Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 19, No. 2, (April 2020) pp. 189–198
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2020.19.2.189

@FMI

On symmetric bi-derivations of BL-algebras

Kyung Ho Kim

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 19, No. 2, April 2020

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 19, No. 2, (April 2020) pp. 189–198
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2020.19.2.189

On symmetric bi-derivations of BL-algebras

Kyung Ho Kim

Received 22 November 2019; Revised 9 December 2019; Accepted 16 December 2019

Abstract. In this paper, we introduce the notion of two types of symmetric bi-derivations in BL-algebra and obtain some results. We study $(\otimes, \vee)$-symmetric bi-derivations on *Gödel* BL-algebras and consider isotone symmetric bi-derivations on BL-algebra A.

2010 AMS Classification: 08A05, 08A30, 20L05, Primary 16Y30

Keywords: BL-algebra, Symmetric mapping, Symmetric bi-derivation, Isotone, $Fix_a(A)$, Boolean center.

1. Introduction

The notion of BL-algebra was introduced by P. *Häjek* [2] in order to provide an algebraic proof of the completeness theorem of Basic Logic. The main example of an BL-algebra is an interval $[0, 1]$ endowed with the structure induced by a continuous t-norm. MV-algebra, *Gödel* algebras, and product algebras are the most known classes of BL-algebras. In this paper, we introduce the notion of two types of symmetric bi-derivations in BL-algebra and obtain some results. We study $(\otimes, \vee)$-symmetric bi-derivations on *Gödel* BL-algebras and consider isotone symmetric bi-derivations on BL-algebra A.

2. Preliminary

An BL-algebra is a structure $(A, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, \otimes, \rightarrow$ and two constants $0, 1$ such that

(BL1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,
(BL2) $(A, \otimes, 1)$ is a commutative monoid,
(BL3) $\otimes$ and $\rightarrow$ form a adjoint pair, i.e., $c \leq a \rightarrow b$ if and only if $a \otimes c \leq b$ for all $a, b, c \in A$,
(BL4) $a \wedge b = a \otimes (a \rightarrow b)$ for any $a, b \in A$,
(BL6) $(a \rightarrow b) \vee (b \rightarrow a) = 1$ for any $a, b \in A$ (see [2]).

For any $a \in A$, we define $a^* = x \to 0$ for any $a \in A$ and denote $(a^*)^* = a^{**}$. Also, we denote the set of natural numbers by ω and define $a^0 = 1$ and $a^n = a^{n-1} \otimes a$ for $a \in \omega \backslash \{0\}$.

We define the binary operations $\oplus$ and $\ominus$ by

$$x \oplus y = (x^* \otimes y^*)^*, \qquad x \ominus y = x \otimes y^*$$

for any $x, y \in A$.

Theorem 2.1. *In any BL-algebra A, the following properties hold for any $x, y, z \in A$,*

(1) $x \leq y$ *if and only if* $x \to y = 1$,
(2) $x \to (y \to z) = (x \otimes y) \to z = y \to (x \to z)$,
(3) *If* $x \leq y$, *then* $y \to z \leq x \to z, z \to x \leq z \to y, x \otimes z \leq y \otimes z$ *and* $y^* \leq x^*$,
(4) $x, y \leq (y \to x) \to x$ *and* $x \vee y = ((x \to y) \to y) \vee ((y \to x) \to x)$,
(5) $x \otimes y \leq x, y$, $x \otimes y \leq x \wedge y, x \otimes 0 = 0$ *and* $x \otimes x^* = 0$,
(6) $1 \to x = x, x \to x = 1, x \leq y \to x, x \to 1 = 1$ *and* $0 \to x = 1$,
(7) $x \otimes y = 0$ *if and only if* $x \leq y^*$,
(8) $x \otimes (y \wedge z) = (x \otimes y) \wedge (x \otimes z)$ *and* $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$ (see [1]).

For a BL-algebra A, if we define

$$B(A) = \{x \in A \mid x \oplus x = x\} = \{x \in A \mid x \otimes x = x\},$$

then $(B(A), \oplus, *, 0)$ is both a largest subalgebra of A and a Boolean algebra. Elements of $B(A)$ are called *Boolean center* of A. If $e \in B(A)$, then $e \otimes x = e \wedge x$ for any $x \in A$.

