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Abstract. This paper proposes the notions of characteristic and Frattini fuzzy submultigroups of fuzzy multigroups. Some properties of characteristic fuzzy submultigroups of fuzzy multigroups are outlined and some related results are obtained. It is shown that a fuzzy submultigroup is characteristic if and only if its alpha-cut is a characteristic subgroup, and every characteristic fuzzy submultigroup is a normal fuzzy submultigroup. Also, we put forward the ideas of maximal fuzzy submultigroups and Frattini fuzzy submultigroups of fuzzy multigroups as extensions of maximal subgroups and Frattini subgroups of classical groups. It is presented that every Frattini fuzzy submultigroup is both characteristic and normal, respectively. Finally, some results are established in connection to level subgroups and alpha-cuts of fuzzy multigroups.

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Keywords: Fuzzy multiset, Fuzzy multigroup, Fuzzy submultigroup, Normal fuzzy submultigroup, Characteristic fuzzy submultigroup, Frattini fuzzy submultigroup.

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## 1. Introduction

Fuzzy set theory proposed by Zadeh [43] revolutionized the entire mathematics with its capacity to adequately tackle uncertainties. For a classical set $X$, a fuzzy set over $X$, or a fuzzy subset of $X$, is characterized by a membership function $\mu$ which associates values from the closed unit interval $I = [0, 1]$ to members of $X$. The theory of fuzzy sets has grown impressively over the years giving birth to fuzzy groups proposed in [31]. Frankly speaking, the concept of fuzzy groups is an application of fuzzy set to group theory. The notion of fuzzy groups caught the attentions of algebraists like wild fire, and there seems to be no end to its ramifications. Some properties of fuzzy groups were explicated as analogs to some group theoretical notions (see [1, 2, 3, 5, 6, 8, 20, 21, 22, 23, 24, 28, 29, 30, 32, 34, 41] for some details). In similar vein, the notions of multigroups and multirings have been proposed in [9, 19, 39] by using multisets.

The idea of multiset [36, 37, 40], which is an extension of set with repeated elements in a collection was instrumental in the introduction of fuzzy multiset by Yager [42]. That is, fuzzy multiset allows repetition of membership degrees of elements in multiset framework. Roughly speaking, fuzzy multiset generalizes fuzzy set. With this, it is meet to say that, every fuzzy set is a fuzzy multiset but the converse is not true. Fuzzy multiset theory has been extensively studied and applied in real-life problems (see [7, 10, 25, 26, 27, 35, 38]).

Applying the idea of fuzzy multisets to group theory, Shinoj et al. [33] proposed fuzzy multigroups as algebraic structure of fuzzy multisets which generalize fuzzy groups. The notion of fuzzy submultigroups of fuzzy multigroups and some properties of fuzzy multigroups were explicated in [11]. The ideas of abelian fuzzy multigroups and order of fuzzy multigroups have been studied [4, 12], and the notion of normal fuzzy submultigroups of fuzzy multigroups was proposed with some number of results in [13]. The concept of homomorphism in fuzzy multigroup setting was studied and some of its properties were explored in [14]. The idea of direct product of fuzzy multigroups was proposed and a number of results were established [15]. Alpha-cuts of fuzzy multigroups were introduced and some of their properties were characterized in [16], and the homomorphic properties of alpha-cuts of fuzzy multigroups were studied in [17]. Some group's theoretic notions in fuzzy multigroup framework were discussed in [18].

Due to the lack of *infrastructural facilities*, the notions of characteristic and Frattini fuzzy submultigroups, could not be discussed satisfactorily. The need to establish maximal fuzzy submultigroup is germane for the introduction of Frattini fuzzy submultigroups. The notion of normal fuzzy submultigroups [13] has been established as a precursor to this present study. The motivation of this paper is to propose characteristic and Frattini fuzzy submultigroups since the ideas of fuzzy submultigroups and normal fuzzy submultigroups have been explored. The rest of the paper are as follow: In Section 2, some mathematical preliminaries paramount to the paper are presented. Section 3 introduces characteristic fuzzy submultigroups with some number of results. In Section 4, the ideas of maximal fuzzy submultigroups and Frattini fuzzy submultigroups are explored with some number of results. Meanwhile, Section 5 draws conclusion to the paper and suggest areas of future works.

## 2. Preliminaries

**Definition 2.1** ([42])**.** Assume $X$ is a set of elements. Then, a fuzzy bag/multiset $A$ drwan from $X$ can be characterized by a count membership function, $CM_A$ such that

$$CM_A : X \to Q,$$

where $Q$ is the set of all crisp bags or multisets from the unit interval $I = [0, 1]$.

From [38], a fuzzy multiset can also be characterized by a high-order function. In particular, a fuzzy multiset, $A$ can be characterized by a function

$$CM_A : X \to N^I \text{ or } CM_A : X \to [0,1] \to N,$$

where $I = [0,1]$ and $N = \mathbb{N} \cup \{0\}$.

By [26], it implies that $CM_A(x)$ for $x \in X$ is given as

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ...\},$$

where $\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x), ... \in [0,1]$ such that

$$\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x) \geq ...,$$

whereas in a finite case, we write

$$CM_A(x) = \{\mu_A^1(x), \mu_A^2(x), ..., \mu_A^n(x)\},$$

for $\mu_A^1(x) \geq \mu_A^2(x) \geq ... \geq \mu_A^n(x)$.

A fuzzy multiset, $A$ can be represented in the form

$$A = \{\langle \frac{CM_A(x)}{x} \rangle \mid x \in X\} \text{ or } A = \{\langle x, CM_A(x) \rangle \mid x \in X\}.$$

In a simple term, a fuzzy multiset $A$ of $X$ is characterized by the count membership function, $CM_A(x)$ for $x \in X$, that takes the value of a multiset of a unit interval $I = [0,1]$ [7, 27].

We denote the set of all fuzzy multisets by $FMS(X)$.

**Example 2.2.** Assume that $X = \{a, b, c\}$ is a set. Then for $CM_A(a) = \{1, 0.5, 0.5\}$, $CM_A(b) = \{0.9, 0.7, 0\}$, $CM_A(c) = \{0, 0, 0\}$, $A$ is a fuzzy multiset of $X$ written as

$$A = \{\langle \frac{1, 0.5, 0.5}{a} \rangle, \langle \frac{0.9, 0.7, 0}{b} \rangle, \langle \frac{0, 0, 0}{c} \rangle\}.$$

**Definition 2.3** ([25])**.** Let $A, B \in FMS(X)$. Then, $A$ is called a fuzzy submultiset of $B$. written as $A \subseteq B$, if $CM_A(x) \leq CM_B(x)\ \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then $A$ is called a proper fuzzy submultiset of $B$ and denoted as $A \subset B$.

