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Abstract. In this paper, the concept of skew fuzzy ideals (filters) of skew lattices and the concept of fuzzy ideals (filters) of skew lattices are introduced. The characterizations of skew fuzzy ideals (filters) of skew lattices in terms of fuzzy subsets are given. An isomorphism between the class of all crisp ideals of a skew lattice and the class of all fuzzy ideals of a skew lattice (called $\alpha - level$ fuzzy ideal) is established. Further, it is proved that any fuzzy ideal (filter) of a skew lattice is a skew fuzzy ideal (filter).

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Keywords: Skew lattice, Ideal(filter) of a skew lattice, Fuzzy subset, Skew fuzzy ideal(filter) of a skew lattice, Fuzzy ideal(filter) of a skew lattice.

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1. Introduction

The introduction of the concept of a fuzzy subset of a nonempty set S as a function of S into the unit interval [0, 1] by Zadeh [10] initiated several algebraists to take up the study of fuzzy subalgebras of various algebraic systems such as groups, rings, modules, lattices etc. The study of fuzzy algebraic structure has started by Rosenfeld [9] and since then this concept has been applied to a variety of algebraic structures such as the notion of a fuzzy subgroup of a group [5], fuzzy subrings [6], fuzzy ideals of rings [7, 8], and fuzzy ideals of lattices [1].

As skew lattices are non-commutative generalization of lattices introduced by Leech [4], the order structure has an important role in the study of these algebras. Skew lattices can be seen as double regular bands where two different order concepts can be defined: the natural preorder, denoted by $\preceq$, and the natural partial order, denoted by $\leq$, one weaker than the other and both of them motivated by analogous order concepts defined for bands. They generalize the partial order of the correspondent lattice. Two kinds of ideals can be naturally derived from these preorders (ideal of a skew lattice defined using preorder and skew ideal defined using the natural partial order). The strong concept of ideal is naturally derived from the preorder and has been largely studied having an important role in the research centered on the congruences of skew Boolean algebras with intersections, a particular case of the Boolean version of a skew lattice [2]. These ideas motivates the researcher to develop the concept of skew fuzzy ideals and fuzzy ideals of skew lattices.

The organization of this paper is as follows: In section two the author describes preliminary concepts which can be used in proving lemmas, theorems and corollaries in the subsequent section. In section three the notion of fuzzy ideals of skew lattices are introduced, some basic arithmetical properties are presented and different characterizations of fuzzy ideals of skew lattices are given. In section four the notion of skew fuzzy ideals of skew lattices are introduced, the relations between fuzzy ideal and skew fuzzy ideals of skew lattices are established, some basic arithmetical properties are presented and different characterizations of skew fuzzy ideals of skew lattices are given. Finally in section five the conclusion of the research is presented.

2. Preliminaries

In this section, we will briefly review some basic concepts of skew lattices, ideals of skew lattices, fuzzy lattices and fuzzy ideals of lattices which will be used in the next sections.

Definition 2.1 ([4]). An algebra $(S, \vee, \wedge)$ of type (2, 2) is said to be a skew lattice, if the operations $\vee$ and $\wedge$ are associative and the absorption laws hold:
(i) $x \wedge (y \vee x) = x = x \wedge (x \vee y)$,
(ii) $(x \wedge y) \vee x = x = (y \wedge x) \vee x$.

On any skew lattice S two binary relations are defined as follows: the natural order relation denoted $\leq$: $x \leq y$ if and only if $x \wedge y = y \wedge x = x$ (or dually, $x \vee y = y \vee x = y$), and the natural preorder relation denoted $\preceq$: $x \preceq y$ if and only if $x \wedge y \wedge x = x$ (or dually, $y \vee x \vee y = y$).

Definition 2.2 ([3]). A nonempty subset I of a skew lattice S closed under $\vee$ is called an ideal of S, if for all $x \in S$ and $y \in I, x \preceq y$ implies $x \in I$.

Theorem 2.3 ([3]). *All ideals in a skew lattice are sub skew lattices.*

Definition 2.4 ([10]). Let X be a nonempty set. By a fuzzy subset μ of X, we mean a map from X to the interval $[0, 1]$, $\mu : X \longrightarrow [0, 1]$.

If μ is a fuzzy subset of X and $t \in [0, 1]$, then the level subset μ_t is defined as follows: $\mu_t = \{x \in X | \mu(x) \geq t\}$.