Theorem 2.2. *For every element $x \in A$ in any BL-algebra, the following conditions are equivalent:*

(1) $x \in B(A)$,
(2) $x \otimes x = x$ *and* $x^{**} = x$,
(3) $x \otimes x = x$ *and* $x^* \to x = x$,
(4) $x^* \vee x = 1$,
(5) $(x \to y) \to x = x$ *for any* $y \in A$,
(6) $x \wedge y = x \otimes y$ *for any* $y \in A$ (see [1]).

We recall that a t-norm is a function $t : [0, 1] \times [0, 1] \to [0, 1]$ such that

(1) t is commutative and associative,
(2) $t(x, 1) = x$ for any $x \in [0, 1]$,
(3) t is nondecreasing in both components.

The following three structures are main examples of BL-algebras on the real unit interval $[0, 1]$.

Example 2.3. Let A be an BL-algebra and $x, y \in A$.

Lukasiewicz: $x \otimes y = \max\{x + y - 1, 0\}$ and $x \to_L y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$

Example 2.4. Let A be an BL-algebra and $x, y \in A$.

Gödel structure : $x \otimes y = \min\{x, y\}$ and $x \to_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$

It is well-known that $\min\{x, y\}$ is the greatest t-norm on $[0, 1]$.

Example 2.5. Let A be an BL-algebra and $x, y \in A$.

Product: $x \otimes y = xy$ and $x \to_P y = \begin{cases} 1 & \text{if } x \leq y \\ x/y & \text{otherwise} \end{cases}$

Definition 2.6. Let A be an BL-algebra. A mapping $D(.,.) : A \times A \to A$ is called symmetric, if $D(x, y) = D(y, x)$ holds for all $x, y \in A$.

Definition 2.7. Let A be an BL-algebra. A mapping $d(x) = D(x, x)$ is called a trace of $D(.,.)$, where $D(.,.) : A \times A \to A$ is a symmetric mapping.

3. $(\otimes, \vee)$-symmetric bi-derivations of BL-algebras

In what follows, let A denote an BL-algebra unless otherwise specified.

Definition 3.1. Let A be a BL-algebra and $D : A \times A \to A$ be a symmetric mapping. We call D a $(\otimes, \vee)$-symmetric bi-derivation on A, if it satisfies the following condition

$$D(x \otimes y, z) = (D(x, z) \otimes y) \vee (x \otimes D(y, z))$$

for all $x, y, z \in A$.

Obviously, a $(\otimes, \vee)$-symmetric bi-derivation D on A satisfies the relation

$$D(x, y \otimes z) = (D(x, y) \otimes z) \vee (y \otimes D(x, z))$$

for all $x, y, z \in A$.

Example 3.2. Let $A = \{0, a, b, 1\}$ be a set where $0 < a < b < 1$ and "$\otimes$" and "$\to$" are defined by

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(A, \wedge, \vee, \otimes, \to, 0, 1)$ is a BL-algebra. Define a map $D : A \times A \to A$ by

$$D(x, y) = \begin{cases} a & \text{if } (x, y) = (a, a), (a, b), (b, a) \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that D is a $(\otimes, \vee)$-symmetric bi-derivation on A.

Proposition 3.3. *Let D be a $(\otimes, \vee)$-symmetric bi-derivation on A and let d be a trace of D. Then the following properties hold, for all $x, y \in A$:*

(1) $d(0) = 0$,
(2) $d(x) \otimes x^* = x \otimes d(x^*) = 0$,
(3) $d(x) = d(x) \vee (x \otimes D(x, 1))$,
(4) $x \in B(A)$ *implies* $x \leq (D(x, x^*))^*$,

(5) *$x \in B(A)$ implies $D(x,y) \leq x$ and $D(x^*,y) \leq x^*$.*

Proof. (1) For every $x \in A$, we have

$$D(x,0) = D(x, 0 \otimes 0) = (D(x,0) \otimes 0) \vee (0 \otimes D(x,0)) \\ = 0 \vee 0 = 0.$$

Since d is a trace of D, we get

$$d(0) = D(0,0) = D(0 \otimes 0, 0) \vee (0 \otimes D(0,0)) \\ = 0 \vee 0 = 0.$$

(2) For any $x \in A$, we have

$$0 = D(x,0) = D(x, x \otimes x^*) \\ = (D(x,x) \otimes x^*) \vee (x \otimes D(x,x^*)).$$

Then $d(x) \otimes x^* = 0$ and $x \otimes D(x,x^*) = 0$.

