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Abstract. Soft points are first introduced in 2012, and in the same year the soft mapping $f_{pu} : SS(V)_A \to SS(V)_B$ and $pu - continuity$ are introduced. In this paper we re-define a soft set (F, A) by its set of soft points $(\ddot{F}, \ddot{A})$ (will be called *sp*-set); and study *sp*-sets properties. Then we define $sp - function$ and $sp - continuity$ and study their properties. The main result is that any soft mapping f_{pu} is $pu - continuous$ if and only if its corresponding $sp - function$ $\ddot{f_{pu}}$ is $sp - continuous$.

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1. Introduction and Preliminaries

In 1999, Molodtsov [3] introduced the soft set theory with various applications of the new theory. In 2003, Maji et al. [4] defined and studied the notation of soft subset, complement, union and intersections. In 2011, the concept of soft topology was introduced by [6] and [7]. In 2012, Zorlutuna et al. [9] defined soft points and soft continuous functions.

Definition 1.1 ([5])**.** Let U be the initial universe, E be the set of parameters, $P(U)$ be the power set of U, and $A \subset E$.

(i) A soft set (F, A) on the universe U is defined by the set of ordered pairs

$$(F, A) = \{(x, F(x)); x \in E\},$$

where $F : E \to P(U)$ with $F(x) = \varnothing$, if $x \notin A$.
We let $SS(U)_E$ stands for the set of all soft sets on the universe U where E is a fixed set of parameters.

(ii) Let (F, A), (G, B) belong to $SS(U)_E$. Then (F, A) is called a soft subset of (G, B), denoted by $(F, A)\widetilde{\subset}(G, B))$, if $F(x) \subset G(x)$, for every $x \in E$.

(iii) Let $(F,A) \in SS(U)_E$. The soft complement of (F,A), denoted by $(F,A)^{\tilde{c}})$, is the soft set$(F^{\tilde{c}},E)$ such that $F^{\tilde{c}}(x) = U \setminus F(x)$.

(iv) Let $(F,A),\ (G,B) \in SS(U)_E$. The union of (F,A) and (G,B), denoted by $(F,A)\tilde{\cup}(G,B)$, is the soft set $(F\tilde{\cup}G,E)$ such that $(F\tilde{\cup}G)(x) = F(x) \cup G(x)$.

(v) Let $(F,A),\ (G,B) \in SS(U)_E$. The intersection of (F,A) and (G,B), denoted by $(F,A)\tilde{\cap}(G,B)$, is the soft set $(F\tilde{\cap}G,E)$ such that $(F\tilde{\cap}G)(x) = F(x) \cap G(x)$.

(vi) Let $(F,A),\ (G,B) \in SS(U)_E$. The difference of (F,A) and (G,B), denoted by $F\tilde{\setminus}G$, is the soft set $(F\tilde{\setminus}G,E)$ such that $(F\tilde{\setminus}G)(x) = F(x) \setminus G(x)$.

(vii) The empty soft set is the only $(F,A) \in SS(U)_E$ with $F(e) = \varnothing$, for every $e \in E$ and will be denoted by $\varnothing_E$. The universal soft set is the only soft set $(F,A) \in SS(U)_E$ with $F(e) = U$, for every $e \in E$ and will be denoted by U_E.

Definition 1.2 ([9]). The soft set $(F,A) \in SS(U)_E$ is called a soft point in U_E, denoted by e_F, where $e \in A$, if $F(e) \neq \varnothing$ and $F(x) = \varnothing$, for every $x \neq e$.

Definition 1.3 ([9]). The soft point e_F is said to be in the soft set (G,B), denoted by $e_F\tilde{\in}(G,B)$, if $F(e) \subset G(e)$.

Proposition 1.4 ([9]). *If $e_F\tilde{\in}(G,B)$, then $e_F\tilde{\notin}(G,B)^{\tilde{c}}$*

The converse of the above proposition is not true, i.e. there are a soft point e_F and a soft set (G,A) such that $e_F\tilde{\notin}(G,A)$ and $e_F\tilde{\notin}(G,A)^{\tilde{c}}$ (See Example 3.11 in [9]).

Definition 1.5. For any soft set $(F,A) \in SS(U)_A$, the soft power set of (F,A), denoted by $2^{(F,A)}$, is defined as follows: $(F_1,A) \in 2^{(F,A)}$ if and only if $F_1(a) \subset F(a)$, for every $a \in A$.