Definition 2.5 ([8]). A fuzzy subset μ of a ring R is called a fuzzy left (right) ideal of R, if
(i) $\mu(x - y) = min\{\mu(x), \mu(y)\}$,
(ii) $\mu(xy) = \mu(y)(\mu(xy) = \mu(x))$, for all $x, y \in R$.

Theorem 2.6 ([8]). *Let μ be a fuzzy subset of a ring R. If for every $t \in Im(\mu)$, μ_t is a left (right) ideal of R, then μ is a fuzzy left (right) ideal of R.*

Definition 2.7 ([1]). Let μ be a fuzzy set in a lattice L. Then μ is called a fuzzy sublattice of L, if
(i) $\mu(x+y) \geq min(\mu(x), \mu(y))$,
(ii) $\mu(xy) \geq min(\mu(x), \mu(y))$, for all $x, y \in L$.

Definition 2.8 ([1]). Let μ be a fuzzy sublattice of L. Then μ is called a fuzzy ideal, if $x \leq y$ in L implies $\mu(x) \geq \mu(y)$.

3. Fuzzy ideals(filters) of skew lattices

In this section, we introduce the concept of fuzzy ideals(filters) of skew lattices.

The following definition deals on the crisp ideal of a skew lattice derived from the natural preorder $\preceq$.

Definition 3.1. A nonempty subset I of a skew lattice S closed under $\vee$ is called an ideal of S, if $y \wedge x \wedge y \in I$, for all $x \in I, y \in S$.

Through out the rest part of this paper S denotes a skew lattice. Next we define fuzzy ideal of a skew lattice.

Definition 3.2. A fuzzy subset μ of S is called a fuzzy subskew lattice of S, if $\mu(x \vee y) \wedge \mu(x \wedge y) \geq \mu(x) \wedge \mu(y),$ for all $x, y \in S$.

Definition 3.3. A fuzzy subset μ of S is called a fuzzy ideal of S, if
(i) $\mu(x \vee y) = min(\mu(x), \mu(y))$, for all $x, y \in S$,
(ii) $\mu(x \wedge y \wedge x) \geq \mu(y)$, for all $x, y \in S$.

Example 3.4. Consider the set $S = \{a, b, c\}$. Let $\vee$ and $\wedge$ are binary operations on S defined by the tables given below. Then $(S, \vee, \wedge)$ is a skew lattice. Now define a fuzzy subset $\mu : S \longrightarrow [0, 1]$, by $\mu(a) = 0.4 = \mu(b)$ and $\mu(c) = 0.7$. It is easy to check that μ is a fuzzy ideal of S.

$\vee$	a	b	c
a	a	a	a
b	b	b	b
c	a	b	c

$\wedge$	a	b	c
a	a	b	c
b	a	b	c
c	c	c	c

Let $\mathfrak{X}_I$ denote the characteristic function of any subset I of a skew lattice S, i.e.,

$$\mathfrak{X}_I(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.5. *A fuzzy subset μ is a fuzzy ideal of a skew lattice S if and only if the level subset μ_t of S is an ideal of a skew lattice S where $t \in Im\mu$. In particular $\mathfrak{X}_I$ is a fuzzy ideal of S if and only if I is an ideal of S.*

Theorem 3.6. *Let μ be a fuzzy subset of S such that $\mu(x \vee y) = \mu(x) \wedge \mu(y)$. Then the following statements equivalently define a fuzzy ideal μ:*
(1) $\mu(x \wedge y \wedge x) \geq \mu(y)$,

(2) *if* $x \preceq y$ *for all* $x, y \in S$, *then* $\mu(x) \geq \mu(y)$,
(3) $\mu(x \wedge y), \mu(y \wedge x) \geq \mu(y)$.

Proof. Let S be a skew lattice and μ be a fuzzy subset of S such that $\mu(x \vee y) = \mu(x) \wedge \mu(y)$.

