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Abstract. We introduce the notions of weighted statistical convergence and strong weighted summability of order β for fuzzy Bernstein Stancu polynomials of difference operator of rough I-convergent of double sequence, and establish the relationship between them and also study the set of all fuzzy Bernstein Stancu polynomials of difference operator of rough I-limits of a double sequence spaces and relation between analyticness and fuzzy Bernstein Stancu polynomials of difference operator of rough I-core of a double sequence spaces. Finally, as an application, we appy our notion of weighted $\Delta^r(p,q)$-statistical of order β with a view of fuzzy Korovkin-type approximation theorem, discuss example to illustrate our approximation results.

2010 AMS Classification: 03E72, 40A35, 40G15, 46S40.

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1. Introduction

After the pioneering work of Zadeh [55], a huge number of research papers have appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in field of science and engineering. The idea of rough convergence was first introduced by Phu [45, 46, 47] in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r. Aytar [6] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [5] studied that the r-limit set of the sequence is equal to intersection of these sets and that r-core of the sequence is equal to the union of these sets. We refer [3, 4, 7, 13, 14, 15, 32, 33, 43] for details in the area of rough convergence. The notion of I-convergence of a double sequence spaces which is based on the structure of the ideal I of subsets of $\mathbb{N}^2$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

In this paper, we investigate some basic properties of rough I-convergence of a double sequence spaces of fuzzy in two dimensional matrix spaces which are not earlier. We study the set of all rough I-limits of a double sequence spaces of fuzzy and also the relation between analyticness and rough I-core of a double sequence spaces of fuzzy. Let K be a subset of the set of positive integers $\mathbb{N}^2$ and let us denote the set $K_{ij} = \{(m,n) \in K : m \leq i, n \leq j\}$. Then the natural density of K is given by

$$\delta(K) = \lim_{i,j\to\infty} \tfrac{|K_{ij}|}{ij},$$

where $|K_{ij}|$ denotes the number of elements in K_{ij}.

First applied the concept of (p,q)-calculus in approximation theory and introduced the (p,q)-analogue of Bernstein operators. Later, based on (p,q)-integers, some approximation results for Bernstein-Stancu operators, Bernstein-Kantorovich operators, (p,q)-Lorentz operators, Bleimann-Butzer and Hahn operators and Bernstein Shurer operators etc. Very recently, Khalid and Lobiyal [31] have given a nice application in computer-aided geometric design and applied these Bernstein basis for construction of (p,q)-Bezier curves and surfaces based on (p,q)-integers which is further generalization of q-Bezier curves and surfaces. Motivated by the above mentioned work on (p,q)-approximation and its application, in this paper we study statistical approximation properties of Bernstein-Stancu operators based on (p,q)-integers. Now we recall some basic definitions about (p,q)-integers. For any $u, v \in \mathbb{N}$, the (p,q)-integer $[uv]_{p,q}$ is defined by

$$[0]_{p,q} := 0 \text{ and } [uv]_{p,q} = \tfrac{p^{uv} - q^{uv}}{p-q} \text{ if } u, v \geq 1,$$

where $0 < q < p \leq 1$. The (p,q)-factorial is defined by

$$[0]_{p,q}! := 1 \text{ and } [uv]!_{p,q} = [1]!_{p,q}[6]!_{p,q}\Delta\Delta\Delta[uv]!_{p,q}, \text{ if } u, v \geq 1.$$

Also the (p,q)-binomial coefficient is defined by

$$\binom{u}{m}\binom{v}{n}_{p,q} = \tfrac{[uv]!_{p,q}}{[mn]!_{p,q}[(u-m)+(v-n)]!_{p,q}},$$

for all $u, v, m, n \in \mathbb{N}$ with $u \geq m, v \geq n$.

The formula for (p,q)-binomial expansion is as follows:

$$(ax+by)_{p,q}^{uv} = \sum_{m=0}^{u}\sum_{n=0}^{v} p^{\frac{(u-m)(u-m-1)+(v-n)(v-n-1)}{4}} q^{\frac{m(m-1)+n(n-1)+}{4}}$$
$$\binom{u}{m}\binom{v}{n}_{p,q} a^{(u-m)+(v-n)} b^{m+n} x^{(u-m)+(v-n)} y^{m+n},$$
$$(x+y)_{p,q}^{uv} = (x+y)(px+qy)\left(p^4x+q^4y\right)\cdots\left(p^{(u-1)+(v-1)}x+q^{(u-1)+(v-1)}y\right),$$

$(1-x)^{uv}_{p,q} = (1-x)(p-qx)\left(p^4-q^4x\right)\cdots\left(p^{(u-1)+(v-1)} - q^{(u-1)+(v-1)}x\right),$
and
$(x)^{mn}_{p,q} = x(px)\left(p^4x\right)\cdots\left(p^{(u-1)+(v-1)}x\right) = p^{\frac{m(m-1)+n(n-1)}{4}}.$
The Bernstein operator of order $r.s$ is given by
$B_{rs}(f,x) = \sum\limits_{m=0}^{r}\sum\limits_{n=0}^{s} f\left(\frac{mn}{rs}\right)\binom{r}{m}\binom{s}{n} x^{m+n}(1-x)^{(m-r)+(n-s)},$
where f is a continuous (real or complex valued) function defined on $[0,1]$.

The (p,q)-Bernstein operators are defined as follows:

$$B^{(m,n)}_{rs,p,q}(f,x) = \frac{1}{p^{\frac{r(r-1)+s(s-1)}{4}}}\sum_{m=0}^{r}\sum_{n=0}^{s}\binom{r}{m}\binom{s}{n} p^{\frac{m(m-1)+n(n-1)}{4}} x^{m+n}$$

$$\times \prod_{u=0}^{(r-m-1)+(s-n-1)}(p^u - q^u x) f\left(\frac{[mn]_{p,q}}{p^{(m-r)+(n-s)}[rs]_{p,q}}\right), x\in[0,1]. \tag{1.1}$$

Also, we have
$(1-x)^{rs}_{p,q} = \sum\limits_{m=0}^{r}\sum\limits_{n=0}^{s}(-1)^{m+n} p^{\frac{(r-m)(r-m-1)+(s-n)(s-n-1)}{4}} q^{\frac{m(m-1)+n(n-1)}{4}}\binom{r}{m}\binom{s}{n}x^{m+n}.$
(p,q)-Bernstein Stancu operators are defined as follows:

$$S^{(m,n)}_{rs,p,q}(f,x) = \frac{1}{p^{\frac{r(r-1)+s(s-1)}{4}}}\sum_{m=0}^{r}\sum_{n=0}^{s}\binom{r}{m}\binom{s}{n} p^{\frac{m(m-1)+n(n-1)}{4}} x^{m+n}$$

$$\times \prod_{u=0}^{(r-m-1)+(s-n-1)}(p^u - q^u x) f\left(\frac{p^{(r-m)+(s-n)}[mn]_{p,q}+\eta}{[rs]_{p,q}+\mu}\right), x\in[0,1]. \tag{1.2}$$

Note that for $\eta=\mu=0, (p,q)$-Bernstein Stancu operators given by (1.2) reduces into (p,q)-Bernstein operators. Also for $p=1,(p,q)$-Bernstein Stancu operators given by (1.1) turn out to be q-Bernstein Stancu operators.

Throughout the paper, $\mathbb{R}^2$ denotes the real two dimensional space with metric d. Consider a double sequence of Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,x)\right)$ such that $\left(S^{(m,n)}_{rs,p,q}(f,x)\right)\in\mathbb{R}, m,n\in\mathbb{N}$.

Let f be a continuous function defined on the closed interval $[0,1]$. A double sequence of Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,x)\right)$ is said to be statistically convergent to $0\in\mathbb{R}$, written as $st_2 - \lim S^{(m,n)}_{rs,p,q}(f,x) = f(x)$, provided that the set

$$K_\epsilon := \left\{(m,n)\in\mathbb{N}^2 : \left|S^{(m,n)}_{rs,p,q}(f,x) - (f,x)\right| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon>0$. In this case, 0 is called the statistical limit of the double sequence of Bernstein Stancu polynomials. i.e., $\delta_2(K_\epsilon)=0$. That is,

$$\lim_{r,s\to\infty}\frac{1}{rs}\left|\left\{m\leq r, n\leq s : \left|S^{(m,n)}_{rs,p,q}(f,x)-(f,x)\right|\geq\epsilon\right\}\right| = 0.$$

In this case, we write $\delta_2 - \lim S^{(m,n)}_{rs,p,q}(f,x) = (f,x)$ or $S^{(m,n)}_{rs,p,q}(f,x)\xrightarrow{st_2}(f,x)$. We denote χ_A-the characteristic function of $A\subset\mathbb{N}$. A subset A of $\mathbb{N}^2$ is said to have asymptotic density $d(A)$, if

$$d(A) = \lim_{i,j\to\infty} \frac{1}{ij} \sum_{m=1}^{i} \sum_{n=1}^{j} \chi_A(K).$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on. Throughout w, Γ and Λ denote the classes of all, entire and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication. Some initial works on double sequence spaces found in Bromwich [9]. Later on, this notion was investigated by Hardy [22], Moricz [39], Moricz and Rhoades [40], Basarir and Solankan [10], Turkmenoglu [54], Kamthan and Gupta [30] and many others.
We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : \sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : P - \lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \, for\, some\, l \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : P - \lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \quad \text{and} \quad \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t),$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $P - \lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Çolak [17, 18] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma$-duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [56] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [41] and Tripathy [49] have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Basar [1] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the α-duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Basar and Sever [8] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Subramanian and Misra [48] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [42] as an extension of the definition of strongly Cesàro summable sequences. Cannor [11] further extended this definition to a definition of strong A-summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong A-summability with respect to a modulus, and A-statistical convergence. In [44] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [16] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton [19, 20, 21]. We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p. \tag{1.3}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent, if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$.

A sequence $x = (x_{mn})$is said to be double analytic, if $\sup_{m,n} |x_{mn}|^{1/m+n} < \infty$.

The vector space of all double analytic sequences will be denoted by Λ^2.

A sequence $x = (x_{mn})$ is called double gai sequence, if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$.

The double gai sequences will be denoted by χ^2. Let $\phi = \{finite\ sequences\}$.

A fuzzy number X is a fuzzy subset of the real $\mathbb{R}^2$, which is normal fuzzy convex, upper semi-continuous, and the X^0 is bounded where $X^0; = cl\left\{x \in \mathbb{R}^2 : X(x) > 0\right\}$ and cl is the closure operator. These properties imply that for each $\alpha \in (0, 1]$, the α-level set X^α defined by

$$X^\alpha = \left\{x \in \mathbb{R}^2 : X(x) \geq \alpha\right\} = \left[\underline{X}^\alpha, \overline{X}^\alpha\right]$$

is a non empty compact convex subset of $\mathbb{R}^2$.