Definition 1.6 ([2]). Let $SS(U)_A$ and $SS(V)_B$ be two families of soft sets. Let $u : U \to V$ and $p : A \to B$ be mappings. Then the mapping $f_{pu} : SS(U)_A \to SS(V)_B$ is called a soft mapping, if it is defined as follows:

(i) for any soft set $(F,A) \in SS(U)_A$, the image of (F,A) under f_{pu}, denoted by $f_{pu}(F,A)$, is the soft set $(f_{pu}(F),B) \in SS(V)_B$ such that for every $b \in B$,

$$f_{pu}(F)(b) = \bigcup_{x\in p^{-1}(b)} u(F(x)),\ \text{if } b \in p(A) \text{ and } f_{pu}(F)(b) = \varnothing \text{ if } b \in B - p(A),$$

(ii) for any soft set $(G,B) \in SS(V)_B$, the inverse image of (G,B) under f_{pu}, denoted by $f_{pu}^{-1}(G,B)$, is the soft set $(f_{pu}^{-1}(G),A) \in SS(U)_A$ such that for every $a \in A$, $f_{pu}^{-1}(G)(b) = u^{-1}(G(p(b)))$.

The properties of soft functions can be found in [9].

Since for any soft sets (F,A) and (G,B) in $SS(U)_A$ and $SS(V)_B$ respectively, we have $2^{(F,A)} \subset SS(U)_A$ and $2^{(G,B)} \subset SS(V)_B$, we can define the soft mapping $f_{pu} : 2^{(F,A)} \to 2^{(G,B)}$ as in the above definition subject to the following conditions:

(i) for any soft set $(F_1,A) \in 2^{(F,A)}$, we have $f_{pu}(F_1,A) \in 2^{(G,B)}$,

(ii) for any soft set $(G_1,B) \in 2^{(G,B)}$, we have $f_{pu}^{-1}(G_1,B) \in 2^{(F,A)}$.

The above two conditions makes $f_{pu} : 2^{(F,A)} \to 2^{(G,B)}$ well defined.

Restricting soft mapping to mappings between soft power sets of given soft sets is very important for soft topology and soft continuity, since topology, basically, is a sub-collection of the power set of a given set.

Definition 1.7 ([7])**.** Let $(F, A) \in SS(U)_A$. A soft topology on (F, A), denoted by $\widetilde{\tau}$, is a collection of soft subsets of (F, A) having the following three properties:
(i) $\varnothing_A, (F, A) \in \widetilde{\tau}$,
(ii) if $(F_1, A), (F_2, A) \in \widetilde{\tau}$, then $(F_1, A)\widetilde{\cap}(F_2, A) \in \widetilde{\tau}$,
(iii) if $(F_\alpha, A) \in \widetilde{\tau}$ for every $\alpha \in \Delta$, then $\underset{\alpha\in\Delta}{\widetilde{\cup}}(F_\alpha, A) \in \widetilde{\tau}$.

A soft subset (F_1, A_1) of (F, A) is called soft open, if $(F_1, A_1) \in \widetilde{\tau}$, and it is called soft closed, if $(F_1, A_1)^{\widetilde{c}} \in \widetilde{\tau}$.

Definition 1.8 ([9])**.** Let $(F, A, \widetilde{\tau}_1)$ and $(G, B, \widetilde{\tau}_2)$ be two soft topological spaces and let $u : U \to V$ and $p : A \to B$ be mappings. Then the soft mapping f_{pu} : $SS(U)_A \to SS(V)_B$ is said to be:
(i) pu-continuous at $e_F\widetilde{\in}(F, A)$, if for each soft open set $(G_1, B)\widetilde{\in}\widetilde{\tau}_2$ soft containing $f_{pu}(e_F)$ there exists a soft open set $(F_1, A)\widetilde{\in}\widetilde{\tau}_1$ soft containing e_F such that $f_{pu}(F_1, A)\widetilde{\subset}(G_1, B)$,
(ii) pu-continuous on (F, A), if it is pu-continuous at every $e_F\widetilde{\in}(F, A)$.

Theorem 1.9 ([9])**.** *Let $(F, A, \widetilde{\tau}_1)$ and $(G, B, \widetilde{\tau}_2)$ be two soft topological spaces. and $u : U \to V$ and $p : A \to B$ be mappings. Then the following two statements are equivalent:*
(1) *the mapping $f_{pu} : SS(U)_A \to SS(V)_B$ is $pu-continuous$ on U_A,*
(2) *for every $(G_1, B) \in \widetilde{\tau}_2$, we have $f_{pu}^{-1}(G_1, B) \in \widetilde{\tau}_1$,*
(3) *for every soft closed set (G_1, B) over V, we have $f_{pu}^{-1}(G_1, B)$ is soft closed over U.*

2. Soft Points and sp-continuity

Let U be the initial universe, A be a fixed set of parameters. Since any soft point e_F in U_A is a soft set (F, A) where $F(a) = \varnothing$ if $a \neq e$ and $F(a) \neq \varnothing$ only for $a = e$, we can rewrite the soft point e_F as an ordered pair (e, D) where $F(e) = D$. To be more precise we begin by the following definition.