Suppose (1) holds and let $x,\ y \in S$ such that $x \preceq y$. Then

$$\begin{aligned} x = x \wedge y \wedge x \quad &\Rightarrow \quad \mu(x) = \mu(x \wedge y \wedge x) \\ &\Rightarrow \quad \mu(x) \geq \mu(y). \end{aligned}$$

Suppose (2) holds. Then clearly, $x \wedge y \preceq y$ and $y \wedge x \preceq y$. Thus by (2), we have

$$\mu(x \wedge y), \mu(y \wedge x) \geq \mu(y).$$

Suppose (3) holds. Then for all $x,\ y \in S$,

$$\mu(x \wedge y \wedge x) = \mu((x \wedge y) \wedge (y \wedge x)) \geq \mu(x \wedge y) \geq \mu(y).$$

So in all the three cases, μ is a fuzzy ideal of S. □

Theorem 3.7. *Let* μ *be a fuzzy subset of a skew lattice* S *such that* $\mu(x \vee y) = \mu(x) \wedge \mu(y)$. *Then* μ *is a fuzzy ideal of* S *if and only if* $\mu(x \vee y \vee x) = \mu(x) \wedge \mu(y)$.

Proof. Suppose μ be a fuzzy ideal of S. Then

$$\begin{aligned} \mu(x \vee y \vee x) &= \mu((x \vee y) \vee x) \\ &= \mu(x \vee y) \wedge \mu(x) \\ &= \mu(x) \wedge \mu(y) \ \lceil \text{since } \mu(x \vee y) = \mu(x) \wedge \mu(y) \rfloor. \end{aligned}$$

Conversely, suppose $\mu(x \vee y \vee x) = \mu(x) \wedge \mu(y)$. Then clearly,

$$\begin{aligned} x \wedge y \wedge x \preceq x, y \quad &\Rightarrow \quad y \vee (x \wedge y \wedge x) \vee y = y \\ &\Rightarrow \quad \mu(y) = \mu(y \vee (x \wedge y \wedge x) \vee y) = \mu(y) \wedge \mu(x \wedge y \wedge x) \\ &\Rightarrow \quad \mu(x \wedge y \wedge x) \geq \mu(y). \end{aligned}$$

Thus μ is a fuzzy ideal of S. □

Corollary 3.8. *If* μ *is a fuzzy ideal of a skew lattice* S, *then* $\mu(x \vee y) = \mu(x \vee y \vee x)$.

Proof. By definition of a fuzzy ideal, we have $\mu(x \vee y) = \mu(x) \wedge \mu(y)$. Since μ is a fuzzy ideal, we have $\mu(x \vee y \vee x) = \mu(x) \wedge \mu(y)$. Then $\mu(x \vee y) = \mu(x \vee y \vee x)$. □

Theorem 3.9. *If* μ *is a fuzzy ideal of a skew lattice* S *and* $a, x \in S$, *then* $\mu(a \wedge x \wedge a) = \mu(x \wedge a \wedge x)$.

Proof. Let $y = x \wedge a \wedge x$. Then $a \wedge y \wedge a = a \wedge (x \wedge a \wedge x) \wedge a = a \wedge x \wedge a$. Thus $\mu(a \wedge x \wedge a) = \mu(a \wedge y \wedge a) \geq \mu(a), \mu(y)$. So $\mu(a \wedge x \wedge a) \geq \mu(y) = \mu(x \wedge a \wedge x)$. Hence $\mu(a \wedge x \wedge a) \geq \mu(x \wedge a \wedge x)$.

Now let $y = a \wedge x \wedge a$. Then $x \wedge y \wedge x = x \wedge (a \wedge x \wedge a) \wedge x = x \wedge a \wedge x$. Thus $\mu(x \wedge a \wedge x) = \mu(x \wedge y \wedge x) \geq \mu(x), \mu(y)$. So $\mu(x \wedge a \wedge x) \geq \mu(y) = \mu(a \wedge x \wedge a)$. Hence $\mu(x \wedge a \wedge x) = \mu(a \wedge x \wedge a)$. □

Consider $FI(S)$ and $FF(S)$ are the classes of all fuzzy ideals and fuzzy filters of S respectively.

Theorem 3.10. *The set of all fuzzy ideals of a strongly distributive skew lattice S is a complete lattice.*

Proof. Let $\{\mu_i : i \in \Delta\}$ be a class of fuzzy ideals of S. Let μ be the point-wise infimum of $\{\mu_i\}_{i\in\Delta}$, that is $\mu(x) = \bigwedge_{i\in\Delta} \mu_i(x)$. Let $x, y \in S$. Then

$$\begin{aligned} \mu(x \vee y) &= \bigwedge_{i\in\Delta} \mu_i(x \vee y) \\ &= \bigwedge_{i\in\Delta} (\mu_i(x) \wedge \mu_i(y)) \\ &= (\bigwedge_{i\in\Delta} \mu_i(x)) \wedge (\bigwedge_{i\in\Delta} \mu_i(x)) \\ &= \mu(x) \wedge \mu(y). \end{aligned}$$