The supremum metric d on the set $L\left(\mathbb{R}^2\right)$ is defined by

$$d(X, Y) = \sup_{\alpha \in [0,1]} \max\left(\left|\underline{X}^\alpha - \underline{Y}^\alpha\right|, \left|\overline{X}^\alpha - \overline{Y}^\alpha\right|\right).$$

Now, given $X, Y \in L\left(\mathbb{R}^2\right)$, we define $X \leq Y$, if $\underline{X}^\alpha \leq \underline{Y}^\alpha$ and $\overline{X}^\alpha \leq \overline{Y}^\alpha$, for each $\alpha \in [0, 1]$.
We write $X \leq Y$ if $X^\alpha \leq Y^\alpha$ and there exists an $\alpha_0 \in [0, 1]$ such that $\underline{X}^{\alpha_0} \leq \underline{Y}^{\alpha_0}$ or $\overline{X}^{\alpha_0} \leq \overline{Y}^{\alpha_0}$.

A subset E of $L\left(\mathbb{R}^2\right)$ is said to be bounded above, if there exists a fuzzy number μ, called an upper bound of E, such that $X \leq \mu$ for every $X \in E$. μ is called the least upper bound of E, if μ is an upper bound and $\mu \leq \mu'$ for all upper bounds μ'.

A lower bound and the greatest lower bound are defined similarly. E is said to be bounded, if it is both bounded above and below.

The notions of least upper bound and the greatest lower bound have been defined only for bounded sets of fuzzy numbers. If the set $E \subset L\left(\mathbb{R}^2\right)$ is bounded then its supremum and infimum exist.

The limit infimum and limit supremum of a double sequence spaces (X_{mn}) is defined by

$$\liminf_{m,n\to\infty} X_{mn} := \inf\ A_X$$
$$\limsup_{m,n\to\infty} X_{mn} := \inf\ B_X,$$

where
$A_X := \left\{\mu \in L\left(\mathbb{R}^2\right):\ \left\{(m,n)\in\mathbb{N}^2 : X_{mn} < \mu\right\}\ \text{is infinite}\right\}$,
$B_X := \left\{\mu \in L\left(\mathbb{R}^2\right): \left\{(m,n)\in\mathbb{N}^2 : X_{mn} > \mu\right\}\ \text{is infinite}\right\}$.

Now, given two fuzzy numbers $X, Y \in L\left(\mathbb{R}^2\right)$, we define their sum as $Z = X + Y$, where $\underline{Z}^\alpha := \underline{X}^\alpha + \underline{Y}^\alpha$ and $\overline{Z}^\alpha := \overline{X}^\alpha + \overline{Y}^\alpha$ for all $\alpha \in [0,1]$.

To any real number $a \in \mathbb{R}^2$, we can assign a fuzzy number $a_1 \in L\left(\mathbb{R}^2\right)$, which is defied by

$$a_1(x) = \begin{cases} 1, & \text{if } x = a, \\ 0, & \text{otherwise} \end{cases}$$

An order interval in $L\left(\mathbb{R}^2\right)$ is defined by $[X,Y] := \left\{Z \in L\left(\mathbb{R}^2\right) : X \le Z \le Y\right\}$, where $X, Y \in L\left(\mathbb{R}^3\right)$.

A set E of fuzzy numbers is called convex, if $\lambda\mu_1 + (1-\lambda)\mu_2 \in E$ for all $\lambda \in [0,1]$ and $\mu_1, \mu_2 \in E$.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [34] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_∞ denote the classes of convergent,null and bounded scalar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \le p \le \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k\ge 1}|\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p} (1 \le p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\},$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

The difference triple sequence space was introduced by Debnath and Debnath [12] and is defined as

$\Delta x_{mnk} = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} + x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1}$ and $\Delta^0 x_{mnk} = \langle x_{mnk}\rangle$.

Let $n \in \mathbb{N}$ and X be a real vector space of dimension m, where $n \le m$. A real valued function $d_p(x_1,\ldots,x_n) = \|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ on X satisfying the following four conditions:

(i) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p = 0$ if and and only if $d_1(x_1,0),\ldots,d_n(x_n,0)$ are linearly dependent,

(ii) $\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p$ is invariant under permutation,

(iii) $\|(\alpha d_1(x_1,0),\ldots,\alpha d_n(x_n,0))\|_p = |\alpha|\,\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_p, \alpha\in\mathbb{R}$,

(iv) $d_p\left((x_1,y_1),(x_2,y_2)\cdots(x_n,y_n)\right) = \left(d_X(x_1,x_2,\cdots x_n)^p + d_Y(y_1,y_2,\cdots y_n)^p\right)^{1/p}$, for $1\le p<\infty$, (or)

(v) $d\left((x_1,y_1),(x_2,y_2),\cdots(x_n,y_n)\right) := \sup\left\{d_X(x_1,x_2,\cdots x_n), d_Y(y_1,y_2,\cdots y_n)\right\}$, for $x_1,x_2,\cdots x_n\in X, y_1,y_2,\cdots y_n\in Y$

is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X=\mathbb{R}$ equipped with the following Euclidean metric in the product space is the p-norm:

$$\|(d_1(x_1,0),\ldots,d_n(x_n,0))\|_E = sup\left(|det(d_{mn}(x_{mn},0))|\right) =$$
$$sup\left(\begin{vmatrix} d_{11}(x_{11},0) & d_{12}(x_{12},0) & \ldots & d_{1n}(x_{1n},0)\\ d_{21}(x_{21},0) & d_{22}(x_{22},0) & \ldots & d_{2n}(x_{1n},0)\\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1},0) & d_{n2}(x_{n2},0) & \ldots & d_{nn}(x_{nn},0)\end{vmatrix}\right)$$

where $x_i=(x_{i1},\cdots x_{in})\in\mathbb{R}^n$, for each $i=1,2,\cdots n$.

If every Cauchy sequence in X converges to some $L\in X$, then X is said to be complete with respect to the p-metric. Any complete $p-$ metric space is said to be p-Banach metric space.

The notion of ideal convergence was introduced first by Kostyrko et al. [36] as a generalization of statistical convergence which was further studied in topological spaces by Kumar and Kumar [37] and also more applications of ideals can be deals with various authors for instance [23, 24, 25, 26, 27, 28, 29, 38, 50, 51, 52, 53].

1.1. **Weighted statistical convergence in double difference operator.** Let $b=(b_{mn})$ be a double sequence of non-negative real numbers such that $\liminf\limits_{mn} b_{mn} > 0$ and

$$\tau_{pq}(uv) = \sum_m \sum_{n\in[p_{uv}q_{uv}]} b_{mn}; u,v\in\mathbb{N}.$$

The $\liminf b_{mn}$ does not exist if the weighted double sequence (b_{mn}) properly diverges to $+\infty$. So the weighted statistical convergence definition is not well defined when the weighted double sequence (b_{mn}), properly diverges to $+\infty$. Therefore, throughout the article we consider $b=(b_{mn})$ is a double sequence of non-negative real numbers such that $b_{mn}>\gamma$ for all $m,n\in\mathbb{N}$, where γ is a positive real number.

Let γ be a subset of $\mathbb{N}$ and also let $0<\beta\le 1$. The weighted pq-density of order β shortly $\delta^{\beta}_{\bar{N}(pq)}$-density, of V is defined by $\delta^{\beta}_{\bar{N}(pq)}(p,q)(V) = \lim\limits_{u,v}\frac{|V_{\tau_{pq}(uv)}|}{\tau^{\beta}_{pq}(uv)}$ in case the above limit exists, where $V_{\tau_{pq}(uv)} = \{m,n\le\tau_{pq}(uv) : m,n\in V\}$.

2. Definitions and preliminaries

Definition 2.1. Let f be a continuous function defined on the closed interval $[0,1]$. A rough double sequence of fuzzy Bernstein stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ is said to be weighted $\Delta^r(p,q)$-statistically convergent of order β to (f,X) denoted by

$S_{rs,p,q}^{(m,n)}(f,X) \rightarrow^{st\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)-limX} (f,X)$, if for any $\epsilon > 0$ we have $b_{mn} d(A(\epsilon)) = 0$, where

$$A(\epsilon) = \delta_{\bar{N}(p,q)^{\beta}}\left(\left\{(r,s) \in \mathbb{N}^2 : b_{mn}\left|S_{rs,p,q}^{(m,n)}(\Delta^r f, X) - (f,X)\right| \geq r+\epsilon\right\}\right) = 0.$$

or

$$\lim_{u,v} \frac{1}{\tau_{pq}^{\beta}(uv)}\left|\left\{(r,s) \in \mathbb{N}^2 : b_{mn}\left|S_{rs,p,q}^{(m,n)}(\Delta^r f, X) - (f,X)\right| \geq r+\epsilon\right\}\right| = 0.$$

Note that $\beta \in (0,1)$.

Definition 2.2. Let f be a continuous function defined on the closed interval $[0,1]$. A rough double sequence of fuzzy Bernstein stancu polynomials $\left(S_{rs,p,q}^{(m,n)}(f,X)\right)$ is said to be strongly weighted $\Delta^r(p,q)$-statistically convergent of order β to (f,X) denoted by

$S_{rs,p,q}^{(m,n)}(f,X) \rightarrow^{st\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)-limX} (f,X)$, if for any $\epsilon > 0$ we have $b_{mn} d(A(\epsilon)) = 0$, where

$$A(\epsilon) = \delta_{\bar{N}(p,q)^{\beta}}\left(\left\{(r,s) \in \mathbb{N}^2 : b_{mn}\left|S_{rs,p,q}(\Delta^r f, X) - (f,X)\right|\right\}\right) = 0.$$

or

$$\lim_{u,v} \frac{1}{\tau_{pq}^{\beta}(uv)}\left|\left\{(r,s) \in \mathbb{N}^2 : b_{mn}\left|S_{rs,p,q}(\Delta^r f, X) - (f,X)\right|\right\}\right| = 0.$$

Definition 2.3. Let f be a continuous function defined on the closed interval $[0,1]$. A rough double sequence of fuzzy Bernstein stancu polynomials $\left(S_{rs,p,q}^{(m,n)}(f,X)\right)$ is said to be weighted $\Delta^r(p,q)$-statistically convergent of order β to (f,X) denoted by $S_{rs,p,q}^{(m,n)}(f,X) \rightarrow^{st\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)-\lim X} (f,X)$, provided that the set

$$\delta_{\bar{N}(p,q)^{\beta}}\left(\left\{(r,s) \in \mathbb{N}^2 : b_{mn}\left|S_{rs,p,q}^{(m,n)}(\Delta^r f, X) - (f,X)\right| \geq r+\epsilon\right\}\right),$$

has natural density zero for every $\epsilon > 0$. In this case, $(\Delta^r f, X)$ is called the statistical limit of the sequence of fuzzy Berstein Stancu polynomials.

Definition 2.4. Let f be a continuous function defined on the closed interval $[0,1]$. A rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S_{rs,p,q}^{(m,n)}(f,X)\right)$ is said to be weighted $\Delta^r(p,q)$-statistically convergent of order β to (f,X) denoted by $S_{rs,p,q}^{(m,n)}(f,X) \rightarrow^{\beta\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)} (f,X)$, if for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for all $r,s \geq N_\epsilon$ we have

$$\delta_{\bar{N}(p,q)^{\beta}}\left(\left\{(r,s) \in \mathbb{N}^2 : b_{mn}\left|S_{rs,p,q}^{(m,n)}(\Delta^r f, X) - (f,X)\right| < r+\epsilon\right\}\right).$$

In this case, $S_{rs,p,q}^{(m,n)}(f,X)$ is called an $\beta-$ limit of (f,X).