Definition 2.1. Let U be the initial universe, let E be a fixed set of parameters and let $e \in E$ and $\varnothing \neq D \subset U$. Then the ordered pair (e, D) is called a soft point in $SS(U)_E$. The set of all soft point in $SS(U)_E$ will be denoted by $SS(\ddot{U})_E$.

For a soft set (F, A) in $SS(U)_E$, we say that the soft point (e, D) in (F, A), denoted by $(e, D)\widetilde{\in}(F, A)$, if $e \in A$ and $D \subset F(e)$. The set of all soft points of (F, A) will be denoted by $(\ddot{F}, \ddot{A})$ and will be called the $sp-set$ of (F, A).

Example 2.2. Let $A = \{a, b, c\}$ and $U = \{1, 2, 3\}$. Consider the soft set $(F, A) = \{(a, \{1, 3\}), (b, \{1, 2\}), (c, \{2\})\}$. Then the $sp-set$ of (F, A) is

$$(\ddot{F}, \ddot{A}) = \{(a, \{1\}), (a, \{3\}), (a, \{1, 3\}), (b, \{1\}), (b, \{2\}), (b, \{1, 2\}), (c, \{2\})\}.$$

Definition 2.3. For any subset K of $SS(\ddot{U})_E$, we define the soft set $(F, A)_K$ in $SS(U)_E$ as follows:

(i) $A = \{a \in E : (a, D) \in K$ for some $D \subset U\}$,
(ii) $F(x) = \bigcup_{(x,D)\in K} D$, if $x \in A$ and $F(x) = \varnothing$, if $x \in E \setminus A$.

$(F, A)_K$ will be called the soft set generated by the set of soft points K.

Let (F, A) be a soft set, $(\ddot{F}, \ddot{A})$ be its sp-set and $(F, A)_{(\ddot{F},\ddot{A})}$ be the soft set generated by the set of soft points $(\ddot{F}, \ddot{A})$. Then one can easily show that $(F, A) = (F, A)_{(\ddot{F},\ddot{A})}$. But the converse is not true, i.e. if we start with a set of soft point K, then $(\ddot{F}, \ddot{A})_K \neq K$ where $(\ddot{F}, \ddot{A})_K$ is the $sp-set$ of the soft set $(F, A)_K$(See the above definition). It is easy to show that $(\ddot{F}, \ddot{A})_K \supset K$.

sp-sets behave similar to power sets and the following properties of power sets are well-known:
(1) $A = B$ if and only if $\mathcal{P}(A) = \mathcal{P}(B)$,
(2) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$,
(3) $\mathcal{P}(A \cup B) \supset \mathcal{P}(A) \cup \mathcal{P}(B)$.

The following proposition shows that the above properties of power sets have a corresponding versions for $sp-sets$, and its proof analogues to its analogue in set theory.

Proposition 2.4. *Let (F, A) and (G, B) be two soft sets in U_X. Then we have:*
(1) *$(F, A) = (G, B)$ if and only if $(\ddot{F}, \ddot{A}) = (\ddot{G}, \ddot{B})$,*
(2) *if $(F, A)\widetilde{\cap}(G, B) = (H, C)$, then $(\ddot{H}, \ddot{C}) = (\ddot{F}, \ddot{A}) \cap (\ddot{G}, \ddot{B})$,*
(3) *if $(F, A)\widetilde{\cup}(G, B) = (H, C)$, then $(\ddot{H}, \ddot{C}) \supset (\ddot{F}, \ddot{A}) \cup (\ddot{G}, \ddot{B})$.*

Theorem 2.5. *Let (F, A) and (G, B) be two soft set with $(F, A) \nsubseteq (G, B)$, then there exists a soft point $(x, D) \in \ddot{U}_X$ such that $(x, D) \in (\ddot{F}, \ddot{A})$ and $(x, D) \in (\ddot{G}, \ddot{B})^{\widetilde{c}}$ where $(\ddot{G}, \ddot{B})^{\widetilde{c}}$ refers to the sp-set of $(G, B)^{\widetilde{c}}$.*

Proof. Since $(F, A) \nsubseteq (G, B)$, there exists $x \in A$ such that $f(x) \nsubseteq g(x)$, which implies $f(x) \setminus g(x)$ is nonempty. Set $(x, D) = (x, f(x) \setminus g(x))$. It is clear that $(x, D) \in (\ddot{F}, \ddot{A})$ and $(x, D) \in (\ddot{G}, \ddot{B})^{\widetilde{c}}$. □

Definition 2.6. Let (F, A) and (G, B) be two soft sets. A mapping $f : (\ddot{F}, \ddot{A}) \to (\ddot{G}, \ddot{B})$ (sends a soft point $(x, D) \in (\ddot{F}, \ddot{A})$ to a soft point $f(x, D) = (y, C)$ in $(\ddot{G}, \ddot{B})$) is called an sp-mapping, if its image and pre-image for soft subsets of (F, A) and (G, B), respectively, are defined as follows:

(i) for every $(F_1, A)\widetilde{\subset}(F, A)$, we have $f(F_1, A) = (f(F_1), B)\widetilde{\subset}(G, B)$ such that for every $y \in B$, $f(F_1)(y) = \bigcup\{C : (y, C) = f(x, D)$ for some $(x, D) \in (F_1, A)\}$,

(ii) for every $(G_1, B)\widetilde{\subset}(G, B)$, we have $f^{-1}(G_1, B) = (f^{-1}(G_1), A)\widetilde{\subset}(F, A)$ such that $f^{-1}(G_1)(x) = \bigcup\{D : (x, D)\widetilde{\in}(F, A)$ and $f(x, D)\widetilde{\in}(G_1, B)\}$, for every $x \in A$.

That is sp-mapping is a point-set function who sends a soft point to a soft point and its image and pre-images of soft subset are defined as mentioned above.

Definition 2.7. Let $(F, A, \widetilde{\tau})$ and $(G, B, \widetilde{\mu})$ be two soft topological spaces and $f : (\ddot{F}, \ddot{A}) \to (\ddot{G}, \ddot{B})$ be an sp-mapping . Then f is said to be sp-continuous at the soft point $(x, D) \in (\ddot{F}, \ddot{A})$, if for any soft open set $(G_1, B) \in \widetilde{\mu}$ with $f(x, D) \in (\ddot{G}_1, \ddot{B})$, there exists $(F_1, A) \in \widetilde{\tau}$ such that $(x, D) \in (\ddot{F}_1, \ddot{A})$ and $f(F_1, A)\widetilde{\subset}(G_1, B)$. For the sake of simplicity we write $f : (\ddot{F}, \ddot{A}, \widetilde{\tau}) \to (\ddot{G}, \ddot{B}, \widetilde{\mu})$ is an sp-continuous mapping.

Theorem 2.8. *An sp − mapping* $f : (\ddot{F}, \ddot{A}, \widetilde{\tau}) \rightarrow (\ddot{G}, \ddot{B}, \widetilde{\mu})$ *is sp − continuous if and only if for every soft open set* (H, B) *in* $(G, B, \widetilde{\mu})$*,* $f^{-1}(H, B)$ *is a soft open set in* $(G, B, \widetilde{\mu})$*.*

Proof. Suppose that f is sp-continuous and (H, B) be a soft open set in $\widetilde{\mu}$. Let $(x, D)\widetilde{\in} f^{-1}(H, B)$. Then $f(x, D) = (y, M)\widetilde{\in}(H, B)$. Since f is sp-continuous, there exists a soft open set (K, A) in $\widetilde{\tau}$ soft containing (x, D) such that $f(K, A)\widetilde{\subset}(H, B)$. It is clear that $(x, D)\widetilde{\in}(K, A)\widetilde{\subset} f^{-1}(H, B)$, which implies $f^{-1}(H, B)$ is soft open.

Conversely, suppose that for every soft open set (H, B) in $(G, B, \widetilde{\mu})$, $f^{-1}(H, B)$ is a soft open set in $(G, B, \widetilde{\mu})$. Let $(x, D) \in (\ddot{F}, \ddot{A})$ and let (H, B) be a soft open set in $\widetilde{\mu}$ with $f(x, D) \in (H, B)$. Since (H, B) is soft open in $\widetilde{\mu}$, $f^{-1}(H, B)$ is soft open in $\widetilde{\tau}$, and since $f(x, D) \in (H, B)$, $(x, D) \in f^{-1}(H, B)$. But$f(f^{-1}(H, B))\widetilde{\subset}(H, B)$; and this shows f is sp-coninuous. □

For notational purpose, one must note that the following statements are equivalent:

(1) $(x, D) \in (\ddot{F}, \ddot{A})$,
(2) $(x, D)\widetilde{\in}(F, A)$,
(3) (x, D) is soft contained in (F, A).

Example 2.9. Let $A = \{1, 2, 3\}$ and $U = \{a, b, c\}$. Consider the two soft sets (F, A) and (G, A) in $SS(U)_A$ given by:

$$(F, A) = \{(1, \{a\}), (2, \{a, b\}), (3, \{b, c\})\},\ (G, A) = \{(1, \{a, b\}), (2, \{c\}), (3, \{b, c\})\}.$$

Let $\tau_1 = \{\varnothing_A, (F, A), (F_1, A), (F_2, A), (F_3, A)\}$, where

$$(F_1, A) = \{(1, \varnothing), (2, \{a, b\}), (3, \{b, c\})\},$$
$$(F_2, A) = \{(1, \{a\}), (2, \{a\}), (3, \varnothing)\},$$
$$(F_3, A) = \{(1, \varnothing), (2, \{a\}), (3, \varnothing)\}.$$

Then we can easily show that τ_1 is a soft topology on (F, A).