Also for any $x, y \in S$,

$$\begin{aligned} \mu(x \wedge y \wedge x) &= \bigwedge_{i\in\Delta} \mu_i(x \wedge y \wedge x) \\ &\geq \bigwedge_{i\in\Delta} \mu_i(y) \\ &= \mu(y). \end{aligned}$$

Thus $\mu(x \wedge y \wedge x) \geq \mu(y)$. So μ is a fuzzy ideal of S. Hence $\mu \in FI(S)$. This shows that $FI(S)$ is a complete lattice. ☐

Definition 3.11. For any ideal I of a skew lattice S and for each $\alpha \in [0, 1]$, the mapping α_I, $\alpha_I : S \longrightarrow [0, 1]$ defined by

$$\alpha_I(y) = \begin{cases} 1 & \text{if } y \in I \\ \alpha & \text{if } y \notin I \end{cases}$$

is called the α-level fuzzy ideal corresponding to I.

Lemma 3.12. *α_I is a fuzzy ideal of S.*

Proof. From the above definition, we have

$$\alpha_I(x \vee y) = \begin{cases} 1 & \text{if } x \vee y \in I \\ \alpha & \text{if } x \vee y \notin I. \end{cases}$$

Then clearly, for any $x,\ y \in S$, $x,\ y \preceq x \vee y$. Thus if $x \vee y \in I$, then $x, y \in I$. In turn this shows that $\alpha_I(x) = 1, \alpha_I(y) = 1$. So $\alpha_I(x \vee y) = 1 = 1 \wedge 1 = \alpha_I(x) \wedge \alpha_I(y)$.

Consider $x \vee y \notin I$ and assume that both $x,\ y \in I$. Then as I is closed under $\vee$, $x \vee y \in I$ which is a contradiction. Thus either $x \notin I$ or $y \notin I$. This implies $\alpha_I(x) = \alpha$ or $\alpha_I(y) = \alpha$. So $\alpha_I(x \vee y) = \alpha = \alpha \wedge \alpha = \alpha_I(x) \wedge \alpha_I(y)$. Clearly,

$$\alpha_I(x \wedge y \wedge x) = \begin{cases} 1 & \text{if } x \wedge y \wedge x \in I \\ \alpha & \text{if } x \wedge y \wedge x \notin I. \end{cases}$$

Suppose $x \wedge y \wedge x \in I$. Then $\alpha_I(x \wedge y \wedge x) = 1 \geq \alpha_I(x) \vee \alpha_I(y) \geq \alpha_I(y)$. Consider that $x \wedge y \wedge x \notin I$. Suppose either x or y belongs to I. With out loss of generality, let $x \in I$. Since $x \wedge y \wedge x \preceq x$, we obtain that $x \wedge y \wedge x \in I$ which is a contradiction. Thus both $x, y \notin I$. This implies that $\alpha_I(x) = \alpha = \alpha_I(y)$. So $\alpha_I(x \wedge y \wedge x) = \alpha = \alpha \vee \alpha = \alpha_I(x) \vee \alpha_I(y) \geq \alpha_I(y)$. Hence α_I is a fuzzy ideal. □

Theorem 3.13. *I is an ideal of a skew lattice S if and only if α_I is a fuzzy ideal of S.*

Proof. Suppose I be a nonempty subset of S such that α_I is a fuzzy ideal of S.

Let $x,\ y \in I$. Then $\alpha_I(x) = \alpha_I(y) = 1$ and $\alpha_I(x \vee y) = \alpha_I(x) \wedge \alpha_I(y) = 1 \wedge 1 = 1$. Thus $x \vee y \in I$.