Remark 2.5. We consider β-limit set $S_{rs,p,q}^{(m,n)}(f,X)$ which is denoted by $\mathrm{LIM}^{\beta\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)} S_{rs,p,q}^{(m,n)}(f,X)$ and is defined by

$$\mathrm{LIM}^{\beta} S_{rs,p,q}^{(m,n)}(f,X) = \delta_{\bar{N}(p,q)^{\beta}}\left(\left\{f : S_{rs,p,q}^{(m,n)}(f,X) \rightarrow^{\beta\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)} (f,X)\right\}\right).$$

Definition 2.6. Let f be a continuous function defined on the closed interval $[0,1]$. A rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S_{rs,p,q}^{(m,n)}(f,X)\right)$ is said to be weighted $\beta\Delta^r(p,q)$-statistically convergent of order β to (f,X) denoted by

$S^{(m,n)}_{rs,p,q}(f,X) \rightarrow^{st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-\lim X} (f,X)$, if $\mathrm{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(\Delta^r f,X) \neq \phi$ and β is called a rough convergence degree of $S^{(m,n)}_{rs,p,q}(\Delta^r f,X)$. If $\beta = 0$ then it is ordinary convergence of double sequence of fuzzy Bernstein Stancu polynomials.

Definition 2.7. Let f be a continuous function defined on the closed interval $[0,1]$. A rough double sequence of fuzzy Bernstein stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ is said to be weighted $\beta\Delta^r(p,q)$-statistically convergent of order β to (f,X) denoted by $S^{(m,n)}_{rs,p,q}(f,X) \rightarrow^{st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-\lim X} (f,X)$, in a metric space $(X,|.,.|)$ and β be a non-negative real number is said to be β-statistically convergent to (f,X), denoted by $S^{(m,n)}_{rs,p,q}(f,X) \rightarrow^{\beta-st_2\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} (f,X)$, if for any $\epsilon > 0$ we have $d(A(\epsilon)) = 0$, where

$$A(\epsilon) = \delta_{\bar{N}(p,q)^\beta}\left(\left\{(r,s) \in \mathbb{N}^2 : b_{mn}\left|S^{(m,n)}_{rs,p,q}(\Delta^r f,X) - (f,X)\right| \geq r+\epsilon\right\}\right).$$

In this case, (f,X) is called β-statistical limit of $S^{(m,n)}_{rs,p,q}(f,X)$. If $\beta = 0$ then it is ordinary statistical convergent of triple sequence of fuzzy Bernstein Stancu polynomials.

Definition 2.8. A class I of subsets of a nonempty set X is said to be an ideal in X, provided that

(i) $\phi \in I$,
(ii) $A,B \in I$ implies $A\bigcup B \in I$,
(iii) $A \in I, B \subset A$ implies $B \in I$.
I is called a nontrivial ideal, if $X \notin I$.

Definition 2.9. A nonempty class F of subsets of a nonempty set X is said to be a filter in X, provided that

(i) $\phi \in F$, (ii) $A,B \in F$ implies $A\bigcap B \in F$,
(iii) $A \in F, A \subset B$ implies $B \in F$.

Definition 2.10. Let I be a non trivial ideal in X, $X \neq \phi$. Then the class

$$F(I) = \{M \subset X : M = X\backslash A \; for \; some \; A \in I\}$$

is a filter on X, called the filter associated with I.

Definition 2.11. A non trivial ideal I in X is called admissible, if $\{x\} \in I$ for each $x \in X$.

Note. If I is an admissible ideal, then usual convergence in X implies I convergence in X.

Remark 2.12. If I is an admissible ideal, then usual rough convergence implies rough I-convergence.

Definition 2.13. Let f be a continuous function defined on the closed interval $[0,1]$. A rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ is said to be weighted $\Delta^r(p,q)$-statistically convergent of order β to (f,X) denoted by $S^{(m,n)}_{rs,p,q}(f,X) \rightarrow^{st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-limX} (f,X)$, in a metric space $(X,|.,.|)$ and β be a non-negative real number is said to be rough ideal convergent or βI-convergent to (f,X), denoted by $S^{(m,n)}_{rs,p,q}(f,X) \rightarrow^{\beta I\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} (f,X)$, if for any $\epsilon > 0$ we have

$$\delta_{\bar{N}(p,q)^\beta}\left(\left\{(r,s)\in\mathbb{N}^2 : b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-(f,X)\right|\geq\beta+\epsilon\right\}\right)\in I.$$

In this case, $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ is called $\beta I\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)$- convergent to (f,X) and a double sequence of fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ is called rough I-convergent to (f,X) with β as roughness of degree. If $\beta=0$, then it is ordinary I-convergent.

Note. Generally, Let f be a continuous function defined on the closed interval $[0,1]$. A rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ is said to be weighted $\Delta^r(p,q)$-statistically convergent of order β to (f,X) denoted by $S^{(m,n)}_{rs,p,q}(f,X)\rightarrow^{st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-\lim X}(f,X)$, is not I-convergent in usual sense and $\delta_{\bar{N}(p,q)^\beta}\left(b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-S^{(m,n)}_{rs,p,q}\left(\Delta^r g,X\right)\right|\leq\beta\right)$ for all $(r,s)\in\mathbb{N}^2$ or

$$\delta_{\bar{N}(p,q)^\beta}\left(\left\{(r,s)\in\mathbb{N}^2 : b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-S^{(m,n)}_{rs,p,q}\left(\Delta^r g,X\right)\right|\geq\beta\right\}\right)\in I,$$

for some $\beta>0$. Then the rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ is βI-convergent.

Note. It is clear that βI-limit of a sequence $S^{(m,n)}_{rs,p,q}(f,X)$ of fuzzy Bernstein Stancu polynomial is not necessarily unique.

Definition 2.14. Let f be a continuous function defined on the closed interval $[0,1]$. A rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ is said to be weighted $\Delta^r(p,q)$-statistically convergent of order β to (f,X) denoted by $S^{(m,n)}_{rs,p,q}(f,X)\rightarrow^{st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-\lim X}(f,X)$, consider βI-limit set of (f,X), which is denoted by

$$\begin{aligned}&I-\mathrm{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)}S^{(m,n)}_{rs,p,q}(f,X)\\&=\delta_{\bar{N}(p,q)^\beta}\left(b_{mn}\left\{f : S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)\rightarrow^{\beta I\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)}(f,X)\right\}\right).\end{aligned}$$

Then the rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ is said to be βI-convergent, if $I-\mathrm{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)}S^{(m,n)}_{rs,p,q}(f,X)\neq\phi$ and β is called a rough I-convergence degree of $S^{(m,n)}_{rs,p,q}(f,X)$.

Definition 2.15. Let f be a continuous function defined on the closed interval $[0,1]$. A rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ is said to be weighted $\Delta^r(p,q)$-statistically convergent of order β to (f,X) denoted by $S^{(m,n)}_{rs,p,q}(f,X)\rightarrow^{st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-limX}(f,X)$ I-analytic, if there exists a positive real number M such that

$$\delta_{\bar{N}(p,q)^\beta}\left(\left\{(r,s)\in\mathbb{N}^2 : b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)\right|^{1/m+n}\geq M\right\}\right)\in I.$$

Definition 2.16. Let f be a continuous function defined on the closed interval $[0,1]$. A rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$

is said to be weighted $\Delta^r(p,q)$-statistically convergent of order β to (f,X) denoted by $S^{(m,n)}_{rs,p,q}(f,X) \rightarrow^{st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-\lim X}(f,X)$, a point $L \in X$ is said to be an I-accumulation point and let f be a continuous function defined on the closed interval $[0,1]$. A fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ is a metric space (X,d) if and only if for each $\epsilon > 0$, the set

$$\delta_{\bar{N}(p,q)^\beta}\left(\left\{(r,s) \in \mathbb{N}^2 : d\left(S^{(m,n)}_{rs,p,q}(f,X),(f,X)\right)\right.\right.$$
$$\left.\left.= b_{mn}\left|S^{(m,n)}_{rs,p,q}(\Delta^r f,X)-(f,X)\right| < \epsilon\right\} \notin I\right).$$

We denote the set of all I-accumulation points of $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ by

$$I\left(\Gamma\left(S^{(m,n)}_{rs,p,q}(f,X)\right)\right).$$

Definition 2.17. Let f be a continuous function defined on the closed interval $[0,1]$. A rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ is said to be weighted $\Delta^r(p,q)$-statistically convergent of order β to (f,X), denoted by $S^{(m,n)}_{rs,p,q}(f,X) \rightarrow^{st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-\lim X}(f,X)$, is rough I-convergent, if

$$I - LIM^\beta S^{(m,n)}_{rs,p,q}(f,X) \neq \phi.$$

It is clear that if $I - \mathrm{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(\Delta^r f,X) \neq \phi$ for a rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ of real numbers, then we have

$$\begin{aligned} & I - \mathrm{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(f,X) \\ = & \left[I - \limsup S^{(m,n)}_{rs,p,q}(\Delta^r f,X) - \beta, I - \liminf S^{(m,n)}_{rs,p,q}(\Delta^r f,X) + \beta\right]. \end{aligned}$$

Definition 2.18. Let f be a continuous function defined on the closed interval $[0,1]$. A rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ is said to be weighted $\Delta^r(p,q)$-statistically convergent of order β to (f,X) denoted by $S^{(m,n)}_{rs,p,q}(f,X) \rightarrow^{st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-\lim X}(f,X)$ is rough $I-$ core $S^{(m,n)}_{rs,p,q}(\Delta^r f,X)$ is defined to the closed interval $[+\infty,-\infty]$.

3. Main results

Theorem 3.1. *Let f be a continuous function defined on the closed interval $[0,1]$. Let a rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ be weighted $\Delta^r(p,q)$-statistically convergent of order β to (f,X) and let $I \subset 2^{\mathbb{N}}$ be an admissible ideal. Then we have*

$$diam\left(I - LIM^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\right) \leq 2\beta.$$

In general, diam $\left(I - LIM^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}\left(\Delta^r f, X\right)\right)$ *has an upper bound.*

Proof. Assume that diam$\left(\text{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}\left(\Delta^r f, X\right)\right) \neq \phi$. Then

$$\exists\ S^{(m,n)}_{rs,p,q}\left(\Delta^r p, X\right), S^{(m,n)}_{rs,p,q}\left(\Delta^r q, X\right) \in LIM^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}\left(\Delta^r f, X\right) \ni:$$

$$b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r p, X\right) - S^{(m,n)}_{rs,p,q}\left(\Delta^r q, X\right)\right| > 2\beta.$$

Take $\epsilon \in \left(0, \frac{b_{mn}\left|S^{(m,n)}_{rs,p,q}(\Delta^r p, X) - S^{(m,n)}_{rs,p,q}(\Delta^r q, X)\right|}{2} - \beta\right)$. Then

$$S^{(m,n)}_{rs,p,q}\left(\Delta^r p, X\right), S^{(m,n)}_{rs,p,q}\left(\Delta^r q, X\right) \in I - \text{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}\left(\Delta^r f, X\right).$$

Thus we have $A_1(\epsilon) \in I$ and $A_2(\epsilon) \in I$, for every $\epsilon > 0$, where

$$A_1(\epsilon) = \left\{(u, v) \in \mathbb{N}^2 : b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f, X\right) - S^{(m,n)}_{rs,p,q}\left(\Delta^r p, X\right)\right| \geq r + \epsilon\right\}$$

and

$$A_2(\epsilon) = \left\{(u, v) \in \mathbb{N}^2 : b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f, X\right) - S^{(m,n)}_{rs,p,q}\left(\Delta^r q, X\right)\right| \geq r + \epsilon\right\}.$$

Using the properties $F(I)$, we get

$$\left(A_1(\epsilon)^c \bigcap A_2(\epsilon)^c\right) \in F(I).$$

So we write

$$\begin{aligned}
&b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r p, X\right) - S^{(m,n)}_{rs,p,q}\left(\Delta^r q, X\right)\right| \\
&\leq b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f, X\right) - S^{(m,n)}_{rs,p,q}\left(\Delta^r p, X\right)\right| + b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f, X\right) - S^{(m,n)}_{rs,p,q}\left(\Delta^r q, X\right)\right| \\
&< (\beta + \epsilon) + (\beta + \epsilon) < 2(\beta + \epsilon),
\end{aligned}$$

for all $(r, s) \in A_1(\epsilon)^c \bigcap A_2(\epsilon)^c$ which is a contradiction. Hence

$$\text{diam}\left(\text{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S_{rs,p,q}\left(\Delta^r f, X\right)\right) \leq 2\beta.$$

Now, consider a rough double sequence of fuzzy Bernstein Stancu polynomials of $\left(S^{(m,n)}_{rs,p,q}\left(\Delta^r f, X\right)\right)$ of real numbers such that $I - \lim_{r,s\to\infty} b_{mn} S^{(m,n)}_{rs,p,q}\left(\Delta^r f, X\right) = (f, X)$.