Similarly, for any $x \in A$, we have

$$0 = D(x^*,0) = D(x^*, x \otimes x^*) \\ = (D(x^*,x) \otimes x^*) \vee (x \otimes D(x^*,x^*)).$$

Thus $x \otimes D(x^*,x^*) = 0$ for all $x \in A$. So $x \otimes d(x^*) = 0$.

(3) For every $x \in A$, we have

$$d(x) = D(x,x) = D(x, x \otimes 1) = (D(x,x) \otimes 1) \vee (x \otimes D(x,1)) \\ = d(x) \vee (x \otimes D(x,1)).$$

(4) Let $x \in B(A)$. Since $x \otimes D(x,x^*) = 0$, we get $D(x,x^*) \leq x^*$. Then $x \leq (D(x,x^*))^*$.

(5) Let $x \in B(A)$. For all $x, y \in A$, since

$$0 = D(x \otimes x^*, y) = (D(x,y) \otimes x^*) \vee (x \otimes D(x^*,y))$$

we have $D(x,y) \otimes x^* = 0$ and $x \otimes D(x^*,y) = 0$, which implies $D(x,y) \leq x$ and $D(x^*,y) \leq x^*$. □

Proposition 3.4. *Let D be a $(\otimes,\vee)$-symmetric bi-derivation on A and let d be a trace of D. If $x \leq y$ for $x, y \in A$, then the following properties hold:*

(1) *$d(x \otimes y^*) = 0$,*
(2) *$d(y^*) \leq x^*$,*
(3) *$x \in B(A)$ implies $d(x) \otimes d(y^*) = 0$.*

Proof. (1) Let $x \leq y$ for $x, y \in A$. Since $x \otimes y^* \leq y \otimes y^* = 0$, we have $x \otimes y^* = 0$. Since $d(0) = 0$, we obtain $d(x \otimes y^*) = 0$.

(2) Let $x \leq y$ for $x, y \in A$. Since $x \otimes d(y^*) \leq y \otimes y^* = 0$, we have $x \otimes d(y^*) = 0$, which implies $d(y^*) \leq x^*$.

(3) Let $x \in B(A)$. By Proposition 3.3 (5), we have $D(x,y) \leq x$. Replacing y by x in this relation, we have $D(x,x) \leq x$. Then $d(x) \leq x$. Since $d(x) \leq x$, we have $d(x) \leq y$. Thus $d(x) \otimes d(y^*) \leq y \otimes d(y^*) \leq y \otimes y^* = 0$ by part (2). So $d(x) \otimes d(y^*) = 0$. □

Proposition 3.5. *Let D be a $(\otimes, \vee)$-symmetric bi-derivation on A and let d be a trace of D. If $x \in B(A)$, the following properties hold, for all $x \in A$.*

(1) $d(x) \otimes d(x^*) = 0$,

(2) *Let $x \in B(A)$. Then $d(x^*) = (d(x))^*$ if and only if d is an identity map on A.*

Proof. (1) In $d(x)\otimes d(y^*) = 0$, replacing y by x in this relation, we have $d(x)\otimes d(x^*) = 0$.

(2) Since $x \otimes d(y^*) = 0$ for all $x, y \in A$, we obtain $x \otimes d(x^*) = x \otimes (d(x))^* = 0$. Since $x \leq d(x)$ and $d(x) \leq x$, we get $d(x) = x$. Hence d is an identity map on A. If d is an identity map on A, then $d(x^*) = (d(x))^*$ for all $x \in A$. □

Definition 3.6. Let $D : A \times A \to A$ be a bi-symmetric mapping. If $x \leq y$ implies $D(x, z) \leq D(y, z)$ for all $x, y, z \in A$, then D is said to be isotone.