Let $x \in S$ and $y \in I$. Then clearly, $x \wedge y \wedge x \preceq y$ and $\alpha_I(y) = 1$. Thus we claim that $x \wedge y \wedge x \in I$. Since α_I is a fuzzy ideal, $\alpha_I(x \wedge y \wedge x) \geq \alpha_I(y) = 1 \Rightarrow \alpha_I(x \wedge y \wedge x) = 1$. So $x \wedge y \wedge x \in I$. Hence I is an ideal of a skew lattice. The for ward proof is done on the previous lemma. □

Theorem 3.14. *Suppose S be a skew lattice and $0 \neq \alpha \in [0, 1]$. Then $f : I(S) \longrightarrow F\alpha_I(S)$ defined by $f(I) = \alpha_I$ is an isomorphism.*

Proof. Suppose I and J are ideals of S. Then

$$\begin{aligned} \alpha_I(y) \wedge \alpha_J(y) &= \begin{cases} 1 & \text{if } y \in I \\ \alpha & \text{if } y \notin I \end{cases} \wedge \begin{cases} 1 & \text{if } y \in J \\ \alpha & \text{if } y \notin J \end{cases} \\ &= \begin{cases} 1 & \text{if } y \in I \text{ and } y \in J \\ \alpha & \text{if } y \notin I \text{ and } y \notin J \end{cases} \\ &= \begin{cases} 1 & \text{if } y \in I \cap J \\ \alpha & \text{if } y \notin I \cap J \end{cases} \\ &= \alpha_{I \cap J}(y). \end{aligned}$$

Thus we have

$$\alpha_{I \cap J} = \alpha_I \wedge \alpha_J. \tag{3.1}$$

Suppose $\alpha_I \leq \alpha_J$ and let $x \in I$. Since $\alpha_I = \alpha_I \wedge \alpha_J$, we have

$$\alpha_I(x) \wedge \alpha_J(x) = \alpha_I(x) \Rightarrow 1 \wedge \alpha_J(x) = 1 \Rightarrow \alpha_J(x) = 1 \Rightarrow x \in J.$$

Then $I \subseteq J$.

Suppose $I \subseteq J$. If $x \in I$, then $\alpha_I(x) \wedge \alpha_J(x) = 1 \wedge 1 = 1 \Rightarrow \alpha_I(x) = 1 = \alpha_J(x) \Rightarrow \alpha_I = \alpha_J$. If $x \in J$ and $x \notin I$, then $\alpha_I(x) = \alpha, \alpha_J(x) = 1$. Thus

$$\alpha_I(x) \wedge \alpha_J(x) = \alpha \wedge 1 = \alpha = \alpha_I(x) \Rightarrow \alpha_I \leq \alpha_J.$$

If $x \notin I$ and $x \notin J$, then $\alpha_I(x) = \alpha_J(x) = \alpha$. Thus $\alpha_I = \alpha_J$. So from the above three cases, we conclude that $\alpha_I \leq \alpha_J$. Hence

$$\alpha_I \leq \alpha_J \Leftrightarrow I \subseteq J. \tag{3.2}$$

From (3.1), we have

$$f(I \cap J) = \alpha_{I \cap J} = \alpha_I \wedge \alpha_J = f(I) \wedge f(J). \tag{3.3}$$

We recall that $I \vee J = \{x \in S : x \leq a \vee b, a \in I, b \in J\}$ and $(\alpha_I \vee \alpha_J)(x) = \bigvee\{\bigwedge_{i=1}^{n}(\alpha_I(a_i) \vee \alpha_J(a_i)) : x \leq \bigvee_{i=1}^{n} a_i; a_i \in S\}$. Since $I \subseteq I \vee J$, from (3.2), we have $\alpha_I \leq \alpha_{I \vee J}$ and $\alpha_J \leq \alpha_I \vee \alpha_J$. Then

$$\alpha_I \vee \alpha_J \leq \alpha_{I \vee J}. \tag{3.4}$$

Also $\alpha_I(x) = 1$ or $\alpha_I(x) = \alpha$ and $\alpha_J(x) = 1$ or $\alpha_J(x) = \alpha$, for any $x \in S$. Thus $(\alpha_I \vee \alpha_J)(x) \geq \alpha = \alpha_{I \vee J}(x)$, if $x \notin I \vee J$. That is, $\alpha_I \vee \alpha_J \geq \alpha_{I \vee J}$, if $x \notin I \vee J$.

