Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 19, No. 2, (April 2020) pp. 169–178
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2020.19.2.169

@FMI

Lattice of soft topologies

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 19, No. 2, April 2020

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 19, No. 2, (April 2020) pp. 169–178
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2020.19.2.169

@FMI

Lattice of soft topologies

K. A. Ajan, T. P. Johnson

Received 15 August 2019; Revised 9 September 2019; Accepted 20 November 2019

Abstract. In a machine learning environment involving inferential knowledge derived from the association between causes and effects, soft set theory can be employed effectively. In the present paper we study about the set of all soft points on a given universal set and a set of parameters. The relation between this set and the set of all soft sets on a given universal set was investigated. Given a universal set and a set of parameters, it is established that there is a bijection between the family of all topologies on the set of all soft points and the family of all soft topologies. It is also proved that the family of all soft topologies is a complete lattice and the bijection mentioned above is a lattice isomorphism.

2010 AMS Classification: 03E72, 08A72

Keywords: Soft set, Soft topology, Lattice of soft topologies, Lattice of topologies, Soft points.

Corresponding Author: K. A, Ajan (ajananathan79@gmail.com)

1. Introduction

Molodtsov [10] introduced the concept of soft set theory in 1999. It has been presented as a mathematical tool to deal with uncertainty and vagueness associated with many real life problems. Since then several works have been done in the algebraic structure of soft set theory [3, 8] and a number of interesting studies were made in application areas like decision making, data mining, artificial intelligence and machine learning etc [1, 2, 7, 11, 18]. In 2011, Shabir and Naz [15], and Çağman [5] introduced the concept of soft topology as a separate work. Xie [19] introduced the concept of soft points and he studied the notions like soft interior points, soft neighbourhood etc. based on the concept of soft points. He also proved that every soft set can be represented as a soft union of soft points. A comparative work on the concept of soft points was done by Senel [14]. He also studied the theory of Hausdorff spaces using soft sets [13].

In a machine learning environment involving causes and possible effects the soft set theory can be an effective tool. There are many situations in which the association between causes and the possible effects vary depending on the environment or the subject. If we take the universal set (or set of objects) as the set of possible effects and the parameter (or attribute) set as the set of causes then the association between the set of causes and set of possible effects in different environments can be represented by different soft sets on the same universal set. The soft set theory uses the operations of classical set theory as a base for developing operations in soft sets which is a real advantage for the development and application of the theory. The development of the soft set theory will be helpful in such situations where learning is based on the inferential knowledge derived from the association between causes and effects. This paper is an attempt to build connections between soft set theory and classical set theory, which we hope will ultimately boost the development of the soft set theory and will give it some leverage in the application areas.

In this paper we establish the existence of a one to one and onto correspondence between the family of all soft sets on a given set and the power set of the set of soft points on the given set. We further go forward to prove that there is a bijection between the family of all soft topologies on a given set and the family of all (point set) topologies on the set of all soft points on the given set. Theorem **??** provides us a method to connect soft topologies on X and topologies on X, but it is one sided. We cannot always find a soft topology on X using some topology on X. We provide a method to find soft topologies generated from a topology on X. We have established a connection between a soft topology on X and topologies on X and E. It is proved that there exists a one to one correspondence between soft topologies on X and topologies on $E \times X$. Corresponding to a topology on X and a topology on E there exists a product topology on $E \times X$ and corresponding to this product topology we will have a soft topology on X.

In 1936, Birkhoff [4] first explicitly described the family of all topologies on a given set. He proved that this family is a complete lattice with set inclusion as ordering. In 1947, Vaidyanathaswamy [17] has shown that this lattice is atomic and also pointed out the anti-atoms. But the proof for the claim that this lattice is anti-atomic was given by Otto Fröhlich [6] in 1964. The problem of complementation of this lattice was solved by Steiner [16] in 1966. A slightly modified proof of Steiner's result was given by Rooij [12] in 1968.

For this work we approach soft set theory and soft topology in a different way and we got some other interesting results also. We proved that the family of all soft topologies is a complete lattice and that the above mentioned bijection is an order preserving map between the two complete lattices. It is then established that this mapping preserves meet and join. It then leads to the conclusion that the lattice of all soft topologies on a given set is isomorphic to the lattice of all topologies on a set (the set of all soft points on the given set). So it can be inferred that both the lattice of soft topologies and the lattices of topologies have common properties namely, atomic, anti-atomic, complemented etc.

2. Preliminaries

Definition 2.1 ([5])**.** Let X be a universal set and E be the set of parameters, $A \subseteq E$ and $P(X)$ be the power set of X. A soft set $\mathrm{F_A}$ over X is defined by the set of ordered pairs

$$\mathrm{F_A} = \{(\alpha, f_A(\alpha)) | \alpha \in E, f_A(\alpha) \in P(X)\},$$

where $f_A \colon E \to P(X)$ such that $f_A(\alpha) = \varnothing$, for all $\alpha \notin E$.
Here f_A is called the approximate function of $\mathrm{F_A}$.