**Definition 2.4** ([38])**.** Let $\{A_i\}_{i \in I}$ be a family of fuzzy multisets over $X$. Then

(i) $CM_{\bigcap_{i\in I} A_i}(x) = \bigwedge_{i \in I} CM_{A_i}(x)\ \forall x \in X$,

(ii) $CM_{\bigcup_{i\in I} A_i}(x) = \bigvee_{i \in I} CM_{A_i}(x)\ \forall x \in X$,

where $\bigwedge$ and $\bigvee$ denote minimum and maximum operations.

**Definition 2.5** ([25])**.** Let $A, B \in FMS(X)$. Then, $A$ and $B$ are comparable to each other if and only if $A \subseteq B$ or $B \subseteq A$, and $A = B \Leftrightarrow CM_A(x) = CM_B(x)\ \forall x \in X$.

**Definition 2.6** ([13])**.** A fuzzy multiset, $B$ of a set $X$ is said to have sup-property if for any subset, $W \subset X\ \exists\ w_0 \in W$ such that

$$CM_B(w_0) = \bigvee_{w \in W} [CM_B(w)].$$

**Definition 2.7** ([33])**.** Let $X$ be a group. A fuzzy multiset, $A$ over $X$ is called a fuzzy multigroup of $X$ if the count membership function of $A$, that is,

$$CM_A : X \to [0,1]$$

satisfies the following conditions:

(i) $CM_A(xy) \geq CM_A(x) \wedge CM_A(y)\ \forall x, y \in X$,
(ii) $CM_A(x^{-1}) = CM_A(x)\ \forall x \in X$.

By implication, a fuzzy multiset, $A$ over $X$ is called a fuzzy multigroup of a group $X$, if

$$CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y)\ \forall x, y \in X.$$

It follows immediately from the definition that,

$$CM_A(e) \geq CM_A(x)\ \forall x \in X,$$

where $e$ is the identity element of $X$. We denote the set of all fuzzy multigroups of $X$ by $FMG(X)$.

**Example 2.8.** Let $X = \{1, -1, i, -i\}$ be group. Then, the fuzzy multiset, $A$ of $X$, that is,

$$A = \{\langle \frac{1, 0.8}{1} \rangle, \langle \frac{0.7, 0.6}{-1} \rangle, \langle \frac{0.6, 0.5}{i} \rangle, \langle \frac{0.6, 0.5}{-i} \rangle\}$$

is a fuzzy multigroup of $X$ satisfying the conditions of Definition 2.7.

**Definition 2.9** ([33])**.** Let $A$ be a fuzzy multigroup of a group $X$. Then, $A^{-1}$ is defined by $CM_{A^{-1}}(x) = CM_A(x^{-1})\ \forall x \in X$.

Then, we notice that $A \in FMG(X) \Leftrightarrow A^{-1} \in FMG(X)$.

**Definition 2.10** ([11, 33])**.** Let $A, B \in FMG(X)$. Then, the product $A \circ B$ of $A$ and $B$ is defined to be a fuzzy multiset of $X$ as follows:

$$CM_{A \circ B}(x) = \begin{cases} \bigvee_{x=yz}[CM_A(y) \wedge CM_B(z)], & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 2.11** ([11])**.** Let $A \in FMG(X)$. A fuzzy submultiset, $B$ of $A$ is called a fuzzy submultigroup of $A$ denoted by $B \sqsubseteq A$ if $B$ is a fuzzy multigroup. A fuzzy submultigroup, $B$ of $A$ is a proper fuzzy submultigroup, denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.

**Definition 2.12** ([16])**.** Let $A \in FMG(X)$. Then, the sets $A_{[\alpha]}$ and $A_{(\alpha)}$ defined as

(i) $A_{[\alpha]} = \{x \in X \mid CM_A(x) \geq \alpha\}$ and
(ii) $A_{(\alpha)} = \{x \in X \mid CM_A(x) > \alpha\}$

are called strong upper alpha-cut and weak upper alpha-cut of $A$, where $\alpha \in [0, 1]$.

**Definition 2.13** ([16])**.** Let $A \in FMG(X)$. Then, the sets $A^{[\alpha]}$ and $A^{(\alpha)}$ defined as

(i) $A^{[\alpha]} = \{x \in X \mid CM_A(x) \leq \alpha\}$ and
(ii) $A^{(\alpha)} = \{x \in X \mid CM_A(x) < \alpha\}$

are called strong lower alpha-cut and weak lower alpha-cut of $A$, where $\alpha \in [0, 1]$.

**Proposition 2.14** ([16])**.** *Let $A$ be a fuzzy multigroup of $X$ and let $x \in X$. Then, $CM_A(x) = \alpha$ if and only if $x \in A_{[\alpha]}$ and $x \notin A_{[\beta]}\ \forall \beta > \alpha$.*

**Theorem 2.15** ([16])**.** *Let $A \in FMG(X)$. Then, $A_{[\alpha]}$ is a subgroup of $X$ for all $\alpha \leq CM_A(e)$ and $A^{[\alpha]}$ is a subgroup of $X$ for all $\alpha \geq CM_A(e)$, where $e$ is the identity element of $X$ and $\alpha \in [0, 1]$.*

**Proposition 2.16** ([11, 33]). *Let $A \in FMG(X)$. Then, the sets $A_*$ and $A^*$ defined by*

(i) $A_* = \{x \in X \mid CM_A(x) > 0\}$ *and*
(ii) $A^* = \{x \in X \mid CM_A(x) = CM_A(e)\}$, *where $e$ is the identity element of $X$*

*are subgroups of $X$.*

**Definition 2.17** ([12]). Let $A \in FMG(X)$. Then, $A$ is said to be abelian (commutative), if for all $x, y \in X$, $CM_A(xy) = CM_A(yx)$.

If $A$ is a fuzzy multigroup of an abelian group $X$, then $A$ is abelian.

**Definition 2.18** ([13]). Let $A$ be a fuzzy submultigroup of $B \in FMG(X)$. Then, $A$ is called a normal fuzzy submultigroup of $B$, if

$$CM_A(xyx^{-1}) \geq CM_A(y) \, \forall x, y \in X.$$

**Definition 2.19** ([13]). Two fuzzy multigroups $A$ and $B$ of $X$ are conjugate to each other, if for all $x, y \in X$,

$$CM_A(y) = CM_B(xyx^{-1}) \text{ or } CM_A(y) = CM_{B^x}(y)$$

and

$$CM_B(x) = CM_A(yxy^{-1}) \text{ or } CM_B(x) = CM_{A^y}(x).$$

**Proposition 2.20** ([13]). *Let $A$ be a fuzzy submultigroup of $B \in FMG(X)$. Then, the following statements are equivalent:*
(1) *$A$ is a normal fuzzy submultigroup of $B$,*
(2) $CM_A(xyx^{-1}) = CM_A(y) \, \forall x, y \in X$,
(3) $CM_A(xy) = CM_A(yx) \, \forall x, y \in X$.