Let $\epsilon > 0$. Then we can write

$$\left\{(r, s) \in \mathbb{N}^2 : b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f, X\right) - (f, X)\right| \geq \epsilon\right\} \in .I$$

Thus we have

$$\begin{aligned}
&b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f, X\right) - S^{(m,n)}_{rs,p,q}\left(\Delta^r p, X\right)\right| \\
&\leq b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f, X\right) - (f, X)\right| + b_{mn}\left|(f, X) - S^{(m,n)}_{rs,p,q}\left(\Delta^r p, X\right)\right| \\
&\leq b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f, X\right) - (f, X)\right| + \beta \\
&\leq \beta + \epsilon,
\end{aligned}$$

for each $S^{(m,n)}_{rs,p,q}\left(\Delta^r p, X\right) \in \bar{S}_{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)}\left(\left(\Delta^r f, X\right)\right) :=$

$\left\{ S_{rs,p,q}^{(m,n)}\left(\Delta^r p, X\right) \in \mathbb{R} : b_{mn}\left| S_{rs,p,q}^{(m,n)}\left(\Delta^r p, X\right) - \left(\Delta^r f, X\right)\right| \leq \beta \right\}.$
So we get

$$b_{mn}\left| S_{rs,p,q}^{(m,n)}\left(\Delta^r f, X\right) - S_{rs,p,q}^{(m,n)}\left(\Delta^r p, X\right)\right| < \beta + \epsilon,$$

for each $(r,s) \in \left\{ (r,s) \in \mathbb{N}^2 : b_{mn}\left| S_{rs,p,q}^{(m,n)}\left(\Delta^r f, X\right) - (f, X)\right| < \epsilon \right\}$. Because the rough double sequence of fuzzy Bernstein Stancu polynomials of $S_{rs,p,q}^{(m,n)}\left(\Delta^r f, X\right)$ is I-convergent to (f, X), we have

$$\left\{ (r,s) \in \mathbb{N}^2 : b_{mn}\left| S_{rs,p,q}^{(m,n)}\left(\Delta^r f, X\right) - (f, X)\right| < \epsilon \right\} \in F(I).$$

Hence we get $p \in I - \mathrm{LIM}^{r\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)} S_{rs,p,q}^{(m,n)}\left(\Delta^r f, X\right)$. Consequently, we can write

$$I - \mathrm{LIM}^{\beta\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)} S_{rs,p,q}^{(m,n)}\left(\Delta^r f, X\right) = \bar{S}_{\beta\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)}\left(\left(\Delta^r f, X\right)\right). \tag{3.1}$$

Because $\mathrm{diam}\left(\bar{S}_{\beta\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)}\left(\left(\Delta^r f, X\right)\right)\right) = 2\beta$, this shows that in general, the upper bound 2β of the diameter of the set $I - \mathrm{LIM}^{\beta\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)} S_{rs,p,q}^{(m,n)}\left(\Delta^r f, X\right)$ is not lower bound. □

Theorem 3.2. *Let f be a continuous function defined on the closed interval $[0,1]$. Let a rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S_{rs,p,q}^{(m,n)}(f, X)\right)$ be weighted $\Delta^r(p,q)$-statistically convergent of order β to (f, X) and let $I \subset 2^{\mathbb{N}}$ be an admissible ideal. For an arbitrary $(f,c) \in I(\Gamma_X)$. Then we have*

$$b_{mn}\left| S_{rs,p,q}^{(m,n)}\left(\Delta^r f, X\right) - (f, c)\right| \leq \beta,$$

for all $S_{rs,p,q}^{(m,n)}\left(\Delta^r f, X\right) \in I - LIM^{\beta} S_{rs,p,q}^{(m,n)}(f, X)$.

Proof. Assume on the contrary that there exist a point $(f,c) \in I(\Gamma_X)$ and $S_{rs,p,q}^{(m,n)}(f, X)$ $\in I - \mathrm{LIM}^{\beta\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)} S_{rs,p,q}^{(m,n)}\left(\Delta^r f, X\right)$ such that $b_{mn}\left| S_{rs,p,q}^{(m,n)}\left(\Delta^r f, X\right) - (f, c)\right| > \beta$. Define $\epsilon := \frac{\left|S_{rs,p,q}^{(m,n)}(\Delta^r f, X) - (f,c)\right| - \beta}{2}$. Then

$$\left\{ (r,s) \in \mathbb{N}^2 : b_{mn}\left| S_{rs,p,q}^{(m,n)}\left(\Delta^r f, X\right) - (f, c)\right| < \epsilon \right\} \subseteq \left\{ (r,s) \in \mathbb{N}^2 : b_{mn}\left| S_{rs,p,q}^{(m,n)}\left(\Delta^r f, X\right) - (f, X)\right| \geq \beta + \epsilon \right\}. \tag{3.2}$$

Since $(f,c)\, c \in I(\Gamma_X)$, we have

$$\left\{ (r,s) \in \mathbb{N}^2 : b_{mn}\left| S_{rs,p,q}^{(m,n)}\left(\Delta^r f, X\right) - (f, c)\right| < \epsilon \right\} \notin I.$$

But from definition of I-convergence, since

$$\left\{ (r,s) \in \mathbb{N}^2 : b_{mn}\left| S_{rs,p,q}^{(m,n)}\left(\Delta^r f, X\right) - (f, X)\right| \geq \beta + \epsilon \right\} \in I,$$

so by (3.2) we have

$$\left\{(r,s)\in\mathbb{N}^2 : b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-(f,c)\right|<\epsilon\right\}\in I,$$

which contradicts the fact $(f,c)\in I\left(\Gamma_X\right)$. On the other hand, if $(f,c)\in I\left(\Gamma_X\right)$ i.e.,

$$\left\{(r,s)\in\mathbb{N}^2 : b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-(f,c)\right|<\epsilon\right\}\notin I,$$

then

$$\left\{(r,s)\in\mathbb{N}^2 : b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-(f,X)\right|\geq\beta+\epsilon\right\}\notin I,$$

which contradicts the fact $(f,X)\in I-\mathrm{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)}S_{rs,p,q}\left(\Delta^r f,X\right)$. □

Theorem 3.3. *Let f be a continuous function defined on the closed interval $[0,1]$. Let a rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ be weighted $\Delta^r(p,q)$-statistically convergent of order β to (f,X). Then*

$$\left(S^{(m,n)}_{rs,p,q}(f,X)\right)\to^{I\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)}(f,X)\Longleftrightarrow I\text{–}LIM^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)}S^{(m,n)}_{rs,p,q}(f,X)=\bar{S}_\beta\left((f,X)\right).$$

Proof. **Necessity:** It is obvious from Theorem 3.1 .

Sufficiency: Let $I-\mathrm{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)}S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)=\bar{S}_\beta\left((f,X)\right)(\neq\phi)$. Then clearly, the rough double sequence spaces of fuzzy Bernstein Stancu polynomials of $\left(S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)\right)$ is I-analytic. Suppose that (f,X) has another I-cluster point $\left(f',x\right)$ different from (f,X). Then

$$\left(\bar{f},X\right)=(f,X)+\frac{\beta}{\left|(f,X)-\left(f',X\right)\right|}\left((f,X)-\left(f',X\right)\right)$$

$$\left(\bar{f},X\right)-\left(f',X\right)=$$

$$(f,X)-\left(f',X\right)+\frac{\beta}{\left|(f,X)-\left(f',X\right)\right|}\left[\left((f,X)-\left(f',X\right)\right)-\left(\left(f',X\right)-\left(f',X\right)\right)\right]$$

$$\left|\left(\bar{f},X\right)-\left(f',X\right)\right|=\left|(f,X)-\left(f',X\right)\right|+\frac{\beta}{\left|(f,X)-\left(f',X\right)\right|}\left|(f,X)-\left(f',X\right)\right|$$

$$\left|\left(\bar{f},X\right)-\left(f',X\right)\right|=\left|(f,X)-\left(f',X\right)\right|+\beta>\beta.$$

Since $\left(f',X\right)\in I\left(\Gamma_X\right)$, by Theorem 4.2, $\left(\bar{f},X\right)\notin I\text{–}\mathrm{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)}S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)$. It is not possible as $\left|\left(\bar{f},X\right)-(f,X)\right|=\beta$ and $I-\mathrm{LIM}^{r\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)}S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)=\bar{S}_\beta\left((f,X)\right)$. Since (f,X) is the unique I-cluster point of (f,X). Thus

$$S_{rs,p,q}\left(\Delta^r f,X\right)\to^{I\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)}(f,X).$$

□

Corollary 3.4. *Let $(X,|.,.|)$ be a strictly convex spaces and let f be a continuous function defined on the closed interval $[0,1]$. Let a rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ be weighted $\Delta^r(p,q)$-statistically convergent of order β to (f,X). If there exists $y_1,y_2\in I\text{–}LIM^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)}S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)$*

such that $|y_1 - y_2| = 2\beta$, *then this rough double sequence of fuzzy* $(f, X) \rightarrow^{I\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} \frac{y_1+y_2}{2}$.

Proof. Omitted. ☐

Theorem 3.5. *Let f be a continuous function defined on the closed interval* $[0,1]$. *Suppose a rough double sequence of fuzzy Bernstein Stancu polynomials* $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ *is weighted* $\Delta^r(p,q)$*-statistically convergent of order β to* (f,X). *If* $I-LIM^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} \neq \phi$, *then* $I - \limsup S^{(m,n)}_{rs,p,q}(\Delta^r f, X)$ *and* $I - \liminf S^{(m,n)}_{rs,p,q}(\Delta^r f, X)$ *belong to the set* $I - LIM^{2\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(\Delta^r f, X)$.