Notation 2.2. The set of all soft sets over X will be denoted by $S(X)$. The cardinality of $S(X)$ is $2^{|X||E|}$.

Definition 2.3 ([5])**.** Let $\mathrm{F_A} \in S(X)$. If $f_\alpha = \varnothing$, for all $\alpha \in E$, then $\mathrm{F_A}$ is called the soft null set and denoted by $F_\varnothing$.

Definition 2.4 ([5])**.** Let $\mathrm{F_A} \in S(X)$. If $f_A(\alpha) = X$, for all $\alpha \in A$, then $\mathrm{F_A}$ is called an A-universal soft set, denoted by $F_{\widetilde{A}}$. If $A = E$ then the A-universal soft set is called a universal soft set and denoted by $F_{\widetilde{E}}$.

Definition 2.5 ([5])**.** Let $\mathrm{F_A}, \mathrm{F_B} \in S(X)$. Then $\mathrm{F_A}$ is a soft subset of $\mathrm{F_B}$, denoted by $\mathrm{F_A} \mathrel{\widetilde{\subseteq}} \mathrm{F_B}$, if $f_A(\alpha) \subseteq f_B(\alpha)$, for all $\alpha \in E$.

Definition 2.6. Let $\mathrm{F_A}, \mathrm{F_B} \in S(X)$. Then $\mathrm{F_A}$ and $\mathrm{F_B}$ are said to be soft equal if $f_A(\alpha) = f_B(\alpha)$, for all $\alpha \in E$.

Definition 2.7 ([5])**.** Let $\mathrm{F_A}, \mathrm{F_B} \in S(X)$. Then the soft union $\mathrm{F_A} \mathbin{\widetilde{\cup}} \mathrm{F_B}$, soft intersection $\mathrm{F_A} \mathbin{\widetilde{\cap}} \mathrm{F_B}$ and soft difference $\mathrm{F_A} \mathbin{\widetilde{\setminus}} \mathrm{F_B}$ are defined by the approximate functions

$f_{A \widetilde{\cup} B}(\alpha) = f_A(\alpha) \cup f_B(\alpha)$, $f_{A \widetilde{\cap} B}(\alpha) = f_A(\alpha) \cap f_B(\alpha)$ and $f_{A \widetilde{\setminus} B}(\alpha) = f_A(\alpha) \setminus f_B(\alpha)$,

respectively.

Definition 2.8 ([5])**.** Let $\mathrm{F_A} \in S(X)$. Then the soft complement of $\mathrm{F_A}$ is denoted by $\mathrm{F_A}^{\widetilde{c}}$ is defined by the approximate function $f_A^{\widetilde{c}}(\alpha) = X \setminus f_A(\alpha)$, for all $\alpha \in E$.

Remark 2.9. It is easy to see that $(\mathrm{F_A}^{\widetilde{c}})^{\widetilde{c}} = \mathrm{F_A}$ and $F_\varnothing^{\widetilde{c}} = F_{\widetilde{E}}$.

Definition 2.10 ([5, 15])**.** Let X be a universal set and E be the set of parameters. Let $\widetilde{\mathscr{T}} \subset S(X)$. Then $\widetilde{\mathscr{T}}$ is called a soft topology on X, if

(i) $F_\varnothing, F_{\widetilde{E}} \in \widetilde{\mathscr{T}}$,

(ii) $\{F_{A_i} | i \in I \subset \mathbb{N}\} \subset \widetilde{\mathscr{T}} \implies \widetilde{\bigcup\limits_{i \in I}} F_{A_i} \in \widetilde{\mathscr{T}}$,

(iii) $\mathrm{F_A}, \mathrm{F_B} \in \widetilde{\mathscr{T}} \implies \mathrm{F_A} \mathbin{\widetilde{\cap}} \mathrm{F_B} \in \widetilde{\mathscr{T}}$.

The tuple $(X, \widetilde{\mathscr{T}}, E)$ is called a soft topological space.

Definition 2.11. Let $(X, \widetilde{\mathscr{T}}, E)$ be a soft topological space. Then every element $\mathrm{F_A} \in \widetilde{\mathscr{T}}$ is called a soft open set on X.

Theorem 2.12 ([15])**.** *Let $(X, \widetilde{\mathscr{T}}, E)$ be a soft topological space over X. Then the collection $\mathscr{T}_\alpha = \left\{ f_A(\alpha) | \mathrm{F_A} \in \widetilde{\mathscr{T}} \right\}$ for each $\alpha \in E$ defines a topology on X.*

Remark 2.13. The converse of the above theorem is not true. In general, it is not possible to construct a soft topology over X using some topologies on X.

Definition 2.14 ([5]). Let $\widetilde{\mathscr{T}_1}$ and $\widetilde{\mathscr{T}_2}$ be two soft topologies over X. If $\widetilde{\mathscr{T}_1} \subseteq \widetilde{\mathscr{T}_2}$, then we say $\widetilde{\mathscr{T}_2}$ is finer than $\widetilde{\mathscr{T}_1}$.