If d is a trace of D and D is isotone, $x \leq y$ implies $d(x) \leq d(y)$ for all $x, y \in A$.

Example 3.7. Let A be an BL-algebra as Example 3.2. Define a map $D : A\times A \to A$ by

$$D(x) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), (a, 0), (0, a), (0, b), (b, 0), (1, 0), (0, 1), (b, a), (a, b) \\ b & \text{if } (x, y) = (b, b), (b, 1), (1, b) \\ a & \text{if } (x, y) = (a, a), (a, 1), (1, a) \\ 1 & \text{if } (x, y) = (1, 1). \end{cases}$$

Then we can see that D is an isotone $(\otimes, \vee)$-symmetric bi-derivation on A.

Proposition 3.8. *Let D be a $(\otimes, \vee)$-symmetric bi-derivation on A and let d be a trace of D. If $d(x^*) = d(x)$ for all $x \in A$, we have*

(1) $d(1) = 0$,

(2) $d(x) \otimes d(x) = 0$,

(3) *if D is isotone, then $d = 0$.*

Proof. (1) In the relation $d(x) = d(x^*)$, replacing x by 0, we obtain $d(1) = 0$.

(2) For every $x \in A$, $d(x) \otimes d(x) = d(x) \otimes d(x^*) = 0$ by hypothesis.

(3) Let D be isotone. For any $x \in A$, we have $d(1) = 0$ since $d(x) \leq d(1) = 0$. □

Definition 3.9. Let D be a $(\otimes, \vee)$-symmetric bi-derivation on A. If $D(x \otimes y, z) = D(x, z) \otimes D(y, z)$ for all $x, y, z \in A$, then D is called a bi-multiplicative mapping on A.

Theorem 3.10. *Let D be a multiplicative $(\otimes, \vee)$-symmetric bi-derivation on A and let d be a trace of D. Then $d(B(A)) \subseteq B(A)$.*

Proof. Let $y \in d(B(A))$. Then $y = d(x)$ for some $x \in B(A)$. Thus

$$\begin{aligned} y \otimes y = d(x) \otimes d(x) &= D(x, x) \otimes D(x, x) = D(x \otimes x, x) \\ &= D(x, x) = y \end{aligned}$$

So $y \in B(A)$. Hence $d(B(A)) \subseteq B(A)$. □

Theorem 3.11. *Let D be a $(\otimes, \vee)$-symmetric bi-derivation on Gödel BL-algebra A and let d be a trace of D. Then the following conditions hold for all $x, y \in A$.*

(1) $d(x) \leq x$.
(2) *If* $x \leq D(1, x)$, *then* $d(x) = x$.
(3) *If* $x \geq D(1, x)$, *then* $D(1, x) \leq d(x)$.
(4) *If* $x \leq y$, *then* $d(x) = x$ *or* $d(x) \geq D(x, y)$.

Proof. (1) For $x \in A$, we have

$$\begin{aligned} d(x) &= D(x,x) = D(x \otimes x, x) = (D(x,x) \otimes x) \vee (x \otimes D(x,x)) \\ &= D(x,x) \otimes x = d(x) \otimes x = \min\{d(x), x\}. \end{aligned}$$

Then $d(x) \leq x$.

(2) Let $x \leq D(1, x)$ for any $x \in A$. Then

$$\begin{aligned} d(x) &= D(x,x) = D(x \otimes 1, x) = (D(x,x) \otimes 1) \vee (x \otimes D(1,x)) \\ &= d(x) \vee (x \otimes D(1,x)) = d(x) \vee (\min\{x, D(1,x)\}) \\ &= d(x) \vee x = x \end{aligned}$$

by (1).

(3) Let $x \geq D(1, x)$ for any $x \in A$. Then

$$\begin{aligned} d(x) &= D(x,x) = D(x \otimes 1, x) = (D(x,x) \otimes 1) \vee (x \otimes D(1,x)) \\ &= d(x) \vee (x \otimes D(1,x)) = d(x) \vee (\min\{x, D(1,x)\}) \\ &= d(x) \vee D(1,x). \end{aligned}$$

Thus $D(1, x) \leq d(x)$.