**Definition 2.21** ([14]). Let $X$ and $Y$ be groups and let $f : X \to Y$ be a homomorphism. Suppose $A$ and $B$ are fuzzy multigroups of $X$ and $Y$, respectively. Then, $f$ induces a homomorphism from $A$ to $B$ which satisfies:
(i) $CM_A(f^{-1}(y_1y_2)) \geq CM_A(f^{-1}(y_1)) \wedge CM_A(f^{-1}(y_2)) \, \forall y_1, y_2 \in Y$,
(ii) $CM_B(f(x_1x_2)) \geq CM_B(f(x_1)) \wedge CM_B(f(x_2)) \, \forall x_1, x_2 \in X$,
where (a) the image of $A$ under $f$, denoted by $f(A)$, is a fuzzy multiset over $Y$ defined by

$$CM_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} CM_A(x), & f^{-1}(y) \neq \varnothing \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$,
(b) the inverse image of $B$ under $f$, denoted by $f^{-1}(B)$, is a fuzzy multiset over $X$ defined by

$$CM_{f^{-1}(B)}(x) = CM_B(f(x)) \, \forall x \in X.$$

It follows that, a homomorphism $f$ of $X$ onto $Y$ is called an automorphism of $A$ onto $A$ if $f$ is both injective and surjective, that is, bijective.

## 3. Characteristic fuzzy submultigroups of fuzzy multigroups

Recall that, a subgroup $Y$ of a group $X$ is a characteristic subgroup if $Y^\theta = Y$ for every automorphism, $\theta$ of $X$, where $Y^\theta$ denotes $\theta(Y)$. Now, we define the analogue of this notion in fuzzy multigroup setting.

**Definition 3.1.** Let $A$ be a fuzzy submultigroup of $B \in FMG(X)$. Then, $A$ is called a characteristic (f-invariant) fuzzy submultigroup of $B$, if

$$CM_{A^\theta}(x) = CM_A(x)\ \forall x \in X$$

for every automorphism, $\theta$ of $X$. That is, $\theta(A) \subseteq A$, for every $\theta \in Aut(X)$.

**Definition 3.2.** Let $A$ be a fuzzy multigroup of a group $X$ and $\theta$ a function from $X$ into itself. Define the multiset $A^\theta$ of $X$ by

$$CM_{A^\theta}(x) = CM_A(x^\theta),\text{ where } x^\theta = \theta(x) = x\ \forall x \in X.$$

**Proposition 3.3.** *Let $A \in FMG(X)$ and $g \in X$. If $\theta$ is an automorphism of $X$ defined by $\theta(x) = gxg^{-1}\ \forall x \in X$, then $A^g = A^\theta$.*

*Proof.* Let $A \in FMG(X)$ and $g \in X$. Suppose $\theta : X \to X$ is defined by $\theta(x) = gxg^{-1}\ \forall x \in X$. By Definition 2.19, we get

$$CM_{A^g}(x) = CM_A(gxg^{-1}) = CM_A(\theta(x)) = CM_{A^\theta}(x).$$

Thus, the result follows. □

**Proposition 3.4.** *Let $A \in FMG(X)$. If $\theta$ is a homomorphism of $X$ into itself, then $A^\theta$ is a fuzzy multigroup of $X$.*

*Proof.* Let $x, y \in X$. Then, $CM_{A^\theta}(xy) = CM_A((xy)^\theta) = CM_A(x^\theta y^\theta)$, since $\theta$ is a homomorphism. Since $A$ is a fuzzy multigroup of $X$, we have

$$CM_A(x^\theta y^\theta) \geq CM_A(x^\theta) \wedge CM_A(y^\theta) = CM_{A^\theta}(x) \wedge CM_{A^\theta}(y).$$

Thus,

$$CM_{A^\theta}(xy) \geq CM_{A^\theta}(x) \wedge CM_{A^\theta}(y).$$

Also,

$$CM_{A^\theta}(x^{-1}) = CM_A((x^{-1})^\theta) = CM_A((x^\theta)^{-1}) = CM_A(x^\theta) = CM_{A^\theta}(x).$$

So, $A^\theta$ is a fuzzy multigroup of $X$. □

**Theorem 3.5.** *Let $\theta : X \to X$ be an automorphism and $A \in FMS(X)$. Then, $A^\theta \in FMG(X)$ if and only if $A \in FMG(X)$.*

*Proof.* Suppose $A \in FMG(X)$. Then, using the same logic in the proof of Proposition 3.4, it follows that $A^\theta \in FMG(X)$.

Conversely, assume $A^\theta$ is a fuzzy multigroup of $X$. That is,

$$CM_{A^\theta}(xy) \geq CM_{A^\theta}(x) \wedge CM_{A^\theta}(y) \text{ and } CM_{A^\theta}(x^{-1}) = CM_{A^\theta}(x)$$

$\forall x, y \in X$ and for every $\theta \in Aut(X)$. Then,

$$CM_{A^\theta}(xy) = CM_A((xy)^\theta) = CM_A(\theta(xy)) = CM_A(xy).$$

Thus,

$$\Rightarrow CM_A(xy) \geq CM_A(x) \wedge CM_A(y)\ \forall x, y \in X.$$

Also,

$$\begin{aligned} CM_{A^\theta}(x^{-1}) = CM_A((x^{-1})^\theta) &= CM_A((x^\theta)^{-1}) \\ &= CM_A((\theta(x))^{-1}) = CM_A(x^{-1}). \end{aligned}$$

So $CM_A(x^{-1}) = CM_A(x)\,\forall x \in X$. Hence, $A \in FMG(X)$. □

**Proposition 3.6.** *If $B \in FMG(X)$. Then, every characteristic submultigroup of $B$ is a normal fuzzy submultigroup.*

*Proof.* Let $x, y \in X$ and let $A$ is a characteristic fuzzy submultigroup of $B$. To prove that $A$ is a normal fuzzy submultigroup of $B$, we have to show that

$$CM_A(xy) = CM_A(yx)\,\forall x, y \in X.$$

Let $\theta$ be the automorphism of $X$ defined by

$$\theta(y) = x^{-1}yx\,\forall y \in X.$$

Since $A$ is a characteristic fuzzy submultigroup of $B$, $A^\theta = A$. Then, we have

$$\begin{aligned} CM_A(xy) &= CM_{A^\theta}((xy)) = CM_A((xy)^\theta) \\ &= CM_A(x^{-1}(xy)x) \text{ by definition of } \theta \\ &= CM_A(yx). \end{aligned}$$

Thus, $A^\theta$ is normal by Proposition 2.20. □

**Remark 3.7.** Let $A, B, C \in FMG(X)$ such that $C \subseteq B \subseteq A$.

(1) If $C$ is a characteristic fuzzy submultigroup of $B$ and $B$ is a characteristic fuzzy submultigroup of $A$, then $C$ is a characteristic fuzzy submultigroup of $A$.