3. Soft points and soft sets

Definition 3.1 ([19]). Let $x \in X$, $\alpha \in E$ and let $\mathrm{F_A} \in S(X)$ be a soft set such that

$$f_A(e) = \begin{cases} \{x\} & \text{if} \quad e = \alpha \\ \varnothing & \text{if} \quad e \neq \alpha. \end{cases}$$

Then $\mathrm{F_A}$ is called a soft point on X and is denoted by x_α.

Definition 3.2. Let $\mathrm{F_A} \in S(X)$. Then a soft point x_α on X is a soft element of $\mathrm{F_A}$, denoted by $\mathrm{x}_\alpha \,\widetilde{\in}\, \mathrm{F_A}$, if $\mathrm{x}_\alpha \,\widetilde{\subseteq}\, \mathrm{F_A}$.

Remark 3.3. $\mathrm{x}_\alpha \,\widetilde{\in}\, \mathrm{F_A}$ if and only if $x \in f_A(\alpha)$.

Definition 3.4. Let $X_E = \{\mathrm{x}_\alpha \,|\, \mathrm{x}_\alpha$ is a soft point on $X\}$. Then this (point) set is called a soft point set.

Remark 3.5. The cardinality of X_E is $|X_E| = |X|\,|E|$.

Proposition 3.6. *Let X be a universal set of objects and E be the set of parameters. Then there exists a one to one correspondence between the cartesian product $E \times X$ and the soft point set X_E.*

Proof. Define a mapping $p : E \times X \to X_E$ such that $p(\alpha, x) = \mathrm{x}_\alpha$. It is clear that the mapping p is a bijection from $E \times X$ to X_E. □

Proposition 3.7. *Let X be a universal set of objects and E be the set of parameters. Corresponding to every soft set $\mathrm{F_A}$ on X there exists a unique subset A_F of X_E.*

Proof. Let $\mathrm{F_A} \in S(X)$. Define $\mathscr{G} : S(X) \to P(X_E)$ by

$$\mathscr{G}(\mathrm{F_A}) = A_F = \left\{\mathrm{x}_\alpha \,|\, \mathrm{x}_\alpha \,\widetilde{\in}\, \mathrm{F_A}\right\}.$$

Then we have

$$\mathscr{G}(F_\varnothing) = \varnothing, \text{ since } \mathrm{x}_\alpha \,\widetilde{\notin}\, F_\varnothing \quad \text{for all} \quad \mathrm{x}_\alpha \in X_E$$

and

$$\mathscr{G}(F_{\widetilde{E}}) = X_E, \text{ since } \mathrm{x}_\alpha \,\widetilde{\in}\, F_{\widetilde{E}} \quad \text{for all} \quad \mathrm{x}_\alpha \in X_E.$$

We will prove that this mapping is well defined, one to one and onto. Let $\mathrm{F_A} \,\widetilde{=}\, \mathrm{F_B}$. Then

$$\begin{aligned} f_A(\alpha) = f_B(\alpha), &\quad \text{for all} \quad \alpha \in E \\ \text{that is,} \quad x \in f_A(\alpha) \iff x \in f_B(\alpha), &\quad \text{for all} \quad \alpha \in E \\ \text{that is,} \quad \mathrm{x}_\alpha \,\widetilde{\in}\, \mathrm{F_A} \iff \mathrm{x}_\alpha \,\widetilde{\in}\, \mathrm{F_B} & \\ \text{that is,} \quad \left\{\mathrm{x}_\alpha \,|\, \mathrm{x}_\alpha \,\widetilde{\in}\, \mathrm{F_A}\right\} = \left\{\mathrm{x}_\alpha \,|\, \mathrm{x}_\alpha \,\widetilde{\in}\, \mathrm{F_B}\right\} & \\ \text{that is,} \quad \mathscr{G}(\mathrm{F_A}) = \mathscr{G}(\mathrm{F_B}). & \end{aligned}$$

Thus $\mathscr{G}$ is well defined.

Let $\mathscr{G}(\mathrm{F_A}) = \mathscr{G}(\mathrm{F_B})$. Then $\left\{\mathrm{x}_\alpha \,|\, \mathrm{x}_\alpha \,\widetilde{\in}\, \mathrm{F_A}\right\} = \left\{\mathrm{x}_\alpha \,|\, \mathrm{x}_\alpha \,\widetilde{\in}\, \mathrm{F_B}\right\}$. Thus $\mathrm{x}_\alpha \,\widetilde{\in}\, \mathrm{F_A} \iff \mathrm{x}_\alpha \,\widetilde{\in}\, \mathrm{F_B}$. So $\mathrm{F_A} \,\widetilde{=}\, \mathrm{F_B}$, by Remark 3.3. Hence $\mathscr{G}$ is one to one.