Proof. We know that $I - \text{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(\Delta^r f, X) \neq \phi$, since a rough double sequence of fuzzy Bernstein Stancu polynomials of $\left(S^{(m,n)}_{rs,p,q}(\Delta^r f, X)\right)$ is I-analytic. The number $I - \liminf S^{(m,n)}_{rs,p,q}(\Delta^r f, X)$ is an I-cluster point of (f,X). Then we have

$$b_{mn}\left|(f,X) - I - \liminf S^{(m,n)}_{rs,p,q}(\Delta^r f, X)\right| \leq \beta \;\forall\; (f,X) \in I - \text{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)}(f,X).$$

Let $A = \left\{(r,s) \in \mathbb{N}^2 : b_{mn}\left|(f,X) - S^{(m,n)}_{rs,p,q}(\Delta^r f, X)\right| \geq \beta + \epsilon\right\}$. Now if $(r,s) \notin A$, then

$$\begin{aligned}
&b_{mn}\left|S^{(m,n)}_{rs,p,q}(\Delta^r f, X) - \left(I - \liminf S^{(m,n)}_{rs,p,q}(\Delta^r f, X)\right)\right| \\
&\leq b_{mn}\left|S^{(m,n)}_{rs,p,q}(\Delta^r f, X) - (f,X)\right| + b_{mn}\left|(f,X) - \left(I - \liminf S^{(m,n)}_{rs,p,q}(\Delta^r f, X)\right)\right| \\
&< 2\beta + \epsilon.
\end{aligned}$$

Thus $I - \liminf S^{(m,n)}_{rs,p,q}(\Delta^r f, X) \in I - \text{LIM}^{2\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(\Delta^r f, X)$.

Similarly it can be shown that $I - \limsup X_{mn} \in I - \text{LIM}^{2\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} X_{mn}$. ☐

Corollary 3.6. *Let f be a continuous function defined on the closed interval* $[0,1]$. *Let a rough double sequence of fuzzy Bernstein Stancu polynomials* $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ *be weighted* $\Delta^r(p,q)$*-statistically convergent of order β to* (f,X). *If*

$$I - LIM^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(f,X) \neq \phi,$$

then

$$I - core\{(f,X)\} \subseteq I - LIM^{2\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(\Delta^r f, X).$$

Proof. We have $I - \text{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(\Delta^r f, X) = \left[I - \lim\sup\; S^{(m,n)}_{rs,p,q}(\Delta^r f, X) - 2\beta, I - \lim\inf\; S^{(m,n)}_{rs,p,q}(\Delta^r f, X) + 2\beta\right]$. Then the result follows from Theorem 3.5. ☐

Theorem 3.7. *Let f be a continuous function defined on the closed interval* $[0,1]$. *Let a rough double sequence of fuzzy Bernstein Stancu polynomials* $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ *be weighted* $\Delta^r(p,q)$*-statistically convergent of order β to* (f,X). *Then*

$$dim\left(I - core\left\{S^{(m,n)}_{rs,p,q}(f,X)\right\}\right) \text{ of the set } I - core\left\{S^{(m,n)}_{rs,p,q}(\Delta^r f, X)\right\} = \beta$$

$\Longleftrightarrow I-core\{(f,X)\} = I-LIM^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(f,X).$

Proof. We have

$$diam\left(I-\text{core}\left\{S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\right\}\right)=\beta$$

$$\Longleftrightarrow \left(I-\limsup S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\right)-(I-\liminf X_{mn})=\beta \Longleftrightarrow I-\text{core}\{X_{mn}\}$$

$$=\left[I-\lim\ \inf X_{mn}, I-\limsup S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\right]$$

$$=\left[I-\limsup S^{(m,n)}_{rs,p,q}(\Delta^r f,X)-\beta, I-\liminf S^{(m,n)}_{rs,p,q}(\Delta^r f,X)+r\right]$$

$$=I-\text{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(\Delta^r f,X).$$

Also it is easy to see that

(i) $\beta > \text{diam}\left(I-\text{core}\left\{S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\right\}\right)$

$$\Longleftrightarrow I-\text{core}\left\{S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\right\}\subset I-\text{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(\Delta^r f,X),$$

(ii) $\beta < \text{diam}\left(I-\text{core}\left\{S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\right\}\right)$

$$\Longleftrightarrow I-\text{LIM}^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\subset I-\text{core}\left\{S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\right\}.$$ □

Theorem 3.8. *Let f be a continuous function defined on the closed interval $[0,1]$. Let a rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S^{(m,n)}_{rs,p,q}(f,X)\right)$ be weighted $\Delta^r(p,q)$-statistically convergent of order β to (f,X). If*

$$\bar{\beta}=\inf\left\{\beta\geq 0: I-LIM^{\beta\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(f,X)\neq\phi\right\},$$

then

$$\bar{\beta}=radius\left(I-core\left\{S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\right\}\right).$$

Proof. If the set $I-\text{core}\left\{S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\right\}$ is singleton, then radius $\left(I-\text{core}\left\{S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\right\}\right)=0$ and the rough double sequence of fuzzy Bernstein Stancu polynomials is $I-$ convergent, i.e., $I-\text{LIM}^{0\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\neq\phi$. Thus we get $\bar{\beta}=\text{radius}\left(I-\text{core}\left\{S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\right\}\right)=0.$

Now assume that the set $I-\text{core}\left\{S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\right\}$ is not a single ton. We can write $I-\text{core}\left\{S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\right\}=[a,b]$, where $a=I-\liminf S^{(m,n)}_{rs,p,q}(\Delta^r f,X)$ and $b=I-\limsup S^{(m,n)}_{rs,p,q}(\Delta^r f,X)$.

Now let us assume that $\bar{\beta}\neq\text{radius}\left(I-\text{core}\left\{S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\right\}\right)$.

If $\bar{\beta}<\text{radius}(I-\text{core}\{X_{mn}\})$, then define $\bar{\epsilon}=\frac{\frac{b-a}{2}-\bar{\beta}}{3}$. Now, the definition of $\bar{\beta}$ implies that

$$I-\text{LIM}^{\bar{\beta}+\bar{\epsilon}\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)} S^{(m,n)}_{rs,p,q}(\Delta^r f,X)\neq\phi,\text{ given }\epsilon>0\ \exists\, l\in\mathbb{R}: A=$$
$$\left\{(r,s)\in\mathbb{N}^2: b_{mn}\left|S^{(m,n)}_{rs,p,q}(\Delta^r f,X)-(f,X)\right|\geq\left(\bar{\beta}+\bar{\epsilon}\right)+\epsilon\right\}\in I.$$

Since $\bar{\beta}+\bar{\epsilon}<\frac{b-a}{2}$ which is a contradiction of the definition of a and b.

If $\bar{\beta} > radius\left(I-\text{core}\left\{S_{rs,p,q}^{(m,n)}\left(\Delta^{r}f,X\right)\right\}\right)$, then define $\bar{\epsilon}=\frac{\bar{\beta}-\frac{b-a}{2}}{2}$ and $\beta^{'}=\bar{\beta}-2\bar{\epsilon}$. It is clear that $0\leq\beta^{'}\leq\bar{\beta}$ and by the definitions of a and b, the number $\frac{b-a}{2}\in I-\text{LIM}^{\beta^{'}\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)}S_{rs,p,q}^{(m,n)}\left(\Delta^{r}f,X\right)$. Thus we get

$$\bar{\beta}\in\left\{\beta\geq 0: I-\text{LIM}^{\beta\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)}S_{rs,p,q}^{(m,n)}\left(\Delta^{r}f,X\right)\neq\phi\right\},$$

which contradicts the equality

$$\bar{\beta}=\inf\left\{\beta\geq 0: I-\text{LIM}^{\beta\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)}S_{rs,p,q}^{(m,n)}\left(\Delta^{r}f,X\right)\neq\phi\right\}\text{ as }\beta^{'}<\beta.$$

□

Corollary 3.9. *Let f be a continuous function defined on the closed interval $[0,1]$. Let a rough double sequence of fuzzy Bernstein Stancu polynomials $\left(S_{rs,p,q}^{(m,n)}(f,X)\right)$ be weighted $\Delta^{r}(p,q)$-statistically convergent of order β to (f,X). Then*

$$I-core\left\{S_{rs,p,q}^{(m,n)}\left(\Delta^{r}f,X\right)\right\}=I-LIM^{2\bar{\beta}\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)}S_{rs,p,q}^{(m,n)}\left(\Delta^{r}f,X\right).$$

Proof. It follows that Theorems 3.7 and 3.8. □

Theorem 3.10. *Let $S_{rs,p,q}^{(m,n)}\left(\Delta^{r}f,X\right)$ and $S_{rs,p,q}^{(m,n)}\left(\Delta^{r}g,X\right)$ be two rough double sequences of fuzzy numbers such that $st\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)-\lim X=X_0$ and $st\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)-\lim Y=Y_0$. Then*

(1) $st\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)-\lim aX_{mn}=aX_0$, *for* $a\in\mathbb{R}$,

(2) $st\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)-\lim\left(X_{mn}+Y_{mn}\right)=X_0+Y_0$.

Proof. (1) Assume that $st\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)-\lim X_{mn}=X_0$. Let $\epsilon>0$ be given. If $a=0$ then nothing to prove. Suppose that $a\neq 0$. We can easily find that

$$\begin{aligned}&\frac{1}{\tau_{pq}^{\beta}(uv)}\left|\left\{m,n\leq\tau_{pq(uv)}:b_{mn}\left|aS_{rs,p,q}^{(m,n)}\left(\Delta^{r}f,X\right)-(f,X)\right|\geq r+\epsilon\right\}\right|\\ &=\frac{1}{\tau_{pq}^{\beta}(uv)}\left|\left\{m,n\leq\tau_{pq(uvw)}:b_{mn}\left|a\right|\left|S_{rs,p,q}^{(m,n)}\left(\Delta^{r}f,X\right)-(f,X)\right|\geq r+\epsilon\right\}\right|\\ &=\frac{1}{\tau_{pq}^{\beta}(uv)}\left|\left\{m,n\leq\tau_{pq(uv)}:b_{mn}\left|S_{rs,p,q}^{(m,n)}\left(\Delta^{r}f,X\right)-(f,X)\right|\geq\frac{r+\epsilon}{|a|}\right\}\right|.\end{aligned}$$

(2) It is given that $st\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)-\lim X=X_0$ and $st\left(\Delta_{\bar{N}(p,q)}^{r,\beta}\right)-\lim Y=Y_0$. Then we can see that

$$\begin{aligned}&S_{rs,p,q}^{(m,n)}\left(\Delta^{r}f,X+Y\right)-\left(f,X_0+Y_0\right)\\ &\leq S_{rs,p,q}^{(m,n)}\left(\Delta^{r}f,X\right)-\left(f,X_0\right)+S_{rs,p,q}^{(m,n)}\left(\Delta^{r}f,Y\right)-\left(f,Y_0\right).\end{aligned}$$