(2) If $C$ is a characteristic fuzzy submultigroup of $B$ and $B$ is a normal fuzzy submultigroup of $A$, then $C$ is a normal fuzzy submultigroup of $A$.

**Proposition 3.8.** *If $B \in FMG(X)$ and $A$ is a characteristic fuzzy submultigroup of $B$. Then, $A^*$ and $A_*$ are characteristic subgroups of $X$. Also, $A^*$ is a characteristic subgroup of $B^*$ and $A_*$ is a characteristic subgroup of $B_*$.*

*Proof.* We know that $A_*$ and $A^*$ are subgroups of $X$ by Proposition 2.16. Now, we proof that $A_*$ and $A^*$ are characteristic subgroups of $X$. It is sufficient to show that $\theta(A^*) \subseteq A^*\ \forall \theta \in Aut(X)$.

Let $\theta \in Aut(X)$. Then, $CM_{A^\theta}(x) = CM_A(x)$, since $A$ is a characteristic fuzzy submultigroup of $B$. Let $x \in A^*$. Then, $CM_A(x) = CM_A(e)$, which implies that

$$CM_{A^\theta}(x) = CM_A(\theta(x)) = CM_A(x) = CM_A(e).$$

Thus, $\theta(x) \in A^*$. So, $\theta(A^*) \subseteq A^*$. This completes the proof.

Similarly, the proof of the fact that $A_*$ is a characteristic subgroup of $X$ follows.

Recall that, $A$ is a characteristic fuzzy submultigroup of $B$, and $A^*$ and $A_*$ are characteristic subgroups of $X$. Synthesizing these, it implies that $A^*$ is a characteristic subgroup of $B^*$ and $A_*$ is a characteristic subgroup of $B_*$. □

**Theorem 3.9.** *Let $X$ be a finite group and $A$ be a characteristic fuzzy submultigroup of $B \in FMG(X)$. Then, $A_{[\alpha]}$ is a characteristic subgroup of $X\ \forall\ \alpha \leq CM_A(e)$ and $A^{[\alpha]}$ is a characteristic subgroup of $X\ \forall\ \alpha \geq CM_A(e)$, where $e$ is the identity element of $X$ and $\alpha \in [0, 1]$.*

*Proof.* It implies from Theorem 2.15 that, $A_{[\alpha]}$ is a subgroup of $X$ $\forall$ $\alpha \leq CM_A(e)$ and $A^{[\alpha]}$ is a subgroup of $X$ $\forall$ $\alpha \geq CM_A(e)$, where $\alpha \in [0,1]$. Now, we proof first that $A_{[\alpha]}$ is a characteristic subgroups of $X$.

Whenever, we show that $\theta(A_{[\alpha]}) \subseteq A_{[\alpha]}$ $\forall \theta \in Aut(X)$ we are done. Let $\theta \in Aut(X)$. Then, $CM_{A^\theta}(x) = CM_A(x)$, since $A$ is a characteristic fuzzy submultigroup of $B$. Let $x \in A_{[\alpha]}$. Then, $CM_A(x) \geq \alpha$, which implies that

$$CM_{A^\theta}(x) = CM_A(\theta(x)) = CM_A(x) \geq \alpha.$$

Thus, $\theta(x) \in A_{[\alpha]}$. So, $\theta(A_{[\alpha]}) \subseteq A_{[\alpha]}$. Hence, $A_{[\alpha]}$ is a characteristic subgroups of $X$. Similarly, $A^{[\alpha]}$ is a characteristic subgroups of $X$. □

**Remark 3.10.** Since $A$ is a characteristic fuzzy submultigroup of $B$, and $A_{[\alpha]}$ and $A^{[\alpha]}$ are characteristic subgroups of $X$. Synthesizing these, it happens that $A_{[\alpha]}$ is a characteristic subgroup of $B_{[\alpha]}$ and $A^{[\alpha]}$ is a characteristic subgroup of $B^{[\alpha]}$.

Now, we give a statement of the converse of Theorem 3.9.

**Theorem 3.11.** *Let $X$ be a finite group and $A$ be a fuzzy submultigroup of $B \in FMG(X)$. If $A_{[\alpha]}$, for $\alpha \in [0,1]$ (also $A^{[\alpha]}$) is a characteristic subgroup of $X$, then $A$ is a characteristic fuzzy submultigroup of $B$.*

*Proof.* Since $X$ is finite, $|A| < \infty$. Let $Im(A) = \{\alpha_0, ..., \alpha_n\}$ with $\alpha_0 > ... > \alpha_n$. By hypothesis, it follows that

$$A_{[\alpha]} = \{x \in X |\, CM_A(x) \geq \alpha_i\}$$

is a characteristic subgroup of $X$, $\forall i = 0, ..., n$. Let $\theta \in Aut(X)$. As

$$CM_{A^\theta}(x) = CM_A(\theta(x)) = CM_{\theta^{-1}(A)}(x) = CM_A(x),$$

it follows that $Im(A^\theta) = Im(A)$. Moreover, $\forall k = 0, ..., n$, we get $(A^\theta)_{[\alpha_k]} = A_{[\alpha_k]}$, since $x \in (A^\theta)_{[\alpha_k]} \Leftrightarrow CM_{A^\theta}(x) \geq \alpha_k \Leftrightarrow CM_A(\theta(x)) \geq \alpha_k \Leftrightarrow \theta(x) \in A_{[\alpha_k]} \Leftrightarrow x \in \theta^{-1}(A_{[\alpha_k]}) \Leftrightarrow x \in A_{[\alpha_k]}$. Thus, $A^\theta = A$. The result follows. □

In the next theorem, we show that Theorems 3.9 and 3.11 hold if the finiteness on the order of $X$ is discarded.

**Theorem 3.12.** *For a fuzzy submultigroup $A$ of $B \in FMG(X)$, the following statements are equivalent:*

(1) *$A$ is a characteristics fuzzy submultigroup of $B$,*

(2) *$A_{[\alpha]}$ for $\alpha \in [0,1]$ is a characteristic subgroup of $X$ $\forall$ $\alpha \leq CM_A(e)$, where $e$ is the identity element of $X$ (also $A^{[\alpha]}$ $\forall$ $\alpha \geq CM_A(e)$).*

*Proof.* (1)⇒(2): Let $\alpha \in Im(A)$, $\theta \in Aut(X)$ and $x \in A_{[\alpha]}$. Then

$$CM_A(\theta(x)) = CM_A(x) \geq \alpha,$$

since $A$ is a characteristic fuzzy submultigroup of $B$. It follows that $\theta(x) \in A_{[\alpha]}$. Thus, $\theta(A_{[\alpha]}) \subseteq A_{[\alpha]}$. We prove that $A_{[\alpha]} \subseteq \theta(A_{[\alpha]})$ by symmetry.