Also since the cardinality of both $S(X)$ and $P(X_E)$ is $2^{|X||E|}$ and since $\mathscr{G}$ is one to one we have $\mathscr{G}$ is onto. □

Proposition 3.8. *Let* $\mathscr{G} : S(X) \to P(X_E)$ *be such that*

$$\mathscr{G}(\mathrm{F_A}) = \left\{ \mathrm{x}_\alpha \,|\, \mathrm{x}_\alpha \,\widetilde{\in}\, \mathrm{F_A} \right\} = A_F$$

and let $F_{A_i} \in S(X),\ i \in I \subseteq \mathbb{N}$ *and* $\mathscr{G}(F_{A_i}) = A_{F_i}$ *for all* $i \in I$*. Then*

$$\mathscr{G}\left(\widetilde{\bigcup_{i \in I}} F_{A_i} \right) = \bigcup_{i \in I} A_{F_i}.$$

Proof.

$$\mathscr{G}\left(\widetilde{\bigcup_{i \in I}} F_{A_i} \right) = \left\{ \mathrm{x}_\alpha \,|\, \mathrm{x}_\alpha \,\widetilde{\in} \widetilde{\bigcup_{i \in I}} F_{A_i} \right\} = \bigcup_{i \in I} \left\{ \mathrm{x}_\alpha \,|\, \mathrm{x}_\alpha \,\widetilde{\in} F_{A_i} \right\} = \bigcup_{i \in I} A_{F_i}.$$

□

Proposition 3.9. *Let* $\mathscr{G} : S(X) \to P(X_E)$ *be such that*

$$\mathscr{G}(\mathrm{F_A}) = \left\{ \mathrm{x}_\alpha \,|\, \mathrm{x}_\alpha \,\widetilde{\in}\, \mathrm{F_A} \right\} = A_F$$

and let $F_{A_1}, F_{A_2} \in S(X)$*. Then* $\mathscr{G}(F_{A_1} \,\widetilde{\cap}\, F_{A_2}) = A_{F_1} \cap A_{F_2}$*.*

Proof.

$$\begin{aligned} \mathscr{G}(F_{A_1} \,\widetilde{\cap}\, F_{A_2}) &= \left\{ \mathrm{x}_\alpha \,|\, \mathrm{x}_\alpha \,\widetilde{\in} F_{A_1} \,\widetilde{\cap}\, F_{A_2} \right\} \\ &= \left\{ \mathrm{x}_\alpha \,|\, \mathrm{x}_\alpha \,\widetilde{\in} F_{A_1} \right\} \cap \left\{ \mathrm{x}_\alpha \,|\, \mathrm{x}_\alpha \,\widetilde{\in} F_{A_2} \right\} \\ &= A_{F_1} \cap A_{F_2}. \end{aligned}$$

□

Notation 3.10. Let $\widetilde{\sum}(X)$ denote the set of all soft topologies on X and $\sum(X_E)$ denote the set of all topologies on X_E.

Proposition 3.11. *Corresponding to every soft topology on* X*, there exists a unique topology on* X_E*.*

Proof. Define $\mathscr{G}_{\mathscr{T}} : \widetilde{\sum}(X) \to \sum(X)$ such that

$$\mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}}) = \mathscr{T} = \left\{ A_F \in P(X_E) | A_F = \mathscr{G}(\mathrm{F_A}) \text{ for some } \mathrm{F_A} \in \widetilde{\mathscr{T}} \right\}.$$

Then $\mathscr{T} = \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}})$ is a topology on X_E:

(i) Since $F_\varnothing, F_{\widetilde{E}} \in \widetilde{\mathscr{T}}$, by the definition of $\mathscr{G}$ in Proposition 3.7, $\varnothing, X_E \in \mathscr{T}$.

(ii) Let $A_{F_i} \in \mathscr{T},\ i \in I \subseteq \mathbb{N}$. Then corresponding to every A_{F_i}, by the definition of $\mathscr{G}_{\mathscr{T}}$, there exists an $F_{A_i} \in \widetilde{\mathscr{T}}$, i.e., $F_{A_i} \in \widetilde{\mathscr{T}}$ for all $i \in I$. Thus $\widetilde{\bigcup\limits_{i \in I}} F_{A_i} \in \widetilde{\mathscr{T}}$. So $\widetilde{\bigcup\limits_{i \in I}} F_{A_i} \in \widetilde{\mathscr{T}}$ and $\mathscr{G}\left(\widetilde{\bigcup\limits_{i \in I}} F_{A_i} \right) = \bigcup\limits_{i \in I} A_{F_i}$, by Proposition 3.8. Hence $\bigcup\limits_{i \in I} A_{F_i} \in \mathscr{T}$.

