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Abstract. We present a general scheme for defining families of fuzzy languages and the related families of crisp languages. In particular, all the varieties of regular fuzzy languages of T. Petković (2005) together with their associated $\star$-varieties of S. Eilenberg (1976) and varieties of finite monoids are obtained this way, but also pairs of more general families of regular fuzzy and crisp languages can be defined. For the families that are not varieties, we show how to get the greatest varieties contained in them. As examples we consider the varieties of commutative fuzzy and crisp languages, the families of rotation invariant fuzzy and crisp regular languages, which are not varieties, and the varieties of aperiodic fuzzy and crisp languages, which are ultimately defined by a sequence of identities.

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1. Introduction

In this paper we present a general scheme for defining families of fuzzy languages and their associated families of crisp languages by certain identities. In particular, all varieties of regular fuzzy languages as defined by Petković [21] and the associated $\star$-varieties of Eilenberg [11] as well as the corresponding varieties of finite monoids (VFMs) are obtained this way. However, also pairs of more general families of regular fuzzy and crisp languages can be defined. For the families that are not varieties, we show how to get the greatest varieties contained in them.

As examples we consider three pairs of families of fuzzy and crisp languages. In [3], Archana defined a fuzzy language λ to be commutative if (C) $\lambda(suvt) = \lambda(svut)$ for all words s, t, u and v over the given alphabet. Condition (C) may be turned into an identity of the kind our theory is about, and then it defines the fuzzy $\star$-variety $FCom$ of fuzzy commutative regular languages, the $\star$-variety Com of crisp commutative regular languages, and the VFM **Com** of the commutative finite monoids. Archana also proposed five generalizations (C1)-(C5) of condition (C), two of which define the fuzzy and crisp languages that we call rotation invariant. The aperiodic fuzzy languages introduced by Li [16], and considered by Archana [4], provide an example of varieties ultimately defined by a sequence of word identities.

In Section 2, we recall some notions from the theory of finite automata. Section 3 introduces fuzzy deterministic finite recognizers (FDFRs) in which just the set of final states is fuzzy while the transition function and the initial state are crisp. As shown by Mateescu et al. [17] (cf. also [6]), every regular fuzzy language is recognized by such an FDFR. Moreover, many facts about these recognizers can be derived directly from the classical theory of Moore machines [19, 2, 7, 24, 25].

In Section 4, it is shown that Archana's conditions (C1), (C2) and (C3) are actually equivalent to condition (C), while (C4) and (C5) are weaker than (C) but equivalent to each other. Hence, we are left with two families of regular fuzzy languages, the family $FCom$ of commutative regular fuzzy languages, and its proper superfamily $FRot$ of regular fuzzy languages which we call rotation invariant.

Section 5 introduces word identities and the families of crisp and fuzzy languages defined by them. In a word identity $\mathrm{u} \approx \mathrm{v}$, u and v are strings of word variables that take as values words over the alphabet considered. It is shown that the family $\mathcal{L}_I$ of crisp regular languages satisfying such an identity (I) is closed under all Boolean operations and inverse homomorphisms, but not necessarily under the quotient operations. The corresponding results hold also for the family $\mathcal{F}_I$ of regular fuzzy languages that satisfy (I). Moreover, a fuzzy language belongs to $\mathcal{F}_I$ if and only if is a linear combination of the characteristic functions of some members of $\mathcal{L}_I$.

In Section 6, we first recall Eilenberg's [11] $\star$-varieties and their fuzzy counterparts considered by Petković [21]. Then we introduce flanked word identities, which are of the form $\xi\mathrm{u}\pi \approx \xi\mathrm{v}\pi$, where ξ and π are two distinct variables that do not appear in the words u and v. For any flanked identity (J) $\xi\mathrm{u}\pi \approx \xi\mathrm{v}\pi$, the families $\mathcal{L}_J$ and $\mathcal{F}_J$ are closed also under the respective quotient operations, and hence $\mathcal{L}_J$ is a $\star$-variety and $\mathcal{F}_J$ is the corresponding fuzzy $\star$-variety. Moreover, the identity (I) $\mathrm{u} \approx \mathrm{v}$ defines the corresponding VFM. We also show that for any word identity (I) $\mathrm{u} \approx \mathrm{v}$ and the corresponding flanked identity (J) $\xi\mathrm{u}\pi \approx \xi\mathrm{v}\pi$, $\mathcal{L}_J$ is the greatest $\star$-variety contained in the family $\mathcal{L}_I$, and $\mathcal{F}_J$ is the greatest fuzzy $\star$-variety contained in $\mathcal{F}_I$.

In Section 7, the results of Section 6 are extended to families of languages, families of fuzzy languages, and VFMs ultimately defined by sequences word identities. In particular, it is shown that a family of regular fuzzy languages is a fuzzy $\star$-variety if and only if it is ultimately defined by a sequence of flanked word identities.

In the following three sections we illustrate the general results by three different examples of families of fuzzy and crisp regular languages defined by word identities. Archana [3] showed that the family $FCom$ of commutative regular fuzzy languages is a fuzzy $\star$-variety by proving that it has all the required closure properties. In Section 8 we show that this result as well as the facts that $FCom$ corresponds to the $\star$-variety Com of commutative regular languages and the VFM **Com** follow from the results of Section 6. Moreover, we note the fuzzy counterparts of some classical characterizations of commutative regular languages.

In Section 9, we consider the families *Rot* and *FRot* of rotation invariant regular crisp and fuzzy languages. Although they are not varieties, they are linked with each other similarly as a $\star$-variety and the corresponding fuzzy $\star$-variety. In particular, a fuzzy languages belongs to *FRot* if and only if it can be represented as a linear combination of the characteristic functions of some members of *Rot*. We also present some alternative characterizations of rotation invariant crisp and fuzzy languages from which it follows that it is decidable whether a crisp or fuzzy regular language is rotation invariant. We also prove that *Com* is the greatest $\star$-variety contained in *Rot* and that *FCom* the greatest fuzzy $\star$-variety contained on *FRot*. Finally, in Section 10, we consider the families of aperiodic regular fuzzy and crisp languages as examples of varieties ultimately defined by a sequence of identities.

2. Finite automata and regular languages

For any nonnegative integer n, let $\mathbf{n} := \{1, \ldots, n\}$. For a relation $\theta \subseteq A \times B$, we express $(a, b) \in \theta$ also by writing $a\,\theta\,b$. For any $a \in A$, let $a\theta := \{b \in B \mid a\,\theta\,b\}$. If an equivalence relation θ on a set A is known from the context, we may denote the θ-class $a\theta$ of an element $a \in A$ by $[a]$. The index of θ is the cardinality of the quotient set $A/\theta := \{[a] \mid a \in A\}$.

In what follows, X is always a finite nonempty alphabet. The set of all (finite) words over X is denoted by X^* and the empty word by ε. Subsets of X^* are called (crisp) languages. A family of languages $\mathcal{L} = \{\mathcal{L}(X)\}_X$ associates with each alphabet X a set $\mathcal{L}(X)$ of languages over X.

A deterministic finite automaton (DFA) $\mathcal{A} = (A, X, \delta)$ consists of a finite nonempty set A of states, the input alphabet X, and a transition function $\delta : A \times X \to A$. As usual, δ is extended to a map $\delta^* : A \times X^* \to A$ by setting $\delta^*(a, \varepsilon) = a$, and $\delta^*(a, vx) = \delta(\delta^*(a, v), x)$ for all $a \in A$, $v \in X^*$ and $x \in X$. We may write $\delta^*(a, w)$ as $aw^{\mathcal{A}}$, or just aw. The maps $w^{\mathcal{A}} : A \to A, a \mapsto \delta^*(a, w)$, form with the product $u^{\mathcal{A}} \cdot v^{\mathcal{A}} = (uv)^{\mathcal{A}}$ the transition monoid $\mathrm{TM}(\mathcal{A})$ of $\mathcal{A}$. Its identity element is the identity map $1_A = \varepsilon^{\mathcal{A}}$.

A deterministic finite recognizer (DFR) $\mathbf{A} = (A, X, \delta, a_0, F)$ consists of an underlying DFA $\mathcal{A} = (A, X, \delta)$, an initial state $a_0 \in A$, and a set of final states $F \subseteq A$. The language recognized by $\mathbf{A}$ is $L(\mathbf{A}) := \{w \in X^* \mid a_0 w \in F\}$. A language is recognizable or regular, if it is recognized by a DFR. Let $Rec = \{RecX)\}_X$ be the family of all regular languages.

A state $a \in A$ of a DFR $\mathbf{A} = (A, X, \delta, a_0, F)$ is accessible, if $a = a_0 w$ for some $w \in X^*$, and two states $a, b \in A$ are equivalent, if for every $w \in X^*$, $aw \in F$ if and only if $bw \in F$. The DFR $\mathbf{A}$ is *connected* its every state is accessible, it is reduced, if no two distinct states are equivalent, and it is minimal, if it connected and reduced. The transition monoid $\mathrm{TM}(\mathbf{A})$ of $\mathbf{A}$ is the transition monoid of its underlying DFA.

The syntactic congruence σ_L of a language $L \subseteq X^*$ is defined by

$$u\,\sigma_L\,v \iff (\forall s, t \in X^*)(sut \in L \Leftrightarrow svt \in L) \qquad (u, v \in X^*),$$

and $\mathrm{SM}(L) := X^*/\sigma_L$ is its syntactic monoid and $\varphi_L : X^* \to \mathrm{SM}(L), w \mapsto [w]$ its syntactic homomorphism. By Myhill's theorem, L is regular if and only if the index of σ_L is finite, i.e., SM(L) is finite. Moreover, if $\mathbf{A}$ is a minimal DFR recognizing L, then SM(L) is isomorphic to TM($\mathbf{A}$).

A Moore machine $\mathfrak{A} = (A, X, Y, \delta, a_0, \mu)$, introduced by Moore [19], consists of a DFA (A, X, δ), an output alphabet Y, an initial state $a_0 \in A$, and an output map $\mu : A \to Y$. The mapping $f_{\mathfrak{A}} : X^* \to Y$ realized by $\mathfrak{A}$ is defined by $f_{\mathfrak{A}}(w) := \mu(a_0 w)$ $(w \in X^*)$. A mapping $f : X^* \to Y$ is said to be finite-state (computable), if $f = f_{\mathfrak{A}}$, for some Moore machine $\mathfrak{A}$. Two states $a, b \in A$ of a Moore machine $\mathfrak{A} = (A, X, Y, \delta, a_0, \mu)$ are equivalent, if $\mu(aw) = \mu(bw)$, for every $w \in X^*$, and $\mathfrak{A}$ is minimal, if all states are accessible and no two distinct states are equivalent.