Let $x \in A_{[\alpha]}$ and let $y \in X$ such that $\theta(y) = x$. Then,

$$CM_A(y) = CM_A(\theta(y)) = CM_A(x) \geq \alpha.$$

Thus, $y \in A_{[\alpha]}$. So $x \in \theta(A_{[\alpha]})$. Hence, $A_{[\alpha]} \subseteq \theta(A_{[\alpha]})$. Therefore, $A_{[\alpha]}$ is a characteristic subgroup of $X$.

(2)⇒(1): Let $x \in X$, $\theta \in Aut(X)$ and $CM_A(x) = \alpha$. Then, $x \in A_{[\alpha]}$ and $x \notin A_{[\beta]}$ $\forall \beta > \alpha$, by Proposition 2.14. Thus, by hypothesis, $\theta(A_{[\alpha]}) = A_{[\alpha]}$. So $\theta(x) \in A_{[\alpha]}$. Hence, $CM_A(x) = CM_A(\theta(x)) \geq \alpha$.

Let $\beta = CM_A(\theta(x))$. Assume $\beta > \alpha$. Then, $\theta(x) \in A_{[\beta]} = \theta(A_{[\beta]})$. Since $\theta$ is one-to-one, it follows that $x \in A_{[\beta]}$, which is a contradiction. Therfore,

$$CM_A(\theta(x)) = \alpha = CM_A(x),$$

implying that $A$ is a characteristic fuzzy submultigroup of $B$. □

**Definition 3.13.** Let $B \in FMG(X)$. For any fuzzy submultigroup $A$ of $B$, the fuzzy submultiset, $yA$ of $B$ for $y \in X$ defined by

$$CM_{yA}(x) = CM_A(y^{-1}x) \, \forall x \in X$$

is called the left fuzzy comultiset of $A$. Similarly, the fuzzy submultiset, $Ay$ of $B$ for $y \in X$ defined by

$$CM_{Ay}(x) = CM_A(xy^{-1}) \, \forall x \in X$$

is called the right fuzzy comultiset of $A$.

**Definition 3.14.** Suppose $B$ is a fuzzy multigroup of $X$ and $A$ a normal fuzzy submultigroup of $B$. Then, the union of the set of left/right fuzzy comultisets of $A$ such that the fuzzy comultisets satisfy

$$xA \circ yA = xyA \, \forall x, y \in X$$

is called quotient or factor fuzzy multigroup of $B$ determined by $A$, denoted by $B/A$.

**Theorem 3.15.** *Let $A$ be a normal fuzzy submultigroup of $B \in FMG(X)$ and let $\theta$ be a homomorphism of $X$ into itself which leaves invariant the subgroup $A^*$. Then, $\theta$ induces a homomorphism $\bar{\theta}$ of $B/A$ into itself define by*

$$\bar{\theta}(xA) = \theta(x)A \, \forall x \in X.$$

*Proof.* First, we show that $\bar{\theta}$ is well defined. To show this, suppose $x, y \in X$ such that $xA = yA$. Then, it suffices to prove that

$$\theta(x)A = \theta(y)A.$$

Since $xA = yA$, we have

$$CM_{xA}(x) = CM_{yA}(x) \text{ and } CM_{xA}(y) = CM_{yA}(y)$$

⇒

$$CM_A(e) = CM_A(y^{-1}x) \text{ and } CM_A(x^{-1}y) = CM_A(e)$$

⇒

$$CM_A(y^{-1}x) = CM_A(x^{-1}y) = CM_A(e)$$

⇒

$$y^{-1}x, x^{-1}y \in A^*.$$

Since by hypothesis, $\theta(A^*) = A^*$, we say $\theta(y^{-1}x)$ and $\theta(x^{-1}y)$ also belong to $A^*$. Thus, we have

$$CM_A(\theta(y^{-1}x)) = CM_A(\theta(x^{-1}y)).$$

Now, let $g \in X$. Then

$$\begin{aligned} CM_{\theta(x)A}(g) &= CM_A(\theta(x^{-1})g) \\ &= CM_A(\theta(x^{-1})\theta(y)\theta(y^{-1})g) \\ &\geq CM_A(\theta(x^{-1})\theta(y)) \wedge CM_A(\theta(y^{-1})g) \\ &= CM_A(\theta(x^{-1}y)) \wedge CM_A(\theta(y^{-1})g) \\ &= CM_{\theta(y)A}(g). \end{aligned}$$

Thus, $CM_{\theta(x)A}(g) \geq CM_{\theta(y)A}(g)$. Similarly, $CM_{\theta(y)A}(g) \geq CM_{\theta(x)A}(g)$. Since $g \in X$ is arbitrary, we have

$$CM_{\theta(x)A}(g) = CM_{\theta(y)A}(g),$$

that is, $\theta(x)A = \theta(y)A$. So, $\bar{\theta}$ is well defined.

Next, we prove that $\bar{\theta}$ is a homomorphism. Let $x, y \in X$. Since $\theta$ is a homomorphism, $\theta(xy) = \theta(x)\theta(y)$. Then, $\theta(xy)A = \theta(x)A\theta(y)A$. Thus, $\bar{\theta}(xyA) = \theta(x)A\theta(y)A$. So,

$$\bar{\theta}(xAyA) = \bar{\theta}(xA)\bar{\theta}(yA).$$

Hence, $\bar{\theta}$ is a homomorphism. □

**Corollary 3.16.** *With the same hypothesis as in Theorem 3.15, the homomorphism $\bar{\theta}$ is an automorphism if $\theta$ is an automorphism and $X$ is finite.*

*Proof.* Since $X$ has finite order, it is easy to see that $\theta$ has finite order. Suppose that $\theta$ has order $n$. Then, $\theta^n = i$, where $i$ depicts identity map. Now, since by Theorem 3.15, $\bar{\theta}$ is a homomorphism, it remains to prove that $\theta$ is one-to-one.