(iii) Let $A_{F_1}, A_{F_2} \in \mathscr{T}$. Then there exists F_{A_1} and F_{A_2} in $\widetilde{\mathscr{T}}$ such that

$$\mathscr{G}(F_{A_1}) = A_{F_1} \text{ and } \mathscr{G}(F_{A_2}) = A_{F_2}.$$

Since $F_{A_1}, F_{A_2} \in \widetilde{\mathscr{T}}$, $F_{A_1} \,\widetilde{\cap}\, F_{A_2} \in \widetilde{\mathscr{T}}$. Thus by Proposition 3.9, $\mathscr{G}(F_{A_1} \,\widetilde{\cap}\, F_{A_2}) = A_{F_1} \cap A_{F_2}$. So $A_{F_1} \cap A_{F_2} \in \mathscr{T}$.

Now we will prove that $\mathscr{G}_{\mathscr{T}}$ is well defined and bijective.

(i) $\mathscr{G}_{\mathscr{T}}$ is well defined:

Let $\widetilde{\mathscr{T}_1} = \widetilde{\mathscr{T}_2}$. Then

$$\mathrm{F_A} \in \widetilde{\mathscr{T}_1} \iff \mathrm{F_A} \in \widetilde{\mathscr{T}_2}, \text{ i.e.,}$$

$$\mathscr{G}(\mathrm{F_A}) \in \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1}) \iff \mathscr{G}(\mathrm{F_A}) \in \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_2}).$$

Since $\mathrm{F_A} \in \widetilde{\mathscr{T}}$, $A_F = \mathscr{G}(\mathrm{F_A}) \in \mathscr{T} = \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}}))$, i.e., $\mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1}) = \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_2})$.

(ii) $\mathscr{G}_{\mathscr{T}}$ is one to one:

Let $\mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1}) = \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_2})$. Then

$$A_F \in \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1}) \iff A_F \in \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_2}), \text{ i.e.,}$$

$$\mathscr{G}(\mathrm{F_A}) \in \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1}) \iff \mathscr{G}(\mathrm{F_A}) \in \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_2}), \text{ i.e.,}$$

$$\mathrm{F_A} \in \widetilde{\mathscr{T}_1} \iff \mathrm{F_A} \in \widetilde{\mathscr{T}_2}, \text{ i.e.,}$$

$$\widetilde{\mathscr{T}_1} = \widetilde{\mathscr{T}_2}.$$

(iii) $\mathscr{G}_{\mathscr{T}}$ is onto:

Let $\mathscr{T} \in \sum(X_E)$ and let $A_F \in \mathscr{T}$. Since $\mathscr{G} : S(X) \to P(X_E)$ is onto and $A_F \in P(X_E)$, there exists an $\mathrm{F_A} \in S(X)$ such that $\mathscr{G}(\mathrm{F_A}) = A_F$. Let $\widetilde{\mathscr{T}} = \{\mathrm{F_A} \,|\mathscr{G}(\mathrm{F_A}) = A_F \in \mathscr{T}\}$. Then $\mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}}) = \mathscr{T}$. Furthermore,

$\widetilde{\mathscr{T}} = \{\mathrm{F_A} \,|\mathscr{G}(\mathrm{F_A}) = A_F \in \mathscr{T}\}$ is a soft topology on X:

(i) Since $\varnothing, X_E \in \mathscr{T}$. we by the definition of $\mathscr{G}$ in Proposition 3.7 $F_\varnothing, F_{\widetilde{E}} \in \widetilde{\mathscr{T}}$.

(ii) Let $F_{A_i} \in \widetilde{\mathscr{T}}$ for $i \in I \subseteq \mathbb{N}$. Then for all $F_{A_i} \in \widetilde{\mathscr{T}}$ there exist an $A_{F_i} \in \mathscr{T}$. Thus by Proposition 3.8, we have $\bigcup_{i\in I} A_{F_i} \in \mathscr{T}$ and $\mathscr{G}\left(\widetilde{\bigcup}_{i\in I} F_{A_i}\right) = \bigcup_{i\in I} A_{F_i}$. So $\widetilde{\bigcup}_{i\in I} F_{A_i} \in \widetilde{\mathscr{T}}$.

(iii) Let $F_{A_1}, F_{A_2} \in \widetilde{\mathscr{T}}$. Then there exists $A_{F_1}, A_{F_2} \in \mathscr{T}$ respectively such that $\mathscr{G}(F_{A_1}) = A_{F_1}$ and $\mathscr{G}(F_{A_2}) = A_{F_2}$. Since $A_{F_1}, A_{F_2} \in \mathscr{T}$, $A_{F_1} \cap A_{F_2} \in \mathscr{T}$. Thus by by Proposition 3.9, $\mathscr{G}(F_{A_1} \widetilde{\cap} F_{A_2}) = A_{F_1} \cap A_{F_2}$. So $F_{A_1} \widetilde{\cap} F_{A_2} \in \widetilde{\mathscr{T}}$. □

4. Lattice of soft topologies

Theorem 4.1 ([15]). *The intersection of two soft topologies on X is again a soft topology on X.*

Remark 4.2. The union of two soft topologies on X may not be a soft topology on X.