Consequently, for given $\epsilon>0$, we obtain

$$\begin{aligned}&\frac{1}{\tau_{pq}^{\beta}(uv)}\left|\left\{m,n\leq\tau_{pq(uvw)}:b_{mn}\left|S_{rs,p,q}^{(m,n)}\left(\Delta^{r}f,X+Y\right)-\left(f,X_0+Y_0\right)\right|\geq r+\epsilon\right\}\right|\\ &\leq\frac{1}{\tau_{pq}^{\beta}(uv)}\left|\left\{m,n\leq\tau_{pq(uvw)}:b_{mn}\left|S_{rs,p,q}^{(m,n)}\left(\Delta^{r}f,X\right)-\left(f,X_0\right)\right|\geq\frac{r+\epsilon}{2}\right\}\right|\end{aligned}$$

$+\frac{1}{\tau^{\beta}_{pq}(uv)}\left|\left\{m,n\le\tau_{pq(uv)}:b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,Y\right)-\left(f,Y_0\right)\right|\ge\frac{r+\epsilon}{2}\right\}\right|.$

It follows that $st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-\lim\left(X_{mn}+Y_{mn}\right)=X_0+Y_0.$ □

Theorem 3.11. *Assume that* $S^{(m,n)}_{rs,p,q}\left(f,X\right)$ *is a rough double sequence of fuzzy numbers and let* $0<\beta\le1$. *Then*

(1) $S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)\to^{s\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-limX}\left(f,X_0\right)$

$\Longrightarrow S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)\to^{st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-\lim X}\left(f,X_0\right),$

(2) $X\in\Lambda^2\left(\Delta^r\right)$ *and* $S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)\to^{st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-\lim X}\left(f,X_0\right)$

$\Longrightarrow S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)\to^{s\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-\lim X}\left(f,X_0\right),$

where $\Lambda^2\left(\Delta^r\right)$ *is the set of all double analytic difference sequences of fuzzy numbers.*

Proof. Assume that $S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)\to^{s\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-\lim X}\left(f,X_0\right)$. For given $\epsilon>0$, we have

$\sum_m\sum_{n,[p_{uv}q_{uv}]}b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-\left(f,X_0\right)\right|=$

$\sum_m\sum_{n,[p_{uv}q_{uv}]}\Big(b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-\left(f,X_0\right)\right|\ge r+\epsilon\Big)b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-\left(f,X_0\right)\right|+$

$\sum_m\sum_{n,[p_{uv}q_{uv}]}\Big(b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-\left(f,X_0\right)\right|<r+\epsilon\Big)b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-\left(f,X_0\right)\right|$

so that

$\frac{1}{\tau^{\beta}_{pq}(uv)}\sum_m\sum_{n,[p_{uv}q_{uv}]}b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-\left(f,X_0\right)\right|\ge$

$\frac{1}{\tau^{\beta}_{pq}(uv)}\sum_m\sum_{n,[p_{uv}q_{uv}]}\Big(b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-\left(f,X_0\right)\right|\ge r+\epsilon\Big)r+\epsilon\ge$

$\frac{1}{\tau^{\beta}_{pq}(uv)}\left|\left\{m,n\le\tau_{pq(uv)}:b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-\left(f,X_0\right)\right|\ge r+\epsilon\right\}\right|r+\epsilon.$

Assuming limit $u,v\to\infty$ in the last inequality, we obtain

$$S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)\to^{st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-\lim X}\left(f,X_0\right).$$

(2) $S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)\to^{st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right)-\lim X}\left(f,X_0\right)$ and $X\in\Lambda^2\left(\Delta^r\right)$. Since $X\in\Lambda^2\left(\Delta^r\right)$, there exists constant $M>0$ such that

$$\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-\left(f,X_0\right)\right|\le M\left(\forall m,n\in\mathbb{N}\right).$$

For given $\epsilon>0$, we can write

$\frac{1}{\tau^{\beta}_{pq}(uv)}\sum_m\sum_{n,[p_{uv}q_{uv}]}b_{mnk}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-\left(f,X_0\right)\right|$

$=\frac{1}{\tau^{\beta}_{pq}(uv)}\sum_m\sum_{n,[p_{uv}q_{uv}]}\Big(b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-\left(f,X_0\right)\right|\ge r+\epsilon\Big)b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-\left(f,X_0\right)\right|$

$+\frac{1}{\tau^{\beta}_{pq}(uv)}\sum_m\sum_{n,[p_{uv}q_{uv}]}\Big(b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-\left(f,X_0\right)\right|<r+\epsilon\Big)b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-\left(f,X_0\right)\right|$

$\le\frac{1}{\tau^{\beta}_{pq}(uv)}\sum_m\sum_{n,[p_{uv}q_{uv}]}\Big(b_{mn}\left|S^{(m,n)}_{rs,p,q}\left(\Delta^r f,X\right)-\left(f,X_0\right)\right|\ge r+\epsilon\Big)M+$

$\frac{1}{\tau_{pq}^{\beta}(uv)} \sum_{m} \sum_{n,[p_{uv}q_{uv}]} \left(b_{mn} \left| S_{rs,p,q}^{(m,n)} (\Delta^r f, X) - (f, X_0) \right| < r + \epsilon \right) r + \epsilon$
$\leq \frac{1}{\tau_{pq}^{\beta}(uvw)} \left| \left\{ (m,n) \leq \tau_{pq(uv)} : b_{mn} \left| S_{rs,p,q}^{(m,n)} (\Delta^r f, X) - (f, X_0) \right| \geq r + \epsilon \right\} \right| r + \epsilon.$
which yields $S_{rs,p,q}^{(m,n)} (\Delta^r f, X) \to^{s\left(\Delta_{\tilde{N}(p,q)}^{r,\beta}\right) - \lim X} (f, X_0)$. □

4. Application to fuzzy Korovkin type theorems

Korovkin type approximation theorem in classical version was first introduced by Korovkin [35]. Anastassiou [2] discussed the following fuzzy Korovkin theorem for the text function $f_i(z) = z^i$ $(i = 0, 1, 2)$. A fuzzy number valued function $g : [a, b] \to L(\mathbb{R})$ is called fuzzy continuous at Y_0 in $[a, b]$ if and only if $d \left[S_{rs,p,q}^{(m,n)} (g, Y_{mn}), (g, Y_0) \right] \to \infty$ whenever $Y_{mn} \to Y_0$ as $m, n \to \infty$. We use the symbol $C_L [a, b]$ to denote the set of all fuzzy continuous function on $[a, b]$.

The operator $\Omega : C_L [a, b] \to C_L [a, b]$ is called fuzzy linear, if

$$\Omega [\alpha_1 \odot (g_1; Y) \oplus \alpha_2 \odot (g_2; Y)] = \alpha_1 \odot \Omega (g_1; Y) \oplus \alpha_2 \odot \Omega (g_2; Y),$$

for all $g_1, g_2 \in C_L [a, b]$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

Further, the operator Ω is said to be fuzzy positive linear, if it is fuzzy linear and satisfies the following condition:

$$\Omega (g_1; Y) \leq \Omega (g_2; Y) (g_1, g_2 \in C_L [a, b]),$$

for all $Y \in [a, b]$ with $g_1 \leq g_2$. Here, we suppose that $C [a, b]$ denotes the space of all continuous function on $[a, b]$ which is equipped with metric.

Theorem 4.1. *Consider a rough double sequence of fuzzy positive linear operators $S_{rs,p,q}^{(m,n)} (f, X_{mn})$ acting from $C_L [a, b]$ into itself. Suppose that there is a corresponding rough double sequence $\tilde{S}_{rs,p,q} (f, X_{mn})$ of positive linear operator from $C [a, b]$ into itself with the following condition:*
(4.1)
$$\left\{ S_{rs,p,q}^{(m,n)} (f, X) \right\}_{\alpha}^{\pm} = \tilde{S}_{rs,p,q} \left(f_{\alpha}^{\pm}, X \right) (\forall \alpha \in [0, 1], X \in [a, b], f \in C_L [a, b], m, n \in \mathbb{N}).$$
Then we have

$$\lim_{mn} d \left(\tilde{S}_{rs,p,q} (f_i, X_{mn}), (f_i, X) \right) = 0 (i = 0, 1, 2, \cdots).$$

Thus

$$\lim_{mn} d^* (S_{rs,p,q} (g, X), (g, X)) = 0 (\forall g \in C_L [a, b]).$$

Theorem 4.2. *Consider a rough double sequence of fuzzy positive linear operators $S_{rs,p,q}^{(m,n)} (f, X_{mnk})$ acting from $C_L [a, b]$ into itself. Suppose that there is a corresponding rough double sequence $\tilde{S}_{rs,p,q} (f, X_{mn})$ of positive linear operator from $C_L [a, b]$ into itself having condition (4.1):*

$$st \left(\Delta_{\tilde{N}(p,q)}^{r,\beta} \right) - \lim d \left(S_{rs,p,q}^{(m,n)} (\Delta^r f_i, X), (f_i, X) \right) = 0 (i = 0, 1, 2, \cdots). \tag{4.2}$$

Then we have

$$st \left(\Delta_{\tilde{N}(p,q)}^{r,\beta} \right) - \lim d^* \left(S_{rs,p,q}^{(m,n)} (\Delta^r g, X), (g, X) \right) = 0 (\forall g \in C_L [a, b]). \tag{4.3}$$

Proof. Assume that (4.2) holds and let $\alpha \in [0,1], X \in [a,b], g \in C_L[a,b]$. Since $g_\alpha^{\pm} \in C[a,b]$, for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| S_{rs,p,q}^{(m,n)}(g, X-Y) - (g, X-Y) \right| < r + \epsilon \, (\forall x \in [a,b]),$$

whenever $|X - Y| < \delta$. In addition, since g is fuzzy bounded, we have that

$$\left| S_{rs,p,q}^{(m,n)}\left(g_\alpha^{\pm}, Y\right) - \left(g_\alpha^{\pm}, Y\right) \right| \leq B_\alpha^{\pm} \text{ for all } a < Y < b.$$

It follows that

$$\left| S_{rs,p,q}^{(m,n)}\left(g_\alpha^{\pm}, X-Y\right) - \left(g_\alpha^{\pm}, X-Y\right) \right| \leq 2B_\alpha^{\pm} \text{ for all } a < X,\ Y < b.$$

Consequently, we may write

$$\left| S_{rs,p,q}^{(m,n)}\left(g_\alpha^{\pm}, X-Y\right) - \left(g_\alpha^{\pm}, X-Y\right) \right| \leq (r+\epsilon) + \frac{2B_\alpha^{\pm}}{\delta^2}(X-Y)^2 \, (\forall |X-Y| < \delta).$$

Since the operator $\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g, X_{mn})$ is positive and linear, by applying $\tilde{S}_{rs,p,q}^{(m,n)}(1,Y) - (1,Y)$ to the above inequality and additionally noting that Y is fixed so $S_{rs,p,q}(g_\alpha^{\pm}, Y)$ is constant, we have

$$\begin{aligned}
&-(r+\epsilon)\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g(1), Y) - \tfrac{2B_\alpha^{\pm}}{\delta^2}\tilde{S}_{rs,p,q}^{(m,n)}(g(X-Y), Y) \\
&< \tilde{S}_{rs,p,q}^{(m,n)}(g_\alpha^{\pm}(X), Y) - g_\alpha^{\pm}(Y)\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g(1), Y) \\
&< (r+\epsilon)\tilde{S}_{rs,p,q}^{(m,n)}(g(1), Y) + \tfrac{2B_\alpha^{\pm}}{\delta^2}\tilde{S}_{rs,p,q}^{(m,n)}\left(g(X-Y)^2, Y\right).
\end{aligned}$$

