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Applications of interval t-norm fuzzy ideals of hemirings with interval valued characteristic function

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Abstract. This communication reports the notions of interval t-norm fuzzy subhemirings, interval t-norm fuzzy ideals, quasi ideals, bi ideals, interior ideals and generalized bi ideals of hemirings with interval valued characteristic function. Some examples of such ideals are also discussed.

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Keywords: Interval t-norm fuzzy ideals, Interval t-norm fuzzy interior ideal, Interval t-norm fuzzy quasi ideal, Interval t-norm fuzzy bi ideal and generalized fuzzy bi ideal.

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1. Introduction

Associative rings have many generalizations. Semirings are one of them. In 1934, Vandiver [15] introduced semirings. Recently semirings have been intensely studied, especially in relations with applications [8]. Semirings is widely used in perusing in theory of graphs, discrete events in dynamical systems, computational mathematics, matrix theory, fuzzy computation, automata structures, coding theory and in developing computer programmes (See [3, 4, 7, 8]). Hemiring is a structure constructed with an additional property of commutativeness with respect to addition in semiring together with zero element. Its applications involves in understanding the concepts in automata theory, formal languages and in information sciences (See [9, 11]). Fuzzy set was first initiated by Zadeh [16] which was very useful development and is remarkably applicable to some basic notions of algebra. Zadeh [17, 18, 19] proposed the definition of interval valued set which was the generalization of fuzzy set. Fuzzy semirings were first investigated in [1]. Triangular norms were introduced by Schweizer and Sklar [14] and generalization of t-norm for interval values were proposed by Bedregal and Takahashi [5]. Ealier fuzzy bi ideals [6], fuzzy quasi ideals

[2], fuzzy interior ideals [10] and fuzzy generalized bi ideals [12] were defined for semi groups. In this note we are able to define Interval t-norm fuzzy ideals, Interval t-norm fuzzy interior ideal, Interval t-norm fuzzy quasi ideal, Interval t-norm fuzzy bi ideal and generalized fuzzy bi ideal for hemirings. Ideals of hemirings and semirings are helpful in discussing the concepts of structure theory.

2. Preliminaries

A set $R \neq \varnothing$ with binary operations of " $+$ " and " $\cdot$ " is a semiring [13], if " $\cdot$ " distributes over " $+$ ", i.e. $\forall\ r, s, t \in R$,

$$r(s+t) = rs + rt,$$
$$(r+s)t = rt + st.$$

An element $'0'$ is called a zero of R, if it satisfies the conditions:

$$0 \cdot r = r \cdot 0 = 0 \text{ and} 0 + r = r + 0 = 0\ \forall r \in R.$$

An element $'1'$ is taken as of R, if $1 \cdot r = r \cdot 1 = r,\ \forall\ r \in R$. A semiring R is commutative iff $\forall\ r, s \in R,\ r \cdot s = s \cdot r$. A commutative semiring w.r.t addition having zero element is said to be a hemiring. Let $\varphi \neq S \subseteq R$. Then S be a subhemiring of R if $\forall\ s, t \in S$ implies $s + t,\ st \in S$ also $0 \in S$.

For $\varnothing \neq I \subseteq R$, I is called a left (right) ideal of R, if $\forall\ r, s \in I$ and $t \in R$,
(i) $r + s \in I$,
(ii) $tr \in I$ ($rt \in I$).

Note $0 \in I$, clearly. I is called an ideal, if it is both a left ideal and a right ideal of R.

If I and J are ideals of R then $I \cap J$, IJ are ideals of R such that $IJ \subseteq I \cap J$. A non-empty subset Q of R is quasi ideal if Q acquires the closure property of addition, and $QR \cap RQ \subseteq Q$. A non-empty subset B of R is a bi ideal if B acquires the closure property of $+$ and $\cdot$ together with $BRB \subseteq B$. $\varnothing \neq A \subseteq R$ is an interior ideal if it acquires the closure property of $+$ and $\cdot$ together with $RAR \subseteq A$. $\varnothing \neq G \subseteq R$ is a generalized bi ideal, if $GRG \subseteq G$. An interval number $\hat{t}$ (See [17]) means an interval $[t^l, t^u]$, where $0 \leq t^l \leq t^u \leq 1$. For interval numbers $\hat{t}_i = [t_i^l, t_i^u] \in D[0,1]$ (set of all interval numbers), $i \in I$, we define

$$\inf \hat{t}_j = \left[\bigwedge_{i \in I} t_i^l, \bigwedge_{i \in I} t_i^u\right], \sup \hat{t}_j = \left[\bigvee_{i \in I} t_i^l, \bigvee_{i \in I} t_i^u\right]$$

and
(i) $\hat{t} \leq \hat{r} \iff t^l \leq r^l$, and $t^u \leq r^u$,
(ii) $\hat{t} = \hat{r} \iff t^l = r^l$, and $t^u = r^u$,
(iii) $\hat{t} \lessdot \hat{r} \iff \hat{t} \leq \hat{r}$, and $\hat{t} \neq \hat{r}$,
(iv) $\hat{t} \subseteq \hat{r} \iff r^l \leq t^l \leq t^u \leq r^u$,
(v) $k\hat{t}_i = [kt^l, kt^u]$, whenever $0 \leq k \leq 1$.