Presentations of the classical theory of finite automata and regular languages can be found, for example, in [2, 7, 10, 11, 13, 14, 24, 25].

3. Fuzzy languages and automata

A fuzzy language is a mapping $\lambda : X^* \to D$, where D is a given set of degrees of membership. In the literature several kinds of algebraic structures D have been used, but adopting Zadeh's [29] original definition of fuzzy sets, we let D be the real unit interval $[0, 1]$ ordered by the usual $\leq$-relation and equipped with the operations $c \vee d = \max(c, d)$ and $c \wedge d = \min(c, d)$.

For any fuzzy language $\lambda : X^* \to [0, 1]$, the support is the language $\text{supp}(\lambda) := \{w \in X^* \mid \lambda(w) > 0\}$, the range is the set $\text{ran}(\lambda) := \{\lambda(w) \mid w \in X^*\}$, and the kernel is the equivalence $\ker(\lambda) := \{(u, v) \mid u, v \in X^*, \lambda(u) = \lambda(v)\}$ on X^*. If $\text{supp}(\lambda)$ is a finite set $\{w_1, \ldots, w_n\}$, we may give λ in the form $\{w_1/\lambda(w_1), \ldots, w_n/\lambda(w_n)\}$. If $\text{ran}(\lambda) \subseteq \{0, 1\}$, then λ is said to be crisp. The characteristic function of a language $L \subseteq X^*$ is the crisp fuzzy language L^χ such that $L^\chi(w) = 1$ for $w \in L$, and $L^\chi(w) = 0$ for $w \in X^* \setminus L$. On the other hand, each crisp fuzzy language λ is the characteristic function of the language $\text{supp}(\lambda)$, and hence it is customary to call ordinary languages crisp languages. A family of fuzzy languages $\mathcal{F} = \{\mathcal{F}(X)\}_X$ assigns to each alphabet X a set $\mathcal{F}(X)$ of fuzzy languages over X.

Many types of fuzzy automata have been considered (cf. [5, 20, 23], for example). In the fuzzy recognizers to be used here just the set of final states is fuzzy while the initial state and the transition function are crisp, but all regular fuzzy languages are recognized by them [17, 6].

A fuzzy deterministic finite recognizer (FDFR) $\mathbf{F} = (A, X, \delta, a_0, \omega)$ consists of a DFA $\mathcal{A} = (A, X, \delta)$, an initial state $a_0 \in A$ and a fuzzy set $\omega : A \to [0, 1]$ of final states. The fuzzy language $\lambda_{\mathbf{F}} : X^* \to [0, 1]$ recognized by $\mathbf{F}$ is defined by $\lambda_{\mathbf{F}}(w) = \omega(a_0 w)$ $(w \in X^*)$. A fuzzy language is said to be regular or recognizable, if it is recognized by a FDFR. Let $FRec = \{FRec(X)\}_X$ be the family of regular fuzzy languages.

As noted already in [17], such recognizers resemble Moore machines. Indeed, a FDFR $\mathbf{F} = (A, X, \delta, a_0, \omega)$ may be seen as a Moore machine in which the output alphabet is the finite set $Y := \{\omega(a) \mid a \in A\}$ of the possible degrees of acceptance (treated as output symbols). Thus $\mathbf{F}$ can be minimized similarly as a Moore machine by eliminating the inaccessible states and then merging all pairs of equivalent states (cf. [2], [7] or [25], for example). Furthermore, if the Nerode (right) congruence ρ_λ

of a fuzzy language $\lambda : X^* \to [0,1]$, defined by

$$u\,\rho_\lambda\,v \iff (\forall w \in X^*)\lambda(uw) = \lambda(vw) \quad (u, v \in X^*),$$

is of finite index, then λ is regular and $\mathbf{F}_\lambda = (X^*/\rho_\lambda, X, \delta_\lambda, [\varepsilon], \omega_\lambda)$, where δ_λ and ω_λ are defined by $\delta_\lambda([w], x) = [wx]$ and $\omega_\lambda([w]) = \lambda(w)$ $(w \in X^*, x \in X)$, is a minimal FDFR for λ (unique up to isomorphism). The syntactic congruence σ_λ (or Myhill congruence) of λ is defined by

$$u\,\sigma_\lambda\,v \iff (\forall s, t \in X^*)\lambda(sut) = \lambda(svt) \quad (u, v \in X^*),$$

and the quotient monoid $\mathrm{SM}(\lambda) := X^*/\sigma_\lambda$ is the syntactic monoid of λ. The transition monoid $\mathrm{TM}(\mathbf{F})$ of $\mathbf{F}$ is defined as the transition monoid of the DFA $\mathcal{A}$. For a fuzzy language "finite-state computable" means regularity, and hence we get the following facts directly from the theory of Moore machines and finite-state maps [2, 7, 10, 13, 24, 25].

Proposition 3.1. *For any fuzzy language $\lambda : X^* \to [0,1]$, the following conditions are equivalent to each other:*
(1) $\lambda \in FRec(X)$,
(2) ρ_λ *is of finite index,*
(3) σ_λ *is of finite index, i.e.,* $\mathrm{SM}(\lambda)$ *is finite.*
Moreover, for any regular λ,
(4) $\mathbf{F}_\lambda$ *is the minimal FDFR recognizing λ,*
(5) $\mathrm{SM}(\lambda)$ *is isomorphic to the transition monoid* $\mathrm{TM}(\mathbf{F}_\lambda)$.

For any fuzzy languages $\varkappa, \lambda : X^* \to [0,1]$, any $c \in [0,1]$ and any homomorphism $\varphi : Y^* \to X^*$, the complement $\overline{\lambda}$, the union $\varkappa \cup \lambda$, the intersection $\varkappa \cap \lambda$, the scalar product $c\lambda$, the fuzzy c-cut $\lambda_{[c)}$, the fuzzy quotient languages $\varkappa^{-1}\lambda$ and $\lambda\varkappa^{-1}$, and $\varphi^{-1}(\lambda) : Y^* \to [0,1]$ are defined as follows $(w \in X^*,\ s \in Y^*)$:
(i) $\overline{\lambda}(w) = 1 - \lambda(w)$, $(\varkappa \cup \lambda)(w) = \varkappa(w) \vee \lambda(w)$, $(\varkappa \cap \lambda)(w) = \varkappa(w) \wedge \lambda(w)$,
(ii) $(c\lambda)(w) = c \cdot \lambda(w)$,
(iii) $\lambda_{[c)}(w) = 1$ if $\lambda(w) \geq c$, and $\lambda_{[c)}(w) = 0$ if $\lambda(w) < c$,
(iv) $(\varkappa^{-1}\lambda)(w) = \bigvee\{\varkappa(v) \wedge \lambda(vw) \mid v \in X^*\}$,
(v) $(\lambda\varkappa^{-1})(w) = \bigvee\{\lambda(wv) \wedge \varkappa(v) \mid v \in X^*\}$,
(vi) $\varphi^{-1}(\lambda)(s) = \lambda(s\varphi)$.
We shall need the following facts (cf. [17, 20, 21, 27]).

Proposition 3.2. *The family $FRec = \{FRec(X)\}_X$ is closed under the operations $\overline{\lambda}$, $\varkappa \cup \lambda$, $\varkappa \cap \lambda$, $c\lambda$, $\lambda_{[c)}$, $\varkappa^{-1}\lambda$, $\lambda\varkappa^{-1}$, and $\varphi^{-1}(\lambda)$ defined above.*

Next we note some links between *Rec* and *FRec* (cf. [17, 20, 27]).

Lemma 3.3. (1) *A language $L \subseteq X^*$ is regular if and only if L^χ is regular.*
(2) *If a fuzzy language $\lambda : X^* \to [0,1]$ is regular, then so is* $\mathrm{supp}(\lambda)$.

As noted in [17], for example, any recognizable fuzzy language can be written as a finite union $c_1L_1^\chi \cup \cdots \cup c_nL_n^\chi$, where $c_1, \ldots, c_n \in [0,1]$ and $L_1, \ldots, L_n$ are recognizable languages. Weighted languages similarly represented are called recognizable step-functions [9].

Proposition 3.4. *A fuzzy language* $\lambda : X^* \to [0,1]$ *is regular if and only if it can be expressed in the form* $\lambda = c_1 L_1^\chi \cup \ldots \cup c_n L_n^\chi$*, where* $n \geq 1$*,* $c_1, \ldots, c_n \in [0,1]$*, and* $L_1, \ldots, L_n \in Rec(X)$*. Moreover, if* $\lambda \in FRec(X)$*, then the constants* c_i *may be chosen to be pairwise distinct and the languages* L_i *to be pairwise disjoint.*

Proof. We present a proof for later reference. If $\lambda = c_1 L_1^\chi \cup \ldots \cup c_n L_n^\chi$ as in the proposition, then $\lambda \in FRec(X)$ by Lemma 3.3 and Proposition 3.2.

Assume then that $\lambda \in FRec(X)$, and consider a minimal FDFR $\mathbf{F} = (A, X, \delta, a_0, \omega)$ recognizing λ. Obviously, $L(\mathbf{A}) \in Rec(X)$ for every DFR $\mathbf{A} = (A, X, \delta, a_0, F)$ obtained by any choice of $F \subseteq A$. Let $\mathrm{ran}(\lambda) = \omega(A) = \{c_1, \ldots, c_n\}$ ($\mathbf{F}$ is connected!). If $\mathbf{A}_i = (A, X, \delta, a_0, \omega^{-1}(c_i))$ and $L_i := L(\mathbf{A}_i)$ for each $i \in \mathbf{n}$, then $\lambda = c_1 L_1^\chi \cup \ldots \cup c_n L_n^\chi$ is a representation as required. □

4. Commutative and rotation invariant fuzzy languages

We shall now introduce two families of fuzzy languages that will serve as our main examples. In [3] Archana presented the following six commutativity conditions for a fuzzy language $\lambda : X^* \to [0,1]$:

(C) $\lambda(suvt) = \lambda(svut)$ for all $s, t, u, v \in X^*$,
(C1) $\lambda(uvw) = \lambda(vuw)$ for all $u, v, w \in X^*$,
(C2) $\lambda(uvw) = \lambda(uwv)$ for all $u, v, w \in X^*$,
(C3) $\lambda(uvw) = \lambda(wvu)$ for all $u, v, w \in X^*$,
(C4) $\lambda(uvw) = \lambda(vwu)$ for all $u, v, w \in X^*$,
(C5) $\lambda(uvw) = \lambda(wuv)$ for all $u, v, w \in X^*$.