In order to estimate $\tilde{S}_{rs,p,q}^{(m,n)}\left(g(X-Y)^2, Y\right)$, we write

$$\begin{aligned}
&\tilde{S}_{rs,p,q}^{(m,n)}\left(g(X-Y)^2, Y\right) \\
&= \tilde{S}_{rs,p,q}^{(m,n)}\left(\Delta^r g\left(X^2\right), Y\right) - 2Y\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g(X), Y) + Y^2\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g(1), Y) \\
&= \left|\tilde{S}_{rs,p,q}^{(m,n)}\left(f\left(X^2\right), Y\right) - \left(g, Y^2\right)\right| - 2Y\left|\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g(X), Y)\right| \\
&\quad + Y^2\left|\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g(1), Y) - (g,1)\right|.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g_\alpha^{\pm}(X), Y) - (g_\alpha^{\pm}(Y), Y) \\
&= \tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g_\alpha^{\pm}(X), Y) - \tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g_\alpha^{\pm}(1), Y) + \left[\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g_\alpha^{\pm}(1), Y) - (g,1)\right] \\
&< (r+\epsilon)\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g(1), Y) + \tfrac{2B_\alpha^{\pm}}{\delta^2}\tilde{S}_{rs,p,q}^{(m,n)}\left(\Delta^r g(X-Y)^2, Y\right) \\
&\quad + \left[\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g(1), Y) - (g,1)\right] \\
&= (r+\epsilon)\tilde{S}_{rs,p,q}^{(m,n)}(g(1), Y) + \tfrac{2B_\alpha^{\pm}}{\delta^2} \\
&\left\{\begin{array}{c} \left[\tilde{S}_{rs,p,q}^{(m,n)}\left(\Delta^r g\left(X^2\right), Y\right) - \left(g, Y^2\right)\right] - 2Y\left[\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g(X), Y) - (g, Y)\right] + \\ Y^2\left[\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g(1), Y) - (g,1)\right] \end{array}\right\} + \\
&g_\alpha^{\pm}(Y)\left[\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g(1), Y) - (g,1)\right] \\
&= (r+\epsilon) + (r+\epsilon)\left[\tilde{S}_{rs,p,q}^{(m,n)}(g(1), Y) - (g,1)\right] + \tfrac{2B_\alpha^{\pm}}{\delta^2} \\
&\left\{\begin{array}{c} \left[\tilde{S}_{rs,p,q}^{(m,n)}\left(\Delta^r g\left(X^2\right), Y\right) - \left(g, Y^2\right)\right] - 2Y\left[\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g(X), Y) - (g, Y)\right] \\ +Y^2\left[\tilde{S}_{rs,p,q}^{(m,n)}(\Delta^r g(1), Y) - (g,1)\right] \end{array}\right\}
\end{aligned}$$

$+ g_\alpha^\pm (Y) \left[\tilde{S}_{rs,p,q}^{(m,n)} (\Delta^r g(1), Y) - (g,1)\right].$

So we obtain that

$$\tilde{S}_{rs,p,q}^{(m,n)} (\Delta^r g_\alpha^\pm (X), Y) - (g_\alpha^\pm, Y)$$
$$\leq (r+\epsilon) + \left((r+\epsilon) + B_\alpha^\pm + \tfrac{2cB_\alpha^\pm}{\delta^2}\right) \tilde{S}_{rs,p,q}^{(m,n)} (g(1), Y) - (g,1)$$
$$+ \tfrac{4cB_\alpha^\pm}{\delta^2} \left|\tilde{S}_{rs,p,q}^{(m,n)} (g(X), Y) - (g,Y)\right| + \tfrac{2B_\alpha^\pm}{\delta^2} \left|\tilde{S}_{rs,p,q}^{(m,n)} \left(g\left(X^2\right), Y\right) - \left(g, Y^2\right)\right|,$$

where $c = max\{|a|, |b|\}$.

Now we are aking supremum over $Y \in [a,b]$ in the last inequality. Then

(4.4)
$$\tilde{S}_{rs,p,q}^{(m,n)} \left(\Delta^r g_\alpha^\pm\right) - \left(g_\alpha^\pm\right) \leq (r+\epsilon) + M_\alpha^\pm (r+\epsilon)$$
$$\left\{\tilde{S}_{rs,p,q}^{(m,n)} (\Delta^r f, X_0) - (f, X_0) + \tilde{S}_{rs,p,q}^{(m,n)} (\Delta^r f, X_1) - (f, X_1) + \tilde{S}_{rs,p,q}^{(m,n)} (\Delta^r f, X_2) - (f, X_2)\right\},$$

where $M_\alpha^\pm (r+\epsilon) = \max\left\{(r+\epsilon) + B_\alpha^\pm + \frac{2cB_\alpha^\pm}{\delta^2}, \frac{4cB_\alpha^\pm}{\delta^2}, \frac{2cB_\alpha^\pm}{\delta^2}\right\}$.

From (4.1), we have

$$d^*\left(\tilde{S}_{rs,p,q}^{(m,n)} (g,g)\right)$$
$$= \sup_{Y\in[a,b]} d\left(\tilde{S}_{rs,p,q}^{(m,n)} (g,Y) - (g,Y)\right)$$
$$= \sup_{Y\in[a,b]} sup_{\alpha\in[0,1]} Max\left\{\left|\tilde{S}_{rs,p,q}^{(m,n)} (\Delta^r g_\alpha^-, Y) - (g_\alpha^-, Y)\right|, \left|\tilde{S}_{rs,p,q}^{(m,n)} (\Delta^r g_\alpha^+, Y) - (g_\alpha^+, Y)\right|\right\}$$

which yields that

(4.5)
$$d^*\left(\tilde{S}_{rs,p,q}^{(m,n)} (\Delta^r g, g)\right) = \sup_{\alpha\in[0,1]} \max\left\{\left|\tilde{S}_{rs,p,q}^{(m,n)} \left(\Delta^r g_\alpha^-\right) - \left(g_\alpha^-\right)\right|, \left|\tilde{S}_{rs,p,q}^{(m,n)} \left(\Delta^r g_\alpha^+\right) - \left(g_\alpha^+\right)\right|\right\}.$$

Using inequality (4.4) together with (4.5), we obtain

$$d^*\left(\tilde{S}_{rs,p,q}^{(m,n)} (\Delta^r g, g)\right)$$
$$\leq (r+\epsilon) + M(r+\epsilon)$$
$$\left\{\tilde{S}_{rs,p,q}^{(m,n)} (f, X_0) - (f, X_0) + \tilde{S}_{rs,p,q}^{(m,n)} (f, X_1) - (f, X_1) + \tilde{S}_{rs,p,q}^{(m,n)} (f, X_2) - (f, X_2)\right\},$$

where $M(r+\epsilon) = \sup_{\alpha\in[0,1]} \max\{M_\alpha^- (r+\epsilon), M_\alpha^+ (r+\epsilon)\}$.

Thus we have

(4.6)
$$b_{mnk} d^*\left(\tilde{S}_{rs,p,q}^{(m,n)} (\Delta^r g, g)\right) \leq (r+\epsilon) + M(r+\epsilon) b_{mn} \left|\tilde{S}_{rs,p,q}^{(m,n)} (f, X_0) - (f, X_0)\right| +$$
$$M(r+\epsilon) b_{mn} \left|\tilde{S}_{rs,p,q}^{(m,n)} (f, X_1) - (f, X_1)\right| + M(r+\epsilon) b_{mn} \left|\tilde{S}_{rs,p,q}^{(m,n)} (f, X_2) - (f, X_2)\right|$$

For given $u > 0$, let us choose $r, \epsilon > 0$ such that $0 < (r+\epsilon) < u$. Then, by setting

$$H = \left\{(m,n) \in \mathbb{N} : b_{mn} d^*\left(\tilde{S}_{rs,p,q}^{(m,n)} (\Delta^r g, g)\right) \geq u\right\}$$

and

$$H_{ij\ell} = \left\{(m,n) \in \mathbb{N} : b_{mn} \left|\tilde{S}_{rs,p,q}^{(m,n)} (\Delta^r f, Y_{ij}) - (f, Y_{ij}| \geq \frac{u - (r+\epsilon)}{4M(r+\epsilon)}\right|\right\} (i,j = 0,1,2),$$

we can easily find from inequality (4.6) that

$$H = \bigcup_{ij=0}^{2} H_{ij}.$$

So we have

$$\delta^{r,\beta}_{\bar{N}(p,q)}(H) \leq \sum_{i=0}^{2}\sum_{j=0}^{2} \delta^{r,\beta}_{\bar{N}(p,q)}(H_{ij}).$$

Finally, by using assumption (4.2), we obtain

$$st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right) - \lim d^{*}\left(\tilde{S}^{(m,n)}_{rs,p,q}(\Delta^{r} g, g)\right) = 0\,(\forall g \in C_L[a,b])$$

which is required condition (4.3). □

Example 4.3. Let us consider the rough double sequence of fuzzy Bernstein Stancu operators is defined as follows:

$$B^{L}_{mn}(g,Y) = \oplus^{m,n}_{r,s}\binom{m}{r}\binom{n}{s} Y^{r+s}(1-Y)^{(m-r)+(n-s)} \odot g\left(\frac{mn}{rs}\right),$$

where $g \in C_L[a,b]$, $Y \in [0,1]$, $m,\ n \in \mathbb{N}$.
Then we can write

$$\left\{B^{L}_{mn}\right\}^{\pm}_{\alpha} = \tilde{B}_{mn}\left(g^{\pm}_{\alpha}, Y\right)\sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s} Y^{r+s}(1-Y)^{(m-r)+(n-s)} g^{\pm}_{\alpha}\left(\frac{mn}{rs}\right)$$

for all $0 \leq \alpha \leq 1$ and $g^{\pm}_{\alpha} \in C[0,1]$. We now define the following rough double sequence of positive linear operator as

$$\left(\tilde{S}^{(m,n)}_{rs,p,q}(\Delta^{r} g, Y)\right) = \left(1 \oplus S^{(m,n)}_{rs,p,q}(\Delta^{r} f, Y) - (f,Y)\right) \odot B^{L}_{mn}(g,Y)$$

so that

$$\left(\tilde{S}^{(m,n)}_{rs,p,q}\left(g^{\pm}_{\alpha}, Y\right)\right) = \left(1 + S^{(m,n)}_{rs,p,q}(\Delta^{r} f, Y_{mn}) - (f,Y)\right)\tilde{B}_{mn}\left(g^{\pm}_{\alpha}, Y\right),$$

where the rough double sequence (Y_{mn}) of difference operators of fuzzy number is defined as follows:

$$\begin{aligned}
&\left(\tilde{S}^{(m,n)}_{rs,p,q}(\Delta^{r} f_{00}, Y)\right)\\
&= \left(1 + S^{(m,n)}_{rs,p,q}(f, Y_{mn}) - (f,Y), \tilde{S}^{(m,n)}_{rs,p,q}(\Delta^{r} f_{11}, Y)\right)\\
&= \left(1 + \left[S^{(m,n)}_{rs,p,q}(\Delta^{r} f, Y_{mn}) - (f,Y)\right)\right]
\end{aligned}$$

and

$$\left(\tilde{S}^{(m,n)}_{rs,p,q}(\Delta^{r} f_{22}, Y)\right) = \left(1 + S^{(m,n)}_{rs,p,q}(f, Y_{mn}) - (f,Y)\right)\left(Y^{2} + \frac{Y - Y^{2}}{mn}\right).$$