(4) Let $x \leq y$. Then by (1), $d(x) \leq x \leq y$, which implies $d(x) \leq y$. Thus $d(x) = D(x, x) = D(x \otimes y, x) = (D(x, x) \otimes y) \vee (x \otimes D(y, x)) = d(x) \vee (x \otimes D(y, x))$. If $x \leq D(x, y)$, then by (1), $d(x) = x$. If $x \geq D(x, y)$, then $d(x) = d(x) \vee D(x, y)$. So $d(x) \geq D(x, y)$. □

Theorem 3.12. *Let* D *be a* $(\otimes, \vee)$*-symmetric bi-derivation on* A*. If there exist* $a \in A$ *such that* $a \otimes D(x, z) = 1$*, for all* $x, z \in A$*, then we have* $a = 1$*.*

Proof. Let D be a $(\otimes, \vee)$-symmetric bi-derivation on A. Assume that there exist $a \in A$ such that $D(x, z) \otimes a = 1$, for all $x, z \in A$. Since D is a $(\otimes, \vee)$-symmetric bi-derivation on A, we get

$$\begin{aligned} 1 &= D(x \otimes a, z) \otimes a = ((D(x,z) \otimes a) \vee x \otimes (D(a,z))) \otimes a \\ &= (1 \vee (x \otimes D(a,z))) \otimes a = 1 \otimes a = a. \end{aligned}$$

This completes the proof. □

Theorem 3.13. *Let* A *be a* BL*-algebra. Define a mapping* $D : A \times A \to A$ *by* $D(x, z) = x \otimes z$ *for all* $x, z \in A$*. Then* D *is an* $(\otimes, \vee)$*-symmetric bi-derivation on* A*.*

Proof. For every $x, y, z \in A$, we have

$$D(x \otimes y, z) = (x \otimes y) \otimes z.$$

On the other hand,

$$\begin{aligned} (D(x,z) \otimes y) \vee (x \otimes D(y,z)) &= ((x \otimes z) \otimes y) \otimes (x \otimes (y \otimes z)) \\ &= (x \otimes y) \otimes z. \end{aligned}$$

Hence D is an $(\otimes, \vee)$-symmetric bi-derivation on A. □

Proposition 3.14. *Let D be an $(\otimes, \vee)$-symmetric bi-derivation on $B(A)$. Then D is a symmetric bi-derivation on lattice, that is,*

$$D(x \wedge y, z) = (D(x, z) \wedge z) \vee (x \wedge D(y, z))$$

for all $x, y, z \in B(A)$.

Proof. Let $x, y, z \in B(A)$. Then we have

$$\begin{aligned} D(x \wedge y, z) = D(x \otimes y, z) &= (D(x, z) \otimes y) \vee (x \otimes D(y, z)) \\ &= (D(x, z) \wedge y) \vee (x \wedge D(y, z)). \end{aligned}$$

□

Theorem 3.15. *Let A be a BL-algebra and $D : A \times A \to A$ be a symmetric mapping. If D is an $(\otimes, \vee)$-symmetric bi-derivation on $A = B(A)$, then $D(x, z) = D(x, z) \wedge x$ for all $x, z \in B(A)$.*

Proof. Let $x, z \in B(A)$. Then we have $x \otimes x = x$. Thus we get

$$\begin{aligned} D(x, z) = D(x \otimes x, z) &= (D(x, z) \otimes x) \vee (x \otimes D(x, z)) \\ &= D(x, z) \otimes x = D(x, z) \wedge x. \end{aligned}$$

□

Let D be an $(\otimes, \vee)$-symmetric bi-derivation of A and $a \in A$. Define a set $Fix_a(A)$ by

$$Fix_a(A) := \{x \in A \mid D(x, a) = x\}.$$

Proposition 3.16. *Let D be a $(\otimes, \vee)$-symmetric bi-derivation of A. If $x, y \in Fix_a(A)$, then $x \otimes y \in Fix_a(A)$.*