Again, let $\tau_2 = \{\varnothing_A, (G, A), (G_1, A), (G_2, A), (G_3, A)\}$, where

$$(G_1, A) = \{(1, \{a, b\}), (2, \varnothing), (3, \{b, c\})\},$$
$$(G_2, A) = \{(1, \{a\}), (2, \{c\}), (3, \varnothing)\},$$
$$(G_3, A) = \{(1, \{a\}), (2, \varnothing), (3, \varnothing)\}.$$

Then we can easily show that τ_2 is also a soft topology on (G, A).

It is clear that

$$(\ddot{F}, \ddot{A}) = \{(1, \{a\}), (2, \{a\}), (2, \{b\}), (2, \{a, b\}), (3, \{b\}), (3, \{c\}), (3, \{b, c\})\},$$
$$(\ddot{G}, \ddot{A}) = \{(1, \{a\}), (1, \{b\}), (1, \{a, b\}), (2, \{c\}), (3, \{b\}), (3, \{c\}), (3, \{b, c\})\}.$$

Define the sp-function $f : (\ddot{F}, \ddot{A}, \widetilde{\tau_1}) \rightarrow (\ddot{G}, \ddot{A}, \widetilde{\tau_2})$ as follows:

$$f(1, \{a\}) = (2, \{c\}),$$
$$f(2, \{a\}) = (2, \{c\}),\ f(2, \{b\}) = (3, \{b\}),\ f(2, \{a, b\}) = (1, \{a, b\}),$$
$$f(3, \{b\}) = (3, \{c\}),\ f(3, \{c\}) = (3, \{c\}),\ f(3, \{b, c\}) = (1, \{a, b\}).$$

We will show that f is sp-continuous, it suffices to show that for every $(K, A) \in \tau_2$, we have $f^{-1}(K, A) \in \tau_1$, this can be established step by step as follows: we first find the sp-set $(\ddot{K}, \ddot{A})$, then we find its inverse under f; which is the sp-set $f^{-1}(\ddot{K}, \ddot{A})$, after this we find $f^{-1}(K, A)$ which is the soft set generated by the sp-set $f^{-1}(\ddot{K}, \ddot{A})$ and then we show it is in τ_2. For example, for the soft open set $(G_1, A) \in \tau_2$, we have

$$(\ddot{G_1}, \ddot{A}) = \{(1, \{a\}), (1, \{b\}), (1, \{a, b\}), (3, \{b\}), (3, \{c\}), (3, \{b, c\})\},$$

so that $f^{-1}(\ddot{G_1}, \ddot{A}) = \{(2, \{a, b\}), (3, \{b, c\}), (2, \{b\}), (3, \{b\}), (3, \{c\})\}$, which implies that $f^{-1}(G_1, A) = \{(1, \varnothing), (2, \{a, b\}), (3, \{b, c\})\} = (F_1, A) \in \tau_1$. Similarly, we show that $f^{-1}(\varnothing_A) = \varnothing_A \in \tau_1, f^{-1}(G, A) = (F, A) \in \tau_1$, $f^{-1}(G_2, A) = (F_2, A) \in \tau_1$ and $f^{-1}(G_3, A) = \varnothing_A \in \tau_1$. So that f is sp-continuous.

Definition 2.10. Let $(F, A, \widetilde{\tau})$ and $(G, B, \widetilde{\mu})$ be two soft topological spaces and $f : (\ddot{F}, \ddot{A}) \to (\ddot{G}, \ddot{B})$ be a sp-mapping. Then f is said to be sp-open (sp-closed), if for every soft open (soft closed) set (F_1, A) in (F, A), we have $f(F_1, A)$ is soft open (soft closed) in (G, B). For the sake of simplicity we write $f : (\ddot{F}, \ddot{A}, \widetilde{\tau}) \to (\ddot{G}, \ddot{B}, \widetilde{\mu})$ is a sp-open (sp-closed) mapping.

Definition 2.11. For any soft set $(F, A) \in SS(U)_A$ and $(G, B) \in SS(V)_B$, let $f_{pu} : 2^{(F,A)} \to 2^{(G,B)}$ be a soft mapping such that $u : U \to V$ and $p : A \to B$. The sp-mapping corresponding to f_{pu} is $\ddot{f_{pu}} : (\ddot{F}, \ddot{A}) \to (\ddot{G}, \ddot{B})$ such that for every soft point $(a, D) \in (\ddot{F}, \ddot{A})$, we have $\ddot{f_{pu}}(a, D) = (p(a), u(D))$.