A fuzzy language satisfying (C) was said to be commutative, and conditions (C1)–(C5) were proposed as generalizations of (C). The conditions should hold also when some of the subwords are empty. For example, (C4) yields $\lambda(uv) = \lambda(vu)$ for $w = \varepsilon$.

We underlined adjacent subwords whose concatenation is viewed as one subword.

Proposition 4.1. (1) *Conditions (C), (C1), (C2) and (C3) are equivalent.*
(2) *Conditions (C4) and (C5) are equivalent.*

Proof. Let $\lambda : X^* \to [0,1]$ be a fuzzy language and $s, t, u, v, w \in X^*$.

(C) ⇒ (C1) : $\lambda(uvw) = \lambda(\varepsilon uvw) = \lambda(\varepsilon vuw) = \lambda(vuw)$.
(C1) ⇒ (C2) : $\lambda(uvw) = \lambda(\underline{uv}w\varepsilon) = \lambda(wuv\varepsilon) = \lambda(wuv) = \lambda(uwv)$.
(C2) ⇒ (C3) : $\lambda(uvw) = \lambda(uwv) = \lambda(\varepsilon u\underline{wv}) = \lambda(\varepsilon wvu) = \lambda(wvu)$.
(C3) ⇒ (C) : $\lambda(\underline{su}vt) = \lambda(t\underline{vsu}) = \lambda(uv\underline{st}) = \lambda(s\underline{tvu}) = \lambda(\underline{ut}vs) = \lambda(svut)$.
Thus we have (C) ⇒ (C1) ⇒ (C2) ⇒ (C3) ⇒ (C), which proves (1).
Similarly, (2) is proved by
(C4) ⇒ (C5): $\lambda(uvw) = \lambda(vwu) = \lambda(wuv)$, and
(C5) ⇒ (C4): $\lambda(uvw) = \lambda(wuv) = \lambda(vwu)$. □

Hence we are left with the following two families of regular fuzzy languages:

(a) the family $FCom = \{FCom(X)\}_X$ of commutative regular fuzzy languages defined by any one of the conditions (C), (C1), (C2) or (C3),

(b) the family $FRot = \{FRot(X)\}_X$ of rotation invariant regular fuzzy languages defined by (C4) or equivalently by (C5).

Proposition 4.2. $FCom \subset FRot$.

Proof. To prove the inclusion $FCom \subseteq FRot$, we show that (C3) implies (C4): if $\lambda : X^* \to [0,1]$ satisfies (C3), then for all $u, v, w \in X^*$,

$$\lambda(uvw) = \lambda(u\underline{v}\underline{w}\varepsilon) = \lambda(\varepsilon vwu) = \lambda(vwu),$$

i.e., λ satisfies (C4). To prove that the inclusion is proper, let $X = \{x, y, z\}$ and $\lambda = \{xyz/1, yzx/1, zxy/1\}$. Then λ satisfies (C4) but it does not satisfy (C3) because $\lambda(xyz) = 1$ while $\lambda(zyx) = 0$. $\square$

Remark 4.3. In terms of the notation of [3], the above results are summarized by

$$\mathbf{CFL} = \mathbf{P_1IF} = \mathbf{P_2IF} = \mathbf{P_3IF} \subset \mathbf{P_4IF} = \mathbf{P_5IF}.$$

5. Families of languages defined by word identities

The varieties of fuzzy languages defined in [21] correspond bijectively to Eilenberg's [11] $\star$-varieties of crisp languages. We shall now introduce a general scheme for defining similarly linked pairs of families of regular fuzzy languages and regular crisp languages that are not necessarily varieties.

In what follows, $\Xi = \{\xi, \pi\} \cup \Xi_0 = \{\xi, \pi, \mathrm{s}, \mathrm{t}, \mathrm{u}, \mathrm{v}, \mathrm{w}, \ldots\}$, is a countably infinite set of (word) variables that range over the words over the alphabet considered. Here $\Xi_0 = \{\mathrm{s}, \mathrm{t}, \mathrm{u}, \mathrm{v}, \mathrm{w}, \ldots\}$ is the set variables ordinarily used, while the two variables ξ and π are singled out for a special role. A (word) identity is an expression

$$\text{(I)} \qquad \mathbf{u} \approx \mathbf{v},$$

where $\mathbf{u}, \mathbf{v} \in \Xi^*$. A language $L \subseteq X^*$ satisfies (I), if

$$\mathbf{u}\alpha \in L \Leftrightarrow \mathbf{v}\alpha \in L,$$

for every homomorphism $\alpha : \Xi^* \to X^*$, and a fuzzy language $\lambda : X^* \to [0,1]$ satisfies the identity (I), if

$$\lambda(\mathbf{u}\alpha) = \lambda(\mathbf{v}\alpha)$$

for every homomorphism $\alpha : \Xi^* \to X^*$. Let $\mathcal{L}_I = \{\mathcal{L}_I(X)\}_X$ be the family of regular languages satisfying (I), and let $\mathcal{F}_I = \{\mathcal{F}_I(X)\}_X$ be the family of regular fuzzy languages satisfying (I).

Any homomorphism $\alpha : \Xi^* \to X^*$ assigns a word in X^* to each word variable and α is determined by these words. Hence, in concrete cases it is convenient to express the satisfaction condition in terms of words instead of homomorphisms. For example, the identity $\mathrm{uwu} \approx \mathrm{vwv}$, where $\mathrm{u}, \mathrm{v}, \mathrm{w} \in \Xi$, is satisfied by a fuzzy language $\lambda : X^* \to [0,1]$, if $\lambda(uwu) = \lambda(vwv)$, for all words $u, v, w \in X^*$.

Proposition 5.1. *Let* (I) $\mathbf{u} \approx \mathbf{v}$ *be a word identity. If* $K, L \in \mathcal{L}_I(X)$*, then*

(1) $\overline{L} := \{w \in X^* \mid w \notin L\}$, $K \cup L, K \cap L \in \mathcal{L}_I(X)$,

(2) $L\varphi^{-1} \in \mathcal{L}_I(Y)$ *for every homomorphism* $\varphi : Y^* \to X^*$.

Proof. It is well known that these operations preserve regularity. For any homomorphism $\alpha : \Xi^* \to Y^*$,

$$\mathbf{u}\alpha \in L\varphi^{-1} \Leftrightarrow \mathbf{u}(\alpha\varphi) \in L \Leftrightarrow \mathbf{v}(\alpha\varphi) \in L \Leftrightarrow \mathbf{v}\alpha \in L\varphi^{-1},$$

i.e., $L\varphi^{-1} \in \mathcal{L}_I(Y)$. The statements in (1) have equally simple proofs. $\square$

Proposition 5.2. *Let* (I) $\mathbf{u} \approx \mathbf{v}$ *be a word identity. If* $\varkappa, \lambda \in \mathcal{F}_I(X)$*, then*
(1) $\overline{\lambda}, \varkappa \cup \lambda, \varkappa \cap \lambda \in \mathcal{F}_I(X)$,
(2) $c\lambda, \lambda_{[c)} \in \mathcal{F}_I(X)$*, for every* $c \in [0, 1]$,
(3) $\varphi^{-1}(\lambda) \in \mathcal{F}_I(Y)$*, for every homomorphism* $\varphi : Y^* \to X^*$.

Proof. By Proposition 3.2, all these operations preserve recognizability. Moreover, it is easy to verify that the resulting fuzzy languages satisfy (I). For example, for any homomorphism $\alpha : \Xi^* \to X^*$,

$$\lambda_{[c)}(\mathbf{u}\alpha) = 1 \Leftrightarrow \lambda(\mathbf{u}\alpha) \geq c \Leftrightarrow \lambda(\mathbf{v}\alpha) \geq c \Leftrightarrow \lambda_{[c)}(\mathbf{v}\alpha) = 1,$$

which shows that $\lambda_{[c)} \in \mathcal{F}_I$. □

Since *Rec* and *FRec* are defined by any trivial identity u $\approx$ u (u $\in \Xi$), the following results generalize Lemma 3.3 and Proposition 3.4.

Lemma 5.3. *Let* (I) $\mathbf{u} \approx \mathbf{v}$ *be a word identity.*
(1) *A language* $L \subseteq X^*$ *satisfies* (I) *if and only if* L^χ *satisfies* (I)*. In particular,* $L \in \mathcal{L}_I(X)$ *if and only if* $L^\chi \in \mathcal{F}_I(X)$.
(2) *If* $\lambda \in \mathcal{F}_I(X)$*, then* $\mathrm{supp}(\lambda) \in \mathcal{L}_I(X)$.

Proof. The first part of (1) follows from the fact that, for any $w \in X^*$, $L^\chi(w) = 1$ if and only if $w \in L$. The second part of (1) follows then from Lemma 3.3. Also (2) has a very simple proof. □

Proposition 5.4. *Let* (I) $\mathbf{u} \approx \mathbf{v}$ *be a word identity. A fuzzy language* $\lambda : X^* \to [0, 1]$ *belongs to* $\mathcal{F}_I(X)$ *if and only if* $\lambda = c_1 L_1^\chi \cup \ldots \cup c_n L_n^\chi$ *for some* $n \geq 1$, $c_1, \ldots, c_n \in [0, 1]$*, and* $L_1, \ldots, L_n \in \mathcal{L}_I(X)$*. Moreover, if* $\lambda \in \mathcal{F}_I(X)$*, then the constants* c_i *may be chosen to be pairwise distinct and the languages* L_i *to be pairwise disjoint.*

Proof. If $\lambda = c_1 L_1^\chi \cup \ldots \cup c_n L_n^\chi$ as in the proposition, then $\lambda \in \mathcal{F}_I(X)$ by Lemma 5.3 and Proposition 5.2.