Proposition 4.3. *Let $\widetilde{\mathscr{T}_1}$ and $\widetilde{\mathscr{T}_2}$ be two soft topologies on X then the intersection of soft topologies on X containing $\widetilde{\mathscr{T}_1} \cup \widetilde{\mathscr{T}_2}$ is a soft topology on X.*

Proof. Let $T = \left\{\widetilde{\mathscr{T}} \mid \widetilde{\mathscr{T}} \supseteq \widetilde{\mathscr{T}_1} \cup \widetilde{\mathscr{T}_2}\right\}$ and let $\widetilde{\mathscr{T}}^* = \bigcap \left\{\widetilde{\mathscr{T}} \mid \widetilde{\mathscr{T}} \supseteq \widetilde{\mathscr{T}_1} \cup \widetilde{\mathscr{T}_2}\right\}$.

(i) Clearly, we have $F_\varnothing, F_{\widetilde{E}} \in \widetilde{\mathscr{T}}$ for all $\widetilde{\mathscr{T}} \in T$. Then $F_\varnothing$, $F_{\widetilde{E}} \in \widetilde{\mathscr{T}}^*$.

(ii) Let $F_{A_i} \in \widetilde{\mathscr{T}}^*, i \in I \subseteq \mathbb{N}$. Then $F_{A_i} \in \widetilde{\mathscr{T}}$, for all $\widetilde{\mathscr{T}} \in T, i \in I$. Thus $\widetilde{\cup} F_{A_i} \in \widetilde{\mathscr{T}}$, for all $\widetilde{\mathscr{T}} \in T$. So $\widetilde{\cup} F_{A_i} \in \widetilde{\mathscr{T}}^*$.

(iii) Let $\mathrm{F_A}, \mathrm{F_B} \in \widetilde{\mathscr{T}}^*$. Then $\mathrm{F_A}$, $\mathrm{F_B} \in \widetilde{\mathscr{T}}$, for all $\widetilde{\mathscr{T}} \in T$. Thus $\mathrm{F_A} \widetilde{\cap} \mathrm{F_B} \in \widetilde{\mathscr{T}}$, for all $\widetilde{\mathscr{T}} \in T$. So $\mathrm{F_A} \widetilde{\cap} \mathrm{F_B} \in \widetilde{\mathscr{T}}^*$.

Hence $\widetilde{\mathscr{T}}^*$ is a soft topology on X. □

Proposition 4.4. *$\widetilde{\sum}(X)$ is a partially ordered set with inclusion as ordering.*

Proof. From Definition 2.14, there exist a relation $\subseteq$ in $\widetilde{\sum}(X)$. It is clear that the relation $\subseteq$ is reflexive, antisymmetric and transitive, which makes $\widetilde{\sum}(X)$ a partially ordered set. □

Proposition 4.5. *$\widetilde{\sum}(X)$ is a complete lattice with meet and join defined as:*

$$\widetilde{\mathscr{T}_1}\tilde{\wedge}\widetilde{\mathscr{T}_2} = \widetilde{\mathscr{T}_1} \cap \widetilde{\mathscr{T}_2} \quad and \quad \widetilde{\mathscr{T}_1}\tilde{\vee}\widetilde{\mathscr{T}_2} = \widetilde{\mathscr{T}}^*,$$

where $\widetilde{\mathscr{T}}^* = \bigcap\left\{\widetilde{\mathscr{T}} \mid \widetilde{\mathscr{T}} \supseteq \widetilde{\mathscr{T}_1} \cup \widetilde{\mathscr{T}_2}\right\}$.

Proof. From Propositions 4.3, 4.4 and Theorem 4.1, we can easily see that $\widetilde{\sum}(X)$ is a lattice.

Now let $\widetilde{H}(X) \subseteq \widetilde{\sum}(X)$. Then

$$\begin{aligned}\widetilde{\bigvee}\widetilde{H}(X) &= \widetilde{\bigvee}\left\{\widetilde{\mathscr{T}} \mid \widetilde{\mathscr{T}} \in \widetilde{H}(X)\right\}\\ &= \bigcap\left\{\widetilde{\mathscr{T}}' \mid \widetilde{\mathscr{T}}' \supseteq \bigcup_{\widetilde{\mathscr{T}} \in \widetilde{H}(X)} \widetilde{\mathscr{T}}\right\} \text{ is a soft topology on } X.\end{aligned}$$

$$\begin{aligned}\widetilde{\bigwedge}\widetilde{H}(X) &= \widetilde{\bigwedge}\left\{\widetilde{\mathscr{T}} \mid \widetilde{\mathscr{T}} \in \widetilde{H}(X)\right\}\\ &= \bigcap\left\{\widetilde{\mathscr{T}} \mid \widetilde{\mathscr{T}} \in \widetilde{H}(X)\right\} \text{ is also a soft topology on } X.\end{aligned}$$