Theorem 2.12. *For any soft mapping $f_{pu} : 2^{(F,A)} \to 2^{(G,B)}$ the sp $-$ mapping $\ddot{f_{pu}} : (\ddot{F}, \ddot{A}) \to (\ddot{G}, \ddot{B})$ corresponding to f_{pu} is well-defined.*

Proof. It suffices to show that for any soft point $(a, D) \in (\ddot{F}, \ddot{A})$ we have $\ddot{f_{pu}}(a, D) = (p(a), u(D)) \in (\ddot{G}, \ddot{B})$. Since $f_{pu} : 2^{(F,A)} \to 2^{(G,B)}$ is well defined (sends a soft subset of (F, A) to a soft subset of (G, B)), we have $f_{pu}(F, A) = (f_{pu}(F), B)$ is a soft subset of (G, B), which implies that $f_{pu}(F)(p(a)) = \bigcup_{x \in p^{-1}(p(a))} u(F(x)) \subset G(p(a))$. Since $(a, D) \in (\ddot{F}, \ddot{A})$, we have $D \subset F(a)$, but $a \in p^{-1}(p(a))$ so that $u(D) \subset u(F(a)) \subset f_{pu}(F)(p(a)) \subset G(p(a))$, which means $(p(a), u(D)) \in (\ddot{G}, \ddot{B})$. □

The following two lemma are very important to the theorem next to them.

Lemma 2.13. *Let (F, A) and (G, B) be any two soft sets and $f_{pu} : 2^{(F,A)} \to 2^{(G,B)}$ be a soft mapping. If $\ddot{f_{pu}} : (\ddot{F}, \ddot{A}) \to (\ddot{G}, \ddot{B})$ is the sp $-$ mapping corresponding to f_{pu}, then for every soft subset (F_1, A) of (F, A), we have $f_{pu}(F_1, A) = \ddot{f_{pu}}(F_1, A)$.*

Proof. Before we begin our proof, we must call Definition 1.6 and Definition 2.6. From Definition 1.6, we have $f_{pu}(F_1, A) = (f_{pu}(F_1), B)$ such that for every $b \in B$, we have

$$f_{pu}(F_1)(b) = \bigcup_{x \in p^{-1}(b)} u(F_1(x)), \text{ if } b \in p(A) \text{ and } f_{pu}(F_1)(b) = \varnothing, \text{ if } b \in B - p(A).$$

From Definition 2.6, we have $\ddot{f_{pu}}(F_1, A) = (\ddot{f_{pu}}(F_1), B)$ such that for every $b \in B$, $\ddot{f_{pu}}(F_1)(b) = \bigcup\{C : (b, C) = \ddot{f_{pu}}(a, D) \text{ for some } (a, D) \in (\ddot{F_1}, \ddot{A}) \text{ with } p(a) = b\}$. It is suffices to show that for every $b \in B$, we have $f_{pu}(F_1)(b) = \ddot{f_{pu}}(F_1)(b)$.

Let $y \in f_{pu}(F_1)(b)$. Then there exists $x \in p^{-1}(b)$ such that $y \in u(F_1(x))$. It is clear that $(x, F_1(x)) \in (\ddot{F}_1, \ddot{A})$ so $\ddot{f_{pu}}(x, F_1(x)) = (p(x), uF_1(x)) = (b, u(F_1(x)))$. Thus $y \in uF_1(x) \subset \bigcup\{C : (b, C) = \ddot{f_{pu}}(a, D)$ for some $(a, D) \in (\ddot{F}_1, \ddot{A})\}$.

Conversely, suppose that $y \in \ddot{f_{pu}}(F_1)(b)$. Then there exists $(a, D) \in (\ddot{F}_1, \ddot{A})$ such that $\ddot{f_{pu}}(a, D) = (b, C)$ with $p(a) = b$ and $y \in C$. Since $D \subset F_1(a)$, $u(D) \subset u(F_1(a))$, but $a \in p^{-1}(b)$. Thus we have $y \in u(F_1(a)) \subset f_{pu}(F_1)(b) = \bigcup_{x\in p^{-1}(b} u(F_1(x))$. So $\ddot{f_{pu}}(F_1)(b) \subset f_{pu}(F_1)(b)$. Hence the proof is complete. □

Lemma 2.14. *Let (F, A) and (G, B) be any two soft sets and $f_{pu} : 2^{(F,A)} \to 2^{(G,B)}$ be a soft mapping. If $\ddot{f_{pu}} : (\ddot{F}, \ddot{A}) \to (\ddot{G}, \ddot{B})$ is the sp−mapping corresponding to f_{pu}, then for every soft subset (G_1, B) of (G, B), we have $f_{pu}^{-1}(G_1, B) = \ddot{f_{pu}}^{-1}(G_1, B)$.*

Proof. Before we begin our proof, we must call Definition 1.6 and Definition 2.6. From Definition 1.6, we have $f_{pu}^{-1}(G_1, B) = (f_{pu}^{-1}(G_1), A)$ such that for every $a \in A$, $f_{pu}^{-1}(G_1)(a) = u^{-1}(G_1(p(a))$.