Since $st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right) - \lim\left(S^{(m,n)}_{rs,p,q}(\Delta^{r} f, X_{mn}) - (f,X)\right) = 0$, We have

$$st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right) - \lim\left|S^{(m,n)}_{rs,p,q}(\Delta^{r} f_{ij}, X) - (f_{ij}, X)\right| = 0\,(i,j = 0,1,2).$$

It follows from Theorem 4.2 that

$$st\left(\Delta^{r,\beta}_{\bar{N}(p,q)}\right) - limd^{*}\left(\tilde{S}_{rs,p,q}\left(\Delta^{r} g, g\right)\right) = 0,\ \forall g \in C_L\left[a,b\right].$$

By putting $r = 0$ in Definition 2.1, we gives the notion of weighted (p,q)-statistical convergence of order β, denoted by

$$st\left(\Delta^{\beta}_{\bar{N}(p,q)}\right) - \lim\left(S^{(m,n)}_{rs,p,q}\left(f, X_{mn}\right) - \left(f, X\right)\right) = X_0.$$

Then we have

$$\lim_{u,v}\frac{1}{\tau^{\beta}_{pq}\left(uvw\right)}\left|\left\{m,n \le \tau^{\beta}_{pq}\left(uv\right) : b_{mn} d\left|S^{(m,n)}_{rs,p,q}\left(\Delta^{r} f, X_{mn}\right) - \left(f, X_0\right)\right| \ge r+\epsilon\right\}\right| = 0.$$

Corollary 4.4. *Consider a rough double sequence of fuzzy positive linear operators* $\left(S^{(m,n)}_{rs,p,q}\left(\Delta^{r} f, X_{mn}\right) - \left(f, X\right)\right)$ *acting from* $C_L\left[a,b\right]$ *into itself having condition (4.1). Suppose that*

$$st\left(\Delta^{\beta}_{\bar{N}(p,q)}\right) - \lim\left|S^{(m,n)}_{rs,p,q}\left(f_{ij}, X\right) - \left(f_{ij}, X\right)\right| = 0.\left(i,j = 0,1,2\right). \tag{4.7}$$

Then we have

$$st\left(\Delta^{\beta}_{\bar{N}(p,q)}\right) - \lim d^{*}\left(S^{(m,n)}_{rs,p,q}\left(f_{ij}, X\right) - \left(f_{ij}, X\right)\right) = 0.\left(\forall g \in C_L\left[a,b\right]\right).$$

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References

[1] B. Altay and F. Baąar, Some new spaces of double sequences, J. Math. Anal. Appl. 309 (1) (2005) 70–90.
[2] G. A. Anastassiou, Fuzzy Mathematics: Approximation Theory, Springer-Verleg, Berlin 2010.
[3] M. Arslan and E. Dündar, Rough convergence in 2-normed spaces, Bull. Math. Anal. Appl. 10 (3) (2018) 1–9.
[4] M. Arslan and E. Dündar, On Rough Convergence in 2-Normed Spaces and Some Properties, Filomat 33 (16) (2019) 5077–5086.
[5] S. Aytar, The rough limit set and the core of a real sequence , Numer. Funct. Anal. Optimiz 29 (3-4) (2008) 283–290.
[6] S. Aytar, Rough statistical convergence, Numer. Funct. Anal. Optimiz 29 (3-4) (2008) 291–303.
[7] S. Aytar, Rough statistical cluster points, Filomat 31 (16) (2017) 5295–5304.
[8] F. Başar and Y. Sever, The space $\mathcal{L}_p$ of double sequences, Math. J. Okayama Univ. 51 (2009) 149–157.
[9] T.J.I.A. Bromwich, An introduction to the theory of infinite series, Macmillan and Co.Ltd. , New York 1965.
[10] M. Basarir and O. Solancan, On some double sequence spaces, J. Indian Acad. Math. 21 (2) (1999) 193–200.
[11] J. Cannor, On strong matrix summability with respect to a modulus and statistical convergence, Canad. Math. Bull. 32 (2) (1989) 194–198.
[12] S. Debnath and Jayanta Debnath, Some ideal convergent sequence spaces of fuzzy real numbers, Palestine J. Math. 3 (1) (2014) 27–32.
[13] E. Dündar and C. Çakan, Rough $\mathcal{I}$-convergence, Demonstratio Mathematica 47 (3) (2014) 638–651.

[14] E. Dündar and C. Çakan, Rough convergence of double sequences, Gulf J. Math. 2 (1) (2014) 45–51.
[15] E. Dündar, On Rough $\mathcal{I}_2$-Convergence of Double Sequences, Numer. Funct. Anal. Optimiz. 37 (4) (2016) 480–491.
[16] A. Esi, S. Araci and M. Acikgoz, Statistical Convergence of Bernstein Operators, Appl.Math.and Inf.Sci. 10 (6) (2016) 2083–2086.
[17] A. Gökhan and R. Colak, The double sequence spaces $c_2^P(p)$ and $c_2^{PB}(p)$, Appl. Math. Comput. 157 (2) (2004) 491–501.
[18] A. Gökhan and R. Colak, Double sequence spaces ℓ_2^∞, ibid. 160 (1) (2005) 147–153.
[19] H. J. Hamilton, Transformations of multiple sequences, Duke Math. J. 2 (1936) 29–60.
[20] H. J. Hamilton, A Generalization of multiple sequences transformation, Duke Math. J. 4 (1938) 343–358.
[21] H. J. Hamilton, Preservation of partial limits in multiple sequence transformations, Duke Math. J. 4 (1939) 293–297.
[22] G. H. Hardy, On the convergence of certain multiple series, Proc. Camb. Phil. Soc. 19 (1917) 86–95.
[23] B. Hazarika, Fuzzy real valued lacunary I-convergent sequences, Applied Math. Letters 25 (3) (2012) 466–470.
[24] B. Hazarika, On generalized difference ideal convergence in random 2-normed spaces, Filomat 26 (6) (2012) 1265–1274.
[25] B. Hazarika, On Fuzzy Real Valued I-Convergent Double Sequence Spaces defined by Musielak-Orlicz function, J. Intell. Fuzzy Systems 25 (1) (2013) 9–15.
[26] B. Hazarika, Lacunary difference ideal convergent sequence spaces of fuzzy numbers, J. Intell. Fuzzy Systems 25 (1) (2013) 157–166.
[27] B. Hazarika, On σ-uniform density and ideal convergent sequences of fuzzy real numbers, J. Intell. Fuzzy Systems 26 (2) (2014) 793–799.
[28] B. Hazarika and V.Kumar, Fuzzy real valued I-convergent double sequences in fuzzy normed spaces, J. Intell. Fuzzy Systems 26(5) (2014) 2323–2332.
[29] B. Hazarika and E.Savas, Some I-convergent lambda-summable difference sequence spaces of fuzzy real numbers defined by a sequence of Orlicz functions, Math. Comp. Modell. 54 (11-12) (2011) 2986–2998.
[30] P. K. Kamthan and M. Gupta, Sequence spaces and series, Lecture notes, Pure and Applied Mathematics, 65 Marcel Dekker, In c. New York 1981.
[31] K. Khan and D.K. Lobiyal, Algorithms and identities for (p, q)-Beźier curves via (p, q)-Blossom arxiv:160403220v1, 9 April 2016.
[32] Ö. Kişi and E. Dündar, Rough $\mathcal{I}_2$-lacunary statistical convergence of double sequences, J. Inequ. Appl. 2018:230 https://doi.org/10.1186/s13660-018-1831-7.
[33] Ö. Kişi and H. K. Ünal, Rough $\Delta\mathcal{I}_2$-Statistical Convergence of Double Sequences in Normed Linear Spaces, Bulletin of Mathematical Analysis and Applications 12 (1)(2020) 1–11.
[34] H. Kizmaz, On certain sequence spaces, Canad. Math.Bull. 24 (2)(1981) 169–176.
[35] P. P. Korovkin, Linear operators and approximation theory, Hindustan Publ. Corp. Delhi 1960.
[36] P. Kostyrko and T. Salat and W. Wilczynski, I-convergence, Real Anal. Exchange 26 (2) (2000-2001) 669–686.
[37] V. Kumar , On I and I^*-convergence of double sequences, Math. Communi. 12 (2007) 171–181.
[38] V. Kumar and K. Kumar, On the ideal convergence of sequences of fuzzy numbers, Inform. Sci. 178 (24) (2008) 4670–4678.
[39] F. Moricz, Extentions of the spaces c and c_0 from single to double sequences, Acta. Math. Hung. 57 (1-2) (1991) 129–136.
[40] F. Moricz and B. E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, Math. Proc. Camb. Phil. Soc. 104 (1988) 283–294.
[41] M. Mursaleen and O. H. H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl. 288 (1) (2003) 223–231.
[42] I. J. Maddox, Sequence spaces defined by a modulus, Math. Proc. Cambridge Philos. Soc. 100 (1) (1986) 161–166.

[43] Ö. Ölmez and S. Aytar, The relation between rough Wijsman convergence and asymptotic cones, Turkish J. Math. 40 (2016) 1349–1355.

[44] A. Pringsheim, Zurtheorie derzweifach unendlichen zahlenfolgen, Math. Ann. 53 (1900) 289–321.

[45] H. X. Phu, Rough convergence in normed linear spaces , Numer. Funct. Anal. Optimiz 22 (2001) 199–222.

[46] H. X. Phu, Rough continuity of linear operators, Numer. Funct. Anal. Optimiz 23 (2002) 139–146.

[47] H. X. Phu, Rough convergence in infinite dimensional normed spaces, Numer. Funct. Anal. Optimiz 24 (2003) 285–301.

[48] N. Subramanian and U. K. Misra, The semi normed space defined by a double gai sequence of modulus function, Fasciculi Math. 46. (2010).

[49] B. C. Tripathy, On statistically convergent double sequences, Tamkang J. Math. 34 (3) (2003) 231–237.

[50] B. C. Tripathy and A. J. Dutta, On I-acceleration convergence of sequences of fuzzy real numbers, Math. Modell. Analysis 17 (4) (2012) 549–557.

[51] B. C. Tripathy and B. Sarma, On I-convergent double sequences of fuzzy real numbers, Kyungpook Math. J. 52 (2) (2012) 189–200.

[52] B. C. Tripathy and M. Sen, On fuzzy I-convergent difference sequence space, J. Intel. Fuzzy Systems, 25 (3) (2013) 643–647.

[53] B. C. Tripathy, M. Sen and S. Nath, I-convergence in probabilistic n-normed space, Soft Comput. 16 (2012) 1021–1027.

[54] A. Turkmenoglu, Matrix transformation between some classes of double sequences, J. Inst. Math. Comp. Sci. Math. Ser. 12 (1) (1999) 23–31.

[55] L. A. Zadeh, Fuzzy Sets, Infor. Control 8 (1965) 338—353.

[56] M. Zeltser, Investigation of Double Sequence Spaces by Soft and Hard Analitical Methods, Dissertationes Mathematicae Universitatis Tartuensis 25, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu 2001.

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