_t]}) \subseteq \{f(A_{[\alpha]}) \mid \bigvee_{t\in T} \alpha_t > \alpha\}.$$

Considering $\alpha_1 < \alpha_2 \leqslant \bigvee_{t\in T}\alpha_t$, we infer that $y \in f(A_{[\alpha_1]})$, which implies $(f(A))_{[\alpha_2]} \subseteq f(A_{[\alpha_1]})$. In the light of Definition 5.1, f has pre-surjective property.

Conversely, suppose f is pre-surjective. On the one hand, whenever $\alpha_1 < \alpha_2$, by using Propositions 3.4 and 3.6 and Theorem 2.11, $f(A_{[\alpha_2]}) \subseteq f(A_{[\alpha_1]})$ is clear.

On the other hand, for any $y \in \bigcap_{t\in T} f(A_{[\alpha_t]})$ $\exists\, x_t \in A_{[\alpha_t]}$ such that $f(x_t) = y$. Consequently, for arbitrary $t \in T$, we get

$$CM_{f(A)}(y) \geq \bigvee_{t\in T} CM_A(x_t) \geq \bigvee_{t\in T}\alpha_t.$$

Then $y \in (f(A))_{[\bigvee_{t\in T}\alpha_t]}$. Since f is pre-surjective, for $\alpha < \bigvee_{t\in T}\alpha_t$, by Theorem 5.2, we deduce that there exists $x \in A_{[\alpha]}$ such that $f(x) = y$. This implies that $y \in f(A_{[\alpha]})$. Thus

$$y \in \bigcap\{f(A_{[\alpha]}) \mid \alpha < \bigvee_{t\in T}\alpha_t\}.$$

So

$$\bigcap_{t\in T} f(A_{[\alpha]}) \subseteq \bigcap\{f(A_{[\alpha]}) \mid \alpha < \bigvee_{t\in T}\alpha_t\}.$$

Hence by Definition 5.3, $h \in N(Y)$. □

Corollary 5.5. *Let $f : X \to Y$ be a mapping and $A \in FMS(X)$. For every $\alpha \in [0,1]$, we define $h(\alpha) = f(A_{[\alpha]})$, then $h \in N(X)$ if and only if $(f(A))_{[\alpha]} \subseteq f(A_{[\alpha]})$.*

Proof. Take any $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 < \alpha_2$. By Propositions 3.4 and 3.6 and Theorem 2.11, we obtain

$$f(A_{[\alpha_2]}) \subseteq f(A_{[\alpha_1]}).$$

Combined with

$$(f(A))_{[\alpha_2]} \subseteq f(A_{[\alpha_2]}),$$

we get

$$(f(A))_{[\alpha_2]} \subseteq f(A_{[\alpha_1]}),$$

that is, f is pre-surjective. By Theorem 5.4, it follows that $h \in N(X)$.

Conversely, suppose $h \in N(X)$. The proof follows by adopting a similar method to the proof of the necessity part of Theorem 5.4. □

Corollary 5.6. *Let $f : X \to Y$ be a quasi-surjective mapping, $A \in FMS(X)$ and $h(\alpha) = f(A_{[\alpha]})$ for any $\alpha \in [0,1]$. Then $h \in N(X)$.*

Proof. Combining Lemma 4.8 and Corollary 5.5, the result follows. □

Theorem 5.7. *Let X be a set and $A \in FMS(X)$. Then for arbitrary $\alpha_t \in [0,1], t \in T$,*

$$\bigcap_{t\in T} A_{[\alpha_t]} = A_{[\bigvee_{t\in T}\alpha_t]}.$$

Proof. For any $x \in \bigcap_{t\in T} A_{[\alpha_t]}$, we have $x \in A_{[\alpha_t]} \forall t \in T$. Then we get $CM_A(x) \geq \alpha_t$. Thus

$$CM_A(x) = \bigvee_{t\in T} CM_A(x) \geq \bigvee_{t\in T}\alpha_t.$$

So $x \in A_{[\bigvee_{t\in T}\alpha_t]}$, that is,

$$\bigcap_{t\in T} A_{[\alpha_t]} \subseteq A_{[\bigvee_{t\in T}\alpha_t]}.$$

Again, for all $x \in A_{[\bigvee_{t\in T}\alpha_t]}$ and $t \in T$, we get

$$CM_A(x) \geq \bigvee_{t\in T}\alpha_t \geq \alpha_t.$$

This implies that $CM_A(x) \geq \alpha_t$, that is, $x \in \bigcap_{t\in T} A_{[\alpha_t]}$. Hence

$$A_{[\bigvee_{t\in T}\alpha_t]} \subseteq \bigcap_{t\in T} A_{[\alpha_t]}.$$

Therefore $\bigcap_{t\in T} A_{[\alpha_t]} = A_{[\bigvee_{t\in T}\alpha_t]}$. □

Theorem 5.8. *Let X and Y be groups, $f : X \to Y$ be a homomorphism, where f is quasi-surjective, $A \in FMG(X)$ and $B \in FMG(Y)$. If f is an upper α-cut homomorphic mapping from A to B with the pre-surjective property, then $f(A) \in FMG(Y)$, and f is also an upper α-cut homomorphic mapping from A to $f(A)$.*

Proof. As f is pre-surjective, it follows that $h \in N(Y)$, where $h(\alpha) = f(A_{[\alpha]})$ for every $\alpha \in [0, 1]$. Since f is quasi-surjective, by Lemma 4.8, for arbitrary $\alpha \in [0, 1]$, we have $(f(A))_{[\alpha]} = f(A_{[\alpha]})$. Then $f(A) \in FMG(Y)$. In addition, for $A \in FMG(X)$, it is clear that $A_{[\alpha]}$ is a subgroup of X by Theorem 3.7. Consequently, $f(A_{[\alpha]})$ is a subgroup of Y. Since

$$(f(A))_{[\alpha]} = f(A_{[\alpha]}) \Rightarrow (f(A))_{[\alpha]}$$

is a subgroup of Y. Hence, $f(A) \in FMG(Y)$.

Again, since f is an upper α-cut homomorphic mapping from A and B, we see that f is a homomorphism from $A_{[\alpha]}$ to $B_{[\alpha]}$. Then for every $x, y \in A_{[\alpha]}$, it is clear that $f(x), f(y) \in f(A_{[\alpha]})$. By Theorems 4.5 and 4.9,

$$f(A_{[\alpha]}) = (f(A))_{[\alpha]} \subseteq B_{[\alpha]}.$$

Since f is a homomorphism from X to Y, $f(xy) = f(x)f(y)$ and $(f(A))_{[\alpha]}$ is a subgroup of Y, we have $f(x)f(y) \in (f(A))_{[\alpha]}$, that is, f preserves the operation. Consequently, f is an upper α-cut homomorphism from A to $f(A)$. □

Corollary 5.9. *Let X and Y be groups, $f : X \to Y$ be a homomorphism, where f is quasi-surjective, $A \in FMG(X)$ and $B \in FMG(Y)$. If f is an upper α-cut homomorphic mapping from A to B with the pre-surjective property, then $f^{-1}(B) \in FMG(X)$, and f is also a surjective homomorphism from $f^{-1}(B)$ to B.*

Proof. For any $\alpha \in [0, 1]$, setting $h(\alpha) = f^{-1}(B_{[\alpha]})$, we show that $h \in N(X)$. Given $\alpha_1, \alpha_2 \in [0, 1]$ with $\alpha_1 < \alpha_2$, by Proposition 3.6 and Theorem 2.11, we get

$$f^{-1}(B_{[\alpha_2]}) \subseteq f^{-1}(B_{[\alpha_1]}).$$

By Proposition 4.3, we get

$$\bigcap_{t\in T} f^{-1}(B_{[\alpha_t]}) = \bigcap_{t\in T} (f^{-1}(B))_{[\alpha_t]}.$$

Thus by Theorem 5.7, it follows that

$$\bigcap_{t\in T}(f^{-1}(B))_{[\alpha_t]} = (f^{-1}(B))_{[\bigvee_{t\in T}\alpha_t]}.$$

Since

$$(f^{-1}(B))_{[\bigvee_{t\in T}\alpha_t]} = f^{-1}(B_{[\bigvee_{t\in T}\alpha_t]}),$$

it is obvious that

$$\bigcap_{t\in T} f^{-1}(B_{[\alpha_t]}) \subseteq \{f^{-1}(B_{[\alpha]}) \mid \alpha < \bigvee_{t\in T}\alpha_t\}.$$

Thus $h \in N(X)$. Since $(f^{-1}(B))_{[\alpha]} \neq \varnothing$, for every $\alpha \in [0,1]$, for any $x, y \in (f^{-1}(B))_{[\alpha]} = f^{-1}(B_{[\alpha]})$, there exists $x_0, y_0 \in B_{[\alpha]}$ such that $f(x) = x_0$ and $f(y) = y_0$. As f is upper α-cut homomorphism from A to B, and $B_{[\alpha]}$ is a subgroup of Y, we infer immediately that

$$f(xy^{-1}) = f(x)(f(y))^{-1} = x_0y_0^{-1} \in B_{[\alpha]}.$$

This implies that

$$xy^{-1} \in f^{-1}(B_{[\alpha]}) = (f^{-1}(B))_{[\alpha]}.$$

So $(f^{-1}(B))_{[\alpha]}$ is a subgroup of X. Hence $f^{-1}(B) \in FMG(X)$. Since f is a surjective upper α-cut homomorphic mapping from A to B, from Definition 4.11, we know that f is a surjective homomorphism from $A_{[\alpha]}$ to $B_{[\alpha]}$. For all $x, y \in (f^{-1}(B))_{[\alpha]}$, we notice that $B \in FMG(Y)$, $f(xy) = f(x)f(y) \in B_{[\alpha]}$. Therefore f is a surjective upper α-cut homomorphism from $f^{-1}(B)$ to B. □

Theorem 5.10. *Let $f : X \to Y$ be homomorphism of groups where f is quasi-surjective, $A \in FMG(X)$ and $B \in FMG(Y)$. If f is an upper α-cut isomorphic mapping from A to B with the pre-surjective property, then*

(1) $f(A) \in FMG(Y)$, and f is also an upper α-cut isomorphic mapping from A to $f(A)$,

(2) $f^{-1}(B) \in FMG(X)$, and f is also an upper α-cut isomorphic mapping from $f^{-1}(B)$ to B.

Proof. Combining Theorem 5.8 and Corollary 5.9, the results follow. □

6. Conclusions

We have presented a precise study of α-cuts of fuzzy multigroups to enhance the introduction of α-cuts homomorphism of fuzzy multigroups. The concept of α-cuts homomorphism of fuzzy multigroups was explicated and some related results were deduced. Some properties of α-cuts homomorphism of fuzzy multigroups were discussed in details. In future research, some homomorphic properties of α-cuts of fuzzy multigroups could still be exploited and the analogous of isomorphism theorems could be established in α-cuts homomorphism of fuzzy multigroups.

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P. A. Ejegwa (ocholohi@gmil.com)
Department of Mathematics/Statistics/Computer Science, University of Agriculture, P.M.B. 2373, Makurdi, Nigeria