From Definition 2.6, we have $\ddot{f_{pu}}^{-1}(G_1, B) = (\ddot{f_{pu}}^{-1}(G_1), A)$ such that for every $a \in A$, $\ddot{f_{pu}}^{-1}(G_1)(a) = \bigcup\{D : (a, D) \in (\ddot{F}, \ddot{A})$ with $\ddot{f_{pu}}(a, D) \in (\ddot{G}, \ddot{B})\} = \bigcup\{D : (a, D) \in (\ddot{F}, \ddot{A})$ with $(p(a), u(D)) \in (\ddot{G}, \ddot{B})\}$. It is sufficient to show that for every $a \in A$ we have $f_{pu}^{-1}(G_1)(a) = \ddot{f_{pu}}^{-1}(G_1)(a)$. Let $x \in f_{pu}^{-1}(G_1)(a)$. Since $\ddot{f_{pu}}$ is well-defined (See the above theorem), $(a, \{x\}) \in (\ddot{F}, \ddot{A})$. Since $x \in f_{pu}^{-1}(G_1)(a) = u^{-1}(G_1(p(a)))$, we have $u(x) \in G_1(p(a))$. Then $(p(a), u(\{x\})) \in (\ddot{G}_1, \ddot{B})$ and $\{x\} \subset \bigcup\{D : (a, D) \in (\ddot{F}, \ddot{A})$ with $(p(a), u(D)) \in (\ddot{G}_1, \ddot{B})\} = \ddot{f_{pu}}^{-1}(G_1)(a)$. Thus $f_{pu}^{-1}(G_1)(a) \subset \ddot{f_{pu}}^{-1}(G_1)(a)$.

Conversely, let $x \in \ddot{f_{pu}}^{-1}(G_1)(a)$. Then $x \in D$, for some $(a, D) \in (\ddot{F}, \ddot{A})$ with $(p(a), u(D)) \in (\ddot{G}_1, \ddot{B})$. Thus $u(D) \subset G_1(p(a))$. So $x \in D \subset u^{-1}(G_1(p(a))) = f_{pu}^{-1}(G_1)(a)$. Hence $\ddot{f_{pu}}^{-1}(G_1)(a) \subset f_{pu}^{-1}(G_1)(a)$. This completes the proof. □

The following definition is aboutpu-open and pu-closed functions and can be found in [1] and [8].

Definition 2.15. Let $(F, A, \widetilde{\tau}_1)$ and $(G, B, \widetilde{\tau}_2)$ be two soft topological spaces, and let $u : U \to V$ and $p : A \to B$ be mappings. Then the soft mapping $f_{pu} : SS(U)_A \to SS(V)_B$ is said to be:

(i) [1] pu-open, if $f_{pu}(F_1, A)\widetilde{\in}\widetilde{\tau}_2$ for every $(F_1, A)\widetilde{\in}\widetilde{\tau}_1$,

(ii) [8] pu-closed, if $f_{pu}(F_1, A)$ is soft closed in $\widetilde{\tau}_2$ for every (F_1, A) soft closed in $\widetilde{\tau}_1$.

Theorem 2.16. *Let $(F, A, \widetilde{\tau}_1)$ and $(G, B, \widetilde{\tau}_2)$ be two soft topological spaces. Let $f_{pu} : 2^{(F,A)} \to 2^{(G,B)}$ be a soft mapping where $u : U \to V$ and $p : A \to B$, and $\ddot{f_{pu}} : (\ddot{F}, \ddot{A}) \to (\ddot{G}, \ddot{B})$ is the $sp-mapping$ corresponding to f_{pu}. Then*

(1) *f_{pu} is $pu-continuous$ if and only if $\ddot{f_{pu}}$ is $sp-continuous$,*
(2) *f_{pu} is $pu-closed$ if and only if $\ddot{f_{pu}}$ is $sp-closed$,*
(3) *f_{pu} is $pu-open$ if and only if $\ddot{f_{pu}}$ is $sp-open$.*

Proof. The proof is easy! we just apply the above two lemmas. □

we close our paper by an example to clarify the above concept where we introduce a pu-continuous mapping f_{pu}, then we show that the sp-mapping $\ddot{f_{pu}}$ corresponding to f_{pu} is sp-continuous.