Thus $\widetilde{\sum}(X)$ is a complete lattice. □

Proposition 4.6. *The mapping $\mathscr{G}_{\mathscr{T}} : \widetilde{\sum}(X) \to \sum(X_E)$ defined in Proposition 3.11 is an order preserving map.*

Proof. Let $\widetilde{\mathscr{T}_1} \subseteq \widetilde{\mathscr{T}_2}$. Then

$$\begin{aligned}\mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1}) &= \left\{A_F \in P(X_E) | A_F = \mathscr{G}(\mathrm{F_A}) \text{ for some } \mathrm{F_A} \in \widetilde{\mathscr{T}_1}\right\}\\ &\subseteq \left\{A_F \in P(X_E) | A_F = \mathscr{G}(\mathrm{F_A}) \text{ for some } \mathrm{F_A} \in \widetilde{\mathscr{T}_2}\right\}, \text{ since } \widetilde{\mathscr{T}_1} \subseteq \widetilde{\mathscr{T}_2}\\ &= \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_2}).\end{aligned}$$

□

Proposition 4.7. *Let $\mathscr{G}_{\mathscr{T}}$ be defined as in Proposition 3.11. Then*

$$\mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1} \cup \widetilde{\mathscr{T}_2}) = \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1}) \cup \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_2}) \text{ and } \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1} \cap \widetilde{\mathscr{T}_2}) = \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1}) \cap \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_2}).$$

Proof.

$$\begin{aligned}\mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1} \cup \widetilde{\mathscr{T}_2}) &= \Big\{A_F \in P(X_E) \ \mid \ A_F = \mathscr{G}(\mathrm{F_A}) \text{ for some } \mathrm{F_A} \in \widetilde{\mathscr{T}_1} \cup \widetilde{\mathscr{T}_2}\Big\} \\ &= \Big\{A_F \in P(X_E) \ \mid \ A_F = \mathscr{G}(\mathrm{F_A}) \text{ for some } \mathrm{F_A} \in \widetilde{\mathscr{T}_1}\Big\} \\ &\quad \cup \Big\{A_F \in P(X_E) \ \mid \ A_F = \mathscr{G}(\mathrm{F_A}) \text{ for some } \mathrm{F_A} \in \widetilde{\mathscr{T}_2}\Big\} \\ &= \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1}) \cup \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_2}).\end{aligned}$$

$$\begin{aligned}\mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1} \cap \widetilde{\mathscr{T}_2}) &= \Big\{A_F \in P(X_E) \ \mid \ A_F = \mathscr{G}(\mathrm{F_A}) \text{ for some } \mathrm{F_A} \in \widetilde{\mathscr{T}_1} \cap \widetilde{\mathscr{T}_2}\Big\} \\ &= \Big\{A_F \in P(X_E) \ \mid \ A_F = \mathscr{G}(\mathrm{F_A}) \text{ for some } \mathrm{F_A} \in \widetilde{\mathscr{T}_1}\Big\} \\ &\quad \cap \Big\{A_F \in P(X_E) \ \mid \ A_F = \mathscr{G}(\mathrm{F_A}) \text{ for some } \mathrm{F_A} \in \widetilde{\mathscr{T}_2}\Big\} \\ &= \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1}) \cap \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_2}).\end{aligned}$$

□

Proposition 4.8. *The mapping $\mathscr{G}_{\mathscr{T}} : \widetilde{\sum}(X) \to \sum(X_E)$ defined in Proposition 3.11 is a lattice isomorphism from $\widetilde{\sum}(X)$ to $\sum(X_E)$.*

Proof. The mapping $\mathscr{G}_{\mathscr{T}}$ is bijective and order preserving by Propositions 3.11 and 4.6, respectively. Now it is left to prove that $\mathscr{G}_{\mathscr{T}}$ is a lattice homomorphism. For

$$\begin{aligned}\mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1} \widetilde{\wedge} \widetilde{\mathscr{T}_2}) &= \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1} \cap \widetilde{\mathscr{T}_2}) \\ &= \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1}) \cap \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_2}), \quad \textit{by Proposition } 4.7.\end{aligned}$$