On the other hand, suppose $x \in I \vee J$. Then there exist $a \in I$ and $b \in J$ such that $x \leq a \vee b$,

$$\begin{aligned}
(\alpha_I \vee \alpha_J)(x) &\geq (\alpha_I(a) \vee \alpha_J(a)) \wedge (\alpha_I(b) \vee \alpha_J(b)) \\
&\geq \alpha_I(a) \wedge \alpha_J(b) \\
&= 1 \wedge 1 \\
&= 1 \\
&= \alpha_{I \vee J}(x).
\end{aligned}$$

It follows that $(\alpha_I \vee \alpha_J)(x) \geq \alpha_{I \vee J}(x)$, for all $x \in S$. Thus

$$\alpha_I \vee \alpha_J \geq \alpha_{I \vee J}. \tag{3.5}$$

So by (3.4) and (3.5), we have $\alpha_I \vee \alpha_J = \alpha_{I \vee J}$. Now $f(I \vee J) = \alpha_{I \vee J} = \alpha_I \vee \alpha_J = f(I) \vee f(J)$. Hence f is homomorphism. Suppose $f(I) = f(J)$. Then $\alpha_I = \alpha_J$. Thus $I = J$. Hence f is an embedding of $I(S)$ in to $F\alpha_I(S)$. Therefore f is an isomorphism. □

Theorem 3.15. *Let S be a skew lattice with 0. Let $\{I_\alpha\}_{\alpha \in [0,1]}$ be a class of ideals of S such that $\bigcap_{\alpha \in M} I_\alpha = I_{\bigvee_{\alpha \in M} \alpha}$ for any $M \subseteq [0,1]$. For any $x \in S$ define $\mu(x) = \bigvee\{\alpha \in [0,1] : x \in I_\alpha\}$. Then μ is a fuzzy ideal of S such that I_α is precisely the $\alpha-$cut of μ for any $\alpha \in [0,1]$. Conversely, every fuzzy ideal of S can be obtained as above.*

Definition 3.16. A fuzzy subset μ of S is called a fuzzy filter of S, if the following conditions hold:

(i) $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$,
(ii) $\mu(y \vee x \vee y) \geq \mu(x) \vee \mu(y)$, for all $x, y \in S$.

Theorem 3.17. *Let μ be a fuzzy subset of S such that $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$. Then the following statements equivalently define a fuzzy filter μ:*

(1) $\mu(y \vee x \vee y) \geq \mu(x) \vee \mu(y)$,
(2) *if $x \preceq y$ for all $x, y \in S$, then $\mu(x) \leq \mu(y)$,*
(3) $\mu(x \vee y) \geq \mu(x) \vee \mu(y)$ *and* $\mu(y \vee x) \geq \mu(x) \vee \mu(y)$.

Proof. Let S be a skew lattice and μ be a fuzzy subset of S such that $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$.

Suppose (1) holds and let $x,\ y \in S$ such that $x \preceq y$. Then

$$y = y \vee x \vee y \Rightarrow \mu(y) = \mu(y \vee x \vee y) \geq \mu(x) \vee \mu(y) \Rightarrow \mu(y) \geq \mu(x).$$

Suppose (2) holds. Then clearly, $x,\ y \preceq x \vee y$ and $x,\ y \preceq y \vee x$. This shows that $\mu(x), \mu(y) \leq \mu(x \vee y), \mu(y \vee x) \Rightarrow \mu(x \vee y) \geq \mu(x) \vee \mu(y)$ and $\mu(y \vee x) \geq \mu(x) \vee \mu(y)$.

Suppose (3) is true. Then

$$\mu(y \vee x \vee y) = \mu((y \vee x) \vee (x \vee y)) \geq \mu(x \vee y) \vee \mu(y \vee x) \geq \mu(x) \vee \mu(y).$$

Thus $\mu(y \vee x \vee y) \geq \mu(x) \vee \mu(y)$. So in all the three cases, μ is a fuzzy filter of S. □

Lemma 3.18. *If μ is a fuzzy filter of a skew lattice S, then $\mu(x \wedge y) = \mu(y \wedge x \wedge y)$*

Proof. From the definition of fuzzy filter, we have $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$ and by the theorem, $\mu(y \wedge x \wedge y) = \mu(x) \wedge \mu(y) \Rightarrow \mu(x \wedge y) = \mu(y \wedge x \wedge y)$. □

Lemma 3.19. *Let μ be a fuzzy filter of a skew lattice S. If $x\mathfrak{D}y$, then $\mu(x) = \mu(y)$.*

Proof. Suppose $x\mathfrak{D}y$. Then $x \preceq y$ and $y \preceq x \Rightarrow \mu(x) \leq \mu(y)$ and $\mu(y) \leq \mu(x) \Rightarrow \mu(x) = \mu(y)$. □

4. Skew fuzzy ideals(filters) of skew lattices

In this section, we define a skew fuzzy ideal(filter) of a skew lattice and characterize skew fuzzy ideals (filters) of skew lattices using fuzzy subsets. First, we will define a skew fuzzy ideal of a skew lattice as follows:

Definition 4.1. A fuzzy subset μ of a skew lattice S is said to be a skew fuzzy ideal of S, if the following conditions hold:

(i) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$,
(ii) $\mu(x \wedge y) \wedge \mu(y \wedge x) \geq \mu(x) \vee \mu(y)$.