Proof. Let $x, y \in Fix_a(A)$. Then we have $D(x, a) = x$ and $D(y, a) = y$. Thus

$$\begin{aligned} D(x \otimes y, a) &= (D(x, a) \otimes y) \vee (x \otimes D(y, a)) \\ &= (x \otimes y) \vee (x \otimes y) = x \otimes y. \end{aligned}$$

So we get $x \otimes y \in Fix_a(A)$. This completes the proof. □

Proposition 3.17. *Let D be a $(\otimes, \vee)$-symmetric bi-derivation of A and $A = B(A)$. If $x, y \in Fix_a(A)$, then we have $x \wedge y \in Fix_a(A)$.*

Proof. Let $x, y \in Fix_a(A)$. Then we have $D(x, a) = x$ and $D(y, a) = .$, Thus

$$\begin{aligned} D(x \wedge y, a) = D(x \otimes y, a) &= (D(x, a) \otimes y) \vee (x \otimes D(y, a)) \\ &= (x \otimes y) \vee (x \otimes y) = x \otimes y = x \wedge y. \end{aligned}$$

So we get $x \wedge y \in Fix_a(A)$. This completes the proof. □

4. $(\otimes, \ominus)$-symmetric bi-derivations of BL-algebras

Definition 4.1. Let A be a BL-algebra and $D : A \times A \to A$ be a symmetric mapping. We call D an $(\ominus, \otimes)$-symmetric bi-derivation on A, if it satisfies the following condition

$$D(x \ominus y, z) = (D(x, z) \ominus y) \otimes (x \ominus D(y, z))$$

for all $x, y, z \in A$.

Example 4.2. Let $A = \{0, a, b, 1\}$ be a set where $0 < a < b < 1$ and "$\otimes$" and "$\to$" are defined by

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then $(A, \wedge, \vee, \otimes, \to, 0, 1)$ is a BL-algebra. Define a map $D : A \times A \to A$ by

$$D(x, y) = \begin{cases} a & \text{if } (x, y) = (a, a), (a, 1), (1, a), (1, 1) \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that D is a $(\otimes, \ominus)$-symmetric bi-derivation on A.

Proposition 4.3. *Let D be a $(\otimes, \ominus)$-symmetric bi-derivation on A and let d be a trace of D. Then the following conditions hold:*

(1) *$d(0) = 0$,*
(2) *$D(x, 0) = D(x, 0) \otimes x$ for any $x \in A$,*
(3) *$D(x, 0) \leq x$ for any $x \in A$.*

Proof. (1) Let D be a $(\otimes, \ominus)$-symmetric bi-derivation on A and let d be a trace of D. Then we have

$$\begin{aligned} d(0) = D(0, 0) = D(0 \ominus 0, 0) &= (D(0, 0) \ominus 0) \times (0 \ominus D(0, 0)) \\ &= D(0, 0) \otimes 0 = 0. \end{aligned}$$

(2) For every $x \in A$, we get

$$\begin{aligned} D(x, 0) = D(x \ominus 0, 0) &= (D(x, 0) \ominus 0) \otimes (x \ominus D(0, 0)) \\ &= (D(x, 0) \otimes 1) \otimes (x \otimes 1) = D(x, 0) \otimes x. \end{aligned}$$

(3) From (2), for any $x \in A$, we have

$$D(x, 0) = D(x, 0) \otimes x \leq x.$$

□

Proposition 4.4. *Let D be a $(\otimes, \ominus)$-symmetric bi-derivation on A and let d be a trace of D. Then the following conditions hold:*

(1) *$D(x, 0) = 0$ for any $x \in A$,*
(2) *$D(x^*, 0) = D(1, 0) \otimes x^*$ for any $x \in A$.*

Proof. (1) Let D be a $(\otimes, \ominus)$-symmetric bi-derivation on A and let d be a trace of D. Then we have

$$\begin{aligned} D(x,0) = D(0,x) = D(x \ominus 1, x) &= (D(x,x) \ominus 1) \otimes (x \ominus D(1,0)) \\ &= (d(x) \otimes 0) \otimes (x \otimes (D(1,0))^*) \\ &= 0 \otimes (x \otimes (D(1,0))^*) = 0. \end{aligned}$$