$$\begin{aligned}\mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1} \widetilde{\vee} \widetilde{\mathscr{T}_2}) &= \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}}^*) \\ &= \mathscr{G}_{\mathscr{T}}(\bigcup \Big\{\widetilde{\mathscr{T}} \ \mid \ \widetilde{\mathscr{T}} \supseteq \widetilde{\mathscr{T}_1} \cup \widetilde{\mathscr{T}_2}\Big\}) \\ &= \Big\{A_F \ \mid \ A_F = \mathscr{G}(\mathrm{F_A}) \text{ for some } \mathrm{F_A} \in \bigcap \Big\{\widetilde{\mathscr{T}} \ \mid \ \widetilde{\mathscr{T}} \supseteq \widetilde{\mathscr{T}_1} \cup \widetilde{\mathscr{T}_2}\Big\}\Big\} \\ &= \bigcap_{\widetilde{\mathscr{T}} \supseteq \widetilde{\mathscr{T}_1} \cup \widetilde{\mathscr{T}_2}} \Big\{A_F \ \mid \ A_F \in \mathscr{G}(\mathrm{F_A}) \text{ for some } \mathrm{F_A} \in \widetilde{\mathscr{T}} \supseteq \widetilde{\mathscr{T}_1} \cup \widetilde{\mathscr{T}_2}\Big\} \\ &= \bigcap_{\widetilde{\mathscr{T}} \supseteq \widetilde{\mathscr{T}_1} \cup \widetilde{\mathscr{T}_2}} \Big\{\mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}}) \ \mid \ \widetilde{\mathscr{T}} \supseteq \widetilde{\mathscr{T}_1} \cup \widetilde{\mathscr{T}_2}\Big\} \\ &= \bigcap \{\mathscr{T} \ \mid \ \mathscr{T} \supseteq \mathscr{T}_1 \cup \mathscr{T}_2\}, \textit{since } \mathscr{G}_{\mathscr{T}} \textit{ is an order preserving map} \\ &\qquad \textit{and } \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1} \cup \widetilde{\mathscr{T}_2}) = \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1}) \cup \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_2}) \textit{ by Proposition } 4.7 \\ &= \mathscr{T}_1 \vee \mathscr{T}_2 \\ &= \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_1}) \vee \mathscr{G}_{\mathscr{T}}(\widetilde{\mathscr{T}_2}).\end{aligned}$$

□

5. Conclusion

In this paper we have proved that the lattice of soft topologies on a set is isomorphic to the lattice of topologies on the point set consisting of all soft points of a given set. This leads us to the conclusion that this lattice possesses all the properties of a lattice of topologies like atomic, anti- atomic, complemented etc. This work provide an insight into the connection between the soft topologies and the topologies on a given set. It also opens a way to study the structures in soft topological spaces that corresponds to structures in a point set topology.

6. Acknowledgement

The corresponding author wishes to thank University Grants Commission, India for the award of teacher fellowship under Faculty Development Programme XII plan period.

References

[1] M. Akdag and A. Ozkan, Soft b-open sets and soft b-continuous functions, Math. Sci.(Springer) (2014) 8:124.
[2] A. Aygünoglu and H. Aygün, Some notes on soft topological spaces, Neural comput & Applic 21 (suppl 1) (2012) S113–S119.
[3] K. V. Babitha and J. J. Sunil, Soft set relations and functions,comput. Math. Appl. 60 (2010) 1840–1849.
[4] G. Birkhoff, On the combination of topologies, Fund. Math. 26 (1936) 156–166.
[5] N. Çağman, S. Karataş and S. Enginoglu, Soft topology, comput. Math. Appl. 62 (2011) 351–58.
[6] Otto Fröhlich, Das Halbordnungssystem der topologischen Räume auf einer Menge, Math. Ann. 156 (1964) 79–95 (German)
[7] S. Hussain and B. Ahmad, Some properties of soft topological spaces, comput. Math. Appl. 62 (2011) 4058–4067.
[8] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[9] Won Keun Min, A note on soft topological spaces, comput. Math. Appl. 62 (2011) 3524–3528.
[10] D. Molodtsov, Soft set theory first results, Comput. Math. Appl. 37 (1999) 19–31.
[11] N. Y. Ozgur and N. Tas, A Note on Application of Fuzzy Soft Sets to Investment Decision Making Problem, Journal of New Theory 7 (2015) 1–10.
[12] A.C.M. van Rooij, The lattice of all topologies is complemented, Canad. J. Math. 20 (1968) 805–807.
[13] G. Senel, A new approach to Hausdorff space theory via soft sets, Mathematical Problems in Engineering 2016 (9) 1–6.
[14] G. Senel, A comparative research on the definition of soft point, International Journal of Computer Applications 163 (2) (2017) 1–4.
[15] M. Shabir and M. Naz, On soft topological spaces, comput. Math. Appl. 61 (2011) 1786–1799.
[16] A. K. Steiner, The lattice of topologies: structure and complementation, Trans. Amer. Math. Soc. 122 (1966) 379–397.
[17] R. Vaidyanathaswamy, Treatise on set topology, Indian Math. Soc. 1947.
[18] N. Tas, N. Y. Ozgur and P. Demir, An Application of Soft Set and Fuzzy Soft Set Theories to Stock Management, Suleyman Demirel University Journal of Natural and Applied Sciences 21 (2) (2017) 791–196.
[19] N. Xie, Soft points and the structure of soft topological spaces, Ann. Fuzzy Math. Inform. 10 (2) (2015) 309–322.

Ajan K A (ajananathan79@gmail.com)
Department of Mathematics, Cochin University of Science and Technology,

Kochi-22, India

T P Johnson (tpjohnson@cusat.ac.in)
Division of Applied Sciences and Humanities, School of Engineering,
Cochin University of Science and Technology, Kochi-22, India