Assume then that $\lambda \in \mathcal{F}_I(X)$, and consider a minimal FDFR $\mathbf{F} = (A, X, \delta, a_0, \omega)$ recognizing λ. Similarly as in the proof of Proposition 3.4, let $\omega(A) = \{c_1, \ldots, c_n\}$, and for each $i \in \mathbf{n}$, let $\mathbf{A}_i = (A, X, \delta, a_0, \omega^{-1}(c_i))$ and $L_i := L(\mathbf{A}_i)$. We claim that $\lambda = c_1 L_1^\chi \cup \ldots \cup c_n L_n^\chi$ is a representation of the required kind. For any $i \in \mathbf{n}$ and any homomorphism $\alpha : \Xi^* \to X^*$,

$$\begin{aligned} \mathbf{u}\alpha \in L_i \Leftrightarrow a_0(\mathbf{u}\alpha) \in \omega^{-1}(c_i) \Leftrightarrow \lambda(\mathbf{u}\alpha) = c_i \Leftrightarrow \lambda(\mathbf{v}\alpha) = c_i \Leftrightarrow a_0(\mathbf{v}\alpha) \in \omega^{-1}(c_i) \\ \Leftrightarrow \mathbf{v}\alpha \in L_i \end{aligned}$$

and hence $L_1, \ldots, L_n$ satisfy (I). Of course, $L_i \in Rec(X)$, for every $i \in \mathbf{n}$ and $L_i \cap L_j = \varnothing$, for $i \neq j$. □

6. Word identities and varieties

In Eilenberg's variety theory [11, 1, 22], a family of regular languages $\mathcal{L} = \{\mathcal{L}(X)\}_X$ is called a $\star$-variety, if for all alphabets X and Y,
(i) $\varnothing \neq \mathcal{L}(X) \subseteq Rec(X)$ and if $K, L \in \mathcal{L}(X)$, then $K \cap L, \overline{L} \in \mathcal{L}(X)$,
(ii) if $L \in \mathcal{L}(X)$, then $\mathcal{L}(X)$ contains also the quotient languages $w^{-1}L := \{u \in X^* \mid wu \in L\}$ and $Lw^{-1} := \{u \in X^* \mid wu \in L\}$ for every $w \in X^*$,
(iii) if $L \in \mathcal{L}(Y)$, then $L\varphi^{-1} \in \mathcal{L}(X)$ for every homomorphism $\varphi : X^* \to Y^*$.

A non-empty class $\mathbf{M}$ of finite monoids is a variety of finite monoids (VFM) or a pseudovariety, if it is closed under submonoids, homomorphic images and finite direct products.

For any $\star$-variety $\mathcal{L}$, let $\mathcal{L}^m$ be the VFM generated by the syntactic monoids $\mathrm{SM}(L)$ with $L \in \mathcal{L}(X)$ for some X, and for any VFM $\mathbf{M}$, let $\mathbf{M}^\ell = \{\mathbf{M}^\ell(X)\}_X$ with $\mathbf{M}^\ell(X) := \{L \subseteq X^* \mid \mathrm{SM}(L) \in \mathbf{M}\}$. By Eilenberg's Variety Theorem [11, 1, 22], the maps $\mathcal{L} \mapsto \mathcal{L}^m$ and $\mathbf{M} \mapsto \mathbf{M}^\ell$ are mutually inverse isomorphisms between the lattice of $\star$-varieties and the lattice of VFMs.

We call a family of fuzzy languages $\mathcal{F} = \{\mathcal{F}(X)\}_X$ is a fuzzy $\star$-variety, if for all alphabets X and Y,

(i) $\varnothing \neq \mathcal{F}(X) \subseteq FRec(X)$ and if $\varkappa, \lambda \in \mathcal{F}(X)$, then $\varkappa \cup \lambda, \varkappa \cap \lambda, \overline{\lambda} \in \mathcal{F}(X)$,

(ii) if $\lambda \in \mathcal{F}(X)$, then $c\lambda, \lambda_{[c]} \in \mathcal{F}(X)$ for every $c \in [0,1]$,

(iii) if $\lambda \in \mathcal{F}(X)$, then $\varkappa^{-1}\lambda, \lambda\varkappa^{-1} \in \mathcal{F}(X)$ for every $\varkappa : X^* \to [0,1]$,

(iv) if $\lambda \in \mathcal{F}(Y)$, then $\varphi^{-1}(\lambda) \in \mathcal{F}(X)$ for any homomorphism $\varphi : X^* \to Y^*$.

The fuzzy $\star$-varieties are the 'varieties of fuzzy languages' of [21] but we excluded the empty variety, as we also excluded the empty $\star$-variety and the empty VFM. By Proposition 3.2, $FRec$ is the greatest fuzzy $\star$-variety.

With any fuzzy $\star$-variety $\mathcal{F}$ we associate the family of regular languages $\mathcal{F}^\ell$, where $\mathcal{F}^\ell(X) := \{L \subseteq X^* \mid L^\chi \in \mathcal{F}(X)\}$ and the VFM $\mathcal{F}^m$ generated by the syntactic monoids $\mathrm{SM}(\lambda)$ with $\lambda \in \mathcal{F}(X)$ for some X. For any VFM $\mathbf{M}$, let $\mathbf{M}^f = \{\mathbf{M}^f(X)\}_X$ be the family of regular fuzzy languages, where

$$\mathbf{M}^f(X) := \{\lambda \in FRec(X) \mid \mathrm{SM}(\lambda) \in \mathbf{M}\}.$$

Furthermore, for any $\star$-variety $\mathcal{L} = \{\mathcal{L}(X)\}_X$, let $\mathcal{L}^f = \{\mathcal{L}^f(X)\}_X$ be the family of fuzzy regular languages, where

$$\mathcal{L}^f(X) = \{c_1 L_1^\chi \cup \cdots \cup c_n L_n^\chi \mid n \geq 1, c_1, \ldots, c_n \in [0,1], L_1, \ldots, L_n \in \mathcal{L}(X)\},$$

The main results of Petković [21] can be summarized as follows.

(1) The maps $\mathcal{F} \mapsto \mathcal{F}^\ell$ and $\mathcal{L} \mapsto \mathcal{L}^f$ are mutually inverse isomorphisms between the lattice of fuzzy $\star$-varieties and that of $\star$-varieties. The maps $\mathcal{F} \mapsto \mathcal{F}^m$ and $\mathbf{M} \mapsto \mathbf{M}^f$ are mutually inverse isomorphisms between the lattice of fuzzy $\star$-varieties and that of VFMs.

(2) These maps and Eilenberg's maps $\mathcal{L} \mapsto \mathcal{L}^m$ and $\mathbf{M} \mapsto \mathbf{M}^\ell$ can be composed in the natural way. For example, $\mathcal{F}^{\ell m} := (\mathcal{F}^\ell)^m = \mathcal{F}^m$ for every fuzzy $\star$-variety $\mathcal{F}$.

A flanked (word) identity is an expression

$$\text{(J)} \qquad \xi\mathbf{u}\pi \approx \xi\mathbf{v}\pi,$$

where $\mathbf{u}, \mathbf{v} \in \Xi_0^*$. The variables ξ and π will be used only this way as border symbols in flanked identities. A word identity $\mathbf{u} \approx \mathbf{v}$ in which $\mathbf{u}, \mathbf{v} \in \Xi_0^*$ is said to be unflanked (although it actually may be equivalent to a flanked identity). Thus there is a natural correspondence between the flanked identities $\xi\mathbf{u}\pi \approx \xi\mathbf{v}\pi$ and the unflanked identities $\mathbf{u} \approx \mathbf{v}$.

Proposition 6.1. *Let* (J) $\xi\mathbf{u}\pi \approx \xi\mathbf{v}\pi$ *be a flanked word identity.*

(1) *If* $L \in \mathcal{L}_J(X)$, *then* $w^{-1}L, Lw^{-1} \in \mathcal{L}_J(X)$, *for every* $w \in X^*$.

(2) *If* $\lambda \in \mathcal{F}_J(X)$, *then* $\varkappa^{-1}\lambda, \lambda\varkappa^{-1} \in \mathcal{F}_J(X)$, *for every* $\varkappa : X^* \to [0,1]$.

Proof. All four claims have similar proofs. Let us show that $\varkappa^{-1}\lambda \in \mathcal{F}_J(X)$. For any homomorphism $\alpha : \Xi^* \to X^*$,

$$\begin{aligned}(\varkappa^{-1}\lambda)((\xi\mathbf{u}\pi)\alpha) &= \bigvee\{\varkappa(w) \wedge \lambda(w(\xi\mathbf{u}\pi)\alpha) \mid w \in X^*\} \\ &= \bigvee\{\varkappa(w) \wedge \lambda((\xi\mathbf{u}\pi)\beta) \mid w \in X^*\} \\ &= \bigvee\{\varkappa(w) \wedge \lambda((\xi\mathbf{v}\pi)\beta) \mid w \in X^*\} \\ &= \bigvee\{\varkappa(w) \wedge \lambda(w(\xi\mathbf{v}\pi)\alpha) \mid w \in X^*\} \\ &= (\varkappa^{-1}\lambda)((\xi\mathbf{v}\pi)\alpha),\end{aligned}$$

where $\beta : \Xi^* \to X^*$ is the homomorphism which differs from α only in that $\xi\beta = w(\xi\alpha)$. Note that it is essential that ξ does not appear in $\mathbf{u}\pi$ or $\mathbf{v}\pi$. □

Theorem 6.2. *If* (J) $\xi\mathbf{u}\pi \approx \xi\mathbf{v}\pi$ *is a flanked word identity, then* $\mathcal{L}_J$ *is a* $\star$*-variety and* $\mathcal{F}_J$ *is the corresponding fuzzy* $\star$*-variety.*

Proof. That $\mathcal{L}_J$ is a $\star$-variety and $\mathcal{F}_J$ is a fuzzy $\star$-variety follows from Propositions 5.1, 5.2 and 6.1. Proposition 5.4 means that $\mathcal{L}_J^f = \mathcal{F}_J$, and hence we also have $\mathcal{F}_J^\ell = \mathcal{L}_J$ by the variety theory. □

Any word identity (I) $\mathbf{u} \approx \mathbf{v}$ may also be regarded as a monoid identity: we say that a monoid M satisfies (I), if $\mathbf{u}\alpha = \mathbf{v}\alpha$, for every homomorphism $\alpha : \Xi^* \to M$. Let $\mathbf{M}_I$ denote the class of all finite monoids satisfying (I). Since $\mathbf{M}_I$ is defined by an identity, it is a VFM.