Example 2.17. Let $A = \{1, 2, 3\}$ and $U = \{a, b, c\}$. Consider the soft sets (F, A) and (G, A) given by, respectively:

$$(F, A) = \{(1, \{b, c\}), (2, \{b, c\}), (3, \{a\})\},\ (G, A) = \{(1, \{a, b\}), (2, \{c\}), (3, \{b, c\})\}.$$

Let $\tau_1 = \{\varnothing_A, (F, A), (F_1, A), (F_2, A), (F_3, A)\}$ be a family of soft sets given by:

$$(F_1, A) = \{(1, \varnothing), (2, \{b, c\}), (3, \{a\})\},$$
$$(F_2, A) = \{(1, \{b, c\}), (2, \varnothing), (3, \{a\})\},$$
$$(F_3, A) = \{(1, \varnothing), (2, \varnothing), (3, \{a\})\},$$

and let $\tau_2 = \{\varnothing_A, (G, A), (G_1, A), (G_2, A), (G_3, A)\}$ be a family of soft sets given by:

$$(G_1, A) = \{(1, \{a, b\}), (2, \varnothing), (3, \{b, c\})\},$$
$$(G_2, A) = \{(1, \{a\}), (2, \{c\}), (3, \varnothing)\},$$
$$(G_3, A) = \{(1, \{a\}), (2, \varnothing), (3, \varnothing)\}.$$

Then we can easily show that τ_1 is a soft topology on (F, A) and τ_2 is a soft topology on (G, A).

Define $u : U \to U$ by $u = \{(a, a), (b, c), (c, c)\}$, and define $p : A \to A$ by $p = \{(1, 2), (2, 3), (3, 1)\}$. Since u and p are well-defined, f_{pu} is well-defined as in Definition 1.6. f_{pu} is soft continuous mapping, because one can easily show that: $f_{pu}^{-1}(G, A) = (F, A) \in \tau_1$, $f_{pu}^{-1}(G_1, A) = (F_1, A) \in \tau_1$, $f_{pu}^{-1}(G_2, A) = (F_2, A) \in \tau_1$ and $f_{pu}^{-1}(G_3, A) = (F_3, A) \in \tau_1$. Now to construct the sp-mapping $\ddot{f_{pu}} : (\ddot{F}, \ddot{A}) \to (\ddot{G}, \ddot{A})$ corresponding to f_{pu} we call Definition 2.11 to get $\ddot{f_{pu}}(x, D) = (p(x), u(D))$, for every $(x, D) \in (\ddot{F}, \ddot{A})$. Note that

$$(\ddot{F}, \ddot{A}) = \{(1, \{b\}), (1, \{c\}), (1, \{b, c\}), (2, \{b\}), (2, \{c\}), (2, \{b, c\}), (3, \{a\})\}$$

and

$$(\ddot{G}, \ddot{A}) = \{(1, \{a\}), (1, \{b\}), (1, \{a, b\}), (2, \{c\}), (3, \{b\}), (3, \{c\}), (3, \{b, c\})\}.$$

Then we have

$$\ddot{f_{pu}}(1, \{b\}) = (p(1), u(\{b\})) = (2, \{c\}),$$
$$\ddot{f_{pu}}(1, \{c\}) = (p(1), u(\{c\})) = (2, \{c\}),$$
$$\ddot{f_{pu}}(1, \{b, c\}) = (p(1), u(\{b, c\})) = (2, \{c\}),$$
$$\ddot{f_{pu}}(2, \{b\}) = (p(2), u(\{b\})) = (3, \{c\}),$$
$$\ddot{f_{pu}}(2, \{c\}) = (p(2), u(\{c\})) = (3, \{c\}),$$
$$\ddot{f_{pu}}(2, \{b, c\}) = (p(2), u(\{b, c\})) = (3, \{c\}),$$
$$\ddot{f_{pu}}(3, \{a\}) = (p(3), u(\{a\})) = (1, \{a\}).$$

As in Example 2.9, we can show that $\ddot{f_{pu}}$ is sp-continuous; actually, we have:

$\ddot{f_{pu}}^{-1}(G, A) = (F, A) \in \tau_1$, $\ddot{f_{pu}}^{-1}(G_1, A) = (F_1, A) \in \tau_1$,

$\ddot{f_{pu}}^{-1}(G_2, A) = (F_2, A) \in \tau_1,\ \ddot{f_{pu}}^{-1}(G_3, A) = (F_3, A) \in \tau_1.$

It is worth noting that

$\ddot{f_{pu}}^{-1}(G, A) = f_{pu}^{-1}(G, A) = (F, A),\ \ddot{f_{pu}}^{-1}(G_1, A) = f_{pu}^{-1}(G_1, A) = (F_1, A),$

$f_{pu}^{-1}(G_2, A) = \ddot{f_{pu}}^{-1}(G_2, A) = (F_2, A),\ \ddot{f_{pu}}^{-1}(G_3, A) = f_{pu}^{-1}(G_3, A) = (F_3, A)$ which is consistent with Lemma 2.14.

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