(2) For every $x \in A$, we obtain

$$\begin{aligned} D(x^*,0) = D(1 \ominus x, 0) &= (D(1,0) \ominus x) \otimes (1 \ominus D(x,0)) \\ &= (D(1,0) \otimes x^*) \otimes (1 \otimes 1) \\ &= D(1,0) \otimes x^*. \end{aligned}$$

□

Proposition 4.5. *Let D be a $(\otimes, \ominus)$-symmetric bi-derivation on A. Then D is an isotone $(\otimes, \ominus)$-symmetric bi-derivation on $B(A)$.*

Proof. Let $x, y, z \in B(A)$ and $x \leq y$. Then we have

$$\begin{aligned} D(x,z) = D(y \wedge x, z) = D(y \otimes x, z) &= (D(y \ominus x^*, z) \\ &= (D(y,z) \ominus x^*) \otimes (y \ominus D(x^*, z)) \\ &\leq D(y,z) \ominus x^* = D(y,z) \otimes x \leq D(y,z). \end{aligned}$$

This completes the proof. □

Theorem 4.6. *Let D be a $(\otimes, \ominus)$-symmetric bi-derivation on A. If $D(A,A) \subseteq B(A)$ and $D(x \ominus y, z) = D(x,z) \ominus D(y,z)$ for all $x, y, z \in A$, then D is an isotone mapping on A.*

Proof. Let $D(A,A) \subseteq B(A)$ and $x \leq y$. Then $0 = x \otimes y^*$. Thus

$$\begin{aligned} 0 = D(x \otimes y^*, z) &= D(x \ominus y, z) \\ &= D(x,z) \ominus D(y,z) = D(x,z) \otimes (D(y,z))^*, \end{aligned}$$

for every $z \in A$. So by Theorem 2.1 (7), $D(x,z) \leq (D(y,z))^{**}$. Hence $D(x,z) \leq D(y,z)$. □

Proposition 4.7. *Let D be a $(\otimes, \ominus)$-symmetric bi-derivation on A. Then the following conditions hold:*

(1) $D(x,z) = D(x,z) \otimes x$ *for every* $x, z \in A$,
(2) $D(x,z) \leq x$ *for every* $x, z \in A$.

Proof. (1) Let $x, z \in A$. Then

$$\begin{aligned} D(x,z) = D(x \ominus 0, z) &= (D(x,z) \ominus 0) \otimes (x \ominus D(0,z)) \\ &= (D(x,z) \otimes 1) \otimes (x \otimes 0^*) = D(x,z) \otimes x. \end{aligned}$$

(2) From (1), we obtain $D(x,z) = D(x,z) \otimes x \leq x$ for every $x, z \in A$. □

Proposition 4.8. *Let D be a $(\otimes, \ominus)$-symmetric bi-derivation on A. Then $x = x \otimes x$ for every $x \in Fix_a(A)$.*

Proof. Let $x \in Fix_a(A)$. Then $D(x,a) = x$. Thus by Proposition 4.7, we have $D(x,a) = D(x,a) \otimes x$. So $x = x \otimes x$. □

5. Conclusions

In this work, we first introduced the notion for two types of symmetric bi-derivations in BL-algebra and obtained some results. We also studied $(\otimes, \vee)$-symmetric bi-derivations on *Gödel* BL-algebras. Furthermore, we took into account isotone symmetric bi-derivations on BL-algebra A. In the future, we will study $(\otimes, \vee)$-symmetric bi-f-derivations and $(\otimes, \ominus)$-symmetric bi-f-derivations on A.

References

[1] A. Di Nola, G. Georgescu and A. Iorgulescu, Pseudo BL-algebra: Part I, Multi-valued Log 8(5-6) (2002) 673–714.

[2] P. *Häjek*, Metamathematics of fuzzy logic, Trends in Logic-Studia Logica Library 4(Kluwer Academic Publishers Dordrecht (1998).

Kyung Ho Kim (ghkim@ut.ac.kr)
Department of Mathematics, Korea National University of Transportation Chungju 27469, Korea