Let $x, y \in X$ such that $\bar{\theta}(xA) = \bar{\theta}(yA)$. Then $\theta(x)A = \theta(y)A$. Thus,

$$\bar{\theta}(\theta(x)A) = \bar{\theta}(\theta(y)A).$$

This implies by the definition of $\bar{\theta}$ that $\theta^2(x)A = \theta^2(y)A$. Continue in this process, we obtain

$$i(x)A = \theta^n(x)A = \theta^n(y)A = i(y)A$$

and so $xA = yA$. Hence, $\bar{\theta}$ is one-to-one. The proof is complete. □

**Corollary 3.17.** *With the same hypothesis as in Theorem 3.15, the function $\theta$ is an automorphism of $X$ if $\bar{\theta}$ is an automorphism and $A^* = \{e\}$.*

*Proof.* Let $x, y \in X$ such that $\theta(x) = \theta(y)$. Then, it follows that $\theta(x)A = \theta(y)A$. That is, $\bar{\theta}(xA) = \bar{\theta}(yA)$. Since $\bar{\theta}$ is one-to-one by the hypothesis, $xA = yA$. Thus,

$$CM_{xA}(y) = CM_{yA}(y) \Rightarrow CM_A(x^{-1}y) = CM_A(e).$$

So, $x^{-1}y \in A^*$. Since $A^* = \{e\}$ by the hypothesis, $x^{-1}y = e$. Hence,

$$CM_A(x) = CM_A(y) \Rightarrow x = y.$$

Therefore, $\theta$ is one-to-one. The result follows. □

**Remark 3.18.** Let $B \in FMG(X)$ and $A$ a normal fuzzy submultigroup of $B$. If $\theta$ is an automorphism of $X$ such that $A^\theta = A$, then $\theta$ induces an automorphism $\bar{\theta}$ of $B/A$ define by $\bar{\theta}(xA) = \theta(x)A \, \forall x \in X$.

To see this, let $x, y \in X$. Then, we have

$$xA = yA \Leftrightarrow xA^\theta = yA^\theta$$

$\Leftrightarrow$

$$CM_{xA^\theta}(g) = CM_{yA^\theta}(g)\ \forall g \in X$$

$\Leftrightarrow$

$$CM_{A^\theta}(x^{-1}g) = CM_{A^\theta}(y^{-1}g)\ \forall g \in X$$

$\Leftrightarrow$

$$CM_A(\theta(x^{-1}g)) = CM_A(\theta(y^{-1}g))\ \forall g \in X$$

$\Leftrightarrow$

$$CM_A(\theta(x^{-1})\theta(g)) = CM_A(\theta(y^{-1})\theta(g))\ \forall g \in X$$

$\Leftrightarrow$

$$CM_{\theta(x)A}(\theta(g)) = CM_{\theta(y)A}(\theta(g))\ \forall g \in X$$

$\Leftrightarrow$

$$\theta(x)A = \theta(y)A$$

$\Leftrightarrow$

$$\bar{\theta}(xA) = \bar{\theta}(yA).$$

Thus, $\bar{\theta}$ is well defined and one-to-one. Clearly, $\bar{\theta}$ maps $B/A$ onto itself. The fact that $\bar{\theta}$ is a homomorphism is analog to the corresponding part of the proof of Theorem 3.15. The remark follows.

**Theorem 3.19.** *Let $\theta : X \to Y$ be a homomorphism. Let $A$ and $B$ be fuzzy multigroups of $X$ and $Y$, respectively. Then $\theta(A) \subseteq B$ if and only if $CM_B(f(x)) \geq CM_A(x)\ \forall\ x \in X$.*

*Proof.* Let $\theta(A) \subseteq B$ and $x \in X$. Then, we have

$$\begin{aligned} CM_B(\theta(x)) &\geq CM_{\theta(A)}(\theta(x)) \\ &= \bigvee\{CM_A(z) \mid \theta(z) = \theta(x)\} \\ &= CM_A(x). \end{aligned}$$

Conversely, let $CM_B(\theta(x)) \geq CM_A(x)$ for every $x \in X$. Then, we have

$$\begin{aligned} CM_{\theta(A)}(y) &= \bigvee\{CM_A(x) \mid \theta(x) = y\} \\ &\leq \bigvee\{CM_B(\theta(x)) \mid \theta(x) = y\} \\ &= CM_B(y) \end{aligned}$$

for every $y \in Y$. Hence, $\theta(A) \subseteq B$. □

**Proposition 3.20.** *Let $X$ be a cyclic group and $B \in FMG(X)$. Then, every fuzzy submultigroup of $B$ is f-invariant.*

*Proof.* Let $X =< x >$ and $A$ be a fuzzy submultigroup of $B \in FMG(X)$. Then, $CM_A(x^n) \geq CM_A(x)$, for every $n \in \mathbb{Z}^+$. This implies that

$$CM_A(\theta(x)) \geq CM_A(x),$$

for every $\theta \in Aut(X)$. By Theorem 3.19, it follows that $A$ is f-invariant. □

## 4. Frattini fuzzy submultigroups of fuzzy multigroups

In this section, we propose the ideas of maximal subgroups and Frattini subgroups in fuzzy multigroup setting by redefining some concepts in the light of fuzzy multigroups.

### 4.1. Maximal fuzzy submultigroups.

**Definition 4.1.** Let $B \in FMG(X)$. A proper fuzzy submultigroup $A$ of $B$ is said to be a maximal fuzzy submultigroup, if there exists other proper fuzzy submultigroups $C_i$, for $i = 1, ..., n$ of $B$ such that $CM_{C_i}(x) \leq CM_A(x)$ and $CM_{C_i}(x) \neq CM_A(x)$ $\forall x \in X$. That is, a maximal fuzzy submultigroup $A$ of $B$ is a proper fuzzy submultigroup that contains all the other proper fuzzy submultigroups of $B$.

**Example 4.2.** Let $X = \{e, a, b, c\}$ be a Klein 4-group and

$$B = \{\langle\frac{1, 0.9}{e}\rangle, \langle\frac{0.3, 0.1}{a}\rangle, \langle\frac{0.4, 0.2}{b}\rangle, \langle\frac{0.3, 0.1}{c}\rangle\}$$

be a fuzzy multigroup of $X$. Then, the following are maximal fuzzy submultigroups of $A$:

$$A_1 = \{\langle\frac{1, 0.8}{e}\rangle, \langle\frac{0.2, 0.1}{a}\rangle, \langle\frac{0.4, 0.3}{b}\rangle, \langle\frac{0.2, 0.1}{c}\rangle\},$$
$$A_2 = \{\langle\frac{1, 0.9}{e}\rangle, \langle\frac{0.3, 0.1}{a}\rangle, \langle\frac{0.3, 0.3}{b}\rangle, \langle\frac{0.3, 0.1}{c}\rangle\},$$
$$A_3 = \{\langle\frac{0.9, 0.7}{e}\rangle, \langle\frac{0.3, 0.2}{a}\rangle, \langle\frac{0.4, 0.2}{b}\rangle, \langle\frac{0.3, 0.2}{c}\rangle\}.$$