Theorem 4.2. *All skew fuzzy ideals are fuzzy subskew lattices.*

Proof. Suppose μ be a skew fuzzy ideals of a skew lattice S. Then $\mu(x \wedge y) \geq \mu(x) \vee \mu(y) \geq \mu(x) \wedge \mu(y)$ and also by (i) of the above definition, $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$. Thus $\mu(x \wedge y) \wedge \mu(x \vee y) \geq \mu(x) \wedge \mu(y)$. So μ is a fuzzy sub skew lattice of S. □

Theorem 4.3. *A fuzzy ideal μ of a skew lattice S is a skew fuzzy ideal of S.*

Proof. Let S be a skew lattice and μ be a fuzzy ideal of S. Then $\mu(x \vee y) = \mu(x) \wedge \mu(y) \geq \mu(x) \wedge \mu(y)$. Clearly, $x \wedge y,\ y \wedge x \preceq x \wedge y \wedge x$. Thus

$$\mu(x \wedge y), \mu(y \wedge x) \geq \mu(x \wedge y \wedge x) \geq \mu(x) \vee \mu(y).$$

So $\mu(x \wedge y) \wedge \mu(y \wedge x) \geq \mu(x) \vee \mu(y)$. Hence μ is a skew fuzzy ideal of S. □

Theorem 4.4. *A fuzzy subset μ of S is a skew fuzzy ideal of S if and only if the following conditions hold:*

(1) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$,
(2) $\mu(y \wedge x \wedge y) \geq \mu(x) \vee \mu(y)$.

Proof. Suppose μ is a skew fuzzy ideal of a skew lattice S. Then the condition (1) holds directly from the definition. Again from the definition, we obtain that

$$\mu(x \wedge y), \mu(y \wedge x) \geq \mu(x) \vee \mu(y).$$

Thus $\mu(y \wedge x \wedge y) = \mu((y \wedge x) \wedge y) \geq \mu(y \wedge x) \vee \mu(y) \geq \mu(y) \vee \mu(x)$.

Conversely, suppose (1) and (2) hold. Since $x \wedge y = x \wedge y \wedge (y \wedge x \wedge y) \wedge x \wedge y$,

$$\begin{aligned}\mu(x \wedge y) &= \mu((x \wedge y) \wedge (y \wedge x \wedge y) \wedge (x \wedge y)) \\ &\geq \mu(y \wedge x \wedge y) \vee \mu(x \wedge y) \\ &\geq \mu(x) \vee \mu(y) \vee \mu(x \wedge y) \\ &\geq \mu(x) \vee \mu(y).\end{aligned}$$

Similarly, $\mu(y \wedge x) = \mu((y \wedge x) \wedge (x \wedge y \wedge x) \wedge (y \wedge x)) \geq \mu(x) \vee \mu(y)$. Then μ is a skew fuzzy ideal of S. □

Definition 4.5. A fuzzy subset μ of a skew lattice S is said to be a skew fuzzy filter of S, if the following conditions hold:

(i) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$,
(ii) $\mu(x \vee y) \wedge \mu(y \vee x) \geq \mu(x) \vee \mu(y)$.

Theorem 4.6. *A skew fuzzy filter is a fuzzy subskew lattice.*

Proof. Suppose μ be a skew fuzzy filter of a skew lattice S. Then

$$\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$$

and

$$\mu(x \vee y) \geq \mu(x) \vee \mu(y) \geq \mu(x) \wedge \mu(y).$$

Thus $\mu(x \wedge y) \wedge \mu(x \vee y) \geq \mu(x) \wedge \mu(y)$. So μ is a fuzzy subskew lattice. □

Theorem 4.7. *A fuzzy subset μ of a skew lattice S is a skew fuzzy filter of S if and only if*
(1) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$,
(2)$\mu(y \vee x \vee y) \geq \mu(x) \vee \mu(y)$.