Lemma 6.3. *A language* $L \subseteq X^*$ *satisfies a flanked word identity* (J) $\xi\mathbf{u}\pi \approx \xi\mathbf{v}\pi$ *if and only if its syntactic monoid* SM(L) *satisfies the unflanked word identity* (I) $\mathbf{u} \approx \mathbf{v}$*. Similarly, a fuzzy language* $\lambda : X^* \to [0, 1]$ *satisfies* (J) *if and only if* SM(λ) *satisfies* (I).

Proof. Let $\mathbf{u} = \mathrm{u}_1 \ldots \mathrm{u}_k$ and $\mathbf{v} = \mathrm{u}_{k+1} \ldots \mathrm{u}_n$, where $0 \le k \le n$ and $\mathrm{u}_1, \ldots, \mathrm{u}_n \in \Xi_0$. Assume first that L satisfies (J), and let $\alpha : \Xi^* \to \mathrm{SM}(L)$ be any homomorphism. For each $i \in \mathbf{n}$, choose a word $w_i \in X^*$ such that $\mathrm{u}_i\alpha = [w_i]$. If $\mathrm{u}_i = \mathrm{u}_j$ for some $i, j \in \mathbf{n}$, then let $w_i = w_j$. For any words $s, t \in X^*$, there is a homomorphism $\beta : \Xi^* \to X^*$ such that $\mathrm{u}_i\beta = w_i$ for each $i \in \mathbf{n}$, $\xi\beta = s$ and $\pi\beta = t$. Then

$$s(\mathbf{u}\beta)t \in L \;\Leftrightarrow\; (\xi\mathbf{u}\pi)\beta \in L \;\Leftrightarrow\; (\xi\mathbf{v}\pi)\beta \in L \;\Leftrightarrow\; s(\mathbf{v}\beta)t \in L.$$

Since this holds for all $s, t \in X^*$, while $\mathbf{u}\beta$ and $\mathbf{v}\beta$ do not depend on s and t, this means that $\mathbf{u}\beta \,\sigma_L\, \mathbf{v}\beta$. Hence,

$$\mathbf{u}\alpha = [w_1] \cdots [w_k] = [\mathbf{u}\beta] = [\mathbf{v}\beta] = [w_{k+1}] \cdots [w_n] = \mathbf{v}\alpha,$$

which shows that SM(L) satisfies (I) $\mathbf{u} \approx \mathbf{v}$.

Assume then that SM(L) satisfies (I), and consider any homomorphism $\alpha : \Xi^* \to X^*$. Since $\alpha\varphi_L : \Xi^* \to \mathrm{SM}(L)$ is a homomorphism, we have $\mathbf{u}\alpha\varphi_L = \mathbf{v}\alpha\varphi_L$, which means that $\mathbf{u}\alpha \,\sigma_L\, \mathbf{v}\alpha$. This implies that

$$(\xi\mathbf{u}\pi)\alpha \in L \;\Leftrightarrow\; (\xi\alpha)\mathbf{u}\alpha(\pi\alpha) \in L \;\Leftrightarrow\; (\xi\alpha)\mathbf{v}\alpha(\pi\alpha) \in L \;\Leftrightarrow\; (\xi\mathbf{v}\pi)\alpha \in L.$$

Thus L satisfies (J). The statement concerning λ has a similar proof. □

The following result is obtained by comparing Lemma 6.3 with the definitions of the $\star$-variety $\mathbf{M}^\ell$ and the fuzzy $\star$-variety $\mathbf{M}^f$ corresponding to a given VFM $\mathbf{M}$.

Proposition 6.4. *Let* (J) $\xi\mathbf{u}\pi \approx \xi\mathbf{v}\pi$ *be a flanked word identity and let* (I) $\mathbf{u} \approx \mathbf{v}$ *be the corresponding unflanked word identity. Then* $\mathcal{F}_J$ *is the fuzzy* $\star$*-variety and* $\mathcal{L}_J$ *the* $\star$*-variety corresponding to the VFM* $\mathbf{M}_I$*.*

The families of crisp and fuzzy regular languages defined by unflanked word identities $\mathbf{u} \approx \mathbf{v}$ are not always varieties. In such cases, the following proposition is of interest. In its proof, we use the following notation and observation.

For any fuzzy language $\lambda : X^* \to [0,1]$ and word $s \in X^*$, define $s^{-1}\lambda : X^* \to [0,1]$ and $\lambda s^{-1} : X^* \to [0,1]$ by $(s^{-1}\lambda)(w) = \lambda(sw)$ and $(\lambda s^{-1})(w) = \lambda(ws)$ $(w \in X^*)$. Then clearly, $s^{-1}\lambda = \varkappa^{-1}\lambda$ and $\lambda s^{-1} = \lambda\varkappa^{-1}$ for $\varkappa = \{s/1\}$. Thus, if $\lambda \in \mathcal{F}(X)$ for a fuzzy $\star$-variety $\mathcal{F}$, then $s^{-1}\lambda, \lambda s^{-1} \in \mathcal{F}(X)$ for any $s \in X^*$. So

$$s^{-1}\lambda t^{-1} : X^* \to [0,1],\ w \mapsto \lambda(swt)$$

belongs to $\mathcal{F}(X)$ for all $s, t \in X^*$, because $s^{-1}\lambda t^{-1} = (s^{-1}\lambda)t^{-1}$.

Proposition 6.5. *Let* (I) $\mathbf{u} \approx \mathbf{v}$ *be a unflanked word identity and let* (J) $\xi\mathbf{u}\pi \approx \xi\mathbf{v}\pi$ *be the corresponding flanked word identity. Then*

(1) $\mathcal{L}_J$ *is the greatest* $\star$*-variety contained in* $\mathcal{L}_I$*,*

(2) $\mathcal{F}_J$ *is the greatest fuzzy* $\star$*-variety contained in* $\mathcal{F}_I$*.*

Proof. The following three parts (a)-(c) constitute a proof for (2).

(a) $\mathcal{F}_J \subseteq \mathcal{F}_I$. Let $\lambda \in \mathcal{F}_J(X)$ and $\alpha_0 : \Xi_0^* \to X^*$ be any homomorphism. If we extend α_0 to a homomorphism $\alpha : \Xi^* \to X^*$ by $\xi\alpha = \pi\alpha = \varepsilon$, then

$$\lambda(\mathbf{u}\alpha_0) = \lambda((\xi\mathbf{u}\pi)\alpha) = \lambda((\xi\mathbf{v}\pi)\alpha) = \lambda(\mathbf{v}\alpha_0),$$

which shows that $\lambda \in \mathcal{F}_I(X)$.

(b) $\mathcal{F}_J$ is a fuzzy $\star$-variety by Theorem 6.2.

(c) If $\mathcal{F}$ is a fuzzy $\star$-variety contained in $\mathcal{F}_I$, then $\mathcal{F} \subseteq \mathcal{F}_J$. To prove this, consider any $\lambda \in \mathcal{F}(X)$ and any homomorphism $\alpha : \Xi^* \to X^*$ and let $\xi\alpha = s$ and $\pi\alpha = t$. Since $\mathcal{F}$ is a fuzzy $\star$-variety contained in $\mathcal{F}_I$, $s^{-1}\lambda t^{-1}$ satisfies (I). Thus

$$\begin{aligned}\lambda((\xi\mathbf{u}\pi)\alpha) &= \lambda(s(\mathbf{u}\alpha)t) = (s^{-1}\lambda t^{-1})(\mathbf{u}\alpha) = (s^{-1}\lambda t^{-1})(\mathbf{v}\alpha) = \lambda(s(\mathbf{v}\alpha)t)\\ &= \lambda((\xi\mathbf{v}\pi)\alpha).\end{aligned}$$

So $\lambda \in \mathcal{F}_J(X)$.

Statement (1) can be proved in a similar manner. □

7. Sequences of word identities

The definitions and results of the previous two sections can easily be extended to concern sets of word identities; a monoid, language or fuzzy language is said to satisfy a set (E) of word identities, if it satisfies every identity in (E). However, this extension would still not yield all $\star$-varieties, fuzzy $\star$-varieties and VFMs. Instead we make use of a well-known theorem by Eilenberg and Schützenberger [12]. Let

$$\text{(S)} \qquad \langle \mathbf{u}_n \approx \mathbf{v}_n \rangle_{n \geq 1} \;=\; \mathbf{u}_1 \approx \mathbf{v}_1, \mathbf{u}_2 \approx \mathbf{v}_2, \mathbf{u}_3 \approx \mathbf{v}_3, \ldots$$

be a denumerable sequence of word identities ($\mathbf{u}_n, \mathbf{v}_n \in \Xi^*$). A monoid M ultimately satisfies the sequence (S), if there is a $k \geq 1$ such that M satisfies $\mathbf{u}_n \approx \mathbf{v}_n$ for every

$n \geq k$. Let $\mathbf{M}_S$ be the class of all finite monoids ultimately satisfying (S), i.e., the class ultimately defined by (S).

Theorem 7.1. (S. Eilenberg and M.P. Schützenberger [12, 11]) *A class of finite monoids is a VFM if and only if it is ultimately defined by a sequence of identities.*

Let us say that a language L ultimately satisfies the sequence (S), if there is a $k \geq 1$ such that L satisfies every $\mathbf{u}_n \approx \mathbf{v}_n$ with $n \geq k$. The family of regular languages ultimately satisfying (S) is denoted by $\mathcal{L}_S = \{\mathcal{L}_S(X)\}_X$, and $\mathcal{L}_S$ is said to be ultimately defined by (S). Similarly, a fuzzy language λ ultimately satisfies (S), if there is a $k \geq 1$ such that λ satisfies every $\mathbf{u}_n \approx \mathbf{v}_n$ with $n \geq k$, and let $\mathcal{F}_S = \{\mathcal{F}_S(X)\}_X$ be the family of regular fuzzy languages ultimately defined by (S).

Remark 7.2. Since any set of word identities (E) is countable, there exists a sequence (S) of word identities such that a language, fuzzy language or monoid satisfies (E) if and only if it ultimately satisfies (S).

With any sequence of unflanked word identities $\langle \mathbf{u}_n \approx \mathbf{v}_n \rangle_{n \geq 1}$ we associate the sequence of flanked word identities $\langle \xi \mathbf{u}_n \pi \approx \xi \mathbf{v}_n \pi \rangle_{n \geq 1}$.

Proposition 7.3. *Let* (S) $\langle \mathbf{u}_n \approx \mathbf{v}_n \rangle_{n \geq 1}$ *be a sequence of unflanked word identities and* (T) $\langle \xi \mathbf{u}_n \pi \approx \xi \mathbf{v}_n \pi \rangle_{n \geq 1}$ *be the associated sequence of flanked identities. Then*

(1) $\mathbf{M}_S$ *is a VFM,*

(2) $\mathcal{L}_T$ *is the $\star$-variety corresponding to* $\mathbf{M}_S$*, and*

(3) $\mathcal{F}_T$ *is the fuzzy $\star$-variety corresponding to* $\mathbf{M}_S$*.*

Proof. (1) $\mathbf{M}_S$ is a VFM by Theorem 7.1.

(2) We show that $\mathcal{L}_T = \mathbf{M}_S^\ell$. By Lemma 6.3, a language $L \subseteq X^*$ satisfies the flanked identity $\xi \mathbf{u}_n \pi \approx \xi \mathbf{v}_n \pi$ if and only SM(L) satisfies $\mathbf{u}_n \approx \mathbf{v}_n$ ($n \geq 1$). This implies that L ultimately satisfies (T) if and only if SM(L) ultimately satisfies (S). In other words, $L \in \mathcal{L}_T(X)$ if and only if SM(L) $\in \mathbf{M}_S$. Then $\mathcal{L}_T = \mathbf{M}_S^\ell$. A simple modification of this argument yields a proof for (3). □

Now we may characterize the fuzzy $\star$-varieties in terms of word identities.

Theorem 7.4. *A family $\mathcal{F}$ of regular fuzzy languages is a fuzzy $\star$-variety if and only if it is ultimately defined by a sequence of flanked word identities. Similarly, a family $\mathcal{L}$ of regular languages is a $\star$-variety if and only if it is ultimately defined by a sequence of flanked word identities. Moreover, if $\mathcal{F}$ is a fuzzy $\star$-variety and $\mathcal{L}$ is the corresponding $\star$-variety ($\mathcal{F}^\ell = \mathcal{L}$), then $\mathcal{F}$ and $\mathcal{L}$ can be ultimately defined by the same sequence of flanked word identities.*

Proof. If $\mathcal{F}$ is ultimately defined by a sequence $\langle \xi \mathbf{u}_n \pi \approx \xi \mathbf{v}_n \pi \rangle_{n \geq 1}$ of flanked word identities, then $\mathcal{F}$ is by Proposition 7.3 the fuzzy $\star$-variety corresponding to the VFM ultimately defined by the sequence $\langle \mathbf{u}_n \approx \mathbf{v}_n \rangle_{n \geq 1}$.

Suppose $\mathcal{F}$ is a fuzzy $\star$-variety. Then by Theorem 7.1, the VFM $\mathcal{F}^m$ is ultimately defined by a sequence $\langle \mathbf{u}_n \approx \mathbf{v}_n \rangle_{n \geq 1}$ of unflanked identities. Thus $\langle \xi \mathbf{u}_n \pi \approx \xi \mathbf{v}_n \pi \rangle_{n \geq 1}$ ultimately defines $\mathcal{F}$, by Proposition 7.3. The claim concerning $\mathcal{L}$ can be proved the same way. Finally, if $\mathcal{F}$ and $\mathcal{L}$ are varieties such that $\mathcal{F}^\ell = \mathcal{L}$, then $\mathcal{F}$ and $\mathcal{L}$ correspond to the same VFM $\mathbf{M} = \mathcal{F}^m = \mathcal{L}^m$, and both are ultimately defined by the same sequence $\langle \xi \mathbf{u}_n \pi \approx \xi \mathbf{v}_n \pi \rangle_{n \geq 1}$ of flanked identities obtained from any sequence $\langle \mathbf{u}_n \approx \mathbf{v}_n \rangle_{n \geq 1}$ of unflanked identities defining $\mathbf{M}$. □

Proposition 6.5 has the following extension to sequences of identities.

Proposition 7.5. *If* (S) $\langle \mathbf{u}_n \approx \mathbf{v}_n \rangle_{n\geq 1}$ *is a sequence of unflanked word identities and* (T) $\langle \xi\mathbf{u}_n\pi \approx \xi\mathbf{v}_n\pi \rangle_{n\geq 1}$ *is the associated sequence of flanked identities, then*
(1) $\mathcal{L}_T$ *is the greatest* $\star$*-variety contained in* $\mathcal{L}_S$*,*
(2) $\mathcal{F}_T$ *is the greatest fuzzy* $\star$*-variety contained in* $\mathcal{F}_S$*.*

Proof. Statement (1) is proved as follows in three steps.
(a) $\mathcal{L}_T \subseteq \mathcal{L}_S$. If $L \in \mathcal{L}_T(X)$, then there is a $k \geq 1$ such that L satisfies $\xi\mathbf{u}_n\pi \approx \xi\mathbf{v}_n\pi$, for every $n \geq k$. But L also satisfies $\mathbf{u}_n \approx \mathbf{v}_n$, for every $n \geq k$. Thus $L \in \mathcal{L}_S(X)$.
(b) $\mathcal{L}_T$ is a $\star$-variety, by Proposition 7.3.
(c) Let $\mathcal{L}$ be any $\star$-variety contained in $\mathcal{L}_S$. To show that $\mathcal{L} \subseteq \mathcal{L}_T$, consider any $L \in \mathcal{L}(X)$ and any homomorphism $\alpha : \Xi^* \to X^*$. Let $\xi\alpha = s$ and $\pi\alpha = t$. Since $\mathcal{L}$ is a $\star$-variety, $s^{-1}Lt^{-1}$ is in $\mathcal{L}(X)$, and as $\mathcal{L} \subseteq \mathcal{L}_S$, there is a $k \geq 1$ such that $s^{-1}Lt^{-1}$ satisfies $\mathbf{u}_n \approx \mathbf{v}_n$, for every $n \geq k$. Then,

$$(\xi\mathbf{u}_n\pi)\alpha \in L \Leftrightarrow s(\mathbf{u}_n\alpha)t \in L \Leftrightarrow \mathbf{u}_n\alpha \in s^{-1}Lt^{-1} \Leftrightarrow \mathbf{v}_n\alpha \in s^{-1}Lt^{-1} \Leftrightarrow s(\mathbf{v}_n\alpha)t \in L$$
$$\Leftrightarrow (\xi\mathbf{v}_n\pi)\alpha \in L,$$

for every $n \geq k$, which shows that $L \in \mathcal{L}_T(X)$.
Statement (2) has a similar proof. □

8. The varieties of commutative languages

It is clear that condition (C), by which the commutativity of fuzzy languages was defined, is equivalent to the flanked word identity

$$\text{(JC)} \qquad \xi \mathrm{uv}\pi \approx \xi \mathrm{vu}\pi$$

$(\xi, \mathrm{u}, \mathrm{v}, \pi \in \Xi)$, and hence $FCom = \mathcal{F}_{JC}$. Furthermore, a monoid M satisfies the word identity (IC) $\mathrm{uv} \approx \mathrm{vu}$ if and only if it is commutative. Thus $\mathbf{M}_{IC}$ is the VFM **Com** of all finite commutative monoids.

Let us recall that a language $L \subseteq X^*$ is commutative, if for any $n \geq 0$ and $x_1, x_2, \ldots, x_n \in X$, $x_1x_2\ldots x_n \in L$ implies $x_{i_1}x_{i_2}\ldots x_{i_n} \in L$, for every permutation $i_1, i_2, \ldots, i_n$ of $\mathbf{n}$. Let $Com = \{Com(X)\}_X$ be the family of commutative regular languages. It is clear that any commutative language satisfies (JC). On the other hand, if a language $L \subseteq X^*$ satisfies (JC), then it satisfies also the condition

$$\text{(LC)} \qquad (\forall s, t \in X^*)(\forall x, y \in X)(sxyt \in L \Leftrightarrow syxt \in L).$$

Since any permutation on $\mathbf{n}$ can be represented as a composition of the transpositions $(1\,2), (2\,3), \ldots, (n-1\,n)$ (cf. [8], for example), condition (LC) implies that L is commutative. We may conclude that $Com = \mathcal{L}_{JC}$.

By applying Proposition 6.4 to the word identities (JC) and (IC), we get the following (known) facts.

Proposition 8.1. *FCom is the fuzzy* $\star$*-variety and Com the* $\star$*-variety corresponding to the VFM* **Com***, i.e.,* $FCom = \mathbf{Com}^f$ *and* $Com = \mathbf{Com}^\ell$*.*

Moreover, the following connections between Com and $FCom$ are obtained as special cases from Lemma 5.3 and Proposition 5.4.

Proposition 8.2. (1) *A language* $L \subseteq X^*$ *is commutative if and only if* L^χ *is commutative, and* $L \in Com(X)$ *if and only if* $L^\chi \in FCom(X)$

(2) *If a fuzzy language* $\lambda : X^* \to [0,1]$ *is commutative, then so is* $\mathrm{supp}(\lambda)$ *and if* $\lambda \in FCom(X)$*, then* $\mathrm{supp}(\lambda) \in Com(X)$*.*

(3) *A regular fuzzy language* $\lambda : X^* \to [0,1]$ *is commutative if and only if it can be expressed in the form* $\lambda = c_1 L_1^\chi \cup \ldots \cup c_n L_n^\chi$*, where* $n \geq 1$*,* $c_1, \ldots, c_n \in [0,1]$ *and* $L_1, \ldots, L_n \in Com(X)$*. Moreover, if* $\lambda \in FCom(X)$*, the constants* c_i *may be chosen to be pairwise distinct and the languages* L_i *to be pairwise disjoint.*

Let us now consider FDFRs recognizing the regular fuzzy commutative languages.

A DFR $\mathbf{A} = (A, X, \delta, a_0, F)$ is commutative, if $auv = avu$, for all $a \in A$ and $u, v \in X^*$. It is easy to see that a regular language L is commutative if and only if it is recognized by a commutative DFR or if and only if its syntactic monoid $\mathrm{SM}(L)$ is commutative. These facts can be found already, in some form, in [14] (cf. also [13]). Shyr [28] called a DFR $\mathbf{A} = (A, X, \delta, a_0, F)$ quasi-abelian, if $a_0uv = a_0vu$, for all $u, v \in X^*$, and he showed that the languages recognized by quasi-abelian DFRs are exactly the commutative regular languages. This follows also from Lemma 8.3 below.

Let us call a FDFR $\mathbf{F} = (A, X, \delta, a_0, \omega)$ commutative, if $auv = avu$, for all $a \in A$ and $u, v \in X^*$ and quasi-abelian, if $a_0uv = a_0vu$, for all $u, v \in X^*$.

Lemma 8.3. *Any commutative FDFR (DFR) is quasi-abelian and every connected quasi-abelian FDFR (DFR) is commutative.*

Proof. The proofs for FDFRs and for DFRs are identical. The first statement holds trivially. Let $\mathbf{F} = (A, X, \delta, a_0, \omega)$ be a connected quasi-abelian FDFR and consider any $a \in A$ and $u, v \in X^*$. Since $\mathbf{F}$ is connected, $a = a_0w$, for some $w \in X^*$. Then $auv = a_0wuv = a_0u\underline{wv} = a_0wvu = avu$. ☐

The following proposition encompasses the fuzzy forms of the classical facts noted above and those appearing in Propositions 7.10 and 7.12 of [28].

Proposition 8.4. *For any regular fuzzy language* $\lambda : X^* \to [0,1]$*, the following conditions are equivalent to each other:*

(1) λ *is commutative,*
(2) $\mathbf{F}_\lambda$ *is a commutative FDFR,*
(3) λ *is recognized by a commutative FDFR,*
(4) $\mathbf{F}_\lambda$ *is a quasi-abelian FDFR,*
(5) λ *is recognized by a quasi-abelian FDFR,*
(6) $\mathrm{SM}(\lambda)$ *is a commutative monoid,*
(7) $\mathrm{TM}(\mathbf{F}_\lambda)$ *is a commutative monoid.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from the definition of δ_λ and the fact that $\mathbf{F}_\lambda$ recognizes λ. Firstly, if λ is commutative, then

$$\delta_\lambda([s], uv) = [suv] = [svu] = \delta_\lambda([s], vu)$$

for all $s, u, v \in X^*$. Conversely, if $\mathbf{F}_\lambda$ is commutative, then

$$\begin{aligned}\lambda(suvt) &= \omega_\lambda(\delta_\lambda([\varepsilon], suvt) = \omega_\lambda(\delta_\lambda([s], uvt) = \omega_\lambda(\delta_\lambda([s], vut) = \omega_\lambda(\delta_\lambda([\varepsilon], svut) \\ &= \lambda(svut),\end{aligned}$$

for all $s, t, u, v \in X^*$, i.e., λ is commutative.

The equivalences (2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5) hold by Lemma 8.3. The implication (2) $\Rightarrow$ (3) is obvious because $\mathbf{F}_\lambda$ recognizes λ, and the converse (3) $\Rightarrow$ (2) holds because commutativity is preserved when a FDFR is minimized. The equivalence (1) $\Leftrightarrow$ (6) follows from Proposition 8.1. Finally, (6) $\Leftrightarrow$ (7) holds because the two monoids are isomorphic. □

As the commutativity of an FDFR can be decided, we get the following result.

Corollary 8.5. *The commutativity of regular fuzzy languages is decidable.*

9. The families of rotation invariant languages

Clearly, $FRot = \mathcal{F}_{IR}$ for the word identity

$$\text{(IR)} \qquad \mathrm{uvw} \approx \mathrm{vwu}$$

$(\mathrm{u}, \mathrm{v}, \mathrm{w} \in \Xi_0)$. Let us call a language $L \subseteq X^*$ rotation invariant, if it satisfies (IR), i.e., for any words $u, v, w \in X^*$, $uvw \in L$ if and only if $vwu \in L$. The family $\mathcal{L}_{IR}$ is denoted by $Rot = \{Rot(X)\}_X$. Note that (IR) is not flanked and hence Propositions 6.1 and 6.2 do not apply to Rot and $FRot$.

Proposition 9.1. *If $K, L \in Rot(X)$, then*
(1) $\overline{L}, K \cup L, K \cap L \in Rot(X)$,
(2) $L\varphi^{-1} \in Rot(Y)$, *for every homomorphism* $\varphi : Y^* \to X^*$.
On the other hand, $w^{-1}L$ and Lw^{-1} are not necessarily in $Rot(X)$ for every $w \in X^*$.

Proof. Statements (1) and (2) follow from Proposition 5.1, since $Rot = \mathcal{L}_{IR}$.

To see that Rot is not closed under the quotient operations, consider the language $L = \{xyz, yzx, zxy\}$ over $X = \{x, y, z\}$. It is clear that $L \in Rot(X)$, but $x^{-1}L = Lx^{-1} = \{yz\}$ is not rotation invariant. □

Proposition 9.2. *If $\varkappa, \lambda \in FRot(X)$, then*
(1) $\overline{\lambda}, \varkappa \cup \lambda, \varkappa \cap \lambda \in FRot(X)$,
(2) $c\lambda, \lambda_{[c)} \in FRot(X)$ *for every* $c \in [0, 1]$,
(3) $\varphi^{-1}(\lambda) \in FRot(Y)$ *for every homomorphism* $\varphi : Y^* \to X^*$.
On the other hand, $\eta^{-1}\lambda$ and $\lambda\eta^{-1}$ do not necessarily belong to $FRot(X)$ for every $\eta : X^* \to [0, 1]$ (even in case $\eta \in FRot(X)$ is assumed).

Proof. Statements (1)-(3) follow from Proposition 5.2 because $FRot = \mathcal{F}_{IR}$.

To see that $FRot$ is not closed under the quotient operations, consider the fuzzy language $\lambda = \{xyz/1, yzx/1, zxy/1\} \in FRot(X)$, where $X = \{x, y, z\}$. If $\eta = \{x/1\}$, then $\eta(x) = 1$, but $\eta(v) = 0$ for all $v \neq x$. This means that $(\eta^{-1}\lambda)(yz) = \eta(x) \wedge \lambda(xyz) = 1$, but $(\eta^{-1}\lambda)(w) = 0$ for any $w \neq yz$, and hence $\eta^{-1}\lambda = \{yz/1\}$. Obviously $\{yz/1\} \notin FRot(X)$. Similarly, $\lambda\eta^{-1} = \{yz/1\} \notin FRot(X)$. Note that also $\eta \in FRot(X)$. □

From Lemma 5.3 and Proposition 5.4 we get the following links between rotation invariant languages and rotation invariant fuzzy languages.

Proposition 9.3. (1) *A language* $L \subseteq X^*$ *is rotation invariant if and only if* L^χ *is rotation invariant, and* $L \in Rot(X)$ *if and only if* $L^\chi \in FRot(X)$.

(2) *If* $\lambda : X^* \to [0,1]$ *is rotation invariant, then so is* $\mathrm{supp}(\lambda)$*, and if* $\lambda \in FRot(X)$*, then* $\mathrm{supp}(\lambda) \in Rot(X)$.

(3) *A regular fuzzy language* $\lambda : X^* \to [0,1]$ *is rotation invariant if and only if* $\lambda = c_1 L_1^\chi \cup \ldots \cup c_n L_n^\chi$ *for some* $n \geq 0$, $c_1, \ldots, c_n \in [0,1]$ *and* $L_1, \ldots, L_n \in Rot(X)$. *Moreover, if* $\lambda \in FRot(X)$*, the constants* c_i *may be assumed to be distinct and the languages* L_i *to be pairwise disjoint.*

For any word $w = x_1 \ldots x_n$ ($n \geq 0$, $x_1, \ldots, x_n \in X$), we call the set

$$\mathrm{ro}(w) := \{w,\, x_2 \ldots x_n x_1,\, x_3 \ldots x_n x_1 x_2,\, \ldots,\, x_n x_1 x_2 \ldots x_{n-1}\}$$

the rotation orbit of w. Obviously, the three conditions (1) $u \in \mathrm{ro}(v)$, (2) $\mathrm{ro}(u) \cap \mathrm{ro}(v) \neq \varnothing$, and (3) $\mathrm{ro}(u) = \mathrm{ro}(v)$, are equivalent to each other for all $u, v \in X^*$. Hence, the condition

$$u \simeq_X v \;\Leftrightarrow\; \mathrm{ro}(u) = \mathrm{ro}(v) \qquad (u, v \in X^*)$$

defines an equivalence on X^* whose equivalence classes are the rotation orbits. For example, $\mathrm{ro}(xyz) = \{xyz, yzx, zxy\}$ and $\mathrm{ro}(xyxy) = \{xyxy, yxyx\}$ are two $\simeq_X$-classes for $X = \{x, y, z\}$. If $|X| > 1$, then $\simeq_X$ is not a congruence on the monoid X^*. For example, $xyx \simeq_X yxx$, but $xyxy \simeq_X yxxy$ does not hold.

The the following proposition offers some alternative descriptions of the rotation invariant crisp languages.

Proposition 9.4. *For any language* $L \subseteq X^*$*, the following conditions are equivalent to each other.*

(1) L *is rotation invariant.*
(2) $(\forall u, v, w \in X^*)(uvw \in L \Leftrightarrow wuv \in L)$.
(3) $(\forall w \in X^*)(w \in L \Rightarrow \mathrm{ro}(w) \subseteq L)$*, i.e.,* L *is saturated by* $\simeq_X$.
(4) $(\forall x \in X)(\forall w \in X^*)(xw \in L \Leftrightarrow wx \in L)$.
(5) $x^{-1}L = Lx^{-1}$ *for every* $x \in X$.

Proof. Consider any language $L \subseteq X^*$.

(1) $\Leftrightarrow$ (2): This equivalence is the crisp counterpart of Proposition 4.1 (2) and it can be verified in a similar way.

(1) $\Rightarrow$ (3): Consider any word $w = x_1 \ldots x_n$ ($x_1, \ldots, x_n \in X$). By (1),

$$w \in L \;\Rightarrow\; x_2 x_3 \ldots x_n x_1 \in L \;\Rightarrow\; x_3 \ldots x_n x_1 x_2 \in L \;\Rightarrow\; \ldots$$
$$\Rightarrow\; x_n x_1 x_2 \ldots x_{n-1} \in L,$$

which means that $w \in L$ implies $\mathrm{ro}(w) \subseteq L$.

(3) $\Rightarrow$ (4): This follows from $wx \in \mathrm{ro}(xw)$.

(4) $\Leftrightarrow$ (5): Suppose (4) holds. Then for any $w \in X^*$,

$$w \in x^{-1}L \;\Leftrightarrow\; xw \in L \;\Leftrightarrow\; wx \in L \;\Leftrightarrow\; w \in Lx^{-1},$$

which means that $x^{-1}L = Lx^{-1}$.

Suppose (5) holds. Then for any $w \in X^*$,

$$xw \in L \;\Leftrightarrow\; w \in x^{-1}L \;\Leftrightarrow\; w \in Lx^{-1} \;\Leftrightarrow\; wx \in L,$$

which means that (4) holds.

(4) $\Rightarrow$ (1): Suppose L satisfies (4) and consider any words $u, v, w \in X^*$. If $u = x_1 \dots x_n$ with $n \geq 0$ and $x_1, \dots, x_n \in X$, then we get

$$uvw \in L \Leftrightarrow x_2 \dots x_n vwx_1 \in L \Leftrightarrow x_3 \dots x_n vwx_1x_2 \in L \Leftrightarrow \dots \Leftrightarrow vwu \in L$$

by n applications of (4).

All the remaining implications follow from the ones we have verified. □

If a language L is recognized by a given DFR, then condition (4) in Proposition 9.4 is effectively testable: for any $x \in X$, the languages $x^{-1}L$ and Lx^{-1} are effectively regular and their equality can be decided.

Corollary 9.5. *The rotation invariance of regular languages is decidable.*

The following facts about fuzzy languages correspond to Proposition 9.4.

Proposition 9.6. *For any fuzzy language $\lambda : X^* \to [0, 1]$, the following conditions are equivalent to each other:*
(1) *λ is rotation invariant,*
(2) $\simeq_X \subseteq \ker(\lambda)$,
(3) *$\lambda(xw) = \lambda(wx)$, for all $x \in X$ and $w \in X^*$,*
(4) *$x^{-1}\lambda = \lambda x^{-1}$, for every $x \in X$.*

Proof. Let us again prove a set of implications from all the claimed equivalences follow. Let $\lambda : X^* \to [0, 1]$ be any fuzzy language.

(1) $\Rightarrow$ (2): For any word $v = x_1 \dots x_n$ ($x_1, \dots, x_n \in X$), (1) implies that

$$\lambda(v) = \lambda(x_2 \dots x_n x_1) = \lambda(x_3 \dots x_n x_1 x_2) = \dots = \lambda(x_n x_1 \dots x_{n-1}),$$

i.e., that $\lambda(u) = \lambda(v)$, for every $u \in \mathrm{ro}(v)$. Since $u \simeq_X v$ means that $u \in \mathrm{ro}(v)$, this shows that (1) implies (2).

(2) $\Rightarrow$ (3): This holds as $xw \simeq_X wx$, for all $x \in X$ and $w \in X^*$.

(3) $\Rightarrow$ (1): Consider any words $u, v, w \in X^*$. If $u = x_1 \dots x_n$ with $n \geq 0$ and $x_1, \dots, x_n \in X$, then we get

$$\lambda(uvw) = \lambda(x_2 \dots x_n vwx_1) = \lambda(x_3 \dots x_n vwx_1x_2) = \cdots = \lambda(vwu)$$

by n applications of (3).

(3) $\Leftrightarrow$ (4): This equivalence follows easily from the fact that $\lambda(xw) = (x^{-1}\lambda)(w)$ and $\lambda(wx) = (\lambda x^{-1})(w)$, for all $x \in X$ and $w \in X^*$. □

Proposition 9.7. *The rotation invariance of regular fuzzy languages is decidable.*

Proof. Let $\mathbf{F} = (A, X, \delta, a_0, \omega)$ be a given FDFR. We may assume that $\mathbf{F}$ is minimal. Let $\lambda_{\mathbf{F}} = c_1 L_1^{\chi} \cup \dots \cup c_n L_n^{\chi}$ as in the proof of Proposition 5.4, where for each $i \in \mathbf{n}$, $L_i = L(\mathbf{A}_i)$ for the DFR $\mathbf{A}_i = (A, X, \delta, a_0, \omega^{-1}(c_i))$. If every language L_i is rotation invariant, then $\lambda_{\mathbf{F}} \in FRot(X)$, by Propositions 9.3 and 9.2. If some L_i is not rotation invariant, then by Proposition 9.4, there exist some $x \in X$ and $w \in X^*$ such that $xw \in L_i$ but $wx \notin L_i$ (or conversely). Thus $\lambda_{\mathbf{F}}(wx) = c_i$ but $\lambda_{\mathbf{F}}(xw) \neq c_i$ (or conversely), which means by Proposition 9.6 that $\lambda_{\mathbf{F}}$ is not rotation invariant. So $\lambda_{\mathbf{F}} \in FRot(X)$ if and only if $L_i \in Rot(X)$, for every $i \in \mathbf{n}$, and this can be decided by Corollary 9.5. □

As noted above, Rot and $FRot$ are not varieties, but we can describe the greatest varieties contained in them.

Proposition 9.8. *Com is the greatest $\star$-variety contained in Rot, and $FCom$ is the greatest fuzzy $\star$-variety contained in $FRot$.*

Proof. By Proposition 6.5, it suffices to show that $Com = \mathcal{L}_{JR}$ and $FCom = \mathcal{F}_{JR}$ for the flanked version (JR) $\xi\mathrm{uvw}\pi \approx \xi\mathrm{vwu}\pi$ of the identity (IR) $\mathrm{uvw} \approx \mathrm{vwu}$ that defines Rot and $FRot$. Let us verify this for fuzzy languages.

If $\lambda : X^* \to [0,1]$ is commutative, then $\lambda(s\underline{u}\underline{vw}t) = \lambda(svwut)$, for all $s, t, u, v, w \in X^*$. Thus λ satisfies (JR). On the other hand, if λ satisfies (JR), then $\lambda(suvt) = \lambda(suv\varepsilon t) = \lambda(sv\varepsilon ut) = \lambda(svut)$, for all $s, t, u, v \in X^*$, i.e., $\lambda \in FCom(X)$. □

Finally, let us note that since Rot is not a $\star$-variety, $Rot = \mathbf{M}^\ell$ for no VFM $\mathbf{M}$. Similarly, there is no VFM $\mathbf{M}$ such that $FRot = \mathbf{M}^f$. Hence, neither Rot nor $FRot$ can be characterized by syntactic monoids in the sense of [11] or [21], respectively. In [3] Archana introduced the monoid identities

(E1) $\mathrm{uvw} \approx \mathrm{vuw}$ (E2) $\mathrm{uvw} \approx \mathrm{uwv}$ (E3) $\mathrm{uvw} \approx \mathrm{wvu}$

(E4) $\mathrm{uvw} \approx \mathrm{vwu}$ (E5) $\mathrm{uvw} \approx \mathrm{wuv}$

to match conditions (C1)–(C5). The idea was that for each $i \in [5]$, a regular fuzzy language satisfies condition (Ci) exactly in case its syntactic monoid satisfies the identity (Ei). However, by the above remark, this is not possible. In fact, it is easy to show that each one of the identities (E1)–(E5) is equivalent to the commutative law (EC) $\mathrm{uv} \approx \mathrm{vu}$ and thus defines **Com**.

10. The varieties of aperiodic languages

A monoid M is aperiodic [11, 15, 22], if there is an $n \geq 1$ such that $a^{n+1} = a^n$ for every $a \in M$. Obviously, $a^{n+1} = a^n$ implies that $a^{m+1} = a^m$, for all $m \geq n$. Hence the VFM **Ap** of aperiodic finite monoids is ultimately defined by the sequence of identities (SAp) $\langle \mathrm{u}^{n+1} \approx \mathrm{u}^n \rangle_{n\geq 1}$. A famous theorem by Schützenberger [26, 11, 15, 22] shows that **Ap** is the VFM corresponding to the $\star$-variety of star-free languages. On the other hand, it is known that the star-free languages are the same as the aperiodic, or noncounting, languages; a language $L \subseteq X^*$ is aperiodic, if there is an $n \geq 1$ such that, for all $s, t, u \in X^*$, $su^{n+1}t \in L$ if and only if $su^nt \in L$. Let $Ap = \{Ap(X)\}_X$ be the family of regular aperiodic languages. Thus Ap is the $\star$-variety corresponding to the VFM **Ap**. Moreover, Ap is ultimately defined by the flanked version (TAp) $\langle \xi\mathrm{u}^{n+1}\pi \approx \xi\mathrm{u}^n\pi \rangle_{n\geq 1}$ of the sequence (SAp); if a language satisfies $\xi\mathrm{u}^{n+1}\pi \approx \xi\mathrm{u}^n\pi$ for some n, then it satisfies $\xi\mathrm{u}^{m+1}\pi \approx \xi\mathrm{u}^m\pi$ for every $m \geq n$. The many ways in which the family Ap arises are discussed in the book [18] by McNaughton and Papert.

Finally, following Li [16], we call a fuzzy language $\lambda : X^* \to [0,1]$ aperiodic, if there is an $n \geq 1$ such that $\lambda(su^{n+1}t) = \lambda(su^nt)$ for all $s, t, u \in X^*$. Let $FAp = \{FAp(X)\}_X$ be the family regular aperiodic fuzzy languages. In [4] Archana showed that FAp is a fuzzy $\star$-variety by verifying the required closure properties. Obviously, also FAp is ultimately defined by the sequence (TAp). This means that the results of Sections 5 and 7 apply the triple Ap, FAp and **Ap**. In particular,

(1) Ap is the $\star$-variety and FAp the fuzzy $\star$-variety characterized by the VFM **Ap**,

(2) a fuzzy language $\lambda : X^* \to [0,1]$ belongs to $FAp(X)$ if and only if it can be expressed in the form $\lambda = c_1 L_1^\chi \cup \ldots \cup c_n L_n^\chi$, where $n \geq 1$, $c_1, \ldots, c_n \in [0,1]$, and $L_1, \ldots, L_n \in Ap(X)$, and then the constants c_i may be chosen to be pairwise distinct and the languages L_i to be pairwise disjoint.

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