### 4.2. Frattini fuzzy submultigroups.

**Definition 4.3.** Let $B \in FMG(X)$. Suppose $A_1, A_2, ..., A_n$ (or simply $A_i$ for $i = 1, 2, .., n$) are maximal fuzzy submultigroups of $B$. Then, the Frattini fuzzy submultigroup of $B$ denoted by $\Phi(A_i)$ is the intersection of $A_i$ defined by

$$CM_{\Phi(A_i)}(x) \quad = \quad CM_{A_1}(x) \wedge ... \wedge CM_{A_n}(x) \; \forall x \in X,$$

or simply

$$CM_{\Phi(A_i)}(x) = \bigwedge_{i=1}^{n} CM_{A_i}(x) \; \forall x \in X.$$

**Example 4.4.** From Example 4.2, the Frattini fuzzy submultigroup of a fuzzy multigroup $B$ of $X$ is

$$\Phi(A_i) = \{\langle\frac{0.9, 0.7}{e}\rangle, \langle\frac{0.2, 0.1}{a}\rangle, \langle\frac{0.3, 0.2}{b}\rangle, \langle\frac{0.2, 0.1}{c}\rangle\}.$$

**Remark 4.5.** Every Frattini fuzzy submultigroup of fuzzy multigroup is a fuzzy multigroup.

**Proposition 4.6.** *If $\Phi(A_i)$ is a Frattini fuzzy submultigroup of $B \in FMG(X)$. Then, $[\Phi(A_i)]^{-1}$ is a Frattini fuzzy submultigroup of $B \in FMG(X)$.*

*Proof.* By Definitions 2.7 and 2.9, it follows that

$$\begin{aligned} CM_{[\Phi(A_i)]^{-1}}(x) &= CM_{\Phi(A_i)}(x^{-1}) \\ &= CM_{\Phi(A_i)}(x) \; \forall x \in X. \end{aligned}$$

This completes the proof. □

**Proposition 4.7.** *Let $A, B \in FMG(X)$. Then, the following statements are equivalent:*

(1) $A = \bigcap_{i=1}^{n} A_i$, *where $A_i$ are the maximal fuzzy submultigroups of $B$,*
(2) *$A$ is a Frattini fuzzy submultigroup of $B$.*

*Proof.* Straightforward. □

**Theorem 4.8.** *Let $X$ be a finite group. If $A \in FMG(X)$, then a Frattini fuzzy submultigroup $\Phi(A_i)$ of $A$ is a normal fuzzy submultigroup.*

*Proof.* Let $\Phi(A_i)$ be a Frattini fuzzy submultigroup of $A$. By Remark 4.5, we get

$$CM_{\Phi(A_i)}(xy^{-1}) \geq CM_{\Phi(A_i)}(x) \wedge CM_{\Phi(A_i)}(y) \; \forall x, y \in X,$$

implies $\Phi(A_i)$ is a fuzzy multigroup of $X$.

Now, we prove that $CM_{\Phi(A_i)}$ is a normal fuzzy submultigroup of $A$. Let $x, y \in X$, then it follows that

$$\begin{aligned} CM_{\Phi(A_i)}(yxy^{-1}) &= CM_{\Phi(A_i)}((yx)y^{-1}) \\ &= CM_{\Phi(A_i)}(x(yy^{-1})) \\ &= CM_{\Phi(A_i)}(xe) \\ &\geq CM_{\Phi(A_i)}(x). \end{aligned}$$

Then, the result by Definition 2.18. □

**Proposition 4.9.** *Every Frattini fuzzy submultigroup of a fuzzy multigroup is abelian.*

*Proof.* Let $A \in FMG(X)$ and $\Phi(A_i)$ be the Frattini fuzzy submultigroup of $A$. It follows that $\Phi(A_i)$ is a normal fuzzy submultigroup of $A$ by Theorem 4.8. Consequently,

$$CM_{\Phi(A_i)}(xyx^{-1}) = CM_{\Phi(A_i)}(y) \; \forall x, y \in X.$$

Then $CM_{\Phi(A_i)}(xy) = CM_{\Phi(A_i)}(yx) \forall x, y \in X$. Thus, the result follows by Definition 2.17. □

**Theorem 4.10.** *Let $A_i$ be maximal fuzzy submultigroups of $A \in FMG(X)$. Then, a Frattini fuzzy submultigroup $\Phi(A_i)$ of $A$ is a characteristic fuzzy submultigroup.*

*Proof.* Suppose $\Phi(A_i)$ is a Frattini submultigroup of $A$. Then

$$CM_{\Phi(A_i)}(x) = \bigwedge_{i=1}^{n} CM_{A_i}(x) \; \forall x \in X.$$

Since $\Phi(A_i) = \bigcap_{i=1}^{n} A_i$, and $A_i$ for $i = 1, ..., n$ are maximal fuzzy submultigroups of $A$, it follows that, $\Phi(A_i) \subseteq A_i$. Then, by Definition 3.1, $\Phi(A_i)$ is a characteristic fuzzy submultigroup of $A$. □

**Theorem 4.11.** *Let $A \in FMG(X)$ and $\Phi(A_i)$ be a Frattini fuzzy submultigroup of $A$. Then, $[\Phi(A_i)]_*$ and $[\Phi(A_i)]^*$ are normal subgroups of $X$.*

*Proof.* First and foremost, we show that $[\Phi(A_i)]_*$ and $[\Phi(A_i)]^*$ are subgroups of $X$.

Let $x, y \in [\Phi(A_i)]_*$. Then, $\mu_{\Phi(A_i)}(x) > 0$ and $\mu_{\Phi(A_i)}(y) > 0$. Now, by Proposition 2.16, we have

$$CM_{\Phi(A_i)}(xy^{-1}) \geq CM_{\Phi(A_i)}(x) \wedge CM_{\Phi(A_i)}(y) > 0.$$

Then, $xy^{-1} \in [\Phi(A_i)]_*$. Thus, $[\Phi(A_i)]_*$ is a subgroup of X.

Again, let $x, y \in [\Phi(A_i)]^*$. Then, by Proposition 2.16,

$$CM_{\Phi(A_i)}(x) = CM_{\Phi(A_i)}(y) = CM_{\Phi(A_i)}(e).$$

It follows that

$$\begin{aligned} CM_{\Phi(A_i)}(xy^{-1}) &\geq CM_{\Phi(A_i)}(x) \wedge CM_{\Phi(A_i)}(y) \\ &= CM_{\Phi(A_i)}(e) \wedge CM_{\Phi(A_i)}(e) \\ &= CM_{\Phi(A_i)}(e) \\ &\geq CM_{\Phi(A_i)}(xy^{-1}). \end{aligned}$$

Thus, $CM_{\Phi(A_i)}(xy^{-1}) = CM_{\Phi(A_i)}(e)\ \forall x, y \in X$. So $xy^{-1} \in [\Phi(A_i)]^*$. Hence, $[\Phi(A_i)]^*$ is a subgroup of X.