Proof. Suppose μ is a skew fuzzy filter of S. Then clearly, (1) holds directly. From the definition, $\mu(x \vee y),\ \mu(y \vee x) \geq \mu(x) \vee \mu(y)$. Thus

$$\mu(y \vee x \vee y) = \mu((y \vee x) \vee (x \vee y)) \geq \mu(x \vee y) \vee \mu(y \vee x) \geq \mu(x) \vee \mu(y).$$

Conversely, suppose that (1) and (2) hold. Since $y \vee x \vee y \preceq x \vee y$, we have

$$x \vee y = (x \vee y) \vee (y \vee x \vee y) \vee (x \vee y).$$

Thus

$$\begin{aligned}\mu(x \vee y) &= \mu((x \vee y) \vee (y \vee x \vee y) \vee (x \vee y)) \\ &\geq \mu(y \vee x \vee y) \vee \mu(x \vee y) \\ &\geq \mu(x) \vee \mu(y).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mu(y \vee x) &= \mu((y \vee x) \vee (x \vee y \vee x) \vee (y \vee x)) \\ &\geq \mu(x \vee y \vee x) \vee \mu(y \vee x) \\ &\geq \mu(x) \vee \mu(y) \vee \mu(y \vee x) \\ &\geq \mu(x) \vee \mu(y).\end{aligned}$$

So $\mu(x \vee y) \wedge \mu(y \vee x) \geq \mu(x) \vee \mu(y)$. Hence μ is a skew fuzzy filter of S. □

Let S be a skew lattice and $x \in S$. Then
(i) $x^{\downarrow} = S \wedge x \wedge S = \{y \in S : y \preceq x\}$ is called a principal ideal generated by x.

(ii) $x^{\downarrow *} = x \wedge S \wedge x = \{y \in S : y \leq x\}$ is called a principal skew ideal generated by x. Now

$$\begin{aligned} \alpha_{x^{\downarrow}}(y) &= \begin{cases} 1 & \text{if } y \in x^{\downarrow} \\ \alpha & \text{if } y \notin x^{\downarrow} \end{cases} \\ &= \begin{cases} 1 & \text{if } y \in S \wedge x \wedge S \\ \alpha & \text{if } y \notin S \wedge x \wedge S \end{cases} \\ &= \begin{cases} 1 & \text{if } y \in \{z \in S : z \preceq x\} \\ \alpha & \text{if } y \notin \{z \in S : z \preceq x\} \end{cases} \\ &= \begin{cases} 1 & \text{if } y \preceq x \\ \alpha & \text{if } y \notin \preceq x. \end{cases} \end{aligned}$$

Thus $\alpha_{x^{\downarrow}}$ is an $\alpha-$level fuzzy ideal of S corresponding to $x^{\downarrow}$. Also

$$\begin{aligned} \alpha_{x^{\downarrow *}}(y) &= \begin{cases} 1 & \text{if } y \in x^{\downarrow *} \\ \alpha & \text{if } y \notin x^{\downarrow *} \end{cases} \\ &= \begin{cases} 1 & \text{if } y \in x \wedge S \wedge x \\ \alpha & \text{if } y \notin x \wedge S \wedge x \end{cases} \\ &= \begin{cases} 1 & \text{if } y \in \{z \in S : z \leq x\} \\ \alpha & \text{if } y \notin \{z \in S : z \leq x\} \end{cases} \\ &= \begin{cases} 1 & \text{if } y \leq x \\ \alpha & \text{if } y \leq x. \end{cases} \end{aligned}$$

So $\alpha_{x^{\downarrow *}}$ is an $\alpha-$level fuzzy ideal of S corresponding to $x^{\downarrow *}$.

Theorem 4.8. *$\alpha_{x^{\downarrow}}$ and $\alpha_{x^{\downarrow *}}$ are fuzzy ideal and skew fuzzy ideal of S respectively if and only if $x^{\downarrow}$ and $x^{\downarrow *}$ are principal ideals and principal skew ideals of S respectively.*

5. Conclusions

In the present paper the theoretical point of view of skew fuzzy ideal(filter) and fuzzy ideal(filter) of skew lattices are discussed. An isomorphism between the class of all crisp ideals of a skew lattice and the class of all fuzzy ideals of a skew lattice(called $\alpha - level$ fuzzy ideal) is established. The fact that all fuzzy ideals of a skew lattice are skew fuzzy ideals is proved. These concepts are basic supporting structures for the development of fuzzy set theory. One can extend this work by studying other algebraic structures.

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