An interval valued fuzzy set F [6] on X is:

$$F = \left\{ \left(a, \left[\mu^l(a), \mu^u(a)\right]\right) | a \in X \right\},$$

where μ^l and μ^u are fuzzy subsets on X s.t $\mu^l(a) \leq \mu^u(a)$ for all $a \in X$. Putting $\hat{\mu}(a) = [\mu^l(a), \mu^u(a)]$, shows $F = \{(x, \hat{\mu}(a)) \,|a \in X\}$, where $\hat{\mu}$ is defined from X to $D[0,1]$. The set operations of $\cup$, $\cap$ of $\hat{\lambda}$ and $\hat{\mu}$ are, $\forall\ s \in R$

$$\begin{aligned}(\hat{\lambda} \cup \hat{\mu})(s) &= [\lambda^-(s) \vee \mu^-(s), \lambda^+(s) \vee \mu^+(s)],\\ (\hat{\lambda} \cap \hat{\mu})(s) &= [\lambda^-(s) \wedge \mu^-(s), \lambda^+(s) \wedge \mu^+(s)].\end{aligned}$$

For a class of $\hat{\lambda}_i$ the union and intersection $\forall\ s \in R$. are defined as:

$$\begin{aligned}(\cup_i \hat{\lambda}_i)(s) &= [\bigvee_{i \in I} \lambda_i^-(s), \bigvee_{i \in I} \lambda_i^+(s)],\\ (\cap_i \hat{\lambda}_i)(s) &= [\bigwedge_{i \in I} \lambda_i^-(s), \bigwedge_{i \in I} \lambda_i^+(s)].\end{aligned}$$

The sum and product of $\hat{\lambda}$ and $\hat{\mu}$ $\forall\ r \in R$ are defined as: $\forall\ r \in R$,

$$(\hat{\lambda} + \hat{\mu})(r) = \bigvee_{r=s+t} [\lambda^-(s) \wedge \mu^-(t), \lambda^+(s) \wedge \mu^+(t)] \ ,$$

$$\begin{aligned}(\hat{\lambda}\hat{\mu})(r) &= \bigvee_{r=\sum_{i=1}^{n} s_i t_i} \{\bigwedge\nolimits_i [\lambda^-(s_i) \wedge \mu^-(t_i), \lambda^+(s_i) \wedge \mu^+(t_i)]\}\\ &= 0 \text{ if } r \neq \sum_{i=1}^{n} s_i t_i.\end{aligned}$$

$\hat{C}_A$ termed as characteristic/membership(interval valued) functions of the set $A \subseteq R$ and is defined from R to $D[0,1]$: $\forall\ r \in R$,

$$\hat{C}_A(r) = \begin{cases} \hat{I} = [1,1] & \text{if } r \in A \\ \hat{O} = [0,0] & \text{if } r \notin A. \end{cases}$$

$T : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a t-norm [14], if it satisfies the following conditions: for all $p,\ q,\ r,\ t \in [0,1]$,

(i) $T(1,p) = p$,
(ii) $T(p,q) = T(q,p)$,
(iii) $T(p, T(q,r)) = T(T(p,q), r)$,
(iv) if $t \leq p$ and $q \leq r$, then $T(t,q) \leq T(p,r)$.

The first, second and fourth conditions give $T(0,p) \leq T(0,1) = 0$.

Every t-norm T has a useful property:

$$T(p,q) \leq \min(p,q) \text{ for all } p,q \in [0,1].$$

A mapping $\triangle$ defined from $D[0,1] \times D[0,1]$ to $D[0,1]$ is called an interval triangular norm [5], if $\triangle$ satisfies the following conditions:

(i) for each $\hat{t}, \hat{r} \in D[0,1], \hat{t} \triangle \hat{r} = \hat{r} \triangle \hat{t}$,
(ii) for each $\hat{t}, \hat{r}, \hat{s} \in D[0,1], \hat{t} \triangle (\hat{r} \triangle \hat{s}) = (\hat{t} \triangle \hat{r}) \triangle \hat{s}$,
(iii) for each $\hat{t}_1, \hat{r}_1, \hat{t}_2, \hat{r}_2 \in D[0,1]$, if $\hat{t}_1 \subseteq \hat{t}_2$ $(\hat{t}_1 \leq \hat{t}_2)$ and $\hat{r}_1 \subseteq \hat{r}_2$ $(\hat{r}_1 \leq \hat{r}_2)$ then $\hat{t}_1 \triangle \hat{r}_1 \subseteq \hat{t}_2 \triangle \hat{r}_2$ $(\hat{t}_1 \triangle \hat{r}_1 \leq \hat{t}_2 \triangle \hat{r}_2)$,
(iv) for each $\hat{t} \in D[0,1], \hat{t} \triangle [1,1] = \hat{t}$.

Let T_1 and T_2 be t -norms. If $T_1 \leq T_2$, then $\triangle : D[0,1] \times D[0,1] \longrightarrow D[0,1]$ defined by

$$\triangle\left(\hat{t}, \hat{r}\right) = \left[T_1\left(t^l, r^l\right), T_2\left(t^u, r^u\right)\right]$$

is an interval t-norm [5] derived from T_1 and T_2.

On the other hand from each interval t-norm, two t-norms can always be obtained.

Let $\triangle$ be interval t-norm, the function $\triangle^l : [0,1] \times [0,1] \longrightarrow [0,1]$ and $\triangle^u : [0,1] \times [0,1] \longrightarrow [0,1]$ defined by

$$\triangle^l(p_1, q_1) = \pi_1\left(\triangle[p_1, p_1], [q_1, q_1]\right)$$

and

$$\triangle^u(p_1, q_1) = \pi_2\left(\triangle[p_1, p_1], [q_1, q_1]\right)$$

are t-norms and π_1, π_2 are projection maps.