Next, we prove that $[\Phi(A_i)]_*$ and $[\Phi(A_i)]^*$ are normal subgroups of $X$. Now, let $x \in X$ and $y \in [\Phi(A_i)]_*$. By Theorem 4.8, we have

$$CM_{\Phi(A_i)}(xyx^{-1}) = CM_{\Phi(A_i)}(y).$$

It follows that

$$CM_{\Phi(A_i)}(xyx^{-1}) = CM_{\Phi(A_i)}(y) > 0.$$

Thus $xyx^{-1} \in \Phi(A_i)_*$. So $[\Phi(A_i)]_*$ is a normal subgroup of X.

Similarly, let $x \in X$ and $y \in [\Phi(A_i)]^*$. Then, we have

$$CM_{\Phi(A_i)}(xyx^{-1}) = CM_{\Phi(A_i)}(y)$$

by Theorem 4.8. It implies that

$$CM_{\Phi(A_i)}(xyx^{-1}) = CM_{\Phi(A_i)}(y) = CM_{\Phi(A_i)}(e).$$

Thus, $xyx^{-1} \in \Phi(A_i)^*$. So, $[\Phi(A_i)]^*$ is a normal subgroup of X. □

**Proposition 4.12.** *Let $A \in FMG(X)$ and $\Phi(A_i)$ be a Frattini fuzzy submultigroup of $A$. Then, $[\Phi(A_i)]_*$ and $[\Phi(A_i)]^*$ are characteristic subgroups of $X$.*

*Proof.* By Theorem 4.11, $[\Phi(A_i)]_*$ and $[\Phi(A_i)]^*$ are subgroups of $X$. The result follows by combining Theorems 4.10 and Theorem 4.11. □

**Proposition 4.13.** *Let $A \in FMG(X)$ and $\Phi(A_i)$ be a Frattini fuzzy submultigroup of $A$. Then,*

*(1) $[\Phi(A_i)]_{[\alpha]}$ is subgroup of $X$ for $\alpha \leq CM_{\Phi(A_i)}(e)$, where $e$ is the identity element of $\Phi(A_i)$,*

*(2) $[\Phi(A_i)]^{[\alpha]}$ is subgroup of $X$ for $\alpha \geq CM_{\Phi(A_i)}(e)$, where $e$ is the identity element of $\Phi(A_i)$.*

*Proof.* Assume $A \in FMG(X)$ and $\Phi(A_i)$ is a Frattini fuzzy submultigroup of $A$.

(1) Let $x, y \in [\Phi(A_i)]_{[\alpha]}$. Then, we have $CM_{\Phi(A_i)}(x) \geq \alpha$ and $CM_{\Phi(A_i)}(y) \geq \alpha$. Since $\Phi(A_i) \in FMG(X)$, it follows that

$$CM_{\Phi(A_i)}(xy^{-1}) \geq CM_{\Phi(A_i)}(x) \wedge CM_{\Phi(A_i)}(y) \geq \alpha.$$

This implies that $CM_{\Phi(A_i)}(xy^{-1}) \geq \alpha$. Thus, $xy^{-1} \in [\Phi(A_i)]_{[\alpha]}$. So, $[\Phi(A_i)]_{[\alpha]}$ is subgroup of $X$.

(2) By Definition 2.13 and following the same logic in (1), the result follows. □

**Proposition 4.14.** *Let $A \in FMG(X)$ and $\Phi(A_i)$ be a Frattini fuzzy submultigroup of $A$. Then*

*(1) $[\Phi(A_i)]_{[\alpha]}$ is a normal subgroup of $X$ for $\alpha \leq CM_{\Phi(A_i)}(e)$, where $e$ is the identity element of $\Phi(A_i)$,*

*(2) $[\Phi(A_i)]^{[\alpha]}$ is a normal subgroup of $X$ for $\alpha \geq CM_{\Phi(A_i)}(e)$, where $e$ is the identity element of $\Phi(A_i)$.*

*Proof.* Suppose $A \in FMG(X)$ and $\Phi(A_i)$ is a Frattini fuzzy submultigroup of $A$.

(1) By Theorem 4.8, it follows that $\Phi(A_i)$ is a normal fuzzy submultigroup of $A$. By Proposition 4.13, $[\Phi(A_i)]_{[\alpha]}$ is subgroup of $X$. Now, let $x \in X$ and $y \in [\Phi(A_i)]_{[\alpha]}$. Then, by Theorem 4.8,

$$CM_{\Phi(A_i)}(xyx^{-1}) = CM_{\Phi(A_i)}(y).$$

It follows that

$$CM_{\Phi(A_i)}(xyx^{-1}) = CM_{\Phi(A_i)}(y) \geq \alpha.$$

Thus, $xyx^{-1} \in [\Phi(A_i)]_{[\alpha]}$. This completes the proof.

(2) Combining Theorem 4.8, Proposition 4.13 and (1), the proof follows. □

**Proposition 4.15.** *Let $A \in FMG(X)$ and $\Phi(A_i)$ be a Frattini fuzzy submultigroup of $A$. Then*

*(1) $[\Phi(A_i)]_{[\alpha]}$ is a characteristic subgroup of $X$ for $\alpha \leq CM_{\Phi(A_i)}(e)$, where $e$ is the identity element of $\Phi(A_i)$,*

*(2) $[\Phi(A_i)]^{[\alpha]}$ is a characteristic subgroup of $X$ for $\alpha \geq CM_{\Phi(A_i)}(e)$, where $e$ is the identity element of $\Phi(A_i)$.*

*Proof.* Suppose $A \in FMG(X)$ and $\Phi(A_i)$ is a Frattini fuzzy submultigroup of $A$.

(1) Combining Theorem 4.10 and Proposition 4.13, the proof follows.

(2) Synthesizing Theorem 4.10, Proposition 4.13 and (1), the proof follows. □

**Proposition 4.16.** *Let $A, B, C \in FMG(X)$. If $A$ is a Frattini fuzzy submultigroup of $B$, and $B$ is a Frattini fuzzy submultigroup of $C$, then $A$ is a Frattini fuzzy submultigroup of $C$.*

*Proof.* Straightforward. □

**Corollary 4.17.** *With the same hypothesis as in Proposition 4.16,*

*(1) $A$ is a normal fuzzy submultigroup of a fuzzy multigroup $C$,*

*(2) $A$ is a characteristic fuzzy submultigroup of a fuzzy multigroup $C$.*

*Proof.* Suppose $A, B, C \in FMG(X)$.

(1) By Proposition 4.16, it follows that $A$ is a Frattini fuzzy submultigroup of $C$. Synthesizing Theorem 4.8, the result follows.

(2) By Proposition 4.16, it follows that $A$ is a Frattini fuzzy submultigroup of $C$. By synthesizing Theorem 4.10, the proof is complete. □

## 5. Conclusions

We have explicated the notions of characteristic fuzzy submultigroups, maximal fuzzy submultigroups and Frattini fuzzy submultigroups of fuzzy multigroups. A number of results were obtained and investigated. Notwithstanding, more properties of characteristic and Frattini fuzzy submultigroups could still be exploited.

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