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Rough fuzzy sets via multifunction

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Abstract. This paper aims at providing a framework to combine fuzzy sets and rough sets together via multifunction, which gives rise to several interesting new concepts such as rough sets, rough fuzzy sets via multifunction and some properties of them. Although many results reported here are only concerned with basic properties about these new notions, one could see that this study presents a very preliminary, but potentially interesting research direction.

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1. Introduction

Fuzzy set theory and rough set theory are mathematical tools for dealing with uncertainties and are closely related [2]. In 1981, Pawlak [6] introduced the concept of rough sets and studied some of its properties. After then, Dubois and Prade [3] investigated the problem of combining fuzzy sets with rough sets whereas Maritz and Pawlak [5] proposed rough sets in terms of multifunction to obtain a hybrid model called rough sets via multifunction. Alternatively, a rough sets via multifunction instead of an equivalence relation can be used to granulate the universe. This leads to a deviation of Pawlak approximation space called a Maritz approximation space, in which rough approximations, rough sets and rough topology via multifunction [1] can be introduced accordingly. Furthermore, we also consider approximation of a fuzzy set in a Maritz approximation space, and initiate a concept called rough fuzzy sets via multifunction, which extends Dubois and Pradés rough fuzzy sets. Further research will be needed to establish whether the notions put forth in this paper may lead to a fruitful theory.

2. Preliminaries

In this section, we have discussed some definitions and results which are relevant of this paper.

In 1965, the theory of fuzzy sets initiated by Zadeh [8] provides an appropriate framework for representing and processing vague concepts by allowing partial memberships.

A fuzzy set λ in a universe U is defined by a membership function $\lambda : U \to [0,1]$. For $x \in U$, the membership value $\lambda(x)$ essentially specifies the degree to which $x \in U$ belongs to the fuzzy set λ. For any fuzzy sets λ and μ in UI, the intersection and union of λ and μ, and the complement of λ are defined as follows: for each $x \in U$,

$$(\lambda \cap \mu)(x) = \lambda(x) \wedge \mu(x),$$
$$(\lambda \cup \mu)(x) = \lambda(x) \vee \mu(x),$$
$$\lambda^c(x) = 1 - \lambda(x).$$

By $\lambda \subseteq \mu$ we mean that $\lambda(x) \leq \mu(x)$, for all $x \in U$. Clearly $\lambda = \mu$, if both $\lambda \subseteq \mu$ and $\mu \subseteq \lambda$, i.e., $\lambda(x) = \mu(x)$, for all $x \in U$.

By a multifunction $F : T \to U$, we mean a point-to-set correspondence from T into U , and always assume that $F(t) \neq \varnothing$, for all $t \in T$.

Definition 2.1 ([5]). Let $F : T \to U$ be a multifunction. Then $F(t)$ said to be an atom and $\digamma(T) = \{F(t) : t \in T\}$ denotes the collection of all atoms.

$F^+(B) = \{t \in T : F(t) \subset B\}$ and $F^-(B) = \{t \in T : F(t) \cap B \neq \varnothing\}$ called strong(resp.weak) inverse of F.

Definition 2.2 ([5]). Let $F : T \to U$ be a surjective multifunction. Then the pair $(U, \digamma(T))$ is the Approximation space (called a Maritz space).

Definition 2.3 ([5]). Let $\Lambda^M = (U, \digamma(T))$ be a Maritz space and $A \subset U$ then. The upper approximation of A in $(U, \digamma(T))$ is the set $F(F^-(A)) = F(\{t \in T : F(t) \cap A \neq \phi\}) = \cup\{F(t) \in \digamma(T) : F(t) \cap A \neq \phi\}$. The lower approximation of A in $\Lambda^M = (U, \digamma(T))$ is the set $F(F^+(A)) = F(\{t \in T : F(t) \subset A\}) = \cup\{F(t) \in \digamma(T) : F(t) \subset A\}$. A is $F-$ rough set if and only if $F(F^-(A) \neq F(F^+(A)$.

Based on an equivalence relation on the universe of discourse, Dubois and Prade [3] introduced the lower and upper approximations of fuzzy sets in a Pawlak approximation space and obtained a new notion called rough fuzzy sets [4].

Definition 2.4 ([3]). Let (U, R) be a Pawlak approximation space. For a fuzzy set μ, the lower and upper rough approximations of μ in (U, R) are denoted by $\underline{R}(\mu)$ and $\overline{R}(\mu)$, respectively, which are fuzzy sets in U defined as follows: for all $x \in U$,

$$\underline{R}(\mu)(x) = \bigwedge\{\mu(y) : y \in [x]_R\},$$
$$\overline{R}(\mu)(x) = \bigvee\{\mu(y) : y \in [x]_R\}.$$

The operators $\underline{R}$ and $\overline{R}$ are called the lower and upper rough approximation operators on fuzzy sets. If $\underline{R}(\mu) = \overline{R}(\mu)$, then the fuzzy set μ is said to be definable; otherwise, μ is called a rough fuzzy set.

By definition, it is easy to see that if $\mu = X$ is a crisp subset of U. Then we have

$$\underline{R}(X) = R_*X = \{x \in U : [x]_R\} \subseteq X,$$
$$\overline{R}(X) = R^*X = \{x \in U : [x]_R\} \cap X \neq \phi.$$

In this case, it follows that X is a rough fuzzy set if and only if X is a rough set. Thus, one can say that rough fuzzy sets are natural extensions of rough sets. In general, a rough fuzzy set can be seen as an approximation of a fuzzy set in a crisp approximation space. This hybrid model combining the concepts of both fuzzy sets and rough sets may be used to deal with knowledge acquisition in information systems with fuzzy decisions [7].

3. Rough Fuzzy Sets via Multifunction

In this section, we shall consider lower and upper rough approximations of fuzzy sets in Maritz approximation space, and obtain a new hybrid model called rough fuzzy sets via multifunction, which can be seen as an extension of Dubois and Pradés rough fuzzy sets.

Definition 3.1. Let $\Lambda^M = (U, F(T))$ be a Maritz space. For a fuzzy set μ in U, the lower and upper rough approximations via multifunction of μ with respect to Λ^M are denoted by $\underline{R_F}(\mu)(x)$ and $\overline{R_F}(\mu)$, respectively, which are fuzzy sets in U given by: for all $x \in U$,

$$\underline{R_F}(\mu)(x) = \bigwedge\{\mu(y) : F(t) \in F(T)[\{x, y\} \subseteq F(T)]\},$$
$$\overline{R_F}(\mu)(x) = \bigvee\{\mu(y) : F(t) \in F(T)[\{x, y\} \subseteq F(T)]\}.$$

If $\underline{R_F}(\mu) = \overline{R_F}(\mu), \mu$ is said to be definable; otherwise μ is called a rough fuzzy set via multifunction.

Proposition 3.2. *Let $F : T \to U$ be a surjective multifunction and $\Lambda^M = (U, F(T))$ a Maritz space. Suppose $\lambda,\ \mu$ are fuzzy sets in U. Then we have*

(1) $\underline{R_F}(\phi) = \overline{R_F}(\phi) = \phi$,
(2) $\underline{R_F}(U) = \overline{R_F}(U) = U$,
(3) $\underline{R_F}(\lambda) \subseteq \lambda \subseteq \overline{R_F}(\lambda)$,
(4) $\lambda \subseteq \mu \Rightarrow \underline{R_F}(\lambda) \subseteq \underline{R_F}(\mu)$ *and* $\overline{R_F}(\lambda) \subseteq \overline{R_F}(\mu)$.

Proof. (1), (2) The proofs are straightforward.

(3) Let λ be a fuzzy set in U and let $x \in U$. Since $F : T \to U$ is a surjective multifunction over U, there exists some $t_\circ \in T$ such that $x \in F(t_\circ)$. Then by definition, we have

$$\underline{R_F}(\lambda)(x) = \bigwedge\{\lambda(y) : F(t) \in F(T)[\{x, y\} \subseteq F(T)]\},$$
$$\overline{R_F}(\lambda)(x) = \bigvee\{\lambda(y) : F(t) \in F(T)[\{x, y\} \subseteq F(T)]\}.$$

Thus it follows that $\underline{R_F}(\lambda)(x) \leq \lambda(x) \leq \overline{R_F}(\lambda)(x)$. So $\underline{R_F}(\lambda) \subseteq \lambda \subseteq \overline{R_F}(\lambda)$.

(4) Let $\lambda,\ \mu$ be fuzzy sets in U, $x \in U$ and $N(x) = \{y : \exists t \in T[\{x, y\} \subseteq F(t)]\}$.

Suppose $\lambda \subseteq \mu$. Then it is easy to see that: for all $y \in N(x)$,

$$\underline{R_F}(\lambda)(x) = \bigwedge\{\lambda(y) : y \in N(x)\} \leq \lambda(y) \leq \mu(y),$$

$$\overline{R_F}(\mu)(x) = \bigvee\{\mu(y) : y \in N(x)\} \leq \mu(y) \leq \mu(y).$$

□

Proposition 3.3. *Let $F : T \to U$ be a surjective multifunction,and $\Lambda^M = (U, F(T))$ a Maritz space. Suppose $\lambda,\ \mu$ are fuzzy sets in U. Then we have*
(1) $\underline{R_F}(\lambda \cap \mu) \subseteq \underline{R_F}(\lambda) \cap \underline{R_F}(\mu)$,
(2) $\overline{R_F}(\lambda \cup \mu) \supset \overline{R_F}(\lambda) \cup \overline{R_F}(\mu)$,
(3) $\underline{R_F}(\lambda \cup \mu) \supseteq \underline{R_F}(\lambda) \cup \underline{R_F}(\mu)$,
(4) $\overline{R_F}(\lambda \cap \mu) \subseteq \overline{R_F}(\lambda) \cap \overline{R_F}(\mu)$.

Proof. (1) Let $\lambda,\ \mu$ be fuzzy sets in U, $x \in U$ and $N(x) = \{y : \exists t \in T[\{x, y\} \subseteq F(t)]\}$. Note that $\underline{R_F}(\lambda \cap \mu)(x) = \bigwedge\{\lambda(y) \cap \mu(y) : y \in N(x)\}$. Then

$$\underline{R_F}(\lambda \cap \mu)(x) \leq \lambda(y) \wedge \mu(y) \leq \lambda(y),\ \forall y \in N(x).$$

Since $\underline{R_F}(\lambda)(x) = \bigwedge\{\mu(y) : y \in N(x)\}$, it follows that $\underline{R_F}(\lambda \cap \mu)(x) \leq \underline{R_F}(\lambda)(x)$. Similarly, we obtain $\underline{R_F}(\lambda \cap \mu)(x) \leq \underline{R_F}(\mu)(x)$. Thus

$$\underline{R_F}(\lambda \cap \mu)(x) \leq \underline{R_F}(\lambda)(x) \wedge \underline{R_F}(\mu)(x).$$

So $\underline{R_F}(\lambda \cap \mu)(x) \subseteq \underline{R_F}(\lambda)(x) \cap \underline{R_F}(\mu)(x)$.
(2) The proof is similar to (1).
(3) Let $\lambda,\ \mu$ be fuzzy sets in U, $x \in U$ and $N(x) = \{y : \exists t \in T[\{x, y\} \subseteq F(t)]\}$. It is clear that $\underline{R_F}(\lambda)(x) = \bigwedge\{\lambda(y) : y \in N(x)\} \leq \lambda(y) \leq \lambda(y) \vee \mu(y), \forall y \in N(x)$. Then we have $\underline{R_F}(\lambda \cup \mu)(x) = \bigwedge\{\lambda(y) \vee \mu(y) : y \in N(x)\} \geq \underline{R_F}(\lambda)(x)$. Similarly, we obtain that $\underline{R_F}(\lambda \cup \mu)(x) \geq \underline{R_F}(\mu)(x)$. Thus

$$\underline{R_F}(\lambda \cup \mu)(x) \geq \underline{R_F}(\lambda)(x) \vee \underline{R_F}(\mu)(x) = (\underline{R_F}(\lambda) \cup \underline{R_F}(\mu))(x).$$

So $\underline{R_F}(\lambda \cup \mu) \supseteq \underline{R_F}(\lambda) \cup \underline{R_F}(\mu)$.
(4) The proof is similar to (3). □

Remark 3.4. The inclusions in the above proposition may hold strictly as seen in the following example.

Example 3.5. Let $U = \{a, b, c, d, e\}, T = \{1, 2, 3, 4\}$ and $F : T \to U$ defined by: $F(1) = \{e\}, F(2) = \{a, d\}, F(3) = \{a, b, c\}, F(4) = \{c, e\}$. Then $(U, F(T))$ is a Maritz space. Consider two fuzzy sets

$$\lambda = \{0.8/a, 0.5/b, 0.7/c, 0.2/d, 0.3/e\}$$

and

$$\mu = \{0.1/a, 0.3/b, 0.6/c, 0.8/d, 0.5/e\}.$$

Then we have

$$\underline{R_F}(\lambda) = \{0.2/a, 0.5/b, 0.3/c, 0.2/d, 0.3/e\},$$
$$\overline{R_F}(\lambda) = \{0.8/a, 0.8/b, 0.8/c, 0.8/d, 0.7/e\},$$
$$\underline{R_F}(\mu) = \{0.1/a, 0.1/b, 0.1/c, 0.1/d, 0.5/e\},$$
$$\overline{R_F}(\mu) = \{0.8/a, 0.6/b, 0.6/c, 0.8/d, 0.6/e\}.$$

Thus we can easily see that

$$\lambda \cup \mu = \{0.8/a, 0.5/b, 0.7/c, 0.8/d, 0.5/e\},$$

$$\lambda \cap \mu = \{0.1/a, 0.3/b, 0.6/c, 0.2/d, 0.3/e\},$$

$$\underline{R_F}(\lambda \cup \mu) = \{0.5/a, 0.5/b, 0.5/c, 0.8/d, 0.5/e\},$$
$$\overline{R_F}(\lambda \cap \mu) = \{0.6/a, 0.6/b, 0.6/c, 0.2/d, 0.6/e\}.$$

Furthermore, we have

$$\underline{R_F}(\lambda) \cup \underline{R_F}(\mu) = \{0.2/a, 0.5/b, 0.3/c, 0.2/d, 0.5/e\},$$
$$\overline{R_F}(\lambda) \cap \overline{R_F}(\mu) = \{0.8/a, 0.6/b, 0.6/c, 0.8/d, 0.6/e\}.$$

Definition 3.6. A multifunction $F : T \to U$ is called a partition multifunction, if $\{F(t) : t \in T\}$ forms a partition of U.

Theorem 3.7. *Let $F : T \to U$ be a partition multifunction over U and $\Lambda^M = (U, \digamma(T))$ a Maritz space. Define a binary relation R on U as follows: for each $(x, y) \in U \times U$,*

$$(x, y) \in R \Leftrightarrow \exists\ t \in T,\ \{x, y\} \subseteq F(t).$$

Then R is an equivalence relation on U such that

$$\underline{R_F}(\lambda)(x) = \bigwedge\{\lambda(y) : y \in [x]_R\},$$

and

$$\overline{R_F}(\lambda)(x) = \bigvee\{\lambda(y) : y \in [x]_R\},$$

where λ is a fuzzy set in U and $x \in U$. Thus in this case, λ is a rough fuzzy set via multifunction with respect to Maritz approximation space $\Lambda^M = (U, \digamma(T))$ if and only if μ is a rough fuzzy set via multifunction with respect to the approximation space (U, R).

Proof. First, we show that the relation R induced by the partition multifunction $F : T \Rightarrow U$ is an equivalence relation on U.

Let $x \in U$. Then exists $t \in T$ such that $x \in F(T)$. Thus $(x, x) \in R$. So R is reflexive.

Suppose $(x, y) \in R$. then there exists $t \in T$ such that $\{x, y\} = \{y, x\} \subseteq F(t)$. Thus we deduce that $(y, x) \in R$. Sot R is symmetric.

Suppose $(x, y) \in R$ and $(y, z) \in R$. Then there exist $t_1, t_2 \in T$ such that $\{x, y\} \subseteq F(t_1)$ and $\{y, z\} \subseteq F(t_2)$. Thus $F(t_1) \cap F(t_2) \neq \phi$. But $\{F(t) : t \in T\}$ is a partition of U, since $\Lambda^M = (U, \digamma(T))$ is a partition multifunction over U. It follows that $F(t_1) = F(t_2)$. So $(x, z) \in R$. Hence R is transitive.

Now let λ be a fuzzy set in U and $x \in U$. Then by definition,

$$\begin{aligned}\underline{R_F}(\lambda)(x) &= \bigwedge\{\lambda(y) : \exists t \in T, \{x, y\} \subseteq \digamma(T)\}\\ &= \bigwedge\{\lambda(y) : (x, y) \in R\}\\ &= \bigwedge\{\mu(y) : y \in [x]_R\}.\end{aligned}$$

The second assertion $\overline{R_F}(\mu)(x) = \bigvee\{\mu(y) : y \in [x]_R\}$ can be proved in a similar way. □

4. Conclusions

In this work, one can easily see that (classical) rough fuzzy sets can be identified with rough fuzzy sets via multifunction when the underlying multifunction in the maritz approximation space is a partition multifunction. Consequently, every rough fuzzy set may be considered a rough fuzzy set via multifunction. In this sense, the notion of rough fuzzy sets via multifunction can be seen as a natural generalization of rough fuzzy sets.

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