Let T be a t-norm and $\hat{t}$, $\hat{r} \in D[0,1]$. If the interval t-norm derived from T is denoted by $\triangle$, then $\triangle\left(\hat{t}, \hat{r}\right)$ is the smallest interval containing $\left\{T(x,y) : x \in \hat{t} \text{ and } y \in \hat{r}\right\}$.

The following distributive laws hold:

(1) $\hat{s} \triangle (\hat{t} \vee \hat{r}) = (\hat{s} \triangle \hat{t}) \vee (\hat{s} \triangle \hat{r})$,
(2) $\hat{s} \triangle (\hat{t} \wedge \hat{r}) = (\hat{s} \triangle \hat{t}) \wedge (\hat{s} \triangle \hat{r})$.

3. Interval t-norm fuzzy Ideals

A fuzzy subset $\hat{\mu}$ (interval valued) in R is specified as an interval t-norm fuzzy subhemiring, if $\forall\ r,\ s \in R$,

(i) $\hat{\mu}(0) \geq \hat{\mu}(r)$,
(ii) $\hat{\mu}(r+s) \geq \hat{\mu}(r) \triangle \hat{\mu}(s)$,
(iii) $\hat{\mu}(rs) \geq \hat{\mu}(r) \triangle \hat{\mu}(s)$

$\hat{\mu}$ is specified as an interval t-norm fuzzy left (right) ideal in R iff

$$\hat{\mu}(r+s) \geq \hat{\mu}(r) \triangle \hat{\mu}(s),\ \forall r,\ s \in R,$$

and

$$\hat{\mu}(ks) \geq \hat{\mu}(s) \qquad (\hat{\mu}(ks) \geq \hat{\mu}(k)),\quad \forall\ k \in R.$$

An interval t-norm fuzzy left and right ideal of R is called an ideal of R.

4. Interval t-norm fuzzy bi ideals

Definition 4.1. An interval t-norm fuzzy subhemiring $\hat{\lambda}$ is called an interval t-norm fuzzy bi ideal, if $\forall\ s,\ r,\ t \in R$

$$\hat{\lambda}(rst) \geq \hat{\lambda}(r) \triangle \hat{\lambda}(t) \quad \text{ for every } s \in R$$

Example 4.2. Let $R = \{0, d, e, f\}$. Then R be a hemiring w.r.t following addition $(+)$ and multiplication $(\cdot)$:

$+$	0	d	e	f
0	0	d	e	f
d	d	d	e	f
e	e	e	e	f
f	f	f	f	e

and

$\cdot$	0	d	e	f
0	0	0	0	0
d	0	d	d	d
e	0	d	d	d
f	0	d	d	d

$\hat{\lambda}$ is defined as

$$\hat{\lambda}(0) = [0.6, 0.9], \hat{\lambda}(d) = [0.4, 0.5], \hat{\lambda}(e) = [0.4, 0.5], \hat{\lambda}(f) = [0.4, 0.5].$$

Corresponding to the Lukasiewicz t-norm

$$T(p, q) = (p + q - 1) \vee 0.$$

Interval t-norm $\triangle$ is defined as

$$\hat{t} \triangle \hat{r} = \left[(t^l + r^l - 1) \vee 0, (t^u + r^u - 1) \vee 0\right] \quad \forall\, \hat{t}, \hat{r} \in D[0, 1].$$

Lemma 4.3. *Let $\varphi \neq A \subseteq R$ is a bi ideal iff $\hat{C}_A$ is interval t-norm fuzzy bi ideal.*

Proof. Follows from Theorem 2.13 in [13]. □

Lemma 4.4. *An interval t-norm fuzzy subhemiring $\hat{\lambda}$ is an interval t-norm fuzzy bi ideal iff*

$$\hat{\lambda}\hat{C}_R\hat{\lambda} \subseteq \hat{\lambda}.$$

Proof. Consider $\hat{\lambda}$ is an interval t-norm fuzzy bi ideal. Let a in R.

Case-1 Suppose a is not expressible as $a = \sum_{i=1}^{p} y_i z_i$ for all $y_i, z_i \in R$ and $p \in \mathbf{N}$. Then

$$\hat{\lambda}\hat{C}_R\hat{\lambda}\,(a) = 0 \leq \hat{\lambda}(a).$$

Case-2 Suppose a is expressible, there exist $x_i, y_i, y\acute{}_j, z\acute{}_j$ of R such that $a = \sum_{i=1}^{p} x_i y_i$ and $x_i = \sum_{j=1}^{q} y\acute{}_j z\acute{}_j$. Since $\hat{\lambda}$ is an interval t-norm-fuzzy bi ideal of R,

$$\hat{\lambda}(pqy) \geq \hat{\lambda}(p) \triangle \hat{\lambda}(y), \text{ for every } p, q, y \in R.$$

Then

$$\begin{aligned}
\left(\hat{\lambda}\hat{C}_R\hat{\lambda}\right)(a) &= \bigvee_{a=\sum_{i=1}^{p} x_i y_i} \left\{\triangle_{\mathbf{i}} \left[\left(\hat{\lambda}\hat{C}_R\right)(x_i) \triangle \hat{\lambda}(y_i)\right]\right\} \\
&= \bigvee_{a=\sum_{i=1}^{p} x_i y_i} \left\{\triangle_{\mathbf{i}} \left[\bigvee_{x_i=\sum_{j=1}^{q} y\acute{}_j z\acute{}_j} \left[\triangle_{\mathbf{j}} \left(\hat{\lambda}(y\acute{}_j) \triangle \hat{C}_R(z\acute{}_j)\right)\right] \triangle \hat{\lambda}(y_i)\right]\right\} \\
&= \bigvee_{a=\sum_{i=1}^{p} x_i y_i} \left\{\triangle_{\mathbf{i}} \left[\bigvee_{x_i=\sum_{j=1}^{q} y\acute{}_j z\acute{}_j} \left[\triangle_{\mathbf{j}} \left(\hat{\lambda}(y\acute{}_j) \triangle [1, 1]\right)\right] \triangle \hat{\lambda}(y_i)\right]\right\} \\
&= \bigvee_{a=\sum_{i=1}^{p}\left(x_i=\sum_{j=1}^{q} y\acute{}_j z\acute{}_j\right) y_i} \left\{\triangle_{\mathbf{i}} \left[\triangle_{\mathbf{j}} \left(\hat{\lambda}(y\acute{}_j) \triangle \hat{\lambda}(y_i)\right)\right]\right\} \\
&\leq \bigvee_{a=\sum_{i=1}^{p}\left(x_i=\sum_{j=1}^{q} y\acute{}_j z\acute{}_j\right) y_i} \left\{\triangle_{\mathbf{i}} \left[\triangle_{\mathbf{j}} \hat{\lambda}\,(y\acute{}_j z\acute{}_j y_i)\right]\right\} \\
&\leq \bigvee_{a=\sum_{i=1}^{p}\left(x_i=\sum_{j=1}^{q} y\acute{}_j z\acute{}_j\right) y_i} \hat{\lambda}\left(\sum_{i=1}^{p}\sum_{j=1}^{q} y\acute{}_j z\acute{}_j y_i\right) = \hat{\lambda}(a).
\end{aligned}$$

Thus we have $\hat{\lambda}\hat{C}_R\hat{\lambda} \subseteq \hat{\lambda}$

Conversely, suppose $\hat{\lambda}\hat{C}_R\hat{\lambda} \subseteq \hat{\lambda}$. Assume x, y, z are elements in R. Then we have

$$\begin{aligned}
\hat{\lambda}(xyz) &\geq \left(\hat{\lambda}\hat{C}_R\hat{\lambda}\right)(xyz) \\
&= \bigvee_{xyz=\sum_{i=1}^{p} b_i c_i} \left\{\triangle_{\mathbf{i}}\left[\left(\hat{\lambda}\hat{C}_R\right)(b_i) \triangle \hat{\lambda}(c_i)\right]\right\} \\
&\geq \left(\hat{\lambda}\hat{C}_R\right)(xy) \triangle \hat{\lambda}(z) \\
&= \bigvee_{xy=\sum_{j=1}^{q} s_j t_j} \left\{\triangle_{\mathbf{j}}\left[\hat{\lambda}(s_j) \triangle \hat{C}_R(t_j)\right]\right\} \triangle \hat{\lambda}(z) \\
&\geq \left[\hat{\lambda}(x) \triangle \hat{C}_R(y)\right] \triangle \hat{\lambda}(z) \\
&= \left[\hat{\lambda}(x) \triangle [1,1]\right] \triangle \hat{\lambda}(z) \\
&= \hat{\lambda}(x) \triangle \hat{\lambda}(z).
\end{aligned}$$

□

Proposition 4.5. *Every interval t-norm fuzzy left ideal is an interval t-norm fuzzy bi ideal.*

Proof. Consider $\hat{\lambda}$ is an interval t-norm fuzzy left ideal and $r, s, w \in R$. Then

$$\hat{\lambda}(rs) \geq \hat{\lambda}(s) \geq \hat{\lambda}(r) \triangle \hat{\lambda}(s)$$

and

$$\hat{\lambda}(rws) = \hat{\lambda}((rw)\,s) \geq \hat{\lambda}(s) \geq \hat{\lambda}(r) \triangle \hat{\lambda}(s),$$

since $[1,1] \geq \hat{\lambda}(r)$ and $\hat{\lambda}(s) = [1,1] \triangle \hat{\lambda}(s) \geq \hat{\lambda}(r) \triangle \hat{\lambda}(s)$. □

A counter example for the converse is as under:
Consider $R = \{0, \alpha, 1\}$ w.r.t the operations given below:

$+$	0	α	1
0	0	α	1
α	α	α	α
1	1	α	1

$\cdot$	0	α	1
0	0	0	0
α	0	α	α
1	0	α	1

$\hat{\lambda}$ is defined as:

$$\hat{\lambda}(0) = [0.6, 0.8], \hat{\lambda}(\alpha) = [0.4, 0.6], \hat{\lambda}(1) = [0.5, 0.7].$$

Corresponding to the t-norm

$$pTq = \begin{cases} p \wedge q \text{ if } p \vee q = 1 \\ 0 \qquad \text{otherwise.} \end{cases}$$

Interval t-norm $\triangle$ is defined as: for any $\hat{t},\ \hat{r} \in D[0,1]$,

$$\hat{t} \triangle \hat{r} = \begin{cases} [t^l \wedge r^l, t^u \wedge r^u] \text{ if } [t^l \vee r^l, t^u \vee r^u] = 1 \\ [0,0] \qquad\qquad\quad \text{otherwise.} \end{cases}$$

Routine calculations shows that $\hat{\lambda}$ is an interval t-norm fuzzy bi ideal of R but, neither left nor right interval t-norm fuzzy ideal of R, because

$$[0.4, 0.6] = \hat{\lambda}(\alpha) = \hat{\lambda}(\alpha 1) \ngeq \hat{\lambda}(1) = [0.5, 0.7]$$

and

$$[0.4, 0.6] = \hat{\lambda}(\alpha) = \hat{\lambda}(1\alpha) \ngeq \hat{\lambda}(1) = [0.5, 0.7].$$

5. Interval t-norm fuzzy interior ideals

Definition 5.1. Interval t-norm fuzzy subhemiring $\hat{\lambda}$ is specified as interval t-norm fuzzy interior ideal, if $\forall\ r, s, t \in R$,

$$\hat{\lambda}(rst) \geq \hat{\lambda}(s).$$

The following is an immediate result of Theorem 2.13 in [13].

Lemma 5.2. *$\varnothing \neq A \subseteq R$ is an interior ideal iff $\hat{C}_A$ is an interval t-norm fuzzy interior ideal.*

The following is an immediate result of Lemma 2.12 in [13].

Lemma 5.3. *Interval t-norm fuzzy subhemiring $\hat{\lambda}$ is termed as interval t-norm fuzzy interior ideal iff*

$$\hat{C}_R \hat{\lambda} \hat{C}_R \subseteq \hat{\lambda}.$$

Note that every interval t-norm fuzzy two-sided ideal is an interval t-norm fuzzy interior ideal. A counter example of the converse is as under:

Let $R = \{0, 1, r, s, t\}$. Then R be a hemiring w.r.t the operations given below:

$+$	0	1	r	s	t
0	0	1	r	s	t
1	1	s	1	r	1
r	r	1	r	s	r
s	s	r	s	1	s
t	t	1	r	s	t

and

$\cdot$	0	1	r	s	t
0	0	0	0	0	0
1	0	1	r	s	t
r	0	r	r	r	t
s	0	s	r	1	t
t	0	t	t	t	0

$\hat{\lambda}$ is defined as:

$\hat{\lambda}(0) = [0.8, 0.9], \hat{\lambda}(1) = [0.3, 0.4], \hat{\lambda}(r) = [0.6, 0.7], \hat{\lambda}(s) = [0.5, 0.6], \hat{\lambda}(t) = [0.4, 0.5].$

Corresponding to the t-norm

$$pTq = \begin{cases} p \wedge q \text{ if } p \vee q = 1 \\ 0 \qquad \text{otherwise.} \end{cases}$$

Interval t-norm $\triangle$ is defined as: for any $\hat{t},\ \hat{r} \in D[0, 1]$,

$$\hat{t} \triangle \hat{r} = \begin{cases} [t^l \wedge r^l, t^u \wedge r^u] \text{ if } [t^l \vee r^l, t^u \vee r^u] = 1 \\ [0, 0] \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Routine calculations shows that $\hat{\lambda}$ is an interior ideal(interval t-norm fuzzy), however neither left nor right interval t-norm fuzzy ideal of R as:

$[0.3, 0.4] = \hat{\lambda}(1) = \hat{\lambda}(ss) \ngeq \hat{\lambda}(s) = [0.5, 0.6]$ and $[0.4, 0.5] = \hat{\lambda}(t) = \hat{\lambda}(ts) \ngeq \hat{\lambda}(s) = [0.5, 0.6].$

6. Interval t-norm fuzzy quasi ideals

Definition 6.1. A fuzzy subset $\hat{\lambda}$(interval valued) is called an interval t-norm fuzzy quasi ideal, if $\forall\ u, v \in R$,

(i) $\hat{\lambda}(u+v) \geq \hat{\lambda}(u) \bigtriangleup \hat{\lambda}(u)$,

(ii) $\left(\left(\hat{\lambda}\hat{C}_R\right) \bigtriangleup \left(\hat{C}_R\hat{\lambda}\right)\right)(u) \leq \hat{\lambda}\,(u)\,.$

Let $R = \{0, \alpha, \beta\}$,be a hemiring w.r.t the operations below:

$+$	0	α	β
0	0	α	β
α	α	α	β
β	β	β	β

$\cdot$	0	α	β
0	0	0	0
α	0	0	0
β	0	0	β

$\hat{\lambda}$ is defined as:

$$\hat{\lambda}(0) = [1, 1], \hat{\lambda}(\alpha) = [0.6, 0.9], \hat{\lambda}(\beta) = [0.4, 0.7].$$

Corresponding to the t-norm

$$T(p, q) = pq.$$

Interval t-norm $\bigtriangleup$ is defined as

$$\hat{t} \bigtriangleup \hat{r} = [t^l r^l, t^u r^u],\ \text{for all}\ \hat{t}, \hat{r} \in D[0, 1].$$

Lemma 6.2. *Let $\varphi \neq A \subseteq R$ is a quasi ideal iff $\hat{C}_A$ is an interval t-norm fuzzy quasi ideal.*

Proof. Suppose A is a quasi-ideal of R.

If $\hat{C}_A(r) = [0, 0]$ or $\hat{C}_A(s) = [0, 0]$, then

$$\begin{aligned} \hat{C}_A(r+s) &\geq [0,0] \\ &= \hat{C}_A(r) \bigtriangleup [0,0] \\ &= \hat{C}_A(r) \bigtriangleup \hat{C}_A(s). \end{aligned}$$

If for $r, s \in A$, $\hat{C}_A(r) = [1, 1]$ and $\hat{C}_A(s) = [1, 1]$, then $r + s \in A$ and

$$\begin{aligned} \hat{C}_A(r+s) &= [1,1] \\ &= [1,1] \bigtriangleup [1,1] \\ &= \hat{C}_A(r) \bigtriangleup \hat{C}_A(s). \end{aligned}$$

Now let $r \in A$. Then

$$\left(\left(\hat{C}_A\hat{C}_R\right) \bigtriangleup \left(\hat{C}_R\hat{C}_A\right)\right)(r) \leq [1, 1] = \hat{C}_A(r).$$

If $r \notin A$, then $\hat{C}_A(r) = [0, 0]$. On the other way, if

$$\left(\left(\hat{C}_A\hat{C}_R\right) \bigtriangleup \left(\hat{C}_R\hat{C}_A\right)\right)(r) = [1, 1],$$

then

$$\left(\hat{C}_A\hat{C}_R\right)(r) = \bigvee_{r=\sum_{i=1}^{n} s_i t_i} \{\bigtriangleup_{\mathbf{i}}[C_A^-(s_i) \bigtriangleup C_R^-(t_i), C_A^+(s_i) \bigtriangleup C_R^+(t_i)]\} = [1, 1]$$

and

$$\left(\hat{C}_R\hat{C}_A\right)(r) = \bigvee_{r=\sum_{i=1}^{n} s_i t_i} \{\triangle_{\mathbf{i}}[C_R^-(s_i) \triangle C_A^-(t_i), C_R^+(s_i) \triangle C_A^+(t_i)]\} = [1,1].$$

Thus there exist $u_i, v_i, w_i, z_i \in R$ with $r = \sum_{i=1}^{n} u_i v_i = \sum_{i=1}^{n} w_i z_i$ s. t.

$$\hat{C}_A(u_i) = [1,1], \hat{C}_A(z_i) = [1,1].$$

So $u_i, z_i \in A$. Hence $r = \sum_{i=1}^{n} u_i v_i = \sum_{i=1}^{n} w_i z_i \in AR \cap RA \subseteq A$. Which contradicts that $r \notin A$. Therefore, we have

$$\left(\left(\hat{C}_A\hat{C}_R\right) \triangle \left(\hat{C}_R\hat{C}_A\right)\right) \subseteq \hat{C}_A.$$

Conversely, consider $\hat{C}_A$ is an interval t-norm fuzzy quasi ideal. Let $r, s \in A$. Then

$$\hat{C}_A(r) = \hat{C}_A(s) = [1,1].$$

Since

$$\begin{aligned}\hat{C}_A(r+s) &\geq \hat{C}_A(r) \triangle \hat{C}_A(s) \\ &= [1,1] \triangle [1,1] \\ &= [1,1],\end{aligned}$$

$r + s \in A$.

Now let $r \in AR \cap RA$. Then there exist $u_i, v_i \in R$ and $w_i, z_i \in A$, such that

$$r = \sum_{i=1}^{n} w_i u_i = \sum_{i=1}^{n} v_i z_i.$$

Thus we have

$$\begin{aligned}\left(\hat{C}_A\hat{C}_R\right)(r) &= \bigvee_{r=\sum_{i=1}^{n} p_i q_i} \{\triangle_{\mathbf{i}}[C_A^-(p_i) \triangle C_R^-(q_i), C_A^+(p_i) \triangle C_R^+(q_i)]\} \\ &\geq \{\triangle_{\mathbf{i}}[C_A^-(w_i) \triangle C_R^-(u_i), C_A^+(w_i) \triangle C_R^+(u_i)]\} \\ &= [1,1] \triangle [1,1].\end{aligned}$$

So $(\hat{C}_A\hat{C}_R)(r) = [1,1]$. Similarly, $\left(\hat{C}_R\hat{C}_A\right)(r) = [1,1]$. This implies that

$$\left(\left(\hat{C}_A\hat{C}_R\right) \triangle \left(\hat{C}_R\hat{C}_A\right)\right)(r) = [1,1] \triangle [1,1] = [1,1].$$

Since $\hat{C}_A$ is an interval t-norm fuzzy quasi-ideal,

$$\hat{C}_A(r) \geq \left(\left(\hat{C}_A\hat{C}_R\right) \triangle \left(\hat{C}_R\hat{C}_A\right)\right)(r) = [1,1].$$

This implies that $\hat{C}_A(r) = [1,1]$. Hence $r \in A$. Therefore $AR \cap RA \subseteq A$. □

Lemma 6.3. *A fuzzy subset $\hat{\lambda}$(interval valued) is interval t-norm fuzzy quasi ideal iff*

(1) $\hat{\lambda} + \hat{\lambda} \subseteq \hat{\lambda}$,
(2) $\left(\left(\hat{\lambda}\hat{C}_R\right) \triangle \left(\hat{C}_R\hat{\lambda}\right)\right) \subseteq \hat{\lambda}$.

Proof. The proof is obvious. □

Proposition 6.4. *Every interval t-norm fuzzy left ideal is interval t-norm fuzzy quasi ideal.*

Proof. Let $\hat{\lambda}$ be interval t-norm fuzzy left ideal. Then $\hat{\lambda}+\hat{\lambda}\subseteq\hat{\lambda}$, $\hat{C}_R\hat{\lambda}\subseteq\hat{\lambda}$. Thus $\left(\left(\hat{\lambda}\hat{C}_R\right)\triangle\left(\hat{C}_R\hat{\lambda}\right)\right)\subseteq\hat{C}_R\hat{\lambda}\subseteq\hat{\lambda}$. □

A counter example for the converse is as under:
Consider $R=\{0,\alpha,\beta\}$ w.r.t the following addition $(+)$ and multiplication $(\cdot)$:

$+$	0	α	β
0	0	α	β
α	α	α	β
β	β	β	β

$\cdot$	0	α	β
0	0	0	0
α	0	0	0
β	0	0	β

$\hat{\lambda}$ is defined as:

$$\hat{\lambda}(0)=[0.2,0.8],\hat{\lambda}(\alpha)=[0.6,0.9],\hat{\lambda}(\beta)=[0.4,0.7].$$

Corresponding to the t-norm

$$T(p,q)=(p+q-1)\vee 0.$$

Interval t-norm $\triangle$ is defined as

$$\hat{t}\triangle\hat{r}=\left[(t^l+r^l-1)\vee 0,(t^u+r^u-1)\vee 0\right]\quad\forall\ \hat{t},\hat{r}\in D[0,1].$$

Routine calculations shows that $\hat{\lambda}$ is an interval t-norm fuzzy quasi ideal, however it is neither a left nor a right interval t-norm fuzzy of R as:

$$[0.2,0.8]=\hat{\lambda}(0)=\hat{\lambda}(\alpha 0)\not\geq\hat{\lambda}(\alpha)=[0.6,0.9]$$

and

$$[0.2,0.8]=\hat{\lambda}(0)=\hat{\lambda}(0\alpha)\not\geq\hat{\lambda}(\alpha)=[0.6,0.9].$$

Proposition 6.5. *Every interval t-norm fuzzy quasi ideal is interval t-norm fuzzy bi ideal.*

Proof. Let $\hat{\lambda}$ be an interval t-norm fuzzy quasi-ideal and $r\in R$. If $r\neq\sum_{i=1}^{n}s_it_i$, then $\hat{\lambda}^2(r)=[0,0]\leqslant\hat{\lambda}(r)$. Otherwise, since $\hat{\lambda}$ is a quasi-ideal,

$$\begin{aligned}
&\hat{\lambda}^2(r)\\
&=(\hat{\lambda}\hat{\lambda})(r)\\
&=\bigvee_{r=\sum_{i=1}^{n}s_it_i}\{\triangle_{\mathbf{i}}[\lambda^-(s_i)\triangle\lambda^-(t_i),\lambda^+(s_i)\triangle\lambda^+(t_i)]\}\\
&=\bigvee_{r=\sum_{i=1}^{n}s_it_i}\{\triangle_{\mathbf{i}}[\lambda^-(s_i),\lambda^+(s_i)]\triangle[\lambda^-(t_i),\lambda^+(t_i)]\}\\
&=\bigvee_{r=\sum_{i=1}^{n}s_it_i}\{\triangle_{\mathbf{i}}[(\hat{\lambda}(s_i))\triangle(\hat{\lambda}(t_i))]\}\\
&=\bigvee_{r=\sum_{i=1}^{n}s_it_i}\{\triangle_{\mathbf{i}}[(\hat{\lambda}(s_i)\triangle[1,1])\triangle(\hat{\lambda}(t_i)\triangle[1,1])]\}\\
&=\bigvee_{r=\sum_{i=1}^{n}s_it_i}\{\triangle_{\mathbf{i}}[\left(\hat{\lambda}(s_i)\triangle\hat{C}_R(t_i)\right)\triangle\left(\hat{\lambda}(s_i)\triangle\hat{C}_R(t_i)\right)]\}\\
&\leq\bigvee_{r=\sum_{i=1}^{n}s_it_i}\left\{\triangle_{\mathbf{i}}[\left(\hat{\lambda}(s_i)\triangle\hat{C}_R(t_i)\right)\right\}\triangle\bigvee_{r=\sum_{i=1}^{n}s_it_i}\left\{\triangle_{\mathbf{i}}\left(\hat{\lambda}(s_i)\triangle\hat{C}_R(t_i)\right)\right\}
\end{aligned}$$

$= \left(\left(\hat{\lambda}\hat{C}_R\right) \triangle \left(\hat{C}_R\hat{\lambda}\right)\right)(r)$
$\leq (\hat{\lambda})(r).$

Thus $\hat{\lambda}^2 \subseteq \hat{\lambda}$. So $\hat{\lambda}$ is an interval t-norm fuzzy subhemiring.

Let $\hat{\lambda}$ be an interval t-norm fuzzy quasi ideal and $r \in R$. If $r \neq \sum_{i=1}^{n} y_i z_i$, then $\hat{\lambda}^2(x) = [0,0] \leqslant \hat{\lambda}(x)$. Otherwise,

$$\begin{aligned}
&\left(\hat{\lambda}\hat{C}_R\hat{\lambda}\right)(r) \\
&= \bigvee_{x=\sum_{i=1}^{n} y_i z_i} \{\triangle_{\mathbf{i}}[(\hat{\lambda}\hat{C}_R)(y_i) \triangle \hat{\lambda}(z_i)]\} \\
&= \bigvee_{r=\sum_{i=1}^{n} y_i z_i} \left[\triangle_{\mathbf{i}} \left\{ \bigvee_{y_i=\sum_{j=1}^{m} p_j q_j} \left[\triangle_{\mathbf{j}}(\hat{\lambda}(p_j) \triangle \hat{C}_R(q_j)\right] \triangle \hat{\lambda}(z_i)\right\}\right] \\
&= \bigvee_{r=\sum_{i=1}^{n} y_i z_i} \left[\triangle_{\mathbf{i}} \left\{ \bigvee_{y_i=\sum_{j=1}^{m} p_j q_j} \left[\triangle_{\mathbf{j}}(\hat{\lambda}(p_j) \triangle [1,1]\right] \triangle \left[[1,1] \triangle \hat{\lambda}(z_i)\right]\right\}\right] \\
&= \bigvee_{r=\sum_{i=1}^{n} y_i z_i} \left[\bigvee_{y_i=\sum_{j=1}^{m} p_j q_j} \left[\triangle_{\mathbf{i}} \left[\triangle_{\mathbf{j}}(\hat{\lambda}(p_j) \triangle [1,1]\right] \triangle \left[[1,1] \triangle \hat{\lambda}(z_i)\right]\right]\right] \\
&= \bigvee_{r=\sum_{i=1}^{n} y_i z_i} \bigvee_{,y_i=\sum_{j=1}^{m} p_j q_j} \left[\triangle_{\mathbf{i}} \left\{\triangle_{\mathbf{j}}(\hat{\lambda}(p_j) \triangle [1,1] \triangle \left[[1,1] \triangle \hat{\lambda}(z_i)\right]\right\}\right] \\
&= \bigvee_{r=\sum_{i=1}^{n} (\sum_{j=1}^{m} p_j q_j) z_i} \left[\triangle_{\mathbf{i}} \left\{\triangle_{\mathbf{j}}(\hat{\lambda}(p_j) \triangle [1,1] \triangle \left[[1,1] \triangle \hat{\lambda}(z_i)\right]\right\}\right] \\
&\leq \bigvee_{r=\sum_{i=1}^{n} (\sum_{j=1}^{m} p_j q_j) z_i} \left[\triangle_{\mathbf{j}} \left\{(\hat{\lambda}(p_j) \triangle [1,1]\right\}\right] \\
&\quad \triangle \bigvee_{r=\sum_{i=1}^{n} (\sum_{j=1}^{m} p_j q_j) z_i} \left[\triangle_{\mathbf{j}} \left\{[1,1] \triangle \hat{\lambda}(z_i)\right\}\right] \\
&= \bigvee_{r=\sum_{i=1}^{n} (\sum_{j=1}^{m} p_j q_j) z_i} \left[\triangle_{\mathbf{j}} \left\{(\hat{\lambda}(p_j) \triangle \hat{C}_R(q_j z_i)\right\}\right] \\
&\quad \triangle \bigvee_{r=\sum_{i=1}^{n} (\sum_{j=1}^{m} p_j q_j) z_i} \left[\triangle_{\mathbf{j}} \left\{\hat{C}_R(p_j q_j) \triangle \hat{\lambda}(z_i)\right\}\right] \\
&= \left(\hat{\lambda}\hat{C}_R\right)(r) \triangle \left(\hat{C}_R\hat{\lambda}\right)(r) \\
&= \left(\left(\hat{\lambda}\hat{C}_R\right) \triangle \left(\hat{C}_R\hat{\lambda}\right)\right)(r) \\
&\leq (\hat{\lambda})(r).
\end{aligned}$$

□

7. Interval t-norm generalized fuzzy bi ideals

Definition 7.1. An interval t-norm fuzzy subset $\hat{\lambda}$ is called an interval t-norm generalized fuzzy bi ideal, if $\forall\ r, s \in R$

$$\hat{\lambda}(rws) \geq \hat{\lambda}(r) \triangle \hat{\lambda}(s), \quad \forall\ \ w \in R.$$

Remark 7.2. Note that every interval t-norm fuzzy bi ideal is an interval t-norm generalized fuzzy bi ideal, however following example shows that converse is not true.

Consider $R = \{0, \alpha, \beta, \gamma\}$ w.r.t the operations:

$+$	0	α	β	γ
0	0	α	β	γ
α	α	α	β	γ
β	β	β	β	γ
γ	γ	γ	γ	β

and

$\cdot$	0	α	β	γ
0	0	0	0	0
α	0	α	α	α
β	0	α	α	α
γ	0	α	α	α

$\hat{\lambda}$ is defined as:

$$\hat{\lambda}(0) = [0.4, 0.5], \hat{\lambda}(\alpha) = [0.5, 0.6], \hat{\lambda}(\beta) = [0.4, 0.7], \hat{\lambda}(\gamma) = [0.3, 0.7].$$

Corresponding to the t-norm

$$T(x, y) = xy.$$

Interval t-norm $\triangle$ is defined as

$$\hat{t} \triangle \hat{r} = [t^l r^l, t^u r^u] \quad \forall\, \hat{t}, \hat{r} \in D[0, 1].$$

Routine calculations shows that $\hat{\lambda}$ is an interval t-norm generalized fuzzy bi ideal, however it is not an interval t-norm fuzzy bi ideal as:

$$[0.4, 0.5] = \hat{\lambda}(0) \ngeq \hat{\lambda}(a) = [0.5, 0.6].$$

The following is an immediate result of Theorem 2.13 in [13].

Lemma 7.3. *$\varphi \neq A \subseteq R$ is a generalized bi ideal iff $\hat{C}_A$ is interval t-norm fuzzy generalized bi ideal.*

The following is an immediate result of Lemma 2.12 in [13].

Lemma 7.4. *An interval valued fuzzy subset $\hat{\lambda}$ is an interval t-norm fuzzy generalized bi ideal iff $\hat{C}_R \hat{\lambda} \hat{C}_R \subseteq \hat{\lambda}$.*

8. Conclusion

In present discussion we gave a concept of Interval t-norm fuzzy interior ideal, Interval t-norm fuzzy bi ideal and generalized bi ideal, Interval t-norm fuzzy quasi ideal of hemirings w.r.t interval valued characteristic functions and discuss some properties and suitable counter